ARTIN APPROXIMATION VIA THE MODEL THEORY OF COHEN-MACAULAY RINGS

HANS SCHOUTENS

ABSTRACT. We show the existence of a first order theory $\mathcal{C}\mathfrak{m}_{d,e}$ whose Noetherian models are precisely the local Cohen-Macaulay rings of dimension d and multiplicity e. The completion of a model of $\mathcal{C}\mathfrak{m}_{d,e}$ is again a model and is moreover Noetherian. If R is an equicharacteristic local Gorenstein ring of dimension d and multiplicity e with algebraically closed residue field and if the Artin Approximation Property holds for R, then R is an existentially closed model in the subclass of all Noetherian models of $\mathcal{C}\mathfrak{m}_{d,e}$. In case R is moreover excellent, SPIVAKOVSKI proved that the weaker Henselian assumption implies the Artin Approximation. This suggests an alternative, model theoretic strategy for proving Artin Approximation under the additional assumptions that R is Gorenstein, equicharacteristic and has algebraically closed residue field.

1. ARTIN APPROXIMATION

1.0.1. *Artin Approximation.* Artin Approximation (see 1.1 for a definition) has proven to be a powerful and versatile tool in algebraic geometry and commutative algebra. Its connections with model theory have been realised by several people,[1] but seemingly in a rather ad hoc way. Our present paper wants to provide a (as natural as possible) framework in which Artin Approximation can be studied by model theoretic tools. The key observation to link Artin Approximation with model theory is the following: a Noetherian local ring R admits the Artin Approximation property, if and only if, it is existentially closed, as a ring, in its completion $\widehat{R}$. (With the *completion* of a local ring we always mean its completion with respect to the topology given by the maximal ideal). Hence it would be highly desirable to find a first order theory $\mathcal{T}$ with the following two properties, where $(R, \mathfrak{m})$ is a Noetherian local ring.

(I) If R is a model of the theory $\mathcal{T}$, then so is its completion.
(II) If R, moreover, admits Artin Approximation, then it is an existentially closed model of $\mathcal{T}$.

The theory of rings trivially verifies (I), but in general, complete (sub)theories will not. Hence (II) asks for narrowing down the models for it to become true without violating (I). The requirement (II) can never hold as it stands, as an existentially closed local ring must have an algebraically closed residue field. In

Date: 24.01.99.

[1]To name few: VAN DEN DRIES shows in Chapter 12 of [14] how the application of Artin Approximation by HOCHSTER for the existence of Big Macaulay modules, can be viewed in a model theoretic way; ROBINSON proposes an approach to rigid analytic quantifier elimination through Artin Approximation; the author has used Artin Approximation in the study of etale complexity in [13]. Yet another instance is explained in Historic Note 1.4 below.

other words, we will have to impose further restrictions. We would like to do this in such way that these extra conditions are not only sufficient but also necessary conditions for the model to be existentially closed. However, we are faced with the following unfortunate fact: any axiomatisable class of (local) rings will contain non-Noetherian models, as soon as there is one model with a non-zero divisor. Hence to stay within the Noetherian realm, we had to confine our attention to Artinian local rings in [11], to obtain a manageable model theory. Of course for Artinian local rings Artin Approximation always holds, so nothing further needs to be said here.

The present paper seeks to generalise the results of the Artinian case to higher dimensions, inevitably complicating matters in light of the presence of non-Noetherian models. Therefore the program as proposed above has to be weakened: we will no longer insist that Artin Approximation implies existentially closedness in the class of all models, but only that this is the case in the (non-elementary) subclass of Noetherian models. From the results of [11] we also learn a further necessary condition to existentially closedness, namely the local ring has also to be Gorenstein. One possible definition for a Noetherian local ring $(R, \mathfrak{m})$ to be Gorenstein, is that there exists a regular sequence $(x_1, \dots, x_d)$, such that the quotient $\bar{R} = R/(x_1, \dots, x_d)$ is an Artinian local Gorenstein ring. The latter means that $\bar{R}$ is self-injective, or, equivalently, that its socle $\operatorname{Ann}_{\bar{R}}(\mathfrak{m}\bar{R})$ is one-dimensional (over the residue field). In particular, this implies that R is Cohen-Macaulay of dimension d. In [11] it was also necessary for maintaining Noetherianity to bound the length of the model. Therefore, if there is any hope to tackle the general case, we will need to bound in some way the length of $R/(x_1, \dots, x_d)$ as well. Of course, choosing a different R-regular sequence might alter this length, but the key result 2.1 shows that for a *generic* d-tuple $(x_1, \dots, x_d)$ the length of $R/(x_1, \dots, x_d)$ is constant. This *generic* length is in fact the cohomological invariant called the *multiplicity* of R, derived from the Hilbert polynomial of (the associated graded ring of) R. This result 2.1, called here Generic Length Lemma, is probably well-known to algebraists, but as we couldn't find a reference for it, we included its proof.

From these observations, it should now be clear that it is necessary to bound both the dimension and the multiplicity of the local ring. Of course, both invariants are only well-behaved in the Noetherian case, so we need to find formulae which imply the required bounds in case the model is Noetherian (but might very well mean something different in the non-Noetherian case). In summary, we will describe a theory of local rings of which the Noetherian models are precisely the d-dimensional Cohen-Macaulay rings of multiplicity e. It is a happy consequence of the proposed theory that even in the non-Noetherian case, the maximal ideal is finitely generated (by at most $d+e-1$ elements). From this it immediately follows that the completion of any model is Noetherian. Even more, the completion is again a model, so that `(I)` is fulfilled. To accomplish the goal we had set in the beginning, we need to restrict the possible morphisms between models: every model comes with a choice of a generic d-tuple (a maximal regular sequence), but these choices might not be compatible under an arbitrary morphism. It turns out that it is quite harmless to add some constant symbols to the language which interpret the generic d-tuple and to add a predicate for the ideal they generate. In this expanded language we now have that also `(II)` is verified if we add the two extra conditions that the residue field is algebraically closed and that the ring is also Gorenstein, as shown in Theorem 3.10, at least under the extra assumption that the ring is equicharacteristic. That these conditions are also necessary is very likely to be the case, but at present we do not succeed to prove this for the Gorenstein condition. It seems also reasonable to expect that the mixed characteristic case should hold. To prove this, one has probably to use the theory of Witt vectors together with the techniques of §3 in [11]. In this most general form, this results could be paraphrased as *a generic ring is Gorenstein*. Compare this with Bass's claim [3] that *Gorenstein rings are ubiquitous*.

Our ultimate, but perhaps unattainable goal, was to provide an alternative proof to Artin's conjecture that any excellent Henselian local ring admits Artin Approximation. Whence the question whether the four conditions

- with algebraically closed residue field
- Gorenstein
- Henselian
- excellent

suffice for a Noetherian local ring to be an existentially closed model of the class of Noetherian models of the above theory. For then any such ring also admits the Artin Approximation, settling the conjecture under some additional, but from the point of Artin Approximation rather harmless, conditions.

1.1. Definition. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $\widehat{R}$ denote its $\mathfrak{m}$-adic completion. We say that the *Artin Approximation* holds in R, if, for any system of equations $F = (F_1, \dots, F_s)$, with $F_i \in R[Y]$ and $Y = (Y_1, \dots, Y_N)$ and for any $c \in \mathbb{N}$, whenever there exists a solution $\widehat{y} \in \widehat{R}^N$, i.e., such that $F(\widehat{y}) = 0$, then there exists a solution $y \in R^N$, i.e., $F(y) = 0$, with $y \equiv \widehat{y} \mod \mathfrak{m}^c\widehat{R}$.

We claim that the Artin Approximation holds for R, if and only if, R is existentially closed in $\widehat{R}$, i.e., if and only if, $R \prec_1 \widehat{R}$. Indeed, suppose first that $R \prec_1 \widehat{R}$ and let F, c and $\widehat{y}$ be as above. There exists $z \in R^N$, such that $\widehat{y} \equiv z \mod \mathfrak{m}^c$ and hence there exist $\widehat{a}_i \in \widehat{R}^N$, such that

$$\widehat{y} = z + \sum_{i=1}^{t} \widehat{a}_i x_i \quad , \tag{1}$$

in $\widehat{R}^N$ and where $(x_1, \dots, x_t) = \mathfrak{m}^c$. Let $A = (A_{ij})$ be a set of tN new variables and set $\widehat{a} = (\widehat{a}_1, \dots, \widehat{a}_t)$, considered as an tN-tuple. Finally, let

$$G_j(Y, A) = z_j - Y_j + \sum_{i=1}^{t} A_{ij} x_i \quad , \tag{2}$$

for $j = 1, \dots, N$, so that $G(\widehat{y}, \widehat{a}) = 0$. Since $R \prec_1 \widehat{R}$, there exist $y \in R^n$ and $a \in R^{tN}$, such that $F(y) = 0$ and $G(y, a) = 0$. The latter equations imply that $y \equiv z \mod \mathfrak{m}^c$ and hence $y \equiv \widehat{y} \mod \mathfrak{m}^c$ by choice of z, as we needed to show.

Conversely, assume that the Artin Approximation holds for R. We now want to show that any system of equations $F(Y) = 0$ and inequalities $G(Y) \neq 0$ over R, with $F_i, G_i \in R[Y]$, which has a solution $\widehat{y}$ over $\widehat{R}$, has already one over R. Since all the $G_i(\widehat{y}) \neq 0$, we can find a $c \in \mathbb{N}$ such that none of the $G_i(\widehat{y})$ lie in $\mathfrak{m}^c\widehat{R}$. By the Artin Approximation property applied to the system $F = 0$ and to c, there exists y over R, such that $F(y) = 0$ and $\widehat{y} \equiv y \mod \mathfrak{m}^c\widehat{R}$. The latter equivalence implies that $G_i(y) \notin \mathfrak{m}^c$ and hence, in particular, none of the G_i vanish at y.

1.2. Definition. We say that the *strong Artin Approximation* holds in R, if given $F_i(Y) \in R[Y]$, with $Y = (Y_1, \dots, Y_N)$ and $F = (F_1, \dots, F_s)$, and $y_k \in R^N$, such that $F(y_k) \equiv 0 \mod \mathfrak{m}^k$, for all $k \in \mathbb{N}$, then, for any pair of natural numbers c and k_0, we can find $y \in R^N$ and $k \geq k_0$, such that $F(y) = 0$ and $y \equiv y_k \mod \mathfrak{m}^c$. We say that the *solvability property* holds in R, if in the above we drop the congruence condition, i.e., given $F(y_k) \equiv 0 \mod \mathfrak{m}^k$ for all k, implies that there exists a solution $y \in R^N$.

If the solvability property holds in R, then so does the Artin Approximation. Indeed, by the argument that having Artin Approximation is equivalent with being existentially closed in its completion, we can drop the congruence condition in the definition of Artin Approximation. Now, let $F(Y)$ be a finite number of polynomials in $Y = (Y_1, \dots, Y_N)$ over R having a solution $\widehat{y} \in \widehat{R}^N$. Let $y_k \in R^N$, such that $\widehat{y} \equiv y_k \mod \mathfrak{m}^k\widehat{R}$, so that $F(y_k) \equiv \mathfrak{m}^k$, for all k. By the solvability property, there exists a real solution $y \in R^N$, as wanted.

1.3. *Historic Note.* Artin proved in [1] and [2] that the Henselisation of a polynomial ring in d variables over a field or an excellent discrete valuation ring has the Artin Approximation property.

1.4. *Historic Note.* In [4] the authors derived from Artin's result, using ultraproducts, that the Henselisation of a polynomial ring in d variables over a field κ has actually the strong Artin Approximation property. Using a compactness argument, one sees that an equivalent condition for the strong Artin Approximation to hold, is in the above definition to replace *for all* n by *for some sufficiently big* n, i.e., for some $n \geq k$ where k depends on the system of equations and c. They then showed that one can take a uniform value for such k, only depending on the total degree of F and the numbers d, N and c, when we take the $F(X, Y) \in \kappa[X, Y]$, i.e., with polynomial coefficients, where $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_N)$. It should be observed that considering only polynomials is no restriction, since one can always reduce to this case.

1.5. *Historic Note.* Artin conjectured that the Artin Approximation holds for an arbitrary excellent Henselian local ring, and this is now claimed by Spivakovsky.

1.5.1. *Informal Logic.* With *informal logic*, we will mean a statement (possibly over some finite tuple of variables) in a meta-mathematical language, which, in the appropriate formal language, can be turned into a first order sentence (or formula). In other words, it is left to the reader, if he or she wants to do so, to verify or actually write down the exact formula. Of course, wherever there might be some doubt about the feasibility of this, we will indicate the relevant facts. As an illustration we give some examples of *informal* statements and the corresponding *formal* versions, which will be used below.

Let R be a ring viewed as a structure in the language of rings (consisting of symbols for addition, subtraction, multiplication together with constant symbols for zero and one). An informal statement would be that

R is a local ring.

A formal rendering could be the following sentence

$$\texttt{Loc} = (\forall r, s)[\texttt{Nu}(r) \wedge \texttt{Nu}(s) \rightarrow \texttt{Nu}(r + s)] \tag{3}$$

where $\texttt{Nu}(y) = (\forall z)[yz \neq 1]$ is a formula expressing that

y is not a unit.

In particular, if R is local, then the statement that

x lies in the maximal ideal of R,

becomes formally that $\mathtt{Nu}(x)$ holds. Moreover, we could say that

R is a local ring with algebraically closed residue field.

To turn this into a genuine sentence one needs to add to the sentence $\mathtt{Loc}$ a sentence $\mathtt{Rt}_d$, one for each $d \geq 1$, expressing that

any polynomial of degree d over R has a root modulo its maximal ideal.

The formal version is

$$\mathtt{Rt}_d = (\forall a_0, \dots, a_d)(\exists y)[a_d = 0 \vee \mathtt{Nu}(a_0 + a_1 y + \dots + a_d y^d)] \ . \tag{4}$$

As a last example we take the statement

R is an Artinian local ring of length at most l.

To express this formally we might use a sentence $\mathtt{LenLess}_l$ expressing that

any chain of ideals

$$(a_0) \subset (a_0, a_1) \subset \dots \subset (a_1, \dots, a_l) \subset (a_1, \dots, a_{l+1}) \tag{5}$$

of length $l + 1$ is not proper.

Formally, this is expressed by

$$(\forall a_0, \dots, a_{l+1})(\exists y_0, \dots, y_l)[\bigvee_{i<l} a_{i+1} = a_0 y_0 + \dots + a_i y_i] \ . \tag{6}$$

Apart from these examples, we briefly describe a *reduction technique*, which allows us to make statements not only about the ring, but about any of its homomorphic images as well. Details can be found in Lemma 0.2 of [11]. Namely, let $\mathtt{P}(x)$ be a formula in the free variables $x = (x_1, \dots, x_m)$ and let $a = (a_1, \dots, a_n)$ be another set of variables. Then one can construct a formula $\operatorname{Red}_a \mathtt{P}(x)$, in the free variables x and a, such that, for any ideal $\mathfrak{a}$ of R generated by n elements $a_1, \dots, a_n$ and any $x \in R^m$, we have that

$$R \models \operatorname{Red}_a \mathtt{P}(x) \iff R/\mathfrak{a} \models \mathtt{P}(x) \ . \tag{7}$$

Therefore an informal statement

the quotient $R/(a_1, \dots, a_n)$ is an Artinian local ring of length at least l

can be turned into a formal first order formula

$$\operatorname{Red}_a (\neg \mathtt{LenLess}_{l-1}) \ . \tag{8}$$

2. Generic Length

2.1. Lemma (Generic Length Lemma)**.** *Let $(R, \mathfrak{m})$ be a d-dimensional local Cohen-Macaulay ring with residue field κ. Let $\mathfrak{q} = (u_1, \dots, u_s)$ be an $\mathfrak{m}$-primary ideal of R. Let $\xi = (\xi_{ij})$, for $i = 1, \dots, d$ and $j = 1, \dots, s$, be a set of variables and define linear forms*

$$L_i(\xi) = \sum_{j=1}^{s} \xi_{ij} u_j \ , \tag{9}$$

for $i = 1, \dots, d$. Finally, let $R(\xi)$ be the localisation of $R[\xi]$ at the prime ideal $\mathfrak{m}R[\xi]$ and let $\mathfrak{a}_\xi$ be the ideal of $R(\xi)$ generated by the $L_i(\xi)$, for $i = 1, \dots, d$. Then the following holds.

- (i) *The ideal $\mathfrak{a}_\xi$ is $\mathfrak{m}R(\xi)$-primary. In fact, $\mathfrak{a}_\xi$ is a reduction of $\mathfrak{q}R(\xi)$. Let l be the length of $R(\xi)/\mathfrak{a}_\xi$.*
- (ii) *For any $(d \times s)$-tuple a over R, the ring $R/\mathfrak{a}_a$ has length at least l, where $\mathfrak{a}_a$ is the ideal of R generated by all the $L_i(a)$, for $i = 1, \dots, d$.*
- (iii) *The multiplicity $e_R(\mathfrak{q})$ of $\mathfrak{q}$ is equal to l.*

Moreover, there exists a dense open subset U of the affine space $\mathbb{A}^{d\times s}_\kappa$, such that for all $(d\times s)$-tuples a over R for which $\bar{a} \in U$ (where $\bar{a} = (\bar{a}_{ij})$ means the reduction modulo $\mathfrak{m}$), we have that

- (iv) *The ideal $\mathfrak{a}_a$ is a reduction of $\mathfrak{q}$ (and hence, in particular, $\{L_1(a), \dots, L_d(a)\}$ is a system of parameters for R).*
- (v) *The quotient $R/\mathfrak{a}_a$ has length l.*

2.2. *Remark.* See 2.6 below for the definition of a reduction of an ideal.

Proof. The theorem is trivial for $d = 0$, so assume $d \geq 1$. We review the proof of Theorem 14.14 in [9] where (iv) has been proven. In there, one constructs the so called *ideal of null-forms* $\mathfrak{Q}$ of $\mathfrak{q}$ as the collection of homogeneous forms $f \in \kappa[X]$, where $X = (X_1, \dots, X_s)$, such that some (or, for that matter, any) lifting $F \in R[X]$ of f to a homogeneous form of the same degree, say n, satisfies $F(u) \in \mathfrak{q}^n\mathfrak{m}$. The reader can check that $\mathfrak{Q}(\kappa(\xi)[X])$ is the ideal of null-forms of $\mathfrak{q}R(\xi)$. From this and the proof in loc. cit., we get the existence of a dense open subset $U \subset \mathbb{A}^{d\times s}_\kappa$, such that (iv) holds whenever $\bar{a} \in U$. Moreover, the analogous statement holds after replacing κ by $\kappa(\xi)$ and U by its base change $U_\xi = U \times_\kappa \kappa(\xi)$ viewed as an open of $\mathbb{A}^{d\times s}_{\kappa(\xi)}$. Saying that U is dense is then equivalent with saying that $\xi \in U_\xi$ and hence in particular, we must have that $\mathfrak{a}_\xi$ is $\mathfrak{m}R(\xi)$-primary, proving (i).

Let us show (iii). Because R is Cohen-Macaulay, also $R(\xi)$ is. Therefore, from (i), we know that $\mathfrak{a}_\xi$ is a *parameter ideal* (i.e., an ideal generated by a system of parameters, or, equivalently, an $\mathfrak{m}$-primary ideal generated by d elements). Therefore, by Theorem 17.11 in [9], we obtain that $l = e_{R(\xi)}(\mathfrak{a}_\xi)$. Since $\mathfrak{a}_\xi$ is a reduction of $\mathfrak{q}R(\xi)$, we have by Theorem 14.13 in [9] that

$$e_{R(\xi)}(\mathfrak{a}_\xi) = e_{R(\xi)}(\mathfrak{q}R(\xi)) \quad . \tag{10}$$

On the other hand, one easily calculates that $e_R(\mathfrak{q}) = e_{R(\xi)}(\mathfrak{q}R(\xi))$, see for instance the remark following Theorem 14.14 in [9]. Hence we proved (iii). To prove (v), take any a such that (iv) holds for $\mathfrak{a}_a$. Since $\mathfrak{a}_a$ is then generated by a system of parameters and is a reduction of $\mathfrak{q}$, we have, by loc. cit., that

$$\ell(R/\mathfrak{a}_a) = e_R(\mathfrak{a}_a) = e_R(\mathfrak{q}) = l \quad , \tag{11}$$

proving (v).

Remains to prove (ii). There is nothing to prove if $\ell(R/\mathfrak{a}_a) = \infty$. Hence assume that $R/\mathfrak{a}_a$ has finite length, so that $\{L_1(a), \dots, L_d(a)\}$ is a system of parameters for R. Therefore, by loc. cit., $\ell(R/\mathfrak{a}_a) = e_R(\mathfrak{a}_a)$. Since $\mathfrak{a}_a \subset \mathfrak{q}$, we have by Formula 14.4 in [9] that $e_R(\mathfrak{a}_a) \geq e_R(\mathfrak{q}) = l$. □ □

2.3. Corollary. *Let $(R, \mathfrak{m})$ be a d-dimensional local Cohen Macaulay ring. Let $\mathfrak{q}$ be a $\mathfrak{m}$-primary ideal of R. Then*

$$e_R(\mathfrak{q}) = \min_{a_1,\dots,a_d \in \mathfrak{q}} \ell(R/(a_1,\dots,a_d)) \ . \tag{12}$$

Proof. Immediate from Lemma 2.1. □ □

2.4. Corollary. *Let $(R, \mathfrak{m})$ be a d-dimensional local Cohen Macaulay ring. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R. Then $\ell(R/\mathfrak{q}) \leq e_R(\mathfrak{q})$.*

Proof. (See also Theorem 17.11 in [9]). With notation as in Lemma 2.1, we have that $\ell(R/\mathfrak{q}) = \ell(R(\xi)/\mathfrak{q}R(\xi))$. Since $\mathfrak{a}_\xi \subset \mathfrak{q}R(\xi)$, we have $\ell(R(\xi)/\mathfrak{q}R(\xi)) \leq \ell(R(\xi)/\mathfrak{a}_\xi) = e_R(\mathfrak{q})$, where the last equality follows from Lemma 2.1. □ □

2.5. Corollary. *Let $(R, \mathfrak{m})$ be a d-dimensional local Cohen Macaulay ring. Fix some positive numbers s and e. Then there exists a first order formula $\texttt{Mult}_e(x)$, in the free variables $x = (x_1,\dots,x_s)$, such that for each $\mathfrak{m}$-primary ideal $\mathfrak{q} = (u_1,\dots,u_s)$ generated by s elements, $\mathfrak{q}$ has multiplicity e, if and only if, $\texttt{Mult}_e(u)$ holds in R, where $u = (u_1,\dots,u_s)$.*

Proof. Let $\texttt{Mlt}_e(u)$ be the sentence expressing the following (informal) fact that

> *for any choice of elements $y = (y_{ij})$ in R, with $i = 1,\dots,d$ and $j = 1,\dots,s$, the quotient of R by the ideal generated by the $L_i(u,y) = \sum_j y_{ij}u_j$, for $i = 1,\dots,d$, has length at least e.*

In view of Corollary 2.3, we have that $\texttt{Mlt}_e(u)$ holds in R, if and only if, the ideal $(u_1,\dots,u_s)$ has multiplicity at least e. The wanted formula $\texttt{Mult}_e(u)$ therefore says that $\texttt{Mlt}_e(u)$ holds , but not $\texttt{Mlt}_{e+1}(u)$. □ □

2.6. Definition. The Generic Length Lemma suggests another invariant associated to a d-dimensional Noetherian local ring $(R, \mathfrak{m})$. Recall that an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ is called a *reduction* of $\mathfrak{m}$, if there exists some $r \in \mathbb{N}$, such that

$$\mathfrak{m}^{r+1} = \mathfrak{q}\mathfrak{m}^r \ . \tag{13}$$

Note that if (13) holds for a value r_0 then it holds in fact for all $r \geq r_0$. Let us call the smallest value of r for which (13) holds the *reduction number* of $\mathfrak{q}$. We set the reduction number equal to ∞ if $\mathfrak{q}$ is not a reduction of $\mathfrak{m}$.

From Lemma 2.1 it follows that a *generic* d-tuple in $\mathfrak{m}$ generates a reduction of $\mathfrak{m}$. The next theorem shows that its reduction number is generically constant.

2.7. Theorem. *Let $(R, \mathfrak{m})$ be a d-dimensional local Cohen-Macaulay ring with residue field κ. Let s be the* embedding dimension *of R, i.e., the minimal number of generators of $\mathfrak{m}$. There exists a number $r \in \mathbb{N}$ and a dense open subset W of $\kappa^{d\times s}$, such that for all $(d \times s)$-tuples a over R for which $\bar{a} \in W$ (where $\bar{a} = (\bar{a}_{ij})$ means the reduction modulo $\mathfrak{m}$), we have that the ideal $\mathfrak{a}_a$ as in Lemma 2.1 is a reduction of $\mathfrak{m}$ with reduction number r.*

Proof. We keep the notation from Lemma 2.1. In particular, the ideal $\mathfrak{a}_\xi$ in $R(\xi)$ is a reduction of $\mathfrak{m}R(\xi)$. Let r be its reduction number. From the proof of Theorem 14.14 in [9], it follows that r is the smallest value for which

$$(X_1,\dots,X_s)^{r+1}\kappa(\xi)[X] \subset (\mathfrak{Q}, L_1(\xi),\dots,L_d(\xi))\kappa(\xi)[X] \ . \tag{14}$$

From the same proof, it follows that for each n, there exists a system of equations $p_{1n}(\xi) = \cdots = p_{sn}(\xi)$, such that a point $\bar{a} \in \mathbb{A}_\kappa^{d\times s}$ does not satisfy this system (i.e., some $p_{in}(\bar{a}) \neq 0$), if and only if,

$$(X_1, \dots, X_s)^n \subset (\mathfrak{Q}, L_1(\bar{a}), \dots, L_d(\bar{a}))\kappa[X] \ . \tag{15}$$

In particular, since r is the smallest value for which (14) holds, we must have that all p_{ir} are identical zero but some $p_{i,r+1}$ is not. This means that for every $\bar{a} \in \mathbb{A}_\kappa^{d\times s}$, we have that (15) does not hold for $n = r$ and hence that the reduction number of $\mathfrak{a}_a$ is at least r. Moreover, if one of the $p_{i,r+1}(\bar{a}) \neq 0$, then $\mathfrak{a}_a$ has reduction number r. Hence the $p_{i,r+1}$ define a dense open set on which the reduction number is constant, as required. □ □

2.8. Definition. Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring. We define the *reduction number* of R to be the generic value r for a reduction of $\mathfrak{m}$ as given by Theorem 2.7. It follows from Theorem 2.7 that it is the reduction number of $\mathfrak{a}_\xi$ in $R(\xi)$.

3. Cohen Macaulay Rings of Fixed Dimension and Multiplicity

3.1. Theorem. *For any pair of positive integers d and e, there exists a first order theory $\mathcal{C}\mathfrak{m}_{d,e}$, such that a Noetherian local ring is a model of this theory, if and only if, it is Cohen-Macaulay of dimension d and multiplicity e.*

3.2. *Remark.* The *multiplicity* of a local ring is by definition the multiplicity of its maximal ideal.

Proof. Let us first briefly show how to construct a formula expressing that the sequence $(x_1, \dots, x_d)$ is R-regular, where R is some local ring. It is clear how to express that

the element x_1 lies in the maximal ideal and is R-regular.

Hence to express that $x = (x_1, \dots, x_d)$ is R-regular, we need to express that

$(x_1, \dots, x_{d-1})$ is R-regular and x_d is $R/(x_1, \dots, x_{d-1})$-regular.

This can easily be achieved by induction on the length of a sequence and the reduction technique described in (7). The theory $\mathcal{C}\mathfrak{m}_{d,e}$ is then given by the following informal axioms; use (3) and (8) to make these axioms formal.

(vi) R is local.

(vii) For each ideal $\mathfrak{a} \subset \mathfrak{m}$ generated by d elements, the quotient $R/\mathfrak{a}$ has length at least e, but no such ideal exists with smaller length.

(viii) R admits a regular sequence of length d.

Indeed, let R be a model of $\mathcal{C}\mathfrak{m}_{d,e}$ which is moreover Noetherian. First of all, R is local. Let $\mathfrak{m}$ be its maximal ideal. From the dimension theory for local rings (see for instance Theorem 13.4 in [9]) it follows from (vii) that R has dimension at most d. On the other hand, from (viii) its depth is at least d, showing that R is Cohen-Macaulay of dimension d, as depth can never exceed dimension. As in Corollary 2.5, the multiplicity of R is then e. Conversely, any d-dimensional local Cohen-Macaulay ring of multiplicity e is a model of $\mathcal{C}\mathfrak{m}_{d,e}$. □ □

3.3. *Remark.* Note that any model of $\mathcal{C}\mathfrak{m}_{d,e}$ is a local ring with maximal ideal generated by at most $d + e - 1$ elements. Indeed, there exists a tuple $(x_1, \dots, x_d)$, such that $R/(x_1, \dots, x_d)$ has length e and whence the latter Artinian ring has embedding dimension at most $e - 1$. As a consequence we see that any power

$\mathfrak{m}^n$ admits a minimal set of generators (apply NAKAYAMA's Lemma to the finite R-module $\mathfrak{m}^n$).

3.4. Lemma. *Let R be a local ring with finitely generated maximal ideal $\mathfrak{m}$ and let $\mathfrak{a}$ be an arbitrary ideal of R. Let $\widehat{R}$ denote the completion of R. Then $\widehat{R}$ is Noetherian and we have an isomorphism*

$$\widehat{(R/\mathfrak{a})} \cong \widehat{R}/\mathfrak{a}\widehat{R} \ . \tag{16}$$

Proof. The first statement is well-known, see for instance Theorem 29.4 in [9]. Hence $\widehat{R}/\mathfrak{a}\widehat{R}$ is also a complete Noetherian local ring and it is now an easy exercise to prove that it must be the completion of $R/\mathfrak{a}$. □ □

3.5. Proposition. *Let $(R, \mathfrak{m})$ be a model of $\mathcal{C}\mathfrak{m}_{d,e}$ and let $\widehat{R}$ denote the completion of R. Then $\widehat{R}$ is a Noetherian model of $\mathcal{C}\mathfrak{m}_{d,e}$, i.e., $\widehat{R}$ is a Cohen-Macaulay local ring of dimension d and multiplicity e.*

Proof. By Lemma 3.4, we know already that $\widehat{R}$ is a Noetherian local ring (generated by at most $s \leq d+e-1$ elements). Let $(\mu_1, \dots, \mu_d)$ be an R-regular sequence of R and let $\mathfrak{g} = (\mu_1, \dots, \mu_d)$ be the ideal generated by this sequence. Since $R/\mathfrak{g} \cong \widehat{R}/\mathfrak{g}\widehat{R}$ by Lemma 3.4, (taking into account that $R/\mathfrak{g}$ is Artinian whence automatically complete), we have that $\ell(\widehat{R}/\mathfrak{g}\widehat{R}) = e$. Assume that we have already shown that $(\mu_1, \dots, \mu_d)$ is also $\widehat{R}$-regular, proving that $\widehat{R}$ is a Noetherian complete Cohen-Macaulay ring of dimension d. Let $\widehat{e}$ be the multiplicity of $\widehat{R}$, so that by Lemma 2.1, we must have that $\widehat{e} \leq e$, since $\widehat{R}/\mathfrak{g}\widehat{R}$ has length e. Also, by loc. cit., there exists a dense open subset $U \subset \kappa^{d\times s}$, where κ is the residue field of R, such that for any $\widehat{a} \in \widehat{R}^{d\times s}$, whenever the residue of $\widehat{a}$ in $\kappa^{d\times s}$ lies in U, then $\widehat{R}/\mathfrak{a}_{\widehat{a}}$ has length exactly $\widehat{e}$ (notation as in loc. cit., with $\mathfrak{q} = \mathfrak{m}$).

Let us first assume that κ is infinite, so that U is in fact non-empty. Hence there exists an $a \in R^{d\times s}$, such that its residue belongs to U and therefore $\widehat{R}/\mathfrak{a}_a\widehat{R}$ has length $\widehat{e}$. By Lemma 3.4, the latter is isomorphic with the completion $\widehat{(R/\mathfrak{a}_a)}$ of $R/\mathfrak{a}_a$. In particular, $\widehat{(R/\mathfrak{a}_a)}$ is Artinian and it is not too hard to verify that then the canonical map $R/\mathfrak{a}_a \to \widehat{(R/\mathfrak{a}_a)}$ must be surjective. We claim that then also $R/\mathfrak{a}_a$ is Artinian and hence isomorphic to $\widehat{(R/\mathfrak{a}_a)}$. Indeed, from the above mentioned surjectivity, it follows that there is some t, such that

$$\mathfrak{m}^t \subset \mathfrak{a}_a + \bigcap_n \mathfrak{m}^n \ . \tag{17}$$

Let $M = \mathfrak{a}_a + \mathfrak{m}^t$, so that $M = \mathfrak{a}_a + \mathfrak{m}M$. We know that $\mathfrak{m}$ is finitely generated and clearly also $\mathfrak{a}_a$ is, so that M is finitely generated. Therefore, by NAKAYAMA's Lemma, we conclude that $M = \mathfrak{a}_a$, i.e., that $\mathfrak{m}^t \subset \mathfrak{a}_a$, proving that $R/\mathfrak{a}_a$ has a nilpotent and finitely generated maximal ideal whence is Artinian. In summary, we proved that $\widehat{R}/\mathfrak{a}_a\widehat{R} \cong R/\mathfrak{a}$ and the latter has length at least e by (ii) of Lemma 2.1, so that $\widehat{e} \geq e$. Therefore $\widehat{e} = e$, as we wanted to show.

If κ is finite, then we can replace R by $R(Y)$, where Y is some variable and show that, for an arbitrary local ring S, we have that $S(Y)$ is a model of $\mathcal{C}\mathfrak{m}_{d,e}$, if and only if,

$$\widehat{(S(Y))} \cong \widehat{S}(Y) \ . \tag{18}$$

Therefore, we will be finished, once we show that $(\mu_1,\dots,\mu_d)$ is $\widehat{R}$-regular. First of all, note that $R/\mu_1 R$ is a model of $\mathcal{C}\mathfrak{m}_{d-1,e}$. Since $(\widehat{R/\mu_1 R}) \cong \widehat{R}/\mu_1\widehat{R}$, by Lemma 3.4, we therefore can reduce by an inductive argument to showing that $\mu = \mu_1$ is $\widehat{R}$-regular. Let $y \in \widehat{R}$, such that $\mu y = 0$. Let $y_n \in R$, such that $y \equiv y_n \mod \mathfrak{m}^n\widehat{R}$, for $n = 1, 2, \dots$. Hence $\mu y_n \in \mathfrak{m}^n\widehat{R} \cap R = \mathfrak{m}^n$, since $R/\mathfrak{m}^n \cong \widehat{R}/\mathfrak{m}^n\widehat{R}$, by another application of Lemma 3.4. Since $\mathfrak{m}^e \subset \mathfrak{g}$, we have that $\mu y_{en} \in \mathfrak{g}^n$. Since μ is in particular R-quasi-regular, we must have that $y_{en} \in \mathfrak{g}^{n-1}$, for $n = 1, 2, \dots$. But then

$$y \in \bigcap_n \mathfrak{m}^n\widehat{R} = 0 \ , \tag{19}$$

since the y_n converge to y. □ □

3.6. Definition. In order to obtain a $\forall_2$-theory with the same Noetherian models as $\mathcal{C}\mathfrak{m}_{d,e}$, we have to take an expansion of the ring language. Recall that an algebraic characterisation of a $\forall_2$-theory is that the class of its models is *σ-persistent*, which means that the union of a chain of models is again a model. Let $\mathcal{L}$ be the language of rings with d extra constant symbols $\mu_1,\dots,\mu_d$ and one extra unary predicate **Id**. The interpretation of the μ_i in a model of $\mathcal{C}\mathfrak{m}_{d,e}$ will be an R-regular sequence $(\mu_1,\dots,\mu_d)$ with $\ell(R/(\mu_1,\dots,\mu_d)) = e$ and **Id** will define the ideal they generate. We define the $\mathcal{L}$-theory $\mathcal{C}\mathfrak{m}^+_{d,e}$ as the theory obtained by adding to $\mathcal{C}\mathfrak{m}_{d,e}$ two extra axioms

- (ix) $(\mu_1,\dots,\mu_d)$ is an R-regular sequence and $R/(\mu_1,\dots,\mu_d)$ has length e
- (x) an element $x \in R$ satisfies **Id**, if and only if, it lies in the ideal generated by $(\mu_1,\dots,\mu_d)$.

As the reader can see, we have identified the symbols for the constants μ_i with their actual interpretation in a model. There will be little chance for confusion, as we do not intend to equip the same ring with different interpretations of the constants. If we want to emphasise the ideal generated by the regular sequence, we will write $(R,\mathfrak{m},\mathfrak{g})$ for a model of $\mathcal{C}\mathfrak{m}^+_{d,e}$ where $\mathfrak{g} = (\mu_1,\dots,\mu_d)$. The only reason for adding an extra predicate **Id** is to ensure that if $(R,\mathfrak{m},\mathfrak{g})$ is a substructure of $(S,\mathfrak{n},\mathfrak{h})$, then $\mathfrak{g} = \mathfrak{h} \cap R$.

3.7. Theorem. *A Noetherian ring is a (reduct of a) model of $\mathcal{C}\mathfrak{m}^+_{d,e}$, if and only if, it is a d-dimensional local Cohen-Macaulay ring of multiplicity e. The theory $\mathcal{C}\mathfrak{m}^+_{d,e}$ is a $\forall_2$-theory. Any extension between Noetherian models of $\mathcal{C}\mathfrak{m}^+_{d,e}$ is local and strict.*

3.8. *Remark.* A morphism $\varphi A \to B$ of topological rings is called *strict* if the canonical isomorphism $A/\ker(\varphi) \cong \varphi(A)$ is a homeomorphism, where the former ring is provided with the quotient topology and the latter with the topology inherited from B. Translated to an embedding of local rings $(R,\mathfrak{m}) \hookrightarrow (S,\mathfrak{n})$ (taken with their maximal adic topology) this amounts in saying that the $\mathfrak{m}$-adic topology on R coincides with the topology given by the ideals $\mathfrak{n}^k \cap R$. In fact, we will show the stronger statement that $\mathfrak{m}S$ is $\mathfrak{n}$-primary. A property of strict maps that will be essential to us is that the completion of a strict monomorphism is again a monomorphism by Corollary 1.1.9.6 in [5].

Proof. The first claim is easy to check, using Theorem 3.1. Next, let us prove the last claim in the stronger form that $\mathfrak{m}S$ is $\mathfrak{n}$-primary as explained in Remark 3.8.

Indeed, firstly, since $\mathfrak{m}^e \subset \mathfrak{g}$ and $\mathfrak{g}S = \mathfrak{h} \subset \mathfrak{n}$, we see that $R \to S$ is local. Next, since $\mathfrak{n}^e \subset \mathfrak{h}$ and $\mathfrak{h} = \mathfrak{g}S \subset \mathfrak{m}S$, this is now immediate. To prove the second claim, we will show that the class of models of $\mathcal{C}\mathfrak{m}^+_{d,e}$ is σ-persistent. Let $(R_j, \mathfrak{m}_j, \mathfrak{g}_j)$ be an increasing chain of models of $\mathcal{C}\mathfrak{m}^+_{d,e}$ and let R be their union. Since $\mathtt{Loc}$ is $\forall_2$ (or by direct verification), it follows that R is a local ring with maximal ideal $\mathfrak{m}$ equal to the union of the $\mathfrak{m}_j$. It is clear that $\mathtt{Id}$ defines the ideal generated by μ in R and whence is equal to the union of the $\mathfrak{g}_j$. Therefore, the $R_j/\mathfrak{g}_j$ form an ascending chain of Artinian local rings of length e with union equal to $R/\mathfrak{g}$. It follows then from Remark 1.4 in [11] (which states that the class of Artinian local rings of length at most e is σ-persistent) that the latter has also length e. One verifies that $(\mu_1, \dots, \mu_d)$ is R-regular. Hence the only thing remaining to show is that $\ell(R/(a_1, \dots, a_d)) \geq e$, for any tuple $(a_1, \dots, a_d)$ in R.

We claim that for each n, there exists a $J_n \in \mathbb{N}$, such that for $j \geq J_n$, we have that

$$\mathfrak{m}_j^n = \mathfrak{m}^n \cap R_j \quad . \tag{20}$$

We prove this by induction on n, the case $n = 1$ being clear as the maximal ideal of R is generated by at most $d + e - 1$ elements, which already belong to R_j for $j \gg 0$. Hence assume (20) proven for n and we want to show its validity for $n + 1$. Let $y_1, \dots, y_k$ be a minimal system of generators for $\mathfrak{m}^n$, which exist by Remark 3.3, and choose $J_{n+1} \geq J_n$ big enough so that $y_1, \dots, y_k \in R_j$, for $j \geq J_{n+1}$. Let $s \in \mathfrak{m}^{n+1} \cap R_j$, so that we can write $s = \sum a_i y_i$ with $a_i \in \mathfrak{m}$. By induction, $s \in \mathfrak{m}_j^n$ and we can find $b_i \in R_j$, such that $s = \sum_i b_i y_i$. Comparing both representations of s, we obtain that

$$\sum_{i=1}^{k} (a_i - b_i) y_i = 0 \tag{21}$$

in R. By minimality, we therefore have that $a_i - b_i \in \mathfrak{m}$, and whence $b_i \in \mathfrak{m} \cap R_j = \mathfrak{m}_j$, as required. This proves our claim. As a consequence we get for each n, that

$$R_j/\mathfrak{m}_j^n \subset R_{j+1}/\mathfrak{m}_{j+1}^n \subset \dots \subset R/\mathfrak{m}^n \quad , \tag{22}$$

for $j \geq J_n$. In particular, using Remark 1.4 in [11] again, we obtain that

$$\ell(R_j/\mathfrak{m}_j^n) = \ell(R/\mathfrak{m}^n) \quad . \tag{23}$$

Our next goal is to show that (23) holds in a uniform way. More precisely, we will show that there exists $J \in \mathbb{N}$, such that for all $j \geq J$ and for $n \geq d+e$, we have that (23) holds. Firstly, let $\widehat{R}_j$ be the completion of R_j, which by Proposition 3.5 is again a model. Since each extension $R_j \subset R_{j+1}$ is strict by above, the corresponding morphism $\widehat{R}_j \to \widehat{R}_{j+1}$ is also injective by Remark 3.8. From Proposition 3.5 it follows that $\widehat{R}_j$ is Noetherian. By the Hilbert-Samuel theory, there exists for each j a polynomial P_j in one variable, of degree less than d, such that for all $n \gg 0$, we have

$$\ell(R_j/\mathfrak{m}_j^n) = \frac{e}{d!} n^d + P_j(n) \quad , \tag{24}$$

taking into account that $R_j/\mathfrak{m}_j^n \cong \widehat{R}_j/\mathfrak{m}_j^n \widehat{R}_j$. In fact, using Theorem 4.3.5, Corollary 3.6.14 and the remark after 3.6.19 in [6], this holds already for all $n > e$.

Therefore, take $J \geq J_{e+1}, \dots, J_{d+e}$, so that for $j, k \geq J$, we have by (23) and (24) that

$$P_j(e+i) = P_k(e+i) \ , \tag{25}$$

for all $i = 1, \dots, d$. As two polynomials of degree less than d which agree on d values must be identical, we conclude the proclaimed uniform version of (23).

For the same reason, the completion $\widehat{R}$ is Noetherian and in fact Cohen-Macaulay of dimension d, as $(\mu_1, \dots, \mu_d)$ remains $\widehat{R}$-regular (by an argument similar to the one in the proof of Proposition 3.5). We compute its multiplicity by means of the well-known formula

$$e_R(\mathfrak{m}) = \lim_{n \to \infty} \frac{d!}{n^d} \ell(R/\mathfrak{m}^n) \ , \tag{26}$$

where we used again that $R/\mathfrak{m}^n \cong \widehat{R}/\mathfrak{m}^n\widehat{R}$. The same formula, for $(R, \mathfrak{m})$ replaced by $(R_j, \mathfrak{m}_j)$ equals e, since the $\widehat{R}_j$ have multiplicity e by Proposition 3.5. Both quantities are though the same in view of the uniform version of (23). In other words, we showed that $\widehat{R}$ has multiplicity e. In particular, for any tuple $(a_1, \dots, a_d)$ in R, we have that $\widehat{R}/(a_1, \dots, a_d)\widehat{R}$ has length at least e. By Lemma 3.4, this quotient is isomorphic with the completion of $R/(a_1, \dots, a_d)$. Either the length of the latter is infinite and there is nothing to prove, or, it has finite length and therefore is equal to its own completion, that is to say

$$R/(a_1, \dots, a_d) \cong \widehat{R}/(a_1, \dots, a_d)\widehat{R} \tag{27}$$

has length at least e, as required. □ □

3.9. *Remark.* The above proof tells us that any model $(R, \mathfrak{m}, \mathfrak{g})$ of $\mathcal{C}\mathfrak{m}^+_{d,e}$ has a *Hilbert-Samuel polynomial*, i.e., that there exists a polynomial $\chi_R(Y)$ of degree d, such that for all $n > e$, we have that

$$\ell(R/\mathfrak{m}^n) = \chi_R(n) \ , \tag{28}$$

and, moreover, its leading coefficient is $e/d!$. In particular, e can be calculated by the limit in (26). Any chain of models has an eventually constant Hilbert-Samuel polynomial.

3.10. Theorem. *Let $(R, \mathfrak{m}, \mathfrak{g})$ be an equicharacteristic Noetherian model of $\mathcal{C}\mathfrak{m}^+_{d,e}$. Suppose that*

- (xi) *the residue field κ is algebraically closed,*
- (xii) *R is Gorenstein,*
- (xiii) *the Artin Approximation Property holds for R,*

then R is an existentially closed model in the class of all Noetherian models of $\mathcal{C}\mathfrak{m}^+_{d,e}$.

Proof. Let $(S, \mathfrak{n}, \mathfrak{h})$ be an extension of R in $\mathcal{C}\mathfrak{m}^+_{d,e}$ and let φ be an $\exists_1$-sentence (in the language $\mathcal{L}$) with parameters from R, holding true in S. Without loss of generality, we may assume that S is even complete, using Proposition 3.5. Hence S contains its own residue field λ, by Cohen's Structure Theorem. Suppose first that R is also complete, so that it contains κ. Note that $\mathfrak{g}$ is $\mathfrak{m}$-primary, and hence R (and S) is even $\mathfrak{g}$-adically complete. Since $R/\mathfrak{g} \subset S/\mathfrak{g}S$ and since both rings have length e with the former being moreover Gorenstein, we have by Theorem 2.3 in [11] that $R/\mathfrak{g} \otimes_\kappa \lambda \cong S/\mathfrak{g}S$. By Cohen's structure theorem for complete

Noetherian local rings, we can write $R = \kappa[[Y]]/I$, where I is some ideal in $\kappa[[Y]]$. Let $\tilde{R} = \lambda[[Y]]/I\lambda[[Y]]$, so that we proved that $\tilde{R}/\mathfrak{g}\tilde{R} \cong S/\mathfrak{g}S$. Moreover, one can easily make S into an $\tilde{R}$-algebra, as λ is contained in S and S is complete. From Theorem 8.4 in [9], we then conclude that also $\tilde{R} \cong S$. Using the infinite dimensional $\mathcal{L}$-variant of Proposition 2.4 in [11], we obtain that φ holds in R, as we wanted to show.

For the general case that R is not complete but has only the Artin Approximation property, let $\widehat{R}$ denote its completion, which by Proposition 3.5 belongs again to $\mathcal{C}\mathfrak{m}_{d,e}$. The latter embeds in S since we assumed S to be complete and any completion of a strict monomorphism is again a monomorphism by Corollary 1.1.9.6 in [5], where we have used Theorem 3.7. As $R/\mathfrak{g} \cong \widehat{R}/\mathfrak{g}\widehat{R}$ by Lemma 3.4 and using Exercise 18.1 in [9], we obtain that $\widehat{R}$ is Gorenstein. By above, $\widehat{R}$ is an existentially closed model of $\mathcal{C}\mathfrak{m}_{d,e}$ and hence φ holds true in it. Let $\varphi = (\exists \vec{x})\chi(\vec{x})$, with χ quantifier free. Analysing the structure of the possible quantifier free formulae in $\mathcal{L}$, we see that we may take χ as a disjunction of a conjunction of formulae of the form $F(x) = 0$ or $\neg\mathtt{Id}(F(x))$, where $F(X)$ is some polynomial over R. Hence, without loss of generality, we assume that χ is of the form

$$F_1(x) = \cdots = F_m(x) = 0 \quad \wedge \quad G_1(x), \dots, G_n(x) \notin \mathfrak{g} \ . \tag{29}$$

Let $\widehat{x}$ be a solution of (29) in $\widehat{R}$. By completeness, there exist $r_i \in R \backslash \mathfrak{g}$ and $\widehat{s}_{ij} \in \widehat{R}$, such that

$$G_i(\widehat{x}) = r_i + \sum_{j=1}^{d} \widehat{s}_{ij}\mu_j \ , \tag{30}$$

for all $i = 1, \dots, n$. Let $H_i(X, Y) = G_i(X) + r_i + \sum_j Y_{ij}\mu_j$, so that the system $F = H = 0$, where $F = (F_1, \dots, F_m)$ and $H = (H_1, \dots, H_n)$, has a solution $(\widehat{x}, \widehat{s})$ over $\widehat{R}$, where $\widehat{s} = (\widehat{s}_{ij})_{i,j}$. By Artin Approximation, we therefore have already a solution (x, s) over R. This simply means that $\chi(x)$ holds in R, as we wanted to show. □ □

3.11. *Remark.* In fact, if R satisfies the assumption of the theorem, then R is even existentially closed in the class of all separated models. Moreover, if S is any model of $\mathcal{C}\mathfrak{m}^+_{d,e}$ extending R, then every $\exists^+$-sentence over R which holds in S, already holds in R.

3.12. *Remark.* By SPIVAKOVSKI's theorem, the Artin Approximation property follows from the Henselian property, provided R is excellent. If one would succeed in giving a proof of the above theorem replacing the Artin Approximation by the Henselian property, one would obtain a new proof of SPIVAKOVSKI's result, under the additional assumptions that R is equicharacteristic and Gorenstein, with algebraically closed residue field.

3.13. *Remark.* Most likely, the converse also holds, i.e., (`xi`)–(`xiii`) hold in any existentially closed model in the class of Noetherian models of $\mathcal{C}\mathfrak{m}^+_{d,e}$. For $d = 0$, it follows from [11]. For general d, this is immediate for (`xiii`) and easy for (`xi`).

In [11] we were able to drop the equicharacteristic condition in the zero-dimensional case. Again it is very likely that we can also do this in the present situation. Probably one needs to use the theory of Witt vectors together with the techniques of loc. cit.

In Definition 1.2 we said that a local ring R has the *solvability property*, if whenever a polynomial system of equations has a solution modulo every power of the maximal ideal, then the system has a solution in R. We then observed that the solvability property implies Artin Approximation. Our last theorem states that the converse holds in the equicharacteristic case when the residue field is algebraically closed.

3.14. Theorem. *Let $(R, \mathfrak{m})$ be an equicharacteristic Noetherian local ring with algebraically closed residue field. The Artin Approximation holds for R, if and only if, the solvability property holds.*

Proof. We only need show the only if part. Firstly, it suffices to prove the theorem only for complete rings. Indeed, assuming the theorem proven for complete rings, let $\widehat{R}$ be the completion of R. Let $F_i(Y) \in R[Y]$, with $i = 1, \dots, m$ and $Y = (Y_1, \dots, Y_N)$ such that there exist, for each $k \in \mathbb{N}$, a tuple $y_k \in R^N$ with $F_i(y_k) \equiv 0 \mod \mathfrak{m}^k$. By our assumption, we can find a solution $\widehat{y} \in \widehat{R}^N$, i.e., $F(\widehat{y}) = 0$. Using Artin Approximation, it then follows that there exists $y \in R^N$, such that $F(y) = 0$, as required.

So assume that R is moreover complete. It is easy to check that if the solvability property holds for a ring R, then it holds for each of its homomorphic images as well. Therefore, it suffices to prove the solvability property for a regular complete local ring, in view of Theorem 29.4 in [9]. Hence we may assume that R is moreover regular, so its multiplicity is 1. Let d be its dimension. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and let R^* denote the ultrapower of R with respect to this ultrafilter. View R as a model of $\mathcal{C}\mathfrak{m}^+_{d,1}$. By Łos' Theorem, R^* is also a model of $\mathcal{C}\mathfrak{m}^+_{d,1}$. Let S be the completion of R^*, so that by Proposition 3.5 also S is a model of $\mathcal{C}\mathfrak{m}^+_{d,1}$. With F and $y_i = (t_{i,1}, \dots, t_{i,N}$ as above, let $t^*_j = [(t_{i,j})_i]$ be the image of the sequence $(t_{i,j})_i$ in R^*, for $j = 1, \dots, N$, and let $y^* = (t^*_1, \dots, t^*_N)$. By our hypothesis we have that

$$F_i(y^*) \in \bigcap_n \mathfrak{m}^n R^* \tag{31}$$

and hence $F_i(y^*) = 0$ in S. Since $R \subset R^*$ and R is Noetherian, we have that $R \subset S$. As R is an existentially closed model of the class of Noetherian models of $\mathcal{C}\mathfrak{m}^+_{d,e}$ by Theorem 3.10, we can find $y \in R^N$, such that $F_i(y) = 0$, for all $i = 1, \dots, m$. □ □

3.15. *Remark.* One can give a more direct proof in the equicharacteristic case, by showing that $\kappa[[X]] \prec_1 \lambda[[X]]$, for κ uncountable and algebraically closed together with the fact that the completion of the ultrapower of $\kappa[[X]]$ is isomorphic to $\kappa^*[[X]]$. However, if we could generalise Theorem 3.10 to the unequal characteristic case, then the above proof would remain valid. I do not know of any direct argument, avoiding the extension of Theorem 3.10, in the unequal characteristic case.

References

[1] Artin, M.: Algebraic approximation of structures over complete local rings. Publ. Math. I.H.E.S. **36** (1969) 23–58

[2] Artin, M.: On the solutions of analytic equations. Invent. Math. **5** (1968) 177–291

[3] Bass, H.: On the ubiquity of Gorenstein rings. Math. Z. **82** (1963) 8–28

[4] Becker, J., Denef, J., Lipshitz, L., van den Dries, L.: Ultraproducts and approximation in local rings I. Invent. Math. **51** (1979) 189–203

[5] Bosch, S., Güntzer, U., Remmert, R.: Non-archimedean analysis. Springer-Verlag, Berlin (1984)
[6] Bruns, W., Herzog, J.: Cohen-Macaulay rings. Cambridge Univ. Press, Cambridge (1993)
[7] Hodges, W.: Model Theory. Cambridge Univ. Press, Cambridge (1993)
[8] Jensen, C.U., Lenzing, H.: Model Theoretic Algebra. Algebra, Logic and Applications Series, Gordon and Breach Science Publishers **2**, Cambridge (1989)
[9] Matsumura, H.: Commutative ring theory. Cambridge University Press, Cambridge (1986)
[10] Robinson, A.: Introduction to model theory and to the metamathematics of algebra. North Holland, Amsterdam (1965)
[11] Schoutens, H.: Existentially Closed Models of the Theory of Artinian Local Rings. J. Symb. Logic (1998) (to appear)
[12] Schoutens, H.: The Model Theory of Artinian Local Gorenstein Rings. (1998) (preprint)
[13] Schoutens, H.: Bounds in cohomology. Israel J. Math. (1998) (to appear)
[14] Strooker, J.: Homological questions in local algebra. LMS Lect. Note Ser. **145**, Cambridge University Press (1990)