ABSOLUTE BOUNDS ON THE NUMBER OF GENERATORS OF COHEN-MACAULAY IDEALS OF HEIGHT TWO

HANS SCHOUTENS

ABSTRACT. For a Noetherian local domain A, there exists an upper bound $N_\tau(A)$ on the minimal number of generators of any height two ideal $\mathfrak{a}$ for which $A/\mathfrak{a}$ is Cohen-Macaulay of type τ. If A contains an infinite field, then we may take $N_\tau(A) := (\tau + 1)e_{\text{hom}}(A)$, where $e_{\text{hom}}(A)$ is the homological multiplicity of A.

1. INTRODUCTION

In this paper, we are interested in finding upper bounds on $\mu_A(\mathfrak{a})$, the minimal number of generators of a height two Cohen-Macaulay ideal $\mathfrak{a}$ in a Noetherian local ring A (here $\mathfrak{a}$ is said to be *Cohen-Macaulay*, if $A/\mathfrak{a}$ is Cohen-Macaulay). The upper bounds that one finds in the literature often depend on invariants of the residue ring $A/\mathfrak{a}$, or are only valid if A is Cohen-Macaulay; see for instance [1, 5, 6, 7, 11, 12, 14, 15, 17, 19]. The goal of this paper is to remove the Cohen-Macaulay assumption on A and to provide absolute bounds, that is to say, bounds which only depend on A. Here are some previously known cases of absolute bounds. In [12], Noether Normalization is used to show that any prime ideal in a two-dimensional affine algebra A (that is to say, a two-dimensional finitely generated algebra over a field) is generated by at most $N(A)$ elements, where $N(A)$ only depends on the algebra. In [8], Gottlieb shows that an ideal $\mathfrak{a}$ for which $A/\mathfrak{a}$ has depth at least $\dim A - 1$ is generated by at most ρ elements, where ρ is the parameter degree of A (see below).

In this paper, we generalize Gottlieb's results to height two Cohen-Macaulay ideals. To state precise results, we need a definition. Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring. We call A *non-degenerate* if A has the same characteristic as any of its irreducible components of maximal dimension, that is to say, $\text{char}(A) = \text{char}(A/\mathfrak{p})$, for every prime $\mathfrak{p}$ of A such that $\dim(A/\mathfrak{p}) = d$. Note that this condition is void if A is equicharacteristic. In mixed characteristic, it means that A/pA has dimension $d - 1$, where p is the characteristic of $A/\mathfrak{m}$. In particular, any Noetherian local domain is non-degenerate. It is easy to see that the class of non-degenerate local rings is closed under completion. We say that an ideal I is *non-degenerate*, if A/I is non-degenerate. An $\mathfrak{m}$-primary ideal I is non-degenerate if, and only if, I contains $p := \text{char}(A/\mathfrak{m})$ if, and only if, A/I is equicharacteristic.

A *parameter ideal* I in A is an ideal generated by a (full) system of parameters. The minimal length of A/I where I runs over all parameter ideals will be called the *parameter degree* of A; if we only let I run over all non-degenerate parameter ideals, the resulting

minimum is called the *equi-parameter degree* of A. The motivation for introducing these notions comes from the following structure theorem due to Cohen.

Theorem 1.1 (Cohen Structure Theorem). *A complete Noetherian local ring A is non-degenerate if, and only if, there exists a finite extension $S \subseteq A$ with S a complete regular local ring. In fact, given any non-degenerate parameter ideal I of A, we may choose S in such way that $\mathfrak{n}A = I$, where $\mathfrak{n}$ is the maximal ideal of S.*

Proof. For the direct implication in the first statement, see [9, Theorem 29.4 and Remark] or [2, IX. Théorème 3]. For the converse, we only have to treat the case that A has mixed characteristic. Let d be the dimension of A and p the characteristic of its residue field. If $S \subseteq A$ is finite with S regular, then S/pS has dimension $d-1$, and hence so does A/pA by base change.

The last statement is clear from the proof given in [9] when A has equal characteristic. So assume A has characteristic zero and its residue field has characteristic p. By assumption, pA has height one and is contained in I. Hence we may choose $x_i \in I$ so that $I = (x_1, \ldots, x_d)A$ and $(p, x_2, \ldots, x_d)$ is a system of parameters in A, where d is the dimension of A. Let $V \subseteq A$ be a coefficient ring of A, that is to say, a complete unramified discrete valuation ring with the same residue field as A. By the proof in [9], the subring $S_0 := V[[x_2, \ldots, x_d]] \subseteq A$ is regular and A is finitely generated as an S_0-module. Let $S := S_0[x_1] \subseteq A$. By the proof of [9, Theorem 29.8], the extension $S_0 \subseteq S$ is Eisenstein, and hence by the same theorem, S is regular. Since the maximal ideal $\mathfrak{n}$ of S is generated by p and all the x_i, we get $\mathfrak{n}A = I$, as required. $\square$

Note that the non-degenerate parameter ideals in A are precisely the ideals of the form $\mathfrak{n}A$ with $\mathfrak{n}$ the maximal ideal of a complete regular subring over which A is finite. We can now state the main result of this paper (Theorem 2.1, Corollaries 2.2, 3.9 and 3.10 and the discussion in §3).

Theorem 1.2. *Let A be a Noetherian local ring and let $\tau \geq 1$. Assume that A is non-degenerate (e.g., A is a domain). If $\mathfrak{a}$ is a height two ideal of A such that $A/\mathfrak{a}$ is Cohen-Macaulay of type τ, then $\mathfrak{a}$ is generated by at most $(\tau+1)\epsilon$ elements, where ϵ can be taken to be the equi-parameter degree of A. Alternatively, we may take ϵ to be the parameter degree of A, in case A is equicharacteristic, or the homological multiplicity of A, in case A is equicharacteristic with infinite residue field, or the (usual) multiplicity of A, in case A is Cohen-Macaulay.*

In particular, the minimal number of generators of a height two Gorenstein ideal is at most 2ϵ.

The case when A is Cohen-Macaulay is well-known ([11, Chapter V, Theorem 3.2 and Corollary 3.3]) and is just added for comparison. Our bounds also improve the ones given in [18, Example 9.5.1]. For the proof of Theorem 1.2, we borrow a technique from [12], except that we replace their use of Noether Normalization by the Cohen Structure Theorem. We even get some estimates without assuming that $\mathfrak{a}$ is a Cohen-Macaulay ideal:

Theorem 1.3. *In a two-dimensional Noetherian local domain A of equi-parameter degree $\bar{\rho}$, every ideal $\mathfrak{a}$ is generated by at most $(\tau+1)\bar{\rho}$ elements, where τ is the type of $A/\mathfrak{a}$.*

Using the Forster-Swan Theorem, we obtain estimates in the global case as well: if A is d-dimensional Noetherian domain which is generated as a module by at most ϵ elements over some regular subring, then any height two Cohen-Macaulay ideal $\mathfrak{a}$ of A can be generated by at most $(\tau+1)\epsilon + d - 2$ elements, where τ is the maximum of the types of

$A_{\mathfrak{m}}/\mathfrak{a}A_{\mathfrak{m}}$, for $\mathfrak{m}$ running over all maximal ideals of A (with a possible exception when $\tau = \epsilon = 1$). In the last section, bounds for affine algebras are shown to be uniform, in the sense that the bounds only depend on the degree of the polynomials representing the affine algebra as a homomorphic image of a polynomial ring (see Theorem 5.1). Here are two special cases that follow from this analysis.

Theorem 1.4. *Let $Y \to X$ be a finite dominant map of degree $\epsilon > 1$ between affine d-dimensional schemes. If X has no singularities, then every codimension two Gorenstein subscheme W of Y is the (ideal-theoretic) intersection of at most $2\epsilon + d - 2$ hypersurfaces.*

Theorem 1.5. *Let C be a (reduced) Gorenstein curve in affine 4-space over an infinite field. If C lies on a quadratic hypersurface, then C is either a set-theoretic complete intersection, or otherwise, the (ideal-theoretic) intersection of exactly five hypersurfaces.*

Acknowledgement. This paper arose in part through several useful and stimulating conversations I had with W. Vasconcelos. I also want to thank the anonymous referee for pointing out an error in Proposition 4.3 and for drawing my attention to the work of Gottlieb.

2. Height Two Gorenstein Ideals

Theorem 2.1. *For a non-degenerate Noetherian local ring A, there is an upper bound on the number of generators of an arbitrary height two ideal $\mathfrak{a}$ for which $A/\mathfrak{a}$ is Gorenstein.*

Proof. Since neither height nor minimal number of generators is affected by taking a faithfully flat extension, we may assume that A is moreover complete (note that the completion of a non-degenerate ring is again non-degenerate). By Theorem 1.1, there exists a regular local subring $S \subseteq A$, such that A is module finite over S. In particular, there exists a surjective linear map

$$\varphi \colon S^N \twoheadrightarrow A. \tag{1}$$

It will suffice to bound the number of generators of $\mathfrak{a}$ viewed as an S-module. To this end, let $H := \varphi^{-1}(\mathfrak{a})$. In particular, we have an exact sequence

$$0 \to H \to S^N \to A/\mathfrak{a} \to 0. \tag{2}$$

Using for instance [3, Exercise 1.2.26] or [9, Exercise 16.7], we get that $A/\mathfrak{a}$ is a Cohen-Macaulay S-module of dimension $d - 2$, where d is the dimension of A. Therefore, by the Auslander-Buchsbaum Theorem, $A/\mathfrak{a}$ has projective dimension 2 as an S-module, and hence H has projective dimension one. Let

$$0 \to S^p \xrightarrow{\ f\ } S^q \to H \to 0 \tag{3}$$

be a minimal free S-resolution of H, so that H is minimally generated by q elements. Taking the (S-)dual of sequence (3), we get an exact sequence

$$S^q \xrightarrow{\ f^*\ } S^p \to \mathrm{Ext}^1_S(H, S) \to 0, \tag{4}$$

where f^* is the *transpose* of f, that is to say, if $\mathbb{A}$ is a matrix defining f, then the matrix transpose of $\mathbb{A}$ gives f^*. In particular, since we took (3) to be minimal, $\mathbb{A}$ has all its entries in the maximal ideal of S. Therefore, the same is true for f^*, so that by Nakayama's Lemma, $\mathrm{Ext}^1_S(H, S)$ is minimally generated by p elements.

Applying [3, Theorem 3.3.7.(b)] to the finite local homomorphism $S \to A/\mathfrak{a}$, we get that

$$\mathrm{Ext}^2_S(A/\mathfrak{a}, S) = \omega_{A/\mathfrak{a}},$$

where $\omega_{A/\mathfrak{a}}$ is the canonical module of $A/\mathfrak{a}$. However, since $A/\mathfrak{a}$ is Gorenstein, we have that $\omega_{A/\mathfrak{a}} \cong A/\mathfrak{a}$. On the other hand, taking the dual of the exact sequence (2) shows that $\mathrm{Ext}^1_S(H, S) \cong \mathrm{Ext}^2_S(A/\mathfrak{a}, S)$. In summary, we obtain an isomorphism

$$\mathrm{Ext}^1_S(H, S) \cong A/\mathfrak{a}.$$

Since this S-module is minimally generated by p elements, we get from (2) that $p \leq N$. Putting (2) and (3) together yields an exact sequence

$$0 \to S^p \to S^q \to S^N$$

from which it follows that $q \leq p + N$. Therefore, $q \leq 2N$, showing that H, and hence a fortiori $\mathfrak{a} = \varphi(H)A$, can be generated by at most $2N$ elements. $\square$

Corollary 2.2. *For a non-degenerate Noetherian local ring A and an arbitrary $\tau \geq 1$, there is an upper bound on the number of generators of an arbitrary height two ideal $\mathfrak{a}$ of A for which $A/\mathfrak{a}$ is Cohen-Macaulay of type τ.*

Proof. Analyzing the proof of Theorem 2.1, we see that the only place were we used that $A/\mathfrak{a}$ is Gorenstein, is to establish the isomorphism $\omega_{A/\mathfrak{a}} \cong A/\mathfrak{a}$. If $A/\mathfrak{a}$ is merely Cohen-Macaulay of type τ, then the canonical module $\omega_{A/\mathfrak{a}}$ is generated as an $A/\mathfrak{a}$-module by τ elements ([3, Proposition 3.3.11]). Therefore, there is an epimorphism $(A/\mathfrak{a})^\tau \twoheadrightarrow \omega_{A/\mathfrak{a}}$. If A is generated as an S-module by N elements, then this implies that $\omega_{A/\mathfrak{a}}$ is generated by at most τN elements as an S-module. Hence from (2) and (4) we get that $p \leq \tau N$ (notation as in that proof), so that $\mu_A(\mathfrak{a}) \leq q \leq p + N \leq (\tau + 1)N$. $\square$

3. NOETHER NORMALIZATION DEGREE

We mentioned in the introduction that it is well-known that one can take ϵ in Theorem 1.2 equal to the multiplicity of A, when A is Cohen-Macaulay. We now will investigate several generalizations of multiplicity which can play the role of ϵ in Theorem 1.2 in absence of the Cohen-Macaulay assumption.

Definition 3.1. We call the *Noether Normalization degree* of a Noetherian ring A the least possible value of $\mu_S(A)$, where S runs over all regular subrings of A (this includes the case that there is no such regular subring over which A is finite, in which case we set its Noether Normalization degree equal to ∞).

By the classical Noether Normalization Theorem, any finitely generated algebra over a field has finite Noether Normalization degree. By Theorem 1.1, a complete Noetherian local ring has finite Noether Normalization degree if, and only if, it is non-degenerate.

For the remainder of this section, $(A, \mathfrak{m})$ denotes a Noetherian local ring, with multiplicity e, parameter degree ρ, equi-parameter degree $\bar{\rho}$ and Noether Normalization degree s. Clearly $\rho \leq \bar{\rho}$, with equality when A is equicharacteristic, for then any system of parameters is non-degenerate. That this inequality can be strict is witnessed by the following example.

Example 3.2. Let $A := R/(X^3 - p^2)R$ with $R := \mathbb{Z}_p[[X]]$ and $\mathbb{Z}_p$ the ring of p-adic integers. Here the only non-degenerate parameter ideal is pA showing that $\bar{\rho} = 3$, whereas A/XA has length two (in fact $\rho = 2$ by the next lemma, as A is Cohen-Macaulay with $e = 2$). Note that in this example pA is not a reduction of the maximal ideal of A.

Lemma 3.3. *We have an inequality $e \leq \rho$. If $A/\mathfrak{m}$ is infinite, then $e = \rho$ if, and only if, A is Cohen-Macaulay.*

Proof. Let I be a parameter ideal such that A/I has length ρ. By [3, Corollary 4.6.11] (=positivity of the first partial Euler characteristic), the multiplicity of I is at most ρ and by [9, Formula 14.4] this multiplicity is at least e, showing that $e \leq \rho$.

To prove the last statement, suppose $e = \rho$, so that A/I has length e, for some parameter ideal I. Since I has multiplicity at least e, we get from [9, Theorem 14.10] that I must have multiplicity e, so that A is Cohen-Macaulay by [9, Theorem 17.11]. Conversely, assume A is Cohen-Macaulay with infinite residue field. By [9, Theorem 14.14], there exists a reduction I of $\mathfrak{m}$ which is a parameter ideal of A. Since I is a reduction of $\mathfrak{m}$, its multiplicity is e. By [9, Theorem 17.11], the length of A/I is e, showing that $\rho \leq e$. $\qquad\square$

Note that we only used the assumption that the residue field of A is infinite for the converse in the last statement. That this assumption is necessary is clear from the next example.

Example 3.4. The local ring $R/(X^2Y + XY^2)R$ with $R := \mathbb{F}_2[[X, Y]]$ and $\mathbb{F}_2$ the two-element field, is Cohen-Macaulay of multiplicity $e = 3$, but parameter degree $\rho = 4$, since no element of degree one is a parameter.

Proposition 3.5. *We have an inequality $\bar{\rho} \leq s$, with equality if s is finite and A is complete.*

Proof. We may assume $s < \infty$. Hence there exists a regular local subring $(S, \mathfrak{n}) \subseteq A$ such that A is generated over S by s elements. By Nakayama's Lemma, s is equal to the vector space dimension of $A/\mathfrak{n}A$ over the residue field of S. In particular, $A/\mathfrak{n}A$ has length at most s. On the other hand, this length is bigger than or equal to $\bar{\rho}$, since $\mathfrak{n}A$ is a non-degenerate parameter ideal of A. In conclusion, we showed that $\bar{\rho} \leq s$.

For the opposite inequality, let I be a non-degenerate parameter ideal in A such that A/I has length $\bar{\rho}$. By Theorem 1.1, there exists a regular local subring $(S, \mathfrak{n}) \subseteq A$ over which A is finitely generated, such that $I = \mathfrak{n}A$. By Nakayama's lemma, $\mu_S(A) = \bar{\rho}$, showing that $s \leq \bar{\rho}$. $\qquad\square$

Observe that in general, A has the same (equi-)parameter degree as its completion $\widehat{A}$, since any $\mathfrak{m}\widehat{A}$-primary ideal is extended from A. In particular, we showed that for a non-degenerate Noetherian local ring, its equi-parameter degree is equal to the Noether Normalization degree of its completion. We next relate these invariants to the homological degree introduced by Vasconcelos in [17, §3] or [18, §9.5].

Proposition 3.6. *Let e_{hom} be the homological multiplicity of A and assume A is complete. If A is equicharacteristic with infinite residue field, then $s \leq e_{\text{hom}}$.*

Proof. In [17, Definition 3.23], the homological multiplicity e_{hom} of A is defined to be the homological degree of A viewed as an A-module. By [9, Theorem 14.14], there exists a parameter ideal I in A whose image in the graded ring of A is generated by elements of degree one. By Theorem 1.1, we can find a regular local subring $(S, \mathfrak{n}) \subseteq (A, \mathfrak{m})$ so that the extension is finite and $\mathfrak{n}A = I$. Such an extension can be used to calculate e_{hom} by [17, Remark 3.11]. It follows that e_{hom} is also the homological degree of A viewed as an S-module. By [17, Proposition 4.1], we then get that e_{hom} is a bound on the number of generators of A as an S-module, whence a fortiori, on the Noether Normalization degree s of A. $\qquad\square$

From [9, Theorem 23.1] and the above results, we get immediately.

Corollary 3.7. *Let A be a complete local Cohen-Macaulay ring of multiplicity e. If A is equicharacteristic and has an infinite residue field, then there is a regular subring $S \subseteq A$ such that A is a free S-module of rank e.*

Since homological multiplicity agrees with multiplicity when A is Cohen-Macaulay, the assumption on the residue field is necessary in the previous proposition and its corollary by Example 3.4, as is the equicharacteristic assumption by Example 3.2. Since A and its completion have the same homological multiplicity by [17, Theorem 3.22], we get the following inequalities:

Corollary 3.8. *Let $\widehat{s}$ be the Noether Normalization degree of the completion of A. If A is equicharacteristic with infinite residue field, then $e \leq \rho = \bar{\rho} = \widehat{s} \leq e_{\mathrm{hom}}$, with equality everywhere if, and only if, A is Cohen-Macaulay.*

We now turn to the bounds proven in the previous section. Inspecting the proofs we get the following explicit upper bounds.

Corollary 3.9. *Let A be a non-degenerate Noetherian local ring and let $\bar{\rho}$ be its equiparameter degree. If $\mathfrak{a}$ is a height two ideal of A for which $A/\mathfrak{a}$ is Cohen-Macaulay of type τ, then $\mathfrak{a}$ can be generated by at most $(\tau + 1)\bar{\rho}$ elements.*

Combining this with Corollary 3.8, we get:

Corollary 3.10. *In an equicharacteristic Noetherian local ring A with an infinite residue field and homological multiplicity e_{hom}, any height two ideal $\mathfrak{a}$ for which $A/\mathfrak{a}$ is Gorenstein (respectively, Cohen-Macaulay of type τ), can be generated by at most $2e_{\mathrm{hom}}$ elements (respectively, $(\tau + 1)e_{\mathrm{hom}}$ elements).*

Remark 3.11. If A is a d-dimensional non-degenerate Noetherian local ring and $\mathfrak{a}$ an ideal in A such that $A/\mathfrak{a}$ has depth at least $d - 1$, then $\mu(\mathfrak{a})$ is at most the equi-parameter degree of A. This was originally proved by Gottlieb in [8], who actually proves a stronger result without the non-degenerate assumption, and with parameter degree instead of equiparameter degree. To prove the above assertion, we may assume A is complete. Since the equi-parameter degree $\bar{\rho}$ of A is equal to its Noether Normalization degree by Proposition 3.5, we get an epimorphism (1) with S regular and $N = \bar{\rho}$. By the Auslander-Buchsbaum Theorem, the S-module $A/\mathfrak{a}$ has projective dimension one, so that $\varphi^{-1}(\mathfrak{a})$ is free, of rank at most $\bar{\rho}$.

In fact, we can incorporate these ideas in the proof of Corollary 2.2, to obtain the following estimate:

Proposition 3.12. *Let $(A, \mathfrak{m})$ be a d-dimensional non-degenerate Noetherian local ring and let $\mathfrak{a}$ be a height one ideal which is almost Cohen-Macaulay (meaning that $A/\mathfrak{a}$ has dimension $d - 1$ and depth $d - 2$). If $\bar{\rho}$ is the equi-parameter degree of A and $\lambda_{\mathfrak{a}}$ the type of the local cohomology $H_{\mathfrak{m}}^{d-2}(A/\mathfrak{a})$ of $A/\mathfrak{a}$, then $\mathfrak{a}$ is generated by at most $(\lambda_{\mathfrak{a}} + 1)\bar{\rho}$ elements.*

Proof. As before, we may take A complete. In the proof of Corollary 2.2, we only used that $\bar{A} := A/\mathfrak{a}$ is Cohen-Macaulay twice. Firstly, it was used to deduce that $\varphi^{-1}(\mathfrak{a})$ (notation as in proof) has projective dimension 2. But this follows in the present situation from the Auslander-Buchsbaum theorem and our assumption that $\bar{A}$ has depth $d - 2$. Secondly, we used the type of $\bar{A}$ to estimate the number of generators λ of $\mathrm{Ext}_S^2(\bar{A}, S)$. By Grothendieck duality, this module is isomorphic to the Matlis (S-)dual of $H_{\mathfrak{n}}^{d-2}(\bar{A})$, where $\mathfrak{n}$ is the maximal ideal of the regular subring $S \subseteq A$. An application of [3, Proposition 3.2.12] then yields that λ is equal to the type of $H_{\mathfrak{n}}^{d-2}(\bar{A}) \cong H_{\mathfrak{m}}^{d-2}(\bar{A})$. $\square$

In the terminology of [3, Remark 3.5.10], Grothendieck duality yields that $\lambda_{\mathfrak{a}}$ is the minimal number of generators of the canonical module $K_{A/\mathfrak{a}}$ of $A/\mathfrak{a}$.

Corollary 3.13. *Let A be a two-dimensional non-degenerate Noetherian local ring and let $\bar{\rho}$ be its equi-parameter degree. Every ideal $\mathfrak{a}$ in A of positive height is generated by at most $(\tau + 1)\bar{\rho}$ elements, where τ is the type of $A/\mathfrak{a}$.*

Proof. If $\mathfrak{a}$ has height two, then it is $\mathfrak{m}$-primary, once in particular Cohen-Macaulay, and we can use Corollary 3.9. So assume $\bar{A} := A/\mathfrak{a}$ has dimension one. If its depth is also one, then $\mathfrak{a}$ is a Cohen-Macaulay ideal and we are done by Remark 3.11 (or Gottlieb's result). So assume its depth is zero. We need to show that $\lambda_{\mathfrak{a}}$ as defined in Proposition 3.12, is equal to the type τ of $\bar{A}$. Since $H^0_{\mathfrak{m}}(\bar{A})$ is equal to $\mathrm{Ann}(\mathfrak{m}^n \bar{A})$ for some sufficiently large n, its socle is equal to $\mathrm{Ann}(\mathfrak{m}\bar{A})$, that is to say, equal to the socle of $\bar{A}$. By [3, Lemma 1.2.19], the dimension of the socle of a depth zero module is its type, showing that $\lambda_{\mathfrak{a}} = \tau$. $\qquad\square$

In particular, we proved Theorem 1.3 from the introduction.

4. The Global Case

To make the reduction to the local case, we use the Forster-Swan Theorem (see for instance [9, Theorem 5.7]).

Theorem 4.1 (Forster-Swan Theorem)**.** *Let A be a Noetherian ring and let M be a finitely generated A-module. For each prime ideal $\mathfrak{p}$ of A, define*

$$f(\mathfrak{p}, M) := \mu_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim(A/\mathfrak{p}).$$

If f is the maximum of all $f(\mathfrak{p}, M)$ for $\mathfrak{p}$ running over all prime ideals in the support of M, then M can be generated by at most f elements.

Corollary 4.2. *Let A be a d-dimensional Noetherian ring and $\mathfrak{a}$ an ideal of A. Let f be a bound on the number of generators of each $\mathfrak{a}A_{\mathfrak{m}}$, where $\mathfrak{m}$ runs over all maximal ideals of A. Then $\mathfrak{a}$ can be generated by at most $\max\{d + 1, f + \dim A/\mathfrak{a}\}$ elements.*

Proof. Let $\mathfrak{p}$ be an arbitrary prime ideal of A. If $\mathfrak{a}$ is not contained in $\mathfrak{p}$, then $\mathfrak{a}A_{\mathfrak{p}} = A_{\mathfrak{p}}$ is generated by a single element, so that $f(\mathfrak{p}, \mathfrak{a}) = 1 + \dim A/\mathfrak{p} \leq d + 1$. If $\mathfrak{a} \subseteq \mathfrak{p}$, then $\dim A/\mathfrak{p} \leq \dim A/\mathfrak{a}$. Choose a maximal ideal $\mathfrak{m}$ of A, containing $\mathfrak{p}$. Since $\mathfrak{a}A_{\mathfrak{p}}$ is a localization of $\mathfrak{a}A_{\mathfrak{m}}$, it is generated by at most f elements. The assertion now follows from the Forster-Swan Theorem. $\qquad\square$

We also need to study the behavior of Noether Normalization degrees under localization and completion.

Proposition 4.3. *Let A be a Noetherian domain with Noether Normalization degree s. For every prime ideal $\mathfrak{p}$ of A, the Noether Normalization degree of the completion $\widehat{A_{\mathfrak{p}}}$ of $A_{\mathfrak{p}}$ is at most s.*

Proof. Let S be a regular subring of A such that $\mu_S(A) = s$ and let $\mathfrak{q} := \mathfrak{p} \cap S$. By base change, the fiber ring $A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$ has dimension at most s over the residue field $k(\mathfrak{q})$. Since $\widehat{A_{\mathfrak{p}}}$ is a direct summand of the $\mathfrak{q}$-adic completion $\widehat{A_{\mathfrak{q}}}$ of $A_{\mathfrak{q}}$ by [9, Theorem 8.15], we get that

$$\dim_{k(\mathfrak{q})}(\widehat{A_{\mathfrak{p}}}/\mathfrak{q}\widehat{A_{\mathfrak{p}}}) \leq \dim_{k(\mathfrak{q})}(\widehat{A_{\mathfrak{q}}}/\mathfrak{q}\widehat{A_{\mathfrak{q}}}) = \dim_{k(\mathfrak{q})}(A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}) \leq s.$$

In particular, $\widehat{A_{\mathfrak{p}}}$ is generated as an $\widehat{S_{\mathfrak{q}}}$-module by at most s elements, by [9, Theorem 8.4]. Since $S_{\mathfrak{q}}$ is regular, whence also its completion, we will have shown that $\widehat{A_{\mathfrak{p}}}$ has Noether Normalization degree at most s provided the natural homomorphism $\widehat{S_{\mathfrak{q}}} \to \widehat{A_{\mathfrak{p}}}$ is injective. At this point, we will need the assumption that A is a domain. By [9, Theorem 9.4], the going-down theorem holds for the inclusion $S \subseteq A$. Hence, $\mathfrak{q}$ and $\mathfrak{p}$ have the same height

by [9, Theorem 15.1]. Therefore, since $\widehat{S_{\mathfrak{q}}} \to \widehat{A_{\mathfrak{p}}}$ is finite homomorphism between rings of the same dimension with $\widehat{S_{\mathfrak{q}}}$ a domain, it must be injective. $\square$

The following counterexample to the Proposition without the domain condition was pointed out to me by a referee of an earlier version of this paper.

Example 4.4. Let $A := R/(XY^n, XZ)R$ with $R := K[[X, Y, Z]]$ and K a field, and let $\mathfrak{p} := (Y, Z)A$. One checks that A is generated by two elements over $S := K[[X - Z, Y]]$, but that $A_{\mathfrak{p}}$ is an Artinian ring of length $n + 1$. In this example $\mathfrak{p}$ has height zero but its contraction to S is YS whence has height one.

Remark 4.5. The theorem also holds if instead of assuming that A is a domain, we require that it is bi-equidimensional (meaning that all minimal primes have the same dimension and all maximal ideals have the same height). Indeed, with $S \subseteq A$ as above, we only need to show that $\mathfrak{p}$ and $\mathfrak{q} := \mathfrak{p} \cap S$ have the same height, for every prime ideal $\mathfrak{p}$ of A. Let d be the dimension of A and h the height of $\mathfrak{p}$. Since $S \subseteq A$ is finite, S also has dimension d. Together with [9, Exercise 9.8], this gives the inequalities $h = \mathrm{ht}(\mathfrak{p}) \le \mathrm{ht}(\mathfrak{q}) \le d$. To prove that the first inequality is an equality, we do downward induction on h, where the case $d = h$ trivially holds. Hence suppose $h < d$, so that by assumption, $\mathfrak{p}$ is not a maximal ideal. Since S is universally catenary ([3, Theorem 2.1.12]), so is A, as it is finite over S. Since A is in particular equidimensional, any maximal chain of prime ideals between two prime ideals $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ in A has length $\mathrm{ht}(\mathfrak{p}_2) - \mathrm{ht}(\mathfrak{p}_1)$. It follows that there exists a prime ideal $\mathfrak{p}'$ of height $h + 1$ containing $\mathfrak{p}$. By our induction hypothesis, $\mathfrak{q}' := \mathfrak{p}' \cap S$ has height $h + 1$ as well. Since A is finite over S, there are no inclusion relations among prime ideals in A lying over the same prime in S. It follows that $\mathfrak{q} \subsetneq \mathfrak{q}'$ so that the former has height at most h, as we wanted to show. $\square$

Let us define the *global type* of a module M over a Noetherian ring A, as the maximal type of any localization $M_{\mathfrak{m}}$ of M at a maximal ideal $\mathfrak{m}$ of A.

Theorem 4.6. *Let A be a d-dimensional Noetherian ring of finite Noether Normalization degree s. Assume that A is either a domain or bi-equidimensional. If $\mathfrak{a}$ is a height two ideal of A for which $A/\mathfrak{a}$ is Cohen-Macaulay of global type τ, then $\mathfrak{a}$ can be generated by at most $(\tau + 1)s + d - 2$ elements (except when $s = \tau = 1$, in which case possibly $d + 1$ generators are needed).*

Proof. Let $\mathfrak{a}$ be a height two Cohen-Macaulay ideal of A. By Corollary 4.2, if we find a bound f on the number of generators of $\mathfrak{a}A_{\mathfrak{m}}$ in each localization with respect to a maximal ideal $\mathfrak{m}$, then $\mathfrak{a}$ itself can be generated by $f + \dim A/\mathfrak{a}$ elements. The statement therefore follows from Corollary 3.9, Proposition 4.3 and Remark 4.5, since we may take $f = (\tau + 1)s$. One just needs to observe that the given bound is at least d and only in the indicated case $s = \tau = 1$ it is equal to it. $\square$

The case $s = \tau = 1$ means that A is regular and $\mathfrak{a}$ is a Gorenstein ideal, whence locally a complete intersection. If A is a polynomial ring over some subring, then the EE-Conjecture proven in [10], states that we may drop the contribution of $f(\mathfrak{p}, \mathfrak{a})$ for all minimal primes $\mathfrak{p}$ of A in the bound in the Forster-Swan Theorem, yielding therefore in this case the upper bound d.

The theorem together with Theorem 1.3 yields immediately:

Corollary 4.7. *Let A be a two-dimensional Noetherian domain of finite Noether Normalization degree s and let $\mathfrak{a}$ be an arbitrary ideal of A. If $A/\mathfrak{a}$ has global type τ, then $\mathfrak{a}$ can be generated by at most $(\tau + 1)s + 1$ elements.*

Proof of Theorem 1.4. If S and A denote the affine algebras of X and Y respectively, then our assumptions imply that $S \subseteq A$ is finite with S regular. By definition, the degree ϵ of $Y \to X$ is the maximal number of points in a closed fiber. In other words, ϵ is the maximum of the dimensions

$$\epsilon(\mathfrak{m}) := \dim_{S/\mathfrak{m}}(A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}),$$

where $\mathfrak{m}$ runs over all maximal ideals of S. By Nakayama's Lemma $\epsilon(\mathfrak{m}) = \mu_{S_\mathfrak{m}}(A_\mathfrak{m})$ and this is also equal to the minimal number of generators of $A_\mathfrak{n}$ over $S_\mathfrak{m}$, for any maximal ideal $\mathfrak{n}$ of A lying over $\mathfrak{m}$. Therefore, if $\mathfrak{a} \subseteq A$ is the ideal defining the subscheme W, then $\mathfrak{a}A_\mathfrak{n}$ is generated by at most 2ϵ elements by Corollary 3.9. The stated bound now follows from Corollary 4.2. $\qquad\square$

5. The Affine Case

Affine rings, that is to say, finitely generated algebras over a field, have the property that their Noether Normalization degree is finite. In fact, we have the following sharper result.

Theorem 5.1. *For each pair (d, n), there exists a bound $\mathsf{E}(d, n)$ with the following property. If A is an affine ring of the form $K[X]/(f_1, \ldots, f_s)K[X]$ with K a field, X a set of d variables and f_i polynomials of degree at most n, then the Noether Normalization degree of A is at most $\mathsf{E}(d, n)$.*

In particular, if $\mathfrak{a}$ is a height two ideal of A for which $A/\mathfrak{a}$ is Cohen-Macaulay of global type τ, then $\mathfrak{a}$ can be generated by at most $(\tau + 1)\mathsf{E}(d, n) + d$ elements.

Proof. To prove the first statement, one just needs to observe that Noether Normalization can be carried out algorithmically from the f_i. The key idea is to make a change of variables so that one of the f_i becomes monic in some variable. If K is infinite, this can be done by a linear change of variables; in the general case, we can still control the degree of this new equation (see [18, §A.5] for details). Assume therefore that all f_i have degree at most n' and that f_1 is monic in X_1 of degree n', where n' only depends on d and n. Hence $K[X]/f_1K[X]$ is generated by $1, X_1, \ldots, X_1^{n'-1}$ over $K[X_2, \ldots, X_d]$. Let

$$I_1 := (f_1, \ldots, f_s)K[X] \cap K[X_2, \ldots, X_d]$$

and put $A_1 := K[X_2, \ldots, X_d]/I_1$. It follows that $A_1 \subseteq A$ is a finite extension, generated by at most n' elements. By [13, Theorem 2.6], the ideal I_1 is generated by polynomials of degree at most n'', where n'' depends only on n', whence only on d and n. Therefore, by an inductive argument, A_1 admits a Noether Normalization $K[Y] \subseteq A_1$ generated by at most n''' elements, where n''' depends only on n'', whence only on d and n. From the composition $K[Y] \subseteq A_1 \subseteq A$ we see that A is generated as a $K[Y]$-module by at most $n'''n''$ elements, a number only depending on d and n.

To prove the second statement, it suffices to show, in view of Corollary 4.2, that $\mathfrak{a}A_\mathfrak{m}$ is generated by at most $(\tau + 1)\mathsf{E}(d, n)$ elements, for each maximal ideal $\mathfrak{m}$ of A (note that A has dimension at most d). To this end, we may first make a faithfully flat base change, and hence assume that K is algebraically closed. After a linear change of variables, we may assume furthermore that $\mathfrak{m} = (X_1, \ldots, X_d)A$. The above argument then shows that $A_\mathfrak{m}$ has Noether Normalization degree at most $\mathsf{E}(d, n)$ and hence the claim follows from Corollary 3.9. $\qquad\square$

Using [8] or Remark 3.11, we can give a similar estimate in the height one case:

Corollary 5.2. *A height one Cohen-Macaulay ideal can be generated by at most $\mathsf{E}(d, n) + d$ elements.*

The above bound holds even uniformly in families in the following sense.

Corollary 5.3. *Let* $s\colon W \to V$ *be map of finite type between schemes of finite type over some field. There exists a bound* $\mathsf{CM}(s)$, *such that for each* $x \in V$ *and each codimension two Cohen-Macaulay subscheme* F *of* $s^{-1}(x)$ *of global type* τ, *the ideal of* F *is generated by at most* $(\tau + 1)\mathsf{CM}(s)$ *elements. If* F *has codimension one, then at most* $\mathsf{CM}(s)$ *generators suffice.*

Proof. Taking a finite affine covering, we may reduce to the case that $V =: \operatorname{Spec} A$ and $W =: \operatorname{Spec} B$ are affine, so that s corresponds to a K-algebra homomorphism $A \to B$. Choose $f_i \in K[X]$ such that $A \cong K[X]/(f_1, \ldots, f_t)K[X]$, and $g_i \in K[X, Y]$ such that $B \cong A[Y]/(g_1, \ldots, g_t)A[Y]$, for some tuples of variables X and Y. Let d be the total number of variables and let n be the maximal degree of the f_i and the g_i. If $\mathfrak{p}$ denotes the prime ideal of A corresponding to the point $x \in V$, then the coordinate ring of the fiber $s^{-1}(x)$ is $B \otimes_A k(\mathfrak{p})$, where $k(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is the residue field of $\mathfrak{p}$. It follows that $B \otimes_A k(\mathfrak{p}) \cong k(\mathfrak{p})[X]/(g_1, \ldots, g_t)k(\mathfrak{p})[X]$. By Theorem 5.1, the ideal of $B \otimes_A k(\mathfrak{p})$ defining F is generated by at most $(\tau + 1)\mathsf{E}(d, n) + d$ elements, where $\mathsf{E}(d, n)$ is as in that Theorem. The height one case follows by a similar argument using Corollary 5.2. $\square$

Corollary 5.4. *For each pair* (d, n), *there exists a bound* $\mathsf{N}(d, n)$ *with the following property. Let* A *be an affine ring of the form* $K[X]/(f_1, \ldots, f_s)K[X]$ *with* K *a field,* X *a set of* d *variables and* f_i *polynomials of degree at most* n. *Let* $\mathfrak{a}$ *be a Cohen-Macaulay ideal of* A *of height* h *and let* τ *be the global type of* $A/\mathfrak{a}$. *If* $\mathfrak{a}$ *contains a height* $h - 2$ *ideal* $I := (g_1, \ldots, g_t)A$, *with the* g_i *of degree at most* n, *then* $\mathfrak{a}$ *can be generated by at most* $(\tau + 1)\mathsf{N}(d, n)$ *elements. If* I *has height* $h - 1$, *then at most* $\mathsf{N}(d, n)$ *generators suffice.*

In particular, every height three ideal $\mathfrak{a}$ *of* A *which contains the image of a polynomial of degree at most* n *not belonging to any minimal prime of* A *and for which* $A/\mathfrak{a}$ *is Gorenstein, can be generated by at most* $2\mathsf{N}(d, n)$ *elements.*

Proof. The second statement is a special case of the first, with $h = 3$ and $\tau = 1$. Let A, $\mathfrak{a}$ and I be as in the first statement. Counting monomials of degree at most n, one easily sees that there is a bound $\mathsf{N}'(d, n)$ on the number of generators of I, only depending on d and n. Let $B := A/I$, so that B is also a homomorphic image of a polynomial ring in d variables by an ideal generated by polynomials of degree at most n. By Theorem 5.1, the height two Cohen-Macaulay ideal $\mathfrak{a}B$ is generated by at most $(\tau + 1)\mathsf{E}(d, n) + d$ elements. Therefore, $\mathfrak{a}$ is generated by at most $(\tau + 1)\mathsf{E}(d, n) + d + \mathsf{N}'(d, n)$ elements. $\square$

Proof of Theorem 1.5. Let $S := K[X_1, \ldots, X_4]$ and let $\mathfrak{a} \subseteq S$ be the one-dimensional radical ideal defining C. By assumption, there is a degree two polynomial $f \in \mathfrak{a}$. As in the proof of Theorem 5.1, we see that S/fS has Noether Normalization degree 2. Therefore, $\mathfrak{a}(S/fS)$ is locally generated by at most 4 elements by Theorem 4.6, and hence $\mathfrak{a}$ is locally generated by at most 5 elements. By [4], a grade three Gorenstein ideal in a local ring is generated by an odd number of elements, so that $\mu(\mathfrak{a}S_{\mathfrak{m}})$ is either 3 or 5 for every maximal ideal $\mathfrak{m}$ of S, and the latter case occurs except when $\mathfrak{a}$ is locally a complete intersection. Since S is regular and $\mathfrak{a}$ is radical, $\mathfrak{a}S_{\mathfrak{p}}$ is generated by 3 elements for every one-dimensional prime $\mathfrak{p}$ containing $\mathfrak{a}$. Hence for such $\mathfrak{p}$, its contribution in the Forster-Swan Theorem is $f(\mathfrak{p}; \mathfrak{a}) = 4$. Therefore, $\mu(\mathfrak{a}) = 5$, if $\mathfrak{a}$ is not locally a complete intersection. On the other hand, if $\mathfrak{a}$ is locally a complete intersection, then $\mu(\mathfrak{a}) \leq 4$ by the EE-conjecture proven in [10]. By work of Ferrand, Boratyński and Mohan Kumar, every locally complete intersection curve in affine space is a set-theoretic complete intersection (see for instance [16, Corollary 1.21]). $\square$

Theorem 5.5. *Let W be a codimension three Gorenstein subvariety in affine d-space over an infinite field. If W lies on a degree e hypersurface, then its ideal $\mathfrak{a} := I(W)$ is generated by at most $2e + d - 2$ elements. If, moreover, W has at most isolated singularities, then $\mathfrak{a}$ is generated by at most $\max\{2e + 1, d\}$ elements.*

Proof. As before, $\mathcal{O}_H$ has Noether Normalization degree at most e, where H is the degree e hypersurface containing W. Therefore, $\mathfrak{a}\mathcal{O}_H$ is generated by at most $2e + d - 3$ elements by Theorem 4.6, whence $\mathfrak{a}$ requires at most one more generator (to wit, the defining equation of H). Suppose now that W has at most isolated singularities. At a non-closed point of W, the ideal $\mathfrak{a}$ is locally generated by at most 3 elements, since it is a complete intersection at such a point. We already observed that at closed points, $\mathfrak{a}\mathcal{O}_H$ requires at most $2e$ local generators, whence $\mathfrak{a}$ requires at most $2e + 1$ local generators. The EE-conjecture ([10]) then yields an upper bound of $\max\{2e + 1, d\}$ global generators for $\mathfrak{a}$. $\square$

References

[1] M. Boratyński, D. Eisenbud, and D. Rees, *On the number of generators of ideals in local Cohen-Macaulay rings*, J. Algebra **57** (1979), 77–81.

[2] N. Bourbaki, *Algèbre commutative*, vol. 8-9, Masson, New York, 1983.

[3] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, 1993.

[4] D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977), 447–485.

[5] J. Elias, *A sharp bound for the minimal number of generators of perfect height two ideals*, Manuscripta Math. **55** (1986), 93–99.

[6] J. Elias and A. Iarrobino, *The Hilbert function of a Cohen-Macaulay local algebra: Extremal Gorenstein algebras*, J. Algebra **110** (1987), 344–356.

[7] J. Elias, L. Robbiano, and G. Valla, *Number of generators of ideals*, Nagoya Math. J. **123** (1991), 39–76.

[8] C. Gottlieb, *Bounding the number of generators for a class of ideals in local rings*, Comm. Algebra **23** (1995), 1499–1502.

[9] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[10] N. Mohan Kumar, *On two conjectures about polynomial rings*, Invent. Math. **46** (1978), 225–236.

[11] J. Sally, *Numbers of generators of ideals in local rings*, Lect. Notes in Pure and Applied Mathematics, Marcel Dekker, Inc., 1978.

[12] J. Sally and W. Vasconcelos, *Stable rings*, J. Pure Appl. Algebra **4** (1974), 319–336.

[13] H. Schoutens, *Bounds in cohomology*, Israel J. Math. **116** (2000), 125–169.

[14] ———, *Number of generators of a Cohen-Macaulay ideal*, J. Algebra **259** (2003), 235–242.

[15] G. Valla, *Generators of ideals and multiplicities*, Comm. Algebra **9** (1981), 1541–1549.

[16] ———, *On set-theoretic complete intersections*, Complete Intersections, Acireale 1983 (Berlin), Springer-Verlag, 1984, pp. 85–101.

[17] W. Vasconcelos, *Cohomological degrees of graded modules*, Six lectures on Commutative Algebra (J. Elias, J.M. Giral, R.M. Miró-Roig, and S. Zarzuela, eds.), Progress in Math., vol. 166, Birkhäuser, 1998, pp. 345–390.

[18] ———, *Computational methods in commutative algebra and algebraic geometry*, Springer-Verlag, Berlin, 1998.

[19] ———, *Multiplicities and the number of generators of Cohen-Macaulay ideals*, Commutative algebra (Grenoble/Lyon, 2001), Contemp. Math., vol. 331, Amer. Math. Soc., Providence, RI, 2003, pp. 343–352.

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF NEW YORK, 365 FIFTH AVENUE, NEW YORK, NY 10016 (USA)

E-mail address: hschoutens@citytech.cuny.edu