On the vanishing of Tor of the absolute integral closure

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Abstract

Let R be an excellent local domain of positive characteristic with residue field k and let R^+ be its absolute integral closure. If $\mathrm{Tor}_1^R(R^+, k)$ vanishes, then R is weakly F-regular. If R has at most an isolated singularity or has dimension at most two, then R is regular.

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1 Introduction

Recall that the *absolute integral closure* A^+ is defined for an arbitrary domain A as the integral closure of A inside an algebraic closure of the field of fractions of A. A key property of the absolute integral closure was discovered by Hochster and Huneke (1992): for R an excellent local domain of positive characteristic, R^+ is a *balanced big Cohen-Macaulay algebra*, that is to say, any system of parameters on R is an R^+-regular sequence. It is well-known that this implies that an excellent local domain R of positive characteristic is regular if, and only if, $R \to R^+$ is flat. Indeed, the direct implication follows since R^+ is a balanced big Cohen-Macaulay algebra of finite projective dimension (use for instance (Schoutens, 2003b, Theorem IV.1)) and the converse follows since $R \to R^+$ and $R^{1/p} \to R^+$ are isomorphic whence both faithfully flat, implying that $R \to R^{1/p}$ is flat, and therefore,

by Kunz's Theorem, that R is regular (here $R^{1/p}$ denotes the extension of R obtained by adding all p-th roots of element of R; for more details see (Huneke, 1996, Theorem 9.1 and Exercise 8.8)).

Huneke (1996, Exercise 8.8) points out that it is not known whether the weaker condition that all *Betti numbers* of R^+ vanish, that is to say, that all $\mathrm{Tor}_n^R(R^+, k)$ vanish for $n \geq 1$, already implies that R is regular. It is not hard to see, using that R^+ is a big Cohen-Macaulay algebra, that this is equivalent with requiring that only $\mathrm{Tor}_1^R(R^+, k)$ vanishes. The main result of this paper is then the following positive solution for isolated singularities.

Theorem 1.1 *Let $(R, \mathfrak{m})$ be an excellent local domain of positive characteristic with residue field k. Suppose R has either an isolated singularity or has dimension at most two. If $\mathrm{Tor}_1^R(R^+, k) = 0$, then R is regular.*

For arbitrary domains, we obtain at least the following.

Theorem 1.2 *Let $(R, \mathfrak{m})$ be an excellent local domain of positive characteristic with residue field k. If $\mathrm{Tor}_1^R(R^+, k) = 0$, then R is weakly F-regular. In particular, R is normal, Cohen-Macaulay, pseudo-rational and any finite extension of R is split (i.e., R is a* splinter*).*

We have some more precise information on the vanishing of certain *Tor*'s in terms of the singular locus of R.

Theorem 1.3 *Let $(R, \mathfrak{m})$ be an excellent local domain of positive characteristic and let $\mathfrak{a}$ be an ideal defining the singular locus of R (e.g., $\mathfrak{a}$ is the* Jacobian *of R). If $\mathrm{Tor}_1^R(R^+, k) = 0$, where k is the residue field of R, then $\mathrm{Tor}_n^R(R^+, M) = 0$ for all $n \geq 1$ and all finitely generated R-modules M for which $M/\mathfrak{a}M$ has finite length.*

The key observation in obtaining all these results, is that, in general, the vanishing of $\mathrm{Tor}_1^R(S, k)$ implies that $R \to S$ is *cyclically pure* (or *ideal-pure*), meaning that $IS \cap R = I$, for all ideals I of R. This is explained in Section 2. To prove Theorem 1.1, we need a result from (Schoutens, 2003b): if the first Betti number of a module over an isolated singularity vanishes, then the module has finite projective dimension. Now, the argument which proofs that $R \to R^+$ is flat when R is regular, yields the same conclusion under the weaker assumption that R^+ has finite projective dimension. This proves also the two-dimensional case, since we know already that R is normal.

Balanced big Cohen-Macaulay algebras in characteristic zero exist by the work of Hochster-Huneke, basically by a lifting procedure due to Hochster. However, the balanced big Cohen-Macaulay algebras obtained in (Hochster and Huneke, 1992) are not canonically defined. In (Schoutens, 2003a), I give an alternative but canonical construction $\mathcal{B}(R)$ of a balanced big Cohen-Macaulay algebra for a $\mathbb{C}$-affine

local domain R using ultraproducts and the absolute integral closure in positive characteristic. It follows from the present results that if $\operatorname{Tor}_1^R(\mathcal{B}(R), k) = 0$, where k is the residue field of R, then R is regular provided R has an isolated singularity or has dimension at most two (moreover,without these additional assumptions, R has at most rational singularities). This is the more interesting because it is not clear whether in general flatness of $R \to \mathcal{B}(R)$ implies regularity of R. For a further generalization to arbitrary excellent local domains, see the forthcoming Aschenbrenner and Schoutens (2003).

Acknowledgement

2 Vanishing of Betti numbers and cyclic purity

We derive a simple criterion for a local ring homomorphism to be cyclically pure. We start with an easy lemma, the proof of which is included for sake of completeness.

Lemma 2.1 *Let A be a ring, $\mathfrak{a}$ an ideal in A and M and N two A-modules. If $\mathfrak{a}N = 0$ and $\operatorname{Tor}_1^A(M, N) = 0$, then $\operatorname{Tor}_1^{A/\mathfrak{a}}(M/\mathfrak{a}M, N) = 0$.*

PROOF. One can derive this by aid of spectral sequences, but the following argument is more direct. Put $\bar{A} := A/\mathfrak{a}$. Since N is an $\bar{A}$-module, we can choose an exact sequence of $\bar{A}$-modules

$$0 \to \bar{H} \to \bar{F} \to N \to 0$$

with $\bar{F}$ a free $\bar{A}$-module. Tensoring with the $\bar{A}$-module $\bar{M} := M/\mathfrak{a}M$, we get an exact sequence

$$0 \to \operatorname{Tor}_1^{\bar{A}}(\bar{M}, N) \to \bar{M} \otimes_{\bar{A}} \bar{H} \to \bar{M} \otimes_{\bar{A}} \bar{F}.$$

Since the last two modules are equal to $M \otimes_A \bar{H}$ and $M \otimes_A \bar{F}$ respectively and since $\operatorname{Tor}_1^A(M, N) = 0$, the last morphism in this exact sequence is injective. Therefore, $\operatorname{Tor}_1^{\bar{A}}(\bar{M}, N) = 0$, as required. $\square$

Theorem 2.2 *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and let S be an arbitrary R-algebra. If $\operatorname{Tor}_1^R(S, k) = 0$ and $\mathfrak{m}S \neq S$, then $R \to S$ is*

cyclically pure. Moreover, if $\mathfrak{n}$ is an $\mathfrak{m}$-primary ideal, then

$$(\mathfrak{n} :_R I)S = (\mathfrak{n}S :_S IS)$$

for every ideal I in R.

PROOF. Since $\operatorname{Tor}_1^R(S, k)$ vanishes, so does $\operatorname{Tor}_1^{R/\mathfrak{n}}(S/\mathfrak{n}S, k)$ by Lemma 2.1, for every $\mathfrak{m}$-primary ideal $\mathfrak{n}$. By the Local Flatness Criterion (see (Matsumura, 1986, Theorem 22.3)) applied to the Artinian local ring $R/\mathfrak{n}$, the base change $R/\mathfrak{n} \to S/\mathfrak{n}S$ is flat, whence faithfully flat, since $\mathfrak{m}S \neq S$. In particular, this base change is injective, showing that $\mathfrak{n}S \cap R = \mathfrak{n}$. Since every ideal is the intersection of $\mathfrak{m}$-primary ideals by Krull's Intersection Theorem, the assertion follows.

The final assertion follows from the flatness of $R/\mathfrak{n} \to S/\mathfrak{n}S$ (use for instance (Matsumura, 1986, Theorem 7.4)). $\square$

Remark 2.3 *Note that with notation from the Theorem, we have that the induced map of affine schemes $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective, since the fiber rings $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ are non-zero.*

The following lemma shows that for a local Cohen-Macaulay ring, the vanishing of some Betti number of a big Cohen-Macaulay algebra is equivalent with the vanishing of all of its Betti numbers.

Lemma 2.4 *If $(R, \mathfrak{m})$ is a local Cohen-Macaulay ring with residue field k and if S is a big Cohen-Macaulay R-algebra, such that $\operatorname{Tor}_j^R(S, k) = 0$ for some $j \geq 1$, then $\operatorname{Tor}_n^R(S, k) = 0$, for all $n \geq 1$.*

PROOF. Let $\mathbf{x}$ be a maximal R-regular sequence which is also S-regular. Put $I := \mathbf{x}R$. Since $\operatorname{Tor}_j^R(S, k)$ vanishes, so does $\operatorname{Tor}_j^{R/I}(S/IS, k)$ by (Matsumura, 1986, Lemma 2, p.140), so that S/IS has finite flat dimension over R/I by the Local Flatness Criterion. However, since the finitistic weak dimension is at most the dimension of a ring by (Auslander and Buchsbaum, 1958, Theorem 2.4), it follows that S/IS is flat over R/I. Therefore, $0 = \operatorname{Tor}_n^{R/I}(S/IS, k) = \operatorname{Tor}_n^R(S, k)$, for all $n \geq 1$. $\square$

Therefore, below, we may replace everywhere the condition that $\operatorname{Tor}_1^R(S, k) = 0$ by the weaker condition that some $\operatorname{Tor}_j^R(S, k) = 0$, provided we also assume that R is Cohen-Macaulay. In fact, if j is either 1 or 2, we do not need to assume that R is Cohen-Macaulay, since this then holds automatically.

Proposition 2.5 *If $(R, \mathfrak{m})$ is a Noetherian local ring with residue field k and if S is a big Cohen-Macaulay R-algebra, such that either $\operatorname{Tor}_1^R(S, k)$ or $\operatorname{Tor}_2^R(S, k)$ vanishes, then R is Cohen-Macaulay.*

PROOF. I claim that $IS \cap R = I$, for some parameter ideal I of R. By a standard argument, it then follows that R is Cohen-Macaulay (see for instance the argument in (Schoutens, 2003a, Theorem 4.2)). For $j = 1$, we can use Lemma 2.1 to conclude that $\operatorname{Tor}_1^{R/I}(S/IS, k) = 0$, so that by the argument above, $R/I \to S/IS$ is faithfully flat. For $j = 2$, we reason as follows. Let

$$0 \to M \to F \to S \to 0$$

be a short exact sequence with F free. It follows that $\operatorname{Tor}_1^R(M, k)$ is equal to $\operatorname{Tor}_2^R(S, k)$, whence is zero. Therefore, by the same argument as before, M/IM is flat over R/I. On the other hand, since we may choose I so that it is generated by an S-regular sequence, we get that $\operatorname{Tor}_1^R(S, R/I) = 0$ (indeed, the canonical morphism $I \otimes M \to IM$ is easily seen to be injective). Hence we get an exact sequence

$$0 \to M/IM \to F/IF \to S/IS \to 0$$

showing that S/IS has finite flat dimension, whence is flat, since R/I is Artinian. $\square$

Is there a counterexample in which some $\operatorname{Tor}_j^R(S, k)$ vanishes for some big Cohen-Macaulay algebra S and some $j > 2$, without R being Cohen-Macaulay?

3   Proofs

Recall that an excellent local ring of positive characteristic is called *F-rational*, if some ideal generated by a system of parameters is tightly closed, and *weakly F-regular*, if every ideal is tightly closed. It is well-known that for excellent local rings, weakly F-regular implies splinter, and F-rational implies Cohen-Macaulay and normal ((Huneke, 1996, Theorem 4.2)). By (Smith, 1997, Theorem 3.1), an F-rational ring is pseudo-rational.

Proof of Theorem 1.2

Suppose R is as in the statement of the theorem, so that in particuar $\operatorname{Tor}_1^R(R^+, k)$ vanishes. By Theorem 2.2, the embedding $R \to R^+$ is cyclically pure. In order to show that R is weakly F-regular, it suffices to show by (Huneke, 1996, Theorem 1.5) that every $\mathfrak{m}$-primary ideal is tightly closed. Towards a contradiction, suppose

that $\mathfrak{n}$ is an $\mathfrak{m}$-primary ideal which is not tightly closed. Therefore, we can find a u in the tight closure of $\mathfrak{n}$ such that $(\mathfrak{n} :_R u) = \mathfrak{m}$. By Theorem 2.2, we have

$$(\mathfrak{n}R^+ :_{R^+} u) = \mathfrak{m}R^+. \tag{1}$$

By definition, there is a $c \in R$ not contained in any minimal prime of R such that $cu^q \in \mathfrak{n}^{[q]}$, for all powers $q = p^e$ (as usual, $I^{[q]}$ denotes the ideal generated by the q-th powers of elements in an ideal I). Since therefore $c^{1/q}u \in \mathfrak{n}R^+$, we get from (1) that $c^{1/q} \in \mathfrak{m}R^+$ whence $c \in \mathfrak{m}^q R^+$. By cyclical purity, $c \in \mathfrak{m}^q$ for all q, contradiction.

In particular, R is F-rational whence pseudo-rational, normal and Cohen-Macaulay (in fact, R is Cohen-Macaulay, by Proposition 2.5, and normal, by the cyclic purity of $R \to R^+$). Since R is normal, it follows from (Hochster, 1977) that $R \to R^+$ is pure. Let us give a direct argument for showing that R is a splinter. Let $R \subset S$ be a finite extension. In order to show that this is split, we may factor out a minimal prime of S and hence assume that S is a domain. So $R \subset S$ extends to the pure map $R \to R^+$ and hence is itself pure. Since a pure map with finitely generated cokernel is split ((Matsumura, 1986, Theorem 7.14)), we showed that any finite extension splits. $\square$

Proof of Theorem 1.1

The vanishing of $\mathrm{Tor}_1^R(R^+, k)$ implies that R is Cohen-Macaulay by Theorem 1.2. Since R^+ is a balanced big Cohen-Macaulay algebra and since R has an isolated singularity, we get from (Schoutens, 2003b, Theorem IV.1) that $R \to R^+$ is flat. As already observed, this implies that R is regular. If R has dimension at most 2, then by Theorem 1.2, it is normal and therefore has an isolated singularity, so that the previous argument applies. $\square$

Recall that by the argument at the end of the previous section, the vanishing of a single $\mathrm{Tor}_j^R(R^+, k)$ implies already that R is regular, if apart from being an isolated singularity, we also assume that R is Cohen-Macaulay, when $j \geq 3$. In order to derive a regularity criterion from Theorem 1.1, we need a lemma on flatness over Artinian local Gorenstein rings of embedding dimension one.

Lemma 3.1 *Let $(A, \mathfrak{m})$ be an Artinian local ring of embedding dimension one and let M be an arbitrary A-module. Then M is A-flat if, and only if, $\mathrm{Ann}_M(I) = \mathfrak{m}M$, where I denotes the socle of A, that is to say, $I = \mathrm{Ann}_A(\mathfrak{m})$.*

PROOF. By assumption $\mathfrak{m} = xA$, for some $x \in A$. It follows that the socle I of A is equal to $x^{e-1}A$, where e is the smallest integer for which $x^e = 0$. I claim that $\mathrm{Ann}_M(x^{e-i}) = x^i M$, for all i. We will induct on i, where the case $i = 1$ is just our assumption. For $i > 1$, let $\mu \in M$ be such that $x^{e-i}\mu = 0$. Therefore,

$x^{e-i+1}\mu = 0$, so that by our induction hypothesis, $\mu \in x^{i-1}M$, say, $\mu = x^{i-1}\nu$. Since $0 = x^{e-i}\mu = x^{e-1}\nu$, we get $\nu \in xM$ whence $\mu \in x^i M$, as required.

Flatness now follows by the Local Flatness Criterion (Matsumura, 1986, Theorem 22.3). Indeed, it suffices to show that $A/xA \to M/xM$ is flat and $xA \otimes M \cong xM$. The first assertion is immediate since A/xA is a field. For the second assertion, observe that $xA \cong A/x^{e-1}A$ and by what we just proved $xM \cong M/\operatorname{Ann}_M(x) \cong M/x^{e-1}M$. It follows that $xA \otimes M$ is isomorphic with xM, as required. $\quad\square$

Corollary 3.2 *Let $(R, \mathfrak{m})$ be a d-dimensional excellent local Cohen-Macaulay domain of positive characteristic. Suppose that there exists an ideal I in R generated by a regular sequence such that $\mathfrak{m}/I$ is a cyclic module. Suppose also that R has either an isolated singularity or that $d \leq 2$. If for each finite extension domain $R \subset S$, we can find a finite extension $S \subset T$, such that*

$$(IS :_S (I :_R \mathfrak{m})S) \subset \mathfrak{m}T, \tag{2}$$

then R is regular.

PROOF. Let $(x_1, \ldots, x_i)$ be the regular sequence generating I and write $\mathfrak{m} = I + xR$. If $i < d$ then necessary $i = d - 1$ and $\mathfrak{m}$ is generated by d elements, so R is regular. Hence assume $i = d$, that is to say, I is $\mathfrak{m}$-primary. It follows that $\overline{R} := R/I$ is an Artinian local ring with maximal ideal $x\overline{R}$. Let e be the smallest integer for which $x^e \in I$. Hence the socle of $\overline{R}$ is $x^{e-1}\overline{R}$. Let $\overline{R^+} := R^+/IR^+$. I claim that

$$\operatorname{Ann}_{\overline{R^+}}(x^{e-1}) = x\overline{R^+}.$$

Assuming the claim, Lemma 3.1 yields that $\overline{R^+}$ is $\overline{R}$-flat. Therefore, if k is the residue field of R, then $\operatorname{Tor}_1^{\overline{R}}(\overline{R^+}, k) = 0$. But $(x_1, \ldots, x_d)$ is both R-regular and R^+-regular, so that $\operatorname{Tor}_1^R(R^+, k) = 0$. Regularity of R then follows from Theorem 1.1.

To prove the claim, one inclusion is clear, so assume that $a \in R^+$ is such that $ax^{e-1} \in IR^+$. Choose a finite extension $R \subset S \subset R^+$ containing a and such that we already have a relation $ax^{e-1} \in IS$. By assumption, we can find a finite extension T of S, such that $(IS : x^{e-1}) \subset \mathfrak{m}T$. Hence $a \in \mathfrak{m}T$. Since T maps to R^+, we get $a \in \mathfrak{m}R^+$, and hence $a \in x\overline{R^+}$, as we wanted to show. $\quad\square$

The condition that $\mathfrak{m}$ is cyclic modulo a regular sequence is in this case equivalent with R being Cohen-Macaulay with regularity defect at most one (recall that the *regularity defect* of R is by definition the difference between its embedding dimension and its Krull dimension). If R is regular, then (2) is true for any $\mathfrak{m}$-primary ideal I of R (use the fact that $R \to R^+$ is flat).

Proof of Theorem 1.3

Let $(R, \mathfrak{m})$ be as in the statement of Theorem 1.3. In particular, R is Cohen-Macaulay by Theorem 1.2. Let M be a finitely generated R-module such that $M/\mathfrak{a}M$ has finite length. Let I be the annihilator of M. By Nakayama's Lemma, $M/\mathfrak{a}M$ having finite length implies that $I + \mathfrak{a}$ is $\mathfrak{m}$-primary. We will induct on the dimension e of M. If $e = 0$, so that M has finite length, the vanishing of $\operatorname{Tor}_n^R(R^+, M)$ follows from Lemma 2.4 and a well-known inductive argument on the length of M (see for instance (Schoutens, 2003b, Corollary II.6)). Hence assume $e > 0$ and let H be the largest submodule of finite length in M. The *Tor* long exact sequence obtained from

$$0 \to H \to M \to M/H \to 0$$

shows that it suffices to prove the result for M/H instead of M. Therefore, after modding out H, me may assume that M has positive depth. By prime avoidance and since $I + \mathfrak{a}$ is $\mathfrak{m}$-primary, we can find an M-regular element $x \in \mathfrak{a}$. The short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

gives rise to a long exact sequence

$$\operatorname{Tor}_{n+1}^R(R^+, M/xM) \to \operatorname{Tor}_n^R(R^+, M) \xrightarrow{x} \operatorname{Tor}_n^R(R^+, M),$$

for all $n \geq 1$. Since the left most module is zero by induction on e, multiplication with x on $\operatorname{Tor}_n^R(R^+, M)$ is injective, for all $n \geq 1$. In particular, we have for each n an embedding

$$\operatorname{Tor}_n^R(R^+, M) \subset (\operatorname{Tor}_n^R(R^+, M))_x = \operatorname{Tor}_n^{R_x}((R^+)_x, M_x). \tag{3}$$

Since $x \in \mathfrak{a}$, the localization R_x is regular. Therefore, $R_x \to (R_x)^+$ is flat. An easy calculation shows that $(R_x)^+ = (R^+)_x$ (see (Hochster and Huneke, 1992, Lemma 6.5)). In particular, $\operatorname{Tor}_n^{R_x}((R^+)_x, M_x) = 0$, and hence $\operatorname{Tor}_n^R(R^+, M) = 0$ by (3). $\square$

If R has dimension three, then $\operatorname{Tor}_n^R(R^+, R/\mathfrak{p})$ vanishes for every $n \geq 1$ and every prime ideal $\mathfrak{p}$ of R not in the singular locus of R, since R is normal by Theorem 1.2 and hence $\mathfrak{a}$ has height at least two. On the other hand, we have the following non-vanishing result.

Corollary 3.3 *Let $(R, \mathfrak{m})$ be an excellent local domain of positive characteristic. If $\mathfrak{p}$ is a prime ideal defining an irreducible component of the singular locus of R, then $\operatorname{Tor}_1^R(R^+, R/\mathfrak{p})$ is non-zero.*

PROOF. Assume $\operatorname{Tor}_1^R(R^+, R/\mathfrak{p})$ vanishes. Hence so does $\operatorname{Tor}_1^{R_\mathfrak{p}}((R^+)_\mathfrak{p}, k(\mathfrak{p}))$, where $k(\mathfrak{p})$ is the residue field of $\mathfrak{p}$. Since $(R^+)_\mathfrak{p}$ is equal to $(R_\mathfrak{p})^+$ by (Hochster and Huneke, 1992, Lemma 6.5), and since $R_\mathfrak{p}$ has an isolated singularity, it follows from Theorem 1.1 that $R_\mathfrak{p}$ is regular, contradicting the choice of $\mathfrak{p}$. $\square$

In view of Lemma 2.4 we can generalize this even further: if R is Cohen-Macaulay, then each $\operatorname{Tor}_n^R(R^+, R/\mathfrak{p})$ is non-zero, for $n \geq 1$ and for $\mathfrak{p}$ defining an irreducible component of the singular locus of R.

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