

ASYMPTOTIC HOMOLOGICAL CONJECTURES IN MIXED CHARACTERISTIC

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ABSTRACT. In this paper, various Homological Conjectures are studied for local rings which are locally finitely generated over a discrete valuation ring V of mixed characteristic. Typically, we can only conclude that a particular Conjecture holds for such a ring provided the residual characteristic of V is sufficiently large in terms of the complexity of the data, where the complexity is primarily given in terms of the degrees of the polynomials over V that define the data, but possibly also by some additional invariants such as (homological) multiplicity. Thus asymptotic versions of the Improved New Intersection Theorem, the Monomial Conjecture, the Direct Summand Conjecture, the Hochster-Roberts Theorem and the Vanishing of Maps of Tors Conjecture are given.

That the results only hold asymptotically, is due to the fact that non-standard arguments are used, relying on the Ax-Kochen-Ershov Principle, to infer their validity from their positive characteristic counterparts. A key role in this transfer is played by the Hochster-Huneke canonical construction of big Cohen-Macaulay algebras in positive characteristic via absolute integral closures.

1. INTRODUCTION

In the last three decades, all the so-called Homological Conjectures have been settled completely for Noetherian local rings containing a field by work of Peskine-Szpiro, Hochster-Roberts, Hochster, Evans-Griffith, et. al. (some of the main papers are [11, 15, 16, 22, 28]). More recently, Hochster-Huneke have given more simplified proofs of most of these results by means of their tight closure theory, including their canonical construction of big Cohen-Macaulay algebras in positive characteristic (see [19, 20, 21, 24]; for further discussion and proofs, see [7, §9] or [48]).

In sharp contrast is the development in mixed characteristic, where only sporadic results (often in low dimensions) are known, apart from the break-through [30] by Roberts, settling the New Intersection Theorem for all Noetherian local rings, and the recent work [13] of Heitmann in dimension three. Some attempts have been made by Hochster, either by finding a suitable substitute for tight closure in mixed characteristic [17], or by constructing big Cohen-Macaulay modules in mixed characteristic [14]. These approaches have yet to bear fruit and the best result to date in this direction is the existence of big Cohen-Macaulay algebras in dimension three [18], which in turn relies on the positive solution of the Direct Summand Conjecture in dimension three by Heitmann [13].

In this paper, we will follow the big Cohen-Macaulay algebra approach, but instead of trying to work with rings of Witt vectors, we will use the Ax-Kochen-Ershov Principle [4, 9, 10], linking complete discrete valuation rings in mixed characteristic with

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complete discrete valuation rings in positive characteristic via an equicharacteristic zero (non-discrete) valuation ring (see Theorem 2.3 below). This intermediate valuation ring is obtained by a construction which originates from logic, but is quite algebraic in nature, to wit, the *ultraproduct* construction. Roughly speaking, this construction associates to an infinite collection of rings C_w their ultraproduct C_∞ , which should be thought of as a kind of “limit” or “average” (realized as a certain homomorphic image of the product). An ultraproduct inherits many of the algebraic properties of its components. The correct formulation of this transfer principle is Łos’ Theorem, which makes precise when a property carries over (namely, when it is first order definable in some suitable language). Properties that carry over are those of being *a domain, a field, a valuation ring, local, Henselian*; among the properties that do not carry over is *Noetherianity*, so that almost no ultraproduct is Noetherian (except an ultraproduct of fields or of Artinian rings of bounded length). This powerful tool is used in [32, 33, 35, 44], to obtain uniform bounds in polynomial rings over fields; in [35, 36, 37, 40], to transfer properties from positive to zero characteristic; and in [3, 39, 41, 42, 47], to give an alternative treatment of tight closure theory in equicharacteristic zero. The key fact in the first set of papers is a certain flatness result about ultraproducts (see Theorem 2.2 below for a precise formulation), and in the two last sets, the so-called *Lefschetz Principle for algebraically closed fields* (the ultraproduct of the algebraic closures of the p -element fields \mathbb{F}_p is isomorphic to \mathbb{C}).

The Ax-Kochen-Ershov Principle is a kind of *Lefschetz Principle for Henselian valued fields*, and its most concrete form states that the ultraproduct of all $\mathbb{F}_p[[t]]$, with t a single indeterminate, is isomorphic to the ultraproduct of all rings of p -adic integers \mathbb{Z}_p . We will identify both ultraproducts and denote the resulting ring by \mathfrak{D} . It follows that \mathfrak{D} is an equicharacteristic zero Henselian valuation ring with principal maximal ideal, whose separated quotient (=the reduction modulo the intersection of all powers of the maximal ideal) is an equicharacteristic zero excellent complete discrete valuation ring.

Z -affine algebras. To explain the underlying idea in this paper, we introduce some notation. Let (Z, \mathfrak{p}) be a (not necessarily Noetherian) local ring. A *Z -affine algebra* C is any Z -algebra of the form $C = Z[X]/I$ where X is a finite tuple of indeterminates and I a finitely generated ideal in $Z[X]$. A *local Z -affine algebra* is any localization $R = C_{\mathfrak{m}}$ of a Z -affine algebra C with respect to a prime ideal \mathfrak{m} of C lying above \mathfrak{p} . In particular, the natural homomorphism $Z \rightarrow R$ is local. We denote the category of all local Z -affine algebras by $\mathbf{Aff}(Z)$.

The objective is to transfer algebraic properties (such as the homological Conjectures) from the positive characteristic categories $\mathbf{Aff}(\mathbb{F}_p[[t]])$ to the mixed characteristic categories $\mathbf{Aff}(\mathbb{Z}_p)$. This will be achieved through the intermediate equicharacteristic zero category $\mathbf{Aff}(\mathfrak{D})$. As this latter category consists mainly of non-Noetherian rings, we will have to find analogues in this setting of many familiar notions from commutative algebra, such as dimension, depth, Cohen-Macaulayness or regularity (see §§5 and 6).

The following example is paradigmatic: let X be a finite tuple of indeterminates and let $\mathcal{L}_{\mathfrak{D}}^{\text{eq}}(A)$ be the ultraproduct of all $\mathbb{F}_p[[t]][X]$, and $\mathcal{L}_{\mathfrak{D}}^{\text{mix}}(A)$, the ultraproduct of all $\mathbb{Z}_p[X]$. Note that both rings contain \mathfrak{D} , and in fact, contain $\mathfrak{D}[X]$. The key algebraic fact, which is equivalent to a result on effective bounds by Aschenbrenner ([2]), is that both inclusions $\mathfrak{D}[X] \subseteq \mathcal{L}_{\mathfrak{D}}^{\text{eq}}(A)$ and $\mathfrak{D}[X] \subseteq \mathcal{L}_{\mathfrak{D}}^{\text{mix}}(A)$ are flat. Suppose we have in each $\mathbb{F}_p[[t]][X]$ a polynomial f_p , and let f_∞ be their ultraproduct. A priori, we can only say that $f_\infty \in \mathcal{L}_{\mathfrak{D}}^{\text{eq}}(A)$. However, if all f_p have X -degree d , for some d independent from p , then f_∞ itself is a polynomial over \mathfrak{D} of degree d (since an ultraproduct commutes with finite sums by Łos’ Theorem). Hence, as f_∞ lies in $\mathfrak{D}[X]$, we can also view it as an element in

$\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(A)$. Therefore, there are polynomials $\tilde{f}_p \in \mathbb{Z}_p[X]$ whose ultraproduct is equal to f_{∞} . The choice of the \tilde{f}_p is not unique, but any two choices will be equal for almost all p , by Łos' Theorem. In conclusion, to a collection of polynomials defined over the various $\mathbb{F}_p[[t]]$, of uniformly bounded degree, we can associate, albeit not uniquely, a collection of polynomials defined over the various \mathbb{Z}_p (of uniformly bounded degree), and of course, this also works the other way. Instead of doing this for just one polynomial in each component, we can now do this for a finite tuple of polynomials of fixed length. If at the same time, we can maintain certain algebraic relations among them (characterizing one of the properties we seek to transfer), we will have achieved our goal.

Unfortunately, it is the nature of an ultraproduct that it only captures the “average” property of its components. In the present context, this means that the desired property does not necessarily hold in all $\mathbb{Z}_p[X]$, but only in almost all. In conclusion, we cannot hope for a full solution of the Homological Conjectures by this method, but only an *asymptotic* solution. In view of the above, the following definition is now natural.

Complexity. Let C be a Z -affine algebra, say, of the form $C = Z[X]/I$, with X a finite tuple of indeterminates and I a finitely generated ideal, and let $R = C_{\mathfrak{m}}$ be a local Z -affine algebra (so that $\mathfrak{p} \subseteq \mathfrak{m}$). We say that C has *Z -complexity* at most c , if $|X| \leq c$ and I is generated by polynomials of degree at most c ; we say that R has *Z -complexity* at most c , if, moreover, also \mathfrak{m} is generated by polynomials of degree at most c . An element $r \in C$ is said to have *Z -complexity* at most c , if C has Z -complexity at most c and r is the image of a polynomial in $Z[X]$ of degree at most c . An element $r \in R$ has *Z -complexity* at most c , if R has Z -complexity at most c and if r is (the image of) a quotient P/Q of polynomials of degree at most c with $Q \notin \mathfrak{m}$. We say that a tuple or a matrix has *Z -complexity* at most c , if each of its entries has Z -complexity at most c and the number of entries is also bounded by c . Note that in a Z -affine algebra, the sum of two elements of Z -complexity at most c , has again Z -complexity at most c , whereas in a *local* Z -affine algebra, the sum has Z -complexity at most $2c$.

An ideal J in C or R has *Z -complexity* at most c , if it is generated by a tuple of Z -complexity at most c . A Z -algebra homomorphism $C \rightarrow C'$ or a local Z -algebra homomorphism $R \rightarrow R'$ is said to have *Z -complexity* at most c , if C and C' (respectively, R and R') are (local) Z -affine algebras of Z -complexity at most c and the homomorphism is given by sending each indeterminate X_i to an element of Z -complexity at most c .

Asymptotic properties. Let \mathbf{P} be a property of Noetherian local rings (possibly involving some additional data). We will use the phrase *\mathbf{P} holds asymptotically in mixed characteristic*, to express that for each c , we can find a bound c' , such that if V is a complete discrete valuation ring of mixed characteristic and C a local V -affine algebra of V -complexity at most c (and a similar bound on the additional data), then property \mathbf{P} holds for C , provided the characteristic of the residue field of V is at least c' . Sometimes, we have to control some additional invariants in terms of the bound c . In this paper, we will prove that in this sense, many Homological Conjectures hold asymptotically in mixed characteristic.

A final note. Its asymptotic nature is the main impediment of the present method to carry out Hochster's program of obtaining tight closure and big Cohen-Macaulay algebras in mixed characteristic. For instance, despite the fact that we are able to define an analogue of a balanced big Cohen-Macaulay algebra for \mathfrak{D} -affine domains, this object cannot be realized as an ultraproduct of \mathbb{Z}_p -algebras, so that there is no candidate so far for a big Cohen-Macaulay in mixed characteristic. Although I will not pursue this line of thought in

this paper, one could also define some non-standard closure operation on ideals in \mathfrak{D} -affine algebras, but again, such an operation will only partially descend to any component.

Notation. A *tuple* \mathbf{x} over a ring Z is always understood to be finite. Its length is denoted by $|\mathbf{x}|$ and the ideal it generates is denoted $\mathbf{x}Z$. When we say that (Z, \mathfrak{p}) is *local*, we mean that \mathfrak{p} is its (unique) maximal ideal, but we do not imply that Z has to be Noetherian.

For a survey of the results and methods in this paper, see [38]. In the forthcoming [45] some of the present asymptotic versions will be generalized through a further investigation of the algebraic properties of ultraproducts using the notions introduced in §§5 and 6.

2. ULTRAPRODUCTS

In this preliminary section, I state some generalities about ultraproducts and then briefly review the situation in equicharacteristic zero and the Ax-Kochen-Ershov Principle. The next section lays out the essential tools for conducting the transfer discussed in the introduction, to wit, approximations, restricted ultraproducts and non-standard hulls, whose properties are then studied in §§5 and 6. The subsequent sections contain proofs of various asymptotic results, using these tools.

Whenever we have an infinite index set W , we will equip it with some (unnamed) countably incomplete non-principal ultrafilter; ultraproducts will always be taken with respect to this ultrafilter and we will write

$$\operatorname{ulim}_{w \rightarrow \infty} O_w \quad \text{or simply} \quad O_\infty$$

for the ultraproduct of objects O_w (this will apply to rings, ideals and elements alike). A first introduction to ultraproducts, including Łos' Theorem, sufficient to understand the present paper, can be found in [39, §2]; for a more detailed treatment, see [23]. Łos' Theorem states essentially that if a fixed algebraic relation holds among finitely many elements f_{1w}, \dots, f_{sw} in each ring C_w , then the same relation holds among their ultraproducts $f_{1\infty}, \dots, f_{s\infty}$ in the ultraproduct C_∞ , and conversely, if such a relation holds in C_∞ , then it holds in almost all C_w . Here *almost all* means "for all w in a subset of the index set which belongs to the ultrafilter" (the idea is that sets belonging to the ultrafilter are *large*, whereas the remaining sets are *small*).

An immediate, but important application of Łos' Theorem is that the ultraproduct of algebraically closed fields of different prime characteristics is an (uncountable) algebraically closed field of characteristic zero, and any sufficiently large algebraically closed field of characteristic zero, including \mathbb{C} , can be realized thus.¹ This simple observation, in combination with work of van den Dries on non-standard polynomials (see below), was exploited in [39] to define an alternative version of tight closure for \mathbb{C} -affine algebras, called *non-standard tight closure*, which was then further generalized to arbitrary Noetherian local rings containing the rationals in [3]. The ensuing notions of F-regularity and F-rationality have been proven to be more versatile [41, 42, 47] than those defined by Hochster-Huneke in [21].

Let me briefly recall the results in [33, 49] on non-standard polynomials mentioned above. Let K_w be fields (of arbitrary characteristic) with ultraproduct K_∞ (which is again a field by Łos' Theorem). Let X be a fixed finite tuple of indeterminates and set $A := K_\infty[X]$ and $A_w := K_w[X]$. Let A_∞ be the ultraproduct of the A_w . As in the example

¹To be more precise, any algebraically closed field of characteristic zero whose cardinality is of the form 2^λ for some infinite cardinal λ , is an ultraproduct of algebraically closed fields of prime characteristic; under the generalized continuum hypothesis this means *every* uncountable algebraically closed field of characteristic zero.

discussed in the introduction, we have a canonical embedding of A inside A_∞ . In fact, the following easy observation, valid over arbitrary rings, describes completely the elements in A_∞ that lie in A (the proof is straightforward and left to the reader).

2.1. Lemma. *Let X be a finite tuple of indeterminates. Let C_w be rings and let C_∞ be their ultraproduct. If f_w is a polynomial in $C_w[X]$ of degree at most c , for each w and for some c independent from w , then their ultraproduct in $\text{ulim}_{w \rightarrow \infty} C_w[X]$ belongs already to the subring $C_\infty[X]$, and conversely, every element in $C_\infty[X]$ is obtained in this way.*

This result also motivates the notion of *complexity* from the introduction. Returning to the Schmidt-van den Dries results, the following two properties of the embedding $A \subseteq A_\infty$ do not only imply the uniform bounds from [33, 35], but also play an important theoretical role in the development of non-standard tight closure [3, 39].

2.2. Theorem (Schmidt-van den Dries). *The embedding $A \subseteq A_\infty$ is faithfully flat and every prime ideal in A extends to a prime ideal in A_∞ .*

To carry out the present program, we have to replace the base fields K_w by complete discrete valuation rings \mathfrak{D}_w . Unfortunately, we now have to face the following complications. Firstly, the ultraproduct \mathfrak{D} of the \mathfrak{D}_w is no longer Noetherian, and so in particular the corresponding $A := \mathfrak{D}[X]$ is non-Noetherian. Moreover, the embedding $A \subseteq A_\infty$, where A_∞ is now the ultraproduct of the $A_w := \mathfrak{D}_w[X]$, although flat (see Theorem 4.2 below), is no longer faithfully flat (this is related to Dedekind's problem; see [2] or [46] for details). Furthermore, not every prime ideal extends to a prime ideal. However, by working locally, we can circumvent all the latter complications (see Theorem 4.2 and Remark 4.5).

To obtain the desired transfer, we will realize \mathfrak{D} in two different ways, as an ultraproduct of complete discrete valuation rings in positive characteristic and as an ultraproduct of complete discrete valuation rings in mixed characteristic, and then pass from one set to the other via \mathfrak{D} , as explained in the introduction (for more details, see §6.9 below). This is the celebrated Ax-Kochen-Ershov Principle [4, 9, 10], and I will discuss this now. For each w , let $\mathfrak{D}_p^{\text{mix}}$ be a complete discrete valuation ring of mixed characteristic with residue field κ_p of characteristic p . To each $\mathfrak{D}_p^{\text{mix}}$, we associate a corresponding equicharacteristic complete discrete valuation ring with the same residue field, by letting

$$(1) \quad \mathfrak{D}_p^{\text{eq}} := \kappa_p[[t]]$$

where t is a single indeterminate.

2.3. Theorem (Ax-Kochen-Ershov). *The ultraproduct of the $\mathfrak{D}_p^{\text{eq}}$ is isomorphic (as a local ring) with the ultraproduct of the $\mathfrak{D}_p^{\text{mix}}$.*

2.4. Remark. As stated, we need to assume the continuum hypothesis. Otherwise, by the Keisler-Shelah Theorem [23, Theorem 9.5.7], one might need to take further ultrapowers, that is to say, over a larger index set. In order to not complicate the exposition, I will nonetheless make the set-theoretic assumption, so that our index set can always be taken to be the set of prime numbers. The reader can convince himself that all proofs in this paper can be adjusted so that they hold without any set-theoretic assumption.

To conclude this section, I state a variant of Prime Avoidance which also works in mixed characteristic (note that for non-prime ideals one normally has to assume that the ring contains a field, see for instance [8, Lemma 3.3]).

2.5. Proposition. *Let Z be a local ring with infinite residue field κ . Let C be an arbitrary Z -algebra and let W be a finitely generated Z -submodule of C . If $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ are ideals in C not containing W , then there exists $f \in W$ not contained in any of the \mathfrak{a}_j .*

Proof. We induct on the number t of ideals to be avoided, where the case $t = 1$ holds by assumption. Hence assume $t > 1$. By induction, we can find elements $g_i \in W$, for $i = 1, 2$, which lie outside any \mathfrak{a}_j for $j \neq i$. If either $g_1 \notin \mathfrak{a}_1$ or $g_2 \notin \mathfrak{a}_2$ we are done, so assume $g_i \in \mathfrak{a}_i$. Therefore, every element of the form $g_1 + z g_2$ with z a unit in Z does not lie in \mathfrak{a}_1 nor in \mathfrak{a}_2 . Since κ is infinite, we can find $t - 1$ units z_1, z_2, \dots, z_{t-1} in Z whose residues in κ are all distinct. I claim that at least one of the $g_1 + z_i g_2$ lies outside all \mathfrak{a}_j , so that we found our desired element in W . Indeed, if not, then each $g_1 + z_i g_2$ lies in one of the $t - 2$ ideals $\mathfrak{a}_3, \dots, \mathfrak{a}_t$, by our previous remark. By the Pigeon Hole Principle, for some j and some $l \neq k$, we have that $g_1 + z_k g_2$ and $g_1 + z_l g_2$ lie both in \mathfrak{a}_j . Hence so does their difference $(z_k - z_l)g_2$. However, $z_k - z_l$ is a unit in Z , by choice of the z_i , so that $g_2 \in \mathfrak{a}_j$, contradiction. \square

2.6. Corollary (Controlled Ideal Avoidance). *Let Z be a local ring with infinite residue field and let C be a (local) Z -affine algebra. If I and $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ are ideals in C with I not contained in any \mathfrak{a}_i , then I contains an element outside every \mathfrak{a}_i . More precisely, if c is an upper bound for the Z -complexity of I , then there exists an element $f \in I$ of Z -complexity at most c^2 , not contained in any \mathfrak{a}_i .*

Proof. Let (x_1, \dots, x_n) be a tuple of Z -complexity at most c generating I and let W be the Z -submodule of C generated by (x_1, \dots, x_n) . In particular, W is not contained in any \mathfrak{a}_i , so that we may apply Proposition 2.5 to obtain an element $f \in W$, outside each \mathfrak{a}_i . Write $f = z_1 x_1 + \dots + z_n x_n$ with $z_i \in Z$. After putting on a common denominator, we see that f has Z -complexity at most $cn \leq c^2$ (in case C is not local, the Z -complexity of f is in fact at most c). \square

It is clear from the proof of Proposition 2.5 that in both results, we only need the residue field to have a larger cardinality than the number of ideals to be avoided.

3. APPROXIMATIONS, RESTRICTED ULTRAPRODUCTS AND NON-STANDARD HULLS

In this section, some general results on ultraproducts of finitely generated algebras over discrete valuation rings will be derived. We start with introducing some general terminology, over arbitrary Noetherian local rings, but once we start proving some non-trivial properties in the next sections, we will specialize to the case that the base rings are discrete valuation rings. For some results in the general case, we refer to [44, 45, 46].

For each w , we fix a Noetherian local ring \mathfrak{O}_w and let \mathfrak{D} be its ultraproduct. If the \mathfrak{p}_w are the maximal ideals of the \mathfrak{O}_w , then their ultraproduct \mathfrak{p} is the maximal ideal of \mathfrak{D} . We will write $\mathfrak{I}_{\mathfrak{D}}$ for the ideal of *infinitesimals* of \mathfrak{D} , that is to say, the intersection of all the powers \mathfrak{p}^k (note that in general $\mathfrak{I}_{\mathfrak{D}} \neq (0)$ and therefore, \mathfrak{D} is in particular non-Noetherian).

By saturatedness of ultraproducts, \mathfrak{D} is quasi-complete in its \mathfrak{p} -adic topology in the sense that any Cauchy sequence has a (non-unique) limit. Hence the completion of \mathfrak{D} is $\mathfrak{D}/\mathfrak{I}_{\mathfrak{D}}$ (see also Lemma 5.3 below). Moreover, we will assume that all \mathfrak{O}_w have embedding dimension at most ϵ . Hence so do \mathfrak{D} and $\mathfrak{D}/\mathfrak{I}_{\mathfrak{D}}$. Since a complete local ring with finitely generated maximal ideal is Noetherian ([27, Theorem 29.4]), we showed that $\mathfrak{D}/\mathfrak{I}_{\mathfrak{D}}$ is a Noetherian complete local ring. For more details in the case of interest to us, where each \mathfrak{O}_w is a discrete valuation ring or a field, see [5].

We furthermore fix throughout a tuple of indeterminates $X = (X_1, \dots, X_n)$, and we set $A := \mathfrak{D}[X]$ and $A_w := \mathfrak{O}_w[X]$.

3.1. Definition. The *non-standard \mathfrak{D} -hull* of A is by definition the ultraproduct of the A_w and is denoted $\mathfrak{L}_{\mathfrak{D}}(A)$.

This terminology is a little misleading, because $\mathcal{L}_{\mathfrak{D}}(A)$ does not only depend on \mathfrak{D} but also on the choice of \mathfrak{D}_w whose ultraproduct is \mathfrak{D} . In fact, we will exploit this dependence when applying the Ax-Kochen-Ershov principle, in which case we have to declare more precisely which non-standard \mathfrak{D} -hull is meant. Nonetheless, whenever \mathfrak{D} and \mathfrak{D}_w are clear from the context, we will denote the non-standard \mathfrak{D} -hull of A simply by $\mathcal{L}(A)$.

By Łos' Theorem, we have an inclusion $\mathfrak{D} \subseteq \mathcal{L}(A)$. Let us continue to write X_i for the ultraproduct in $\mathcal{L}(A)$ of the constant sequence $X_i \in A_w$. By Łos' Theorem, the X_i are algebraically independent over \mathfrak{D} . In other words, A is a subring of $\mathcal{L}(A)$. In the next section, we will prove the key algebraic property of the extension $A \subseteq \mathcal{L}(A)$ when the base rings \mathfrak{D}_w are discrete valuation rings, to wit, its flatness. We start with extending the notions of non-standard hull and approximation from [39], to arbitrary local \mathfrak{D} -affine algebras (recall that a *local \mathfrak{D} -affine algebra* is a localization of a finitely presented \mathfrak{D} -algebra at a prime ideal containing \mathfrak{p}).

\mathfrak{D} -approximations and non-standard \mathfrak{D} -hulls. An *\mathfrak{D} -approximation* of a polynomial $f \in A$ is a sequence of polynomials $f_w \in A_w$, such that their ultraproduct is equal to f , viewed as an element in $\mathcal{L}(A)$. Note that according to Lemma 2.1, we can always find such an \mathfrak{D} -approximation. Moreover, any two \mathfrak{D} -approximations are equal for almost all w , by Łos' Theorem. Similarly, an *\mathfrak{D} -approximation* of a finitely generated ideal $I := \mathbf{f}A$ with \mathbf{f} a finite tuple, is a sequence of ideals $I_w := \mathbf{f}_w A_w$, where \mathbf{f}_w is an \mathfrak{D} -approximation of \mathbf{f} (meaning that each entry in \mathbf{f}_w is an \mathfrak{D} -approximation of the corresponding entry in \mathbf{f}). Łos' Theorem gives once more that any two \mathfrak{D} -approximations are almost all equal. Moreover, if I_w is some \mathfrak{D} -approximation of I then

$$(2) \quad \text{ulim}_{w \rightarrow \infty} I_w = I\mathcal{L}(A).$$

Assume now that C is an \mathfrak{D} -affine algebra, say $C = A/I$ with I a finitely generated ideal. We define an *\mathfrak{D} -approximation* of C to be the sequence of finitely generated \mathfrak{D}_w -algebras $C_w := A_w/I_w$, where I_w is some \mathfrak{D} -approximation of I . We define the *non-standard \mathfrak{D} -hull* of C to be the ultraproduct of the C_w and denote it $\mathcal{L}_{\mathfrak{D}}(C)$ or simply $\mathcal{L}(C)$. It is not hard to show that $\mathcal{L}(C)$ is uniquely defined up to C -algebra isomorphism (for more details see [39] or [44]). From (2), it follows that $\mathcal{L}(C) = \mathcal{L}(A)/I\mathcal{L}(A)$. In particular, there is a canonical homomorphism $C \rightarrow \mathcal{L}(C)$ obtained from the base change $A \rightarrow \mathcal{L}(A)$.

When I is not finitely generated, $I\mathcal{L}(A)$ might not be realizable as an ultraproduct of ideals, and consequently, has no \mathfrak{D} -approximation. Although one can find special cases of infinitely generated ideals admitting \mathfrak{D} -approximations, we will never have to do this in the present paper. Similarly, we only define \mathfrak{D} -approximations for \mathfrak{D} -affine algebras.

Although $A \rightarrow \mathcal{L}(A)$ is injective, this is not necessarily the case for $C \rightarrow \mathcal{L}(C)$, if the \mathfrak{D}_w are not fields. For instance, if W is the set of prime numbers, $\mathfrak{D}_p := \mathbb{Z}_p$ for each $p \in W$ and $I = (1 - \pi X, \gamma)A$ where $\pi := \text{ulim}_{p \rightarrow \infty} p$ and $\gamma := \text{ulim}_{p \rightarrow \infty} p^p$, then $I \neq (1)$ but $I\mathcal{L}(A) = (1)$. However, when the \mathfrak{D}_w are discrete valuation rings, we will see shortly, that this phenomenon disappears if we localize at prime ideals containing \mathfrak{p} . Next we define a process which is converse to taking \mathfrak{D} -approximations.

Restricted Ultraproducts. Fix some c . For each w , let I_w be an ideal in A_w of \mathfrak{D}_w -complexity at most c . In other words, we can write $I_w = \mathbf{f}_w A_w$, for some tuple \mathbf{f}_w of \mathfrak{D}_w -complexity at most c . Let \mathbf{f} be the ultraproduct of these tuples. By Lemma 2.1, the tuple \mathbf{f} is already defined over A . We call $I := \mathbf{f}A$ the *restricted ultraproduct* of the I_w . It

follows that the I_w are an \mathfrak{D} -approximation of I and that $I\mathfrak{L}(A)$ is the ultraproduct of the I_w .

With $C_w := A_w/I_w$ and $C := A/I$, we call C the *restricted ultraproduct* of the C_w . The C_w are an \mathfrak{D} -approximation of C and their ultraproduct $\mathfrak{L}(C)$ is the non-standard \mathfrak{D} -hull of C . We can now extend the previous definition to the image in C_w of an element $c_w \in A_w$ (respectively, to the extension $J_w C_w$ of a finitely generated ideal $J_w \subseteq A_w$) of \mathfrak{D}_w -complexity at most c and define similarly their *restricted ultraproduct* $c \in C$ and JC as the image in C of the respective restricted ultraproduct of the c_w and the J_w .

Functoriality. We have a commutative diagram

$$(3) \quad \begin{array}{ccc} C & \xrightarrow{\varphi} & D \\ \downarrow & & \downarrow \\ \mathfrak{L}(C) & \xrightarrow{\mathfrak{L}(\varphi)} & \mathfrak{L}(D) \end{array}$$

where $C \rightarrow D$ is an \mathfrak{D} -algebra homomorphism of finite type between \mathfrak{D} -affine algebras and $\mathfrak{L}(C) \rightarrow \mathfrak{L}(D)$ is its base change over $\mathfrak{L}(A)$. Alternatively, we may view this diagram coming from a sequence of \mathfrak{D}_w -algebras homomorphisms $C_w \rightarrow D_w$ of \mathfrak{D}_w -complexity at most c , for some c independent from w , in which case $C \rightarrow D$ and $\mathfrak{L}(C) \rightarrow \mathfrak{L}(D)$ are the respective restricted ultraproduct and ultraproduct of these homomorphisms.

3.2. Lemma. *Any prime ideal \mathfrak{m} of A containing \mathfrak{p} is finitely generated and its extension $\mathfrak{m}\mathfrak{L}(A)$ is again prime.*

Proof. Since $A/\mathfrak{p}A = \kappa[X]$ is Noetherian, where κ is the residue field of \mathfrak{D} , the ideal $\mathfrak{m}(A/\mathfrak{p}A)$ is finitely generated. Therefore so is \mathfrak{m} , since by assumption \mathfrak{p} is finitely generated. Moreover, $\mathfrak{L}(A)/\mathfrak{p}\mathfrak{L}(A)$ is the ultraproduct of the $\kappa_w[X]$, so that by Theorem 2.2, the extension $\mathfrak{m}(\mathfrak{L}(A)/\mathfrak{p}\mathfrak{L}(A))$ is prime, whence so is $\mathfrak{m}\mathfrak{L}(A)$. \square

In particular, if \mathfrak{m}_w is an \mathfrak{D} -approximation of \mathfrak{m} , then almost all \mathfrak{m}_w are prime ideals. Therefore, the following notions are well-defined (with the convention that we put $B_{\mathfrak{n}}$ equal to zero whenever \mathfrak{n} is not a prime ideal of the ring B). Let R be a local \mathfrak{D} -affine algebra, say, of the form $C_{\mathfrak{m}}$, with C an \mathfrak{D} -affine algebra and \mathfrak{m} a prime ideal containing \mathfrak{p} .

3.3. Definition. We call $\mathfrak{L}(C)_{\mathfrak{m}\mathfrak{L}(C)}$ the *non-standard \mathfrak{D} -hull* of R and denote it $\mathfrak{L}_{\mathfrak{D}}(R)$ or simply $\mathfrak{L}(R)$. Moreover, if C_w and \mathfrak{m}_w are \mathfrak{D} -approximations of C and \mathfrak{m} respectively, then the collection $R_w := (C_w)_{\mathfrak{m}_w}$ is an *\mathfrak{D} -approximation* of R .

One easily checks that the ultraproduct of the \mathfrak{D} -approximations R_w is precisely the non-standard \mathfrak{D} -hull $\mathfrak{L}(R)$.

4. FLATNESS OF NON-STANDARD \mathfrak{D} -HULLS

In this section, we specialize the notions from the previous result to the situation where each \mathfrak{D}_w is a discrete valuation ring. We fix throughout the following notation. For each w , let \mathfrak{D}_w be a discrete valuation ring with uniformizing parameter π_w and with residue field κ_w . Let \mathfrak{D} , π and κ be their respective ultraproducts, so that $\pi\mathfrak{D}$ is the maximal ideal of \mathfrak{D} and κ its residue field. We call any ring of this form an *ultra-DVR*. The intersection

of all $\pi^m \mathfrak{D}$ is called the *ideal of infinitesimals* of \mathfrak{D} and is denoted $\mathfrak{I}_{\mathfrak{D}}$. Using [34], one sees that $\mathfrak{D}/\pi^m \mathfrak{D}$ is an Artinian local Gorenstein κ -algebra of length m .

Fix a finite tuple of indeterminates X and let $A := \mathfrak{D}[X]$. As before, we denote the non-standard \mathfrak{D} -hull of A by $\mathfrak{L}(A)$; recall that it is given as the ultraproduct of the \mathfrak{D} -approximations $A_w := \mathfrak{D}_w[X]$.

4.1. Proposition. *For I an ideal in A , the residue ring A/I is Noetherian if and only if $\mathfrak{I}_{\mathfrak{D}} \subseteq I$. In particular, every maximal ideal of A contains $\mathfrak{I}_{\mathfrak{D}}$ and is of the form $\mathfrak{I}_{\mathfrak{D}}A + J$ with J a finitely generated ideal.*

Proof. Let $C := A/I$ for some ideal I of A . If C is Noetherian, then the intersection of all $\pi^n C$ is zero by Krull's Intersection Theorem. Hence $\mathfrak{I}_{\mathfrak{D}} \subseteq I$. Conversely, if $\mathfrak{I}_{\mathfrak{D}} \subseteq I$, then since $A/\mathfrak{I}_{\mathfrak{D}}A = (\mathfrak{D}/\mathfrak{I}_{\mathfrak{D}})[X]$ is Noetherian, so is C . The last assertion is now clear. \square

In spite of Lemma 3.2, there are even maximal ideals of A (necessarily not containing π) which do not extend to a proper ideal in $\mathfrak{L}(A)$. For instance with X a single indeterminate and $W = \mathbb{N}$, the ideal $\mathfrak{I}_{\mathfrak{D}}A + (1 - \pi X)A$ is maximal (with residue field the field of fractions of $\mathfrak{D}/\mathfrak{I}_{\mathfrak{D}}$), but $\mathfrak{I}_{\mathfrak{D}}\mathfrak{L}(A) + (1 - \pi X)\mathfrak{L}(A)$ is the unit ideal. Indeed, let f_{∞} be the ultraproduct of the

$$f_w := (1 - (\pi_w X)^w)/(1 - \pi_w X).$$

Since $(1 - \pi_w X)f_w \equiv 1$ modulo $(\pi_w)^w A_w$, we get by Łos' Theorem that $(1 - \pi X)f_{\infty} \equiv 1$ modulo $\mathfrak{I}_{\mathfrak{D}}\mathfrak{L}(A)$. Therefore, we cannot hope for $A \rightarrow \mathfrak{L}(A)$ to be faithfully flat. Nonetheless, using for instance a result of Aschenbrenner on bounds of syzygies, we do have this property for local affine algebras. This result will prove to be crucial in what follows.

4.2. Theorem. *The canonical homomorphism $A \rightarrow \mathfrak{L}(A)$ is flat. In particular, the canonical homomorphism of a local \mathfrak{D} -affine algebra to its non-standard \mathfrak{D} -hull is faithfully flat, whence in particular injective.*

Proof. The last assertion is clear from the first, since the homomorphism $R \rightarrow \mathfrak{L}(R)$ is obtained as a base change of $A \rightarrow \mathfrak{L}(A)$ followed by a suitable localization, for any local \mathfrak{D} -affine algebra R . I will provide two different proofs for the first assertion

For the first proof, we use a result of Aschenbrenner [2] in order to verify the equational criterion for flatness, that is to say, given a linear equation $L = 0$, with L a linear form over A , and given a solution \mathbf{f}_{∞} over $\mathfrak{L}(A)$, we need to show that there exist solutions \mathbf{b}_i in A such that \mathbf{f}_{∞} is an $\mathfrak{L}(A)$ -linear combination of the \mathbf{b}_i . Choose L_w and \mathbf{f}_w with respective ultraproducts L and \mathbf{f}_{∞} . In particular, almost all L_w have \mathfrak{D}_w -complexity at most c , for some c independent from w . By Łos' Theorem, \mathbf{f}_w is a solution of the linear equation $L_w = 0$, for almost all w . Therefore, by [2, Corollary 4.27], there is a bound c' , only depending on c , such that \mathbf{f}_w is an A_w -linear combination of solutions $\mathbf{b}_{1w}, \dots, \mathbf{b}_{sw}$ of \mathfrak{D}_w -complexity at most c . Note that s can be chosen independent from w as well by [44, Lemma 1]. In particular, the ultraproduct \mathbf{b}_i of the \mathbf{b}_{iw} lies in A by Lemma 2.1. By Łos' Theorem, each \mathbf{b}_i is a solution of $L = 0$ in $\mathfrak{L}(A)$, whence in A , and \mathbf{f}_{∞} is an $\mathfrak{L}(A)$ -linear combination of the \mathbf{b}_i , proving flatness.

If we want to avoid the use of Aschenbrenner's result, we can reason as follows. By Theorem 2.2, both extensions $A/\pi A \rightarrow \mathfrak{L}(A)/\pi\mathfrak{L}(A)$ and $A \otimes Q \rightarrow \mathfrak{L}(A) \otimes Q$ are faithfully flat, where Q is the field of fractions of \mathfrak{D} . Let M be an A -module. Since π is A -regular, the standard spectral sequence

$$\mathrm{Tor}_p^{A/\pi A}(\mathfrak{L}(A)/\pi\mathfrak{L}(A), \mathrm{Tor}_q^A(M, A/\pi A)) \implies \mathrm{Tor}_{p+q}^A(\mathfrak{L}(A)/\pi\mathfrak{L}(A), M)$$

degenerates into short exact sequences

$$\begin{aligned} \mathrm{Tor}_{i-1}^{A/\pi A}(\mathfrak{L}(A)/\pi\mathfrak{L}(A), (0 :_M \pi)) \rightarrow \mathrm{Tor}_i^A(\mathfrak{L}(A)/\pi\mathfrak{L}(A), M) \rightarrow \\ \mathrm{Tor}_i^{A/\pi A}(\mathfrak{L}(A)/\pi\mathfrak{L}(A), M/\pi M), \end{aligned}$$

for all $i \geq 2$. For $i = 2$, since $A/\pi A \rightarrow \mathfrak{L}(A)/\pi\mathfrak{L}(A)$ is flat, the middle module $\mathrm{Tor}_2^A(\mathfrak{L}(A)/\pi\mathfrak{L}(A), M)$ vanishes. Applying this to the short exact sequence

$$0 \rightarrow \mathfrak{L}(A) \xrightarrow{\pi} \mathfrak{L}(A) \rightarrow \mathfrak{L}(A)/\pi\mathfrak{L}(A) \rightarrow 0$$

we get a short exact sequence

$$(4) \quad 0 = \mathrm{Tor}_2^A(\mathfrak{L}(A)/\pi\mathfrak{L}(A), M) \rightarrow \mathrm{Tor}_1^A(\mathfrak{L}(A), M) \xrightarrow{\pi} \mathrm{Tor}_1^A(\mathfrak{L}(A), M).$$

On the other hand, flatness of $A \otimes Q \rightarrow \mathfrak{L}(A) \otimes Q$ yields

$$(5) \quad \mathrm{Tor}_1^A(\mathfrak{L}(A), M) \otimes Q = \mathrm{Tor}_1^{A \otimes Q}(\mathfrak{L}(A) \otimes Q, M \otimes Q) = 0.$$

In order to prove that $A \rightarrow \mathfrak{L}(A)$ is flat, it suffices by [27, Theorem 7.8] to show that $\mathrm{Tor}_1^A(\mathfrak{L}(A), A/I)$ vanishes, for every finitely generated ideal I of A . Towards a contradiction, suppose that $\mathrm{Tor}_1^A(\mathfrak{L}(A), A/I)$ contains a non-zero element τ . By (5), we have $a\tau = 0$, for some non-zero $a \in \mathfrak{D}$. As observed in [31, Proposition 3], every polynomial ring over a valuation ring is coherent, so that in particular I is finitely presented (namely, since I is torsion-free over \mathfrak{D} , it is \mathfrak{D} -flat, and therefore finitely presented by [29, Theorem 3.4.6]). Hence we have some exact sequence

$$A^{a_2} \xrightarrow{\varphi_2} A^{a_1} \xrightarrow{\varphi_1} A \rightarrow A/I \rightarrow 0.$$

Therefore $\mathrm{Tor}_1^A(\mathfrak{L}(A), A/I)$ is calculated as the homology of the complex

$$\mathfrak{L}(A)^{a_2} \xrightarrow{\varphi_2} \mathfrak{L}(A)^{a_1} \xrightarrow{\varphi_1} \mathfrak{L}(A).$$

Suppose τ is the image of a tuple $\mathbf{x} \in \mathfrak{L}(A)^{a_1}$ with $\varphi_1(\mathbf{x}) = 0$. Hence \mathbf{x} does not belong to $\varphi_2(\mathfrak{L}(A)^{a_2})$ but $a\mathbf{x}$ does. Choose \mathbf{x}_w, a_w and φ_{i_w} with respective ultraproduct \mathbf{x}, a and φ_i . By Łoś' Theorem, almost all \mathbf{x}_w lie in the kernel of φ_{1w} but not in the image of φ_{2w} , yet $a_w\mathbf{x}_w$ lies in the image of φ_{2w} . Choose $n_w \in \mathbb{N}$ maximal such that $\mathbf{y}_w := (\pi_w)^{n_w}\mathbf{x}_w$ does not lie in the image of φ_{2w} . Since almost all a_w are non-zero, this maximum exists for almost all w . Therefore, if \mathbf{y} is the ultraproduct of the \mathbf{y}_w , then $\varphi_1(\mathbf{y}) = 0$ and \mathbf{y} does not lie in $\varphi_2(\mathfrak{L}(A)^{a_2})$, but $\pi\mathbf{y}$ lies in $\varphi_2(\mathfrak{L}(A)^{a_2})$. Therefore, the image of \mathbf{y} in $\mathrm{Tor}_1^A(\mathfrak{L}(A), A/I)$ is a non-zero element annihilated by π , contradicting (4). \square

4.3. Remark. In [46], I exhibit a general connection between the flatness of an ultraproduct over certain canonical subrings and the existence of bounds on syzygies. In particular, using these ideas, the second argument in the above proof of flatness reproves the result in [2]. In fact, the role played here by coherence is not accidental either; see [1] or [46] for more details.

4.4. Theorem. *Let R be a local \mathfrak{D} -affine algebra with non-standard \mathfrak{D} -hull $\mathfrak{L}(R)$ and \mathfrak{D} -approximation R_w .*

- *Almost all R_w are flat over \mathfrak{D}_w if and only if R is torsion-free over \mathfrak{D} if and only if π is R -regular.*
- *Almost all R_w are domains if and only if R is.*

Proof. Suppose first that almost all R_w are flat over \mathfrak{D}_w , which amounts in this case, to almost all R_w being torsion-free over \mathfrak{D}_w . By Łos' Theorem, $\mathfrak{L}(R)$ is torsion-free over \mathfrak{D} , and since $R \subseteq \mathfrak{L}(R)$, so is R . Conversely, assume π is R -regular. By faithful flatness, π is $\mathfrak{L}(R)$ -regular, whence almost all π_w are R_w -regular by Łos' Theorem. Since the \mathfrak{D}_w are discrete valuation rings, this means that almost all $\mathfrak{D}_w \rightarrow R_w$ are flat.

If almost all R_w are domains, then so is $\mathfrak{L}(R)$ by Łos' Theorem, and hence so is R , since it embeds in $\mathfrak{L}(R)$. Conversely, assume R is a domain. If $\pi = 0$ in R , then $\mathfrak{L}(R)$ is a domain by Lemma 3.2, whence so are almost all R_w by Łos' Theorem. So assume π is non-zero in R , whence R -regular. By what we just proved, R is then torsion-free over \mathfrak{D} . Let Q be the field of fractions of \mathfrak{D} . Write R in the form S/\mathfrak{p} , where S is some localization of A at a prime ideal containing π and \mathfrak{p} is a finitely generated prime ideal in S . Since S/\mathfrak{p} is torsion-free over \mathfrak{D} , the extension $\mathfrak{p}(S \otimes_{\mathfrak{D}} Q)$ is again prime and its contraction in S is \mathfrak{p} . By Theorem 2.2, since we are now over a field, $\mathfrak{p}(\mathfrak{L}(S) \otimes_{\mathfrak{D}} Q)$ is a prime ideal, where $\mathfrak{L}(S)$ is the non-standard \mathfrak{D} -hull of S (note that $\mathfrak{L}(S) \otimes_{\mathfrak{D}} Q$ is then the non-standard hull of $S \otimes_{\mathfrak{D}} Q$ in the sense of [39]). Moreover, since S/\mathfrak{p} is torsion-free over \mathfrak{D} , so is $\mathfrak{L}(S)/\mathfrak{p}\mathfrak{L}(S)$ by the first assertion. This in turn means that

$$\mathfrak{p}\mathfrak{L}(S) = \mathfrak{p}(\mathfrak{L}(S) \otimes_{\mathfrak{D}} Q) \cap \mathfrak{L}(S),$$

showing that $\mathfrak{p}\mathfrak{L}(S)$ is prime. It follows then from Łos' Theorem that almost all \mathfrak{p}_w are prime, where \mathfrak{p}_w is an \mathfrak{D} -approximation of \mathfrak{p} , and hence almost all R_w are domains. \square

4.5. Remark. The last assertion is equivalent with saying that any prime ideal in R extends to a prime ideal in $\mathfrak{L}(R)$. Indeed, let \mathfrak{q} be a prime ideal in R with \mathfrak{D} -approximation \mathfrak{q}_w . By the above result (applied to R/\mathfrak{q} and its \mathfrak{D} -approximation R_w/\mathfrak{q}_w), we get that almost all \mathfrak{q}_w are prime, whence so is their ultraproduct $\mathfrak{q}\mathfrak{L}(R)$, by Łos' Theorem.

5. GEOMETRIC DIMENSION

In this and the next section, we will study the local algebra of the category $\underline{\text{Aff}}(\mathfrak{D})$. Although part of the theory can be developed for arbitrary base rings \mathfrak{D} , or even for arbitrary local rings of finite embedding dimension (see [45]), we will only deal with the case that \mathfrak{D} is a local domain of embedding dimension one. Recall that the *embedding dimension* of a local ring (Z, \mathfrak{p}) is by definition the minimal number of generators of \mathfrak{p} , and its *ideal of infinitesimals* \mathfrak{I}_Z is the intersection of all powers \mathfrak{p}^n . Of course, if Z is moreover Noetherian, then its ideal of infinitesimals is zero. In general, we call $\tilde{Z} := Z/\mathfrak{I}_Z$ the *separated quotient* of Z .

For the duration of the next two sections, let \mathfrak{D} denote a local domain of embedding dimension one, with generator of the maximal ideal π , with ideal of infinitesimals $\mathfrak{I}_{\mathfrak{D}}$ and with residue field κ . We will work in the category $\underline{\text{Aff}}(\mathfrak{D})$ of local \mathfrak{D} -affine algebras, that is to say, the category of algebras of the form $R := (A/I)_{\mathfrak{m}}$, where as before $A := \mathfrak{D}[X]$ for some finite tuple of indeterminates X , where I is a finitely generated ideal in A and where \mathfrak{m} is a prime ideal containing π and I . Nonetheless, some results can be stated even for local algebras which are locally finitely generated over \mathfrak{D} , that is, without the assumption that I is finitely generated. We call R a *torsion-free* \mathfrak{D} -algebra if it is torsion-free over \mathfrak{D} (that is to say, if $ar = 0$ for some $r \in R$ and some non-zero $a \in \mathfrak{D}$, then $r = 0$). Recall from Theorem 4.4 that a local \mathfrak{D} -affine algebra R is torsion-free if and only if π is R -regular.

5.1. Lemma. *The separated quotient $\mathfrak{D}/\mathfrak{I}_{\mathfrak{D}}$ of \mathfrak{D} is a discrete valuation ring with uniformizing parameter π .*

Proof. For each element $a \in \mathfrak{D}$ outside $\mathfrak{J}_{\mathfrak{D}}$, there is a smallest $e \in \mathbb{N}$ for which $a \notin \pi^{e+1}\mathfrak{D}$. Hence $a = u\pi^e$ with u a unit in \mathfrak{D} . It is now straightforward to check that the assignment $a \mapsto e$ induces a discrete valuation on $\mathfrak{D}/\mathfrak{J}_{\mathfrak{D}}$. \square

Note that we do not even need \mathfrak{D} to be domain, having positive depth (that is to say, assuming that $\pi\mathfrak{D}$ is not an associated prime of \mathfrak{D} ; see [7, Proposition 9.1.4]) would suffice, for then π is necessarily \mathfrak{D} -regular. However, we do not need this amount of generality as in all our applications \mathfrak{D} will be an ultra-DVR, that is to say, an ultraproduct of discrete valuation rings \mathfrak{D}_w . If we are in this situation, then as before, we let $A_w := \mathfrak{D}_w[X]$ and we let $\mathfrak{L}(A)$ be their ultraproduct. Moreover, for $R = (A/I)_{\mathfrak{m}}$ as above, we let $\mathfrak{L}(R) := (\mathfrak{L}(A)/I\mathfrak{L}(A))_{\mathfrak{m}\mathfrak{L}(A)}$ be its non-standard \mathfrak{D} -hull and we let $R_w := (A_w/I_w)_{\mathfrak{m}_w}$ be an \mathfrak{D} -approximation of R , where I_w and \mathfrak{m}_w are \mathfrak{D} -approximations of I and \mathfrak{m} respectively. Note that \mathfrak{m} is finitely generated, as it contains by definition π .

5.2. Lemma. *Let (R, \mathfrak{m}) be a local ring which is locally finitely generated over \mathfrak{D} . If I is a proper ideal in R containing some power π^m , then the intersection of all I^n for $n \in \mathbb{N}$ is equal to $\mathfrak{J}_{\mathfrak{D}}R$. In particular, $\mathfrak{J}_R = \mathfrak{J}_{\mathfrak{D}}R$ and the separated quotient of R is equal to $\tilde{R} := R/\mathfrak{J}_{\mathfrak{D}}R$ whence is Noetherian.*

Proof. Suppose $\pi^m \in I \subseteq \mathfrak{m}$. Let J be the intersection of all I^n for $n \in \mathbb{N}$. Since $\pi^m \in I$, we get that $\mathfrak{J}_{\mathfrak{D}}R \subseteq J$. Since \tilde{R} is locally finitely generated over the discrete valuation ring $\mathfrak{D}/\mathfrak{J}_{\mathfrak{D}}$ (see Lemma 5.1), it is Noetherian. Applying Krull's Intersection Theorem (see for instance [27, Theorem 8.10]), we get that $J\tilde{R} = (0)$, and hence that $J = \mathfrak{J}_{\mathfrak{D}}R$. The last assertion follows by letting $I := \mathfrak{m}$. \square

5.3. Lemma. *Let \mathfrak{D} be an ultra-DVR. A local \mathfrak{D} -affine algebra (R, \mathfrak{m}) has the same \mathfrak{m} -adic completion as its separated quotient, and this is also isomorphic to $\mathfrak{L}(R)/\mathfrak{J}_{\mathfrak{L}(R)}$. In particular, the completion is Noetherian.*

Proof. Let $\tilde{R} := R/\mathfrak{J}_R$ be the separated quotient. For every n , we have

$$R/\mathfrak{m}^n \cong \tilde{R}/\mathfrak{m}^n\tilde{R} \cong \mathfrak{L}(R)/\mathfrak{m}^n\mathfrak{L}(R),$$

where the second isomorphism follows from the fact that length is a first order invariant (see for instance [34]). Hence R , \tilde{R} and $\mathfrak{L}(R)$ have the same completion \widehat{R} . Noetherianity now follows from Lemma 5.2. By saturatedness of ultraproducts (with respect to a countably incomplete non-principal ultrafilter), $\mathfrak{L}(R)$ is quasi-complete in the sense that every Cauchy sequence has a (non-unique) limit. Therefore, its separated quotient $\mathfrak{L}(R)/\mathfrak{J}_{\mathfrak{L}(R)}$ is complete, whence equal to \widehat{R} . For a more detailed proof, see [45, Lemma 5.2]. \square

Our first goal is to introduce a good notion of dimension. Below, the *dimension* of a ring will always mean its *Krull dimension*. Recall that it is always finite for Noetherian local rings.

5.4. Theorem. *For a local ring (R, \mathfrak{m}) which is locally finitely generated over \mathfrak{D} , the following numbers are all equal:*

- the least possible length d of a tuple in R generating some \mathfrak{m} -primary ideal;
- the dimension \widehat{d} of the completion \widehat{R} ;
- the dimension \tilde{d} of the separated quotient $\tilde{R} := R/\mathfrak{J}_{\mathfrak{D}}R$;
- the degree \underline{d} of the Hilbert-Samuel polynomial χ_R , where χ_R is the unique polynomial with rational coefficients for which $\chi_R(n)$ equals the length of R/\mathfrak{m}^{n+1} for all large n .

If π is R -regular, then $R/\pi R$ has dimension $d - 1$.

If, moreover, \mathfrak{D} is an ultra-DVR and R a torsion-free local \mathfrak{D} -affine algebra with \mathfrak{D} -approximation R_w , then almost all R_w have dimension d .

Proof. By Lemmas 5.2 and 5.3, the separated quotient \tilde{R} is Noetherian, with completion equal to \widehat{R} . Hence $\widehat{d} = \tilde{d}$. Moreover, $\chi_R = \chi_{\widehat{R}}$, so that by the Hilbert-Samuel theory, $\underline{d} = \widehat{d}$.

Let \mathbf{x} be a tuple of length \tilde{d} such that its image in \tilde{R} is a system of parameters of \tilde{R} . Hence, for some n , we have that $\mathfrak{m}^n \subseteq \mathbf{x}R + \mathfrak{J}_{\mathfrak{D}}R$. In particular, since $\mathfrak{J}_{\mathfrak{D}}R \subseteq \pi^{n+1}R$, we can find $x \in \mathbf{x}R$ and $r \in R$, such that $\pi^n = x + r\pi^{n+1}$. Therefore, $\pi^n \in \mathbf{x}R$, since $1 - r\pi$ is a unit. Since $\mathfrak{J}_{\mathfrak{D}} \subseteq \pi^n \mathfrak{D}$, we get that $\mathfrak{m}^n \subseteq \mathbf{x}R$, showing that $\mathbf{x}R$ is an \mathfrak{m} -primary ideal and hence that $\tilde{d} \leq \underline{d}$. On the other hand, if \mathbf{y} is a tuple of length d such that $\mathbf{y}R$ is \mathfrak{m} -primary, then $\mathbf{y}\tilde{R}$ is an $\mathfrak{m}\tilde{R}$ -primary ideal, and hence $\tilde{d} \leq d$. This concludes the proof of the first assertion.

Assume that π is moreover R -regular. I claim that π is \tilde{R} -regular. Indeed, suppose $\pi\tilde{r} = 0$, for some $\tilde{r} \in \tilde{R}$. Take a pre-image $r \in R$, so that $\pi r \in \mathfrak{J}_{\mathfrak{D}}R \subseteq \pi^n R$, for every n . Since π is R -regular, we get that $r \in \pi^{n-1}R$, for all n . Therefore, $r \in \mathfrak{J}_{\mathfrak{D}}R$ by Lemma 5.2, whence $\tilde{r} = 0$ in \tilde{R} , as we needed to show. Since π is \tilde{R} -regular and $\tilde{R}/\pi\tilde{R} = R/\pi R$, the dimension of $R/\pi R$ is $\tilde{d} - 1$.

Suppose finally that \mathfrak{D} is moreover an ultra-DVR. We already observed that $R_w/\pi_w R_w$ is an approximation of $R/\pi R$ in the sense of [39]. In particular, by [39, Theorem 4.5], almost all $R_w/\pi_w R_w$ have dimension $\tilde{d} - 1$. Since π is $\mathfrak{L}(R)$ -regular by flatness, whence π_w is R_w -regular by Los' Theorem, we get that R_w has dimension \tilde{d} , for almost all w . \square

5.5. Geometric dimension. The common value given by the theorem is called the *geometric dimension* of R . We call a tuple \mathbf{x} in R *generic*, if it generates an \mathfrak{m} -primary ideal and has length equal to the geometric dimension of R . Note that if (x_1, \dots, x_d) is a generic sequence, then $R/(x_1, \dots, x_e)R$ has geometric dimension $d - e$.

5.6. Corollary. *In a local ring (R, \mathfrak{m}) which is locally finitely generated over \mathfrak{D} , every \mathfrak{m} -primary ideal contains a generic sequence.*

Proof. Let $\tilde{R} := R/\mathfrak{J}_{\mathfrak{D}}R$ and let d be the geometric dimension of R . Let \mathfrak{n} be an \mathfrak{m} -primary ideal of R . Since $\mathfrak{n}\tilde{R}$ is $\mathfrak{m}\tilde{R}$ -primary and \tilde{R} is Noetherian, we can find a tuple \mathbf{y} with entries in \mathfrak{n} so that its image in \tilde{R} is a system of parameters. In particular, \mathbf{y} has length d by Theorem 5.4. Let $S := R/\mathbf{y}R$ and $\tilde{S} := S/\mathfrak{J}_{\mathfrak{D}}S$. By Theorem 5.4, the geometric dimension of S is equal to the dimension of \tilde{S} , whence is zero since $\tilde{S} = \tilde{R}/\mathbf{y}\tilde{R}$. In particular, $\mathbf{y}R$ is \mathfrak{m} -primary. Since \mathbf{y} has length equal to the geometric dimension of R , it is therefore a generic sequence. \square

In fact the above proof shows that there is a one-one correspondence between generic sequences in R and systems of parameters in $R/\mathfrak{J}_{\mathfrak{D}}R$. In general, the last assertion in Theorem 5.4 is false when R is not torsion-free. For instance, let $R := \mathfrak{D}/a\mathfrak{D}$ with a a non-zero infinitesimal, so that each $R_w = \mathfrak{D}_w/a_w\mathfrak{D}_w$ has dimension zero, but R/\mathfrak{J}_R is the (one-dimensional) discrete valuation ring $\mathfrak{D}/\mathfrak{J}_{\mathfrak{D}}$.

In the following definition, let \mathfrak{D} be an ultra-DVR and let R be a local \mathfrak{D} -affine algebra of geometric dimension d , with \mathfrak{D} -approximation R_w . Note that the R_w have almost all dimension at most d . Indeed, if \mathbf{y} has length d and generates an \mathfrak{m} -primary ideal, then almost all \mathbf{y}_w are \mathfrak{m}_w -primary by Los' Theorem, for \mathbf{y}_w an \mathfrak{D} -approximation of \mathbf{y} .

5.7. Definition. We say that R is *isodimensional* if almost all R_w have dimension equal to the geometric dimension of R .

Theorem 5.4 shows that every torsion-free local \mathfrak{D} -affine algebra is isodimensional. In particular, over an ultra-DVR, the restricted ultraproduct R of domains R_w of uniformly bounded \mathfrak{D}_w -complexity is isodimensional, since $\mathfrak{L}(R)$ is then a domain by Łos' Theorem, whence so is R as it embeds in $\mathfrak{L}(R)$. The next result shows that generic sequences in an isodimensional ring are the analog of systems of parameters.

5.8. Corollary. *Let \mathfrak{D} be an ultra-DVR and R an isodimensional local \mathfrak{D} -affine algebra with \mathfrak{D} -approximation R_w . Let \mathbf{x} be a tuple in R with \mathfrak{D} -approximation \mathbf{x}_w .*

If \mathbf{x} is generic, then \mathbf{x}_w is a system of parameters of R_w , for almost all w . Conversely, if $(\pi_w)^c \in \mathbf{x}_w R_w$, for some c and almost all w , then \mathbf{x} is generic.

Proof. Let \mathfrak{m} be the maximal ideal of R , with \mathfrak{D} -approximation \mathfrak{m}_w . Let d be the geometric dimension of R , so that almost all R_w have dimension d . Suppose first that \mathbf{x} is generic, so that $|\mathbf{x}| = d$ and $\mathbf{x}R$ is \mathfrak{m} -primary. Since $\mathbf{x}\mathfrak{L}(R)$ is then $\mathfrak{m}\mathfrak{L}(R)$ -primary, $\mathbf{x}_w R_w$ is \mathfrak{m}_w -primary by Łos' Theorem, showing that \mathbf{x}_w is a system of parameters for almost all w .

Conversely, suppose \mathbf{x}_w is a system of parameters of R_w , generating an ideal containing $(\pi_w)^c$. By Łos' Theorem and faithful flatness, $\pi^c \in \mathbf{x}R$. Applying [44, Corollary 4] to the Artinian base ring $\mathfrak{D}_w/(\pi_w)^c$, we can find a bound c' , only depending on c , such that $(\mathfrak{m}_w)^{c'} \subseteq \mathbf{x}_w R_w$, for almost all w . Hence $\mathfrak{m}^{c'}\mathfrak{L}(R) \subseteq \mathbf{x}\mathfrak{L}(R)$, so that by faithful flatness, $\mathbf{x}R$ is \mathfrak{m} -primary. This shows that \mathbf{x} is generic. \square

The additional requirement in the converse is necessary: indeed, for arbitrary $n_w > 0$, the element $(\pi_w)^{n_w}$ is a parameter in \mathfrak{D}_w and has \mathfrak{D}_w -complexity zero, but if n_w is unbounded, its ultraproduct is an infinitesimal whence not generic. To characterize isodimensional rings, we use the following notion introduced in [43].

5.9. Definition (Parameter degree). The *parameter degree* of a Noetherian local ring C is by definition the smallest possible length of a residue ring $C/\mathbf{x}C$, where \mathbf{x} runs over all systems of parameters of C .

In general, the parameter degree is larger than the multiplicity, with equality precisely when C is Cohen-Macaulay, provided the residue field is infinite (see [27, Theorem 17.11]). The homological degree of C is an upper bound for its parameter degree (see [43, Corollary 4.6]). A priori, being isodimensional is a property of the \mathfrak{D} -approximations of R , of for that matter, of its non-standard \mathfrak{D} -hull. However, the last equivalent condition in the next result shows that it is in fact an intrinsic property.

5.10. Proposition. *Let \mathfrak{D} be an ultra-DVR and let R be a local \mathfrak{D} -affine algebra with \mathfrak{D} -approximation R_w . The following are equivalent:*

- (5.10.1) R is isodimensional;
- (5.10.2) there exists a $c \in \mathbb{N}$, such that for almost all w , we can find a system of parameters \mathbf{x}_w of R_w of \mathfrak{D}_w -complexity at most c , generating an ideal containing $(\pi_w)^c$;
- (5.10.3) there exists an $e \in \mathbb{N}$, such that almost all R_w have parameter degree at most e ;
- (5.10.4) for every generic sequence in R of the form (π, \mathbf{y}) , the contracted ideal $\mathbf{y}R \cap \mathfrak{D}$ is zero.

Proof. Let \mathfrak{m} be the maximal ideal of R , with \mathfrak{D} -approximation \mathfrak{m}_w . Let d be the geometric dimension of R and let d' be the dimension of almost all R_w . Suppose first that $d = d'$. Let \mathbf{x} be any generic sequence in R with \mathfrak{D} -approximation \mathbf{x}_w . By Łos' Theorem, almost all \mathbf{x}_w generate an \mathfrak{m}_w -primary ideal. Since their length is equal to the dimension of R_w , they are almost all systems of parameters of R_w . Choose c large enough so that $\pi^c \in \mathbf{x}R$. Enlarging c if necessary, we may moreover assume by Lemma 2.1 that almost all \mathbf{x}_w have \mathfrak{D}_w -complexity at most c . By Łos' Theorem, $(\pi_w)^c \in \mathbf{x}_w R_w$, so that (5.10.2) holds.

Assume next that c and the \mathbf{x}_w are as in (5.10.2). Let $\overline{R}_w := R_w/(\pi_w)^c R_w$. We can apply [44, Corollary 2] over $\mathfrak{D}_w/(\pi_w)^c \mathfrak{D}_w$ to the $\mathfrak{m}_w \overline{R}_w$ -primary ideal $\mathbf{x}_w \overline{R}_w$, to conclude that there is some c' , depending only on c , such that $\overline{R}_w/\mathbf{x}_w \overline{R}_w$ has length at most c' . Since the latter residue ring is just $R_w/\mathbf{x}_w R_w$ by assumption, the parameter degree of R_w is at most c' , and hence (5.10.3) holds.

To show that (5.10.3) implies (5.10.1), assume that almost all R_w have parameter degree at most e . Let \mathbf{y}_w be a system of parameters of R_w such that $R_w/\mathbf{y}_w R_w$ has length at most e , for almost all w . It follows that $(\mathfrak{m}_w)^e$ is contained in $\mathbf{y}_w R_w$. Let \mathbf{y}_∞ be the ultraproduct of the \mathbf{y}_w . By Łos' Theorem, $\mathfrak{m}^e \mathcal{L}(R) \subseteq \mathbf{y}_\infty \mathcal{L}(R)$ whence $\mathfrak{m}^e \widehat{R} \subseteq \mathbf{y}_\infty \widehat{R}$, by Lemma 5.3, showing that $\mathbf{y}_\infty \widehat{R}$ is $\mathfrak{m} \widehat{R}$ -primary. Since \mathbf{y}_∞ has length at most d' (some entries might be zero in \widehat{R}), the dimension of \widehat{R} is at most d' . Since we already remarked that $d' \leq d$, we get from Theorem 4.4 that $d' = d$.

So remains to show that (5.10.4) is equivalent to the other conditions. Assume first that it holds but that R is not isodimensional. Since we have inequalities $d - 1 \leq d' \leq d$, this means that $d' = d - 1$. Moreover, $R/\pi R$ must have geometric dimension also equal to $d - 1$, for if not, its geometric dimension would be d , whence almost all $R_w/\pi_w R_w$ would have dimension d by [39, Theorem 4.5], which is impossible. Since there is a uniform bound c on the \mathfrak{D}_w -complexity of each R_w , we can choose, using Corollary 2.6, a system of parameters \mathbf{y}_w of R_w of \mathfrak{D}_w -complexity at most c^2 . In particular, some power of π_w lies in $\mathbf{y}_w R_w$. Let $a \in \mathfrak{D}$ be the ultraproduct of these powers. If \mathbf{y} is the ultraproduct of the \mathbf{y}_w , then \mathbf{y} is already defined over R by Lemma 2.1. By Łos' Theorem, $a \in \mathbf{y} \mathcal{L}(R)$, whence by faithful flatness, a is a non-zero element in $\mathbf{y}R \cap \mathfrak{D}$. Therefore, to reach the desired contradiction with (5.10.4), we only need to show that (π, \mathbf{y}) is generic. As we already established, $R_w/\pi_w R_w$ has dimension $d - 1$, so that \mathbf{y}_w is also a system of parameters in that ring. Therefore, \mathbf{y} is a system of parameters in $R/\pi R$ by [39, Theorem 4.5]. This in turn implies that (π, \mathbf{y}) generates an \mathfrak{m} -primary ideal in R . Since this tuple has length d , it is therefore generic, as we wanted to show.

Finally, assume R is isodimensional, and suppose (π, \mathbf{y}) is generic. Let $a \in \mathbf{y}R \cap \mathfrak{D}$ and choose \mathfrak{D} -approximations a_w and \mathbf{y}_w of a and \mathbf{y} respectively. By Łos' Theorem, $a_w \in \mathbf{y}_w R_w$. However, if a is non-zero, then a_w is, up to a unit, a power of π_w , which contradicts the assertion in Corollary 5.8 that (π_w, \mathbf{y}_w) is a system of parameters. So $a = 0$, as we needed to show. \square

5.11. Corollary. *For each c , there exists a bound $PD(c)$ with the following property. Let V be a discrete valuation ring and let C be a local V -affine algebra of V -complexity at most c . If C is torsion-free over V , then the parameter degree of C is at most $PD(c)$.*

Proof. If the statement is false for some c , then we can find for each w a discrete valuation ring \mathfrak{D}_w and a torsion-free local \mathfrak{D}_w -affine algebra R_w of \mathfrak{D}_w -complexity at most c , whose parameter degree is at least w . Let R be the restricted ultraproduct of the R_w and let $\mathcal{L}(R)$ be their ultraproduct. Since π_w is R_w -regular, π is $\mathcal{L}(R)$ -regular, whence R -regular. Hence

R is isodimensional by Theorem 5.4. Therefore, there is a bound on the parameter degree of almost all R_w by Proposition 5.10, contradicting our assumption. \square

Our next goal is to introduce a notion similar to height. Let I be an arbitrary ideal of R .

5.12. Definition (Geometric height). We call the *geometric height* of I the maximum of all h such that there exists a generic sequence whose first h entries belong to I .

For Noetherian rings, we cannot expect a good relationship between the height of an ideal and the dimension of its residue ring, unless the ring is a catenary domain; the following is the analogue over ultra-DVR's.

5.13. Theorem. *Let \mathfrak{D} be an ultra-DVR and let R be a local \mathfrak{D} -affine domain with \mathfrak{D} -approximation R_w . Let I be a finitely generated ideal in R with \mathfrak{D} -approximation I_w .*

If R/I is isodimensional, then the geometric height of I is equal to the geometric dimension of R minus the geometric dimension of R/I , and this is also equal to the height of almost all I_w .

Proof. Let d be the geometric dimension of R and e the geometric dimension of R/I . Since a domain is isodimensional, almost all R_w have dimension d by Theorem 5.4, and by assumption, almost all R_w/I_w have dimension e . Let h be the geometric height of I . Let \mathbf{z} be a generic sequence in R with its first h entries in I , and let \mathbf{z}_w be an \mathfrak{D} -approximation of \mathbf{z} . By Corollary 5.8, almost all \mathbf{z}_w are a system of parameters in R_w . Since by Łos' Theorem the first h entries of \mathbf{z}_w lie in I_w , we get that R_w/I_w has dimension at most $d-h$. In other words, $h \leq d - e$. Since almost all R_w are catenary domains, almost all I_w have height $d - e$.

So remains to show that $d - e \leq h$. By Lemma 5.2, the separated quotient of R/I is equal to $\tilde{R}/I\tilde{R}$. Therefore, by the remark following Corollary 5.6, we can find a generic sequence (x_1, \dots, x_d) in R such that (the image of) (x_1, \dots, x_e) is a generic sequence in R/I . By definition of generic sequence, $S := R/(x_1, \dots, x_e)R$ has geometric dimension $d - e$. If x_{iw} is an \mathfrak{D} -approximation of x_i , then almost each $\mathbf{x}_w := (x_{1w}, \dots, x_{ew})$ is a system of parameters in R_w/I_w by Corollary 5.8. Since \mathbf{x}_w is therefore part of a system of parameters in R_w , almost each $S_w := R_w/\mathbf{x}_w R_w$ has dimension $d - e$ by [27, Theorem 14.1]. By choice of the x_i , the ideal $I + (x_1, \dots, x_e)R$ is \mathfrak{m} -primary and hence IS is $\mathfrak{m}S$ -primary. Therefore, by Corollary 5.6, we can find a tuple \mathbf{y} of length $d - e$ in I , so that its image in S is a generic sequence. It follows that $(x_1, \dots, x_e)R + \mathbf{y}R$ is \mathfrak{m} -primary. Since $(\mathbf{y}, x_1, \dots, x_e)$ has length d , it is a generic sequence, showing that $d - e \leq h$. \square

6. PSEUDO SINGULARITIES

In this section, we maintain the notation introduced in the previous section. Our goal is to extend several singularity notions of Noetherian local rings to the category of local \mathfrak{D} -affine algebras.

Grade and depth. Let B be an arbitrary ring and $I := (x_1, \dots, x_n)B$ a finitely generated ideal. The *grade* of I , denoted $\text{grade}(I)$, is by definition equal to $n - h$, where h is the largest value i for which the i -th Koszul homology $H_i(x_1, \dots, x_n)$ is non-zero. For a local ring R of finite embedding dimension, we define its *depth* as the grade of its maximal ideal.

If B is moreover Noetherian, then we can define the grade of I alternatively as the minimal i for which $\text{Ext}_B^i(B/I, B)$ is non-zero (for all this see for instance [7, §9.1]). An

arbitrary local ring has positive depth if and only if its maximal ideal is not an associated prime. Grade, and hence depth, *deforms well*, in the sense that the

$$\text{grade}(I(B/\mathbf{x}B)) = \text{grade}(I) - |\mathbf{x}|$$

for every B -regular sequence \mathbf{x} in I . For a locally finitely generated \mathfrak{D} -algebra (R, \mathfrak{m}) , its depth never exceeds its geometric dimension. Indeed, by definition, the grade of a finitely generated ideal never exceeds its minimal number of generators, and by [7, Proposition 9.1.3], the depth of R is equal to the grade of any of its \mathfrak{m} -primary ideals. It follows that the depth of R is at most its geometric dimension.

In general, the grade of a finitely generated ideal might be positive without it containing a B -regular element. However, the next lemma shows that this is not the case for ultraproducts of Noetherian local rings.

6.1. Lemma. *Let C_∞ be the ultraproduct of Noetherian local rings C_w and let I_∞ be a finitely generated ideal of C_∞ obtained as the ultraproduct of ideals $I_w \subseteq C_w$.*

If I_∞ has grade n , then there exists a C_∞ -regular sequence of length n with all of its entries in I_∞ . Moreover, any permutation of a C_∞ -regular sequence is again C_∞ -regular.

Proof. By [7, Proposition 9.1.3], there exists a finite tuple of indeterminates Y and a $C_\infty[Y]$ -regular sequence \mathbf{f}_∞ of length n , with all of its entries in $I_\infty C_\infty[Y]$. Choose tuples \mathbf{f}_w in $C_w[Y]$ so that their ultraproduct is \mathbf{f}_∞ . By Łos' Theorem, \mathbf{f}_w is $C_w[Y]$ -regular and has all of its entries in $I_w C_w[Y]$, for almost all w . This shows that $I_w C_w[Y]$ has grade at least n . Since $C_w \rightarrow C_w[Y]$ is faithfully flat, I_w has grade at least n by [7, Proposition 9.1.2]. Hence, since C_w is Noetherian, we can find a C_w -regular sequence \mathbf{x}_w of length n with all of its entries in I_w . By Łos' Theorem, the ultraproduct \mathbf{x}_∞ of the \mathbf{x}_w is C_∞ -regular and has all of its entries in I_∞ .

The last assertion follows from Łos' Theorem and the fact that in a Noetherian local ring, any permutation of a regular sequence is again regular ([27, Theorem 16.3]). \square

Recall that a Noetherian local ring for which its dimension and its depth (respectively, its dimension and its embedding dimension) coincide is Cohen-Macaulay (respectively, regular). We will shortly see that upon replacing dimension by geometric dimension, we get equally well behaved notions. Let us therefore make the following definitions, for R a local \mathfrak{D} -affine algebra.

6.2. Definition. We say that R is *pseudo-Cohen-Macaulay*, if its geometric dimension is equal to its depth, and *pseudo-regular*, if its geometric dimension is equal to its embedding dimension.

6.3. Theorem. *Let \mathfrak{D} be an ultra-DVR and let R be an isodimensional local \mathfrak{D} -affine algebra with \mathfrak{D} -approximation R_w . In order for R to be pseudo-Cohen-Macaulay it is necessary and sufficient that almost all R_w are Cohen-Macaulay.*

Proof. Let d be the geometric dimension of R and δ its depth. Suppose first that $d = \delta$. Since $R \rightarrow \mathfrak{L}(R)$ is faithfully flat, $\mathfrak{L}(R)$ has depth δ as well by [7, Proposition 9.1.2]. By Lemma 6.1, there exists an $\mathfrak{L}(R)$ -regular sequence \mathbf{x}_∞ of length d . If \mathbf{x}_w is an \mathfrak{D} -approximation of \mathbf{x}_∞ , then almost each \mathbf{x}_w is R_w -regular by Łos' Theorem. Since almost all R_w have dimension d by isodimensionality, almost all are Cohen-Macaulay.

Conversely, assume almost all R_w are Cohen-Macaulay. It follows by reversing the above argument that $\mathfrak{L}(R)$ has depth d and hence, so has R , by faithful flatness. \square

Since every system of parameters is a regular sequence in a local Cohen-Macaulay ring, we expect a similar behavior for generic sequences, and this indeed holds.

6.4. Theorem. *Let \mathfrak{D} be an ultra-DVR and let R be an isodimensional local \mathfrak{D} -affine algebra. If R is pseudo-Cohen-Macaulay, then any generic sequence is R -regular.*

Proof. Let \mathbf{x} be a generic sequence with \mathfrak{D} -approximation \mathbf{x}_w . Almost each \mathbf{x}_w is a system of parameters in R_w , by Corollary 5.8. Since almost all R_w are Cohen-Macaulay by Theorem 6.3, almost each \mathbf{x}_w is R_w -regular. Hence \mathbf{x} is $\mathfrak{L}(R)$ -regular, by Łos' Theorem, whence R -regular, by faithful flatness. \square

6.5. Theorem. *Let \mathfrak{D} be an ultra-DVR. An isodimensional local \mathfrak{D} -affine algebra R with \mathfrak{D} -approximation R_w is pseudo-regular if and only if almost all R_w are regular local rings.*

Proof. Let \mathfrak{m} be the maximal ideal of R , with \mathfrak{D} -approximation \mathfrak{m}_w . Let $\mathfrak{L}(R)$ be the non-standard \mathfrak{D} -hull of R . Let ϵ be the embedding dimension of R and d its geometric dimension. Suppose that R is pseudo-regular, that is to say, that $\epsilon = d$. Hence $\mathfrak{m} = \mathbf{x}R$ for some d -tuple \mathbf{x} , necessarily generic. Since $\mathfrak{m}\mathfrak{L}(R) = \mathbf{x}\mathfrak{L}(R)$, Łos' Theorem yields that $\mathfrak{m}_w = \mathbf{x}_w R_w$, where \mathbf{x}_w is an \mathfrak{D} -approximation of \mathbf{x} . Since almost all R_w have dimension d , almost all are regular local rings.

Conversely, suppose almost all R_w are regular. Since the \mathfrak{D}_w -complexity of almost all R_w is at most c , for some c , we can find a regular system of parameters \mathbf{x}_w of \mathfrak{D}_w -complexity at most c (as part of a minimal system of generators of \mathfrak{m}_w). By Lemma 2.1, their ultraproduct \mathbf{x} belongs to R , and is a generic sequence by Corollary 5.8. By Łos' Theorem and faithful flatness, $\mathbf{x}R = \mathfrak{m}$ whence $\epsilon \leq d$. Since geometric dimension never exceeds embedding dimension, $\epsilon = d$ and R is pseudo-regular. \square

The following is now immediate from the previous result and Theorem 4.4.

6.6. Corollary. *Let \mathfrak{D} be an ultra-DVR. If R is a pseudo-regular local \mathfrak{D} -affine algebra, then R is a domain if and only if it is isodimensional. Moreover, if this is the case, then every localization of R with respect to a prime ideal containing π is again pseudo-regular.*

In fact, the restricted ultraproduct R of regular local \mathfrak{D}_w -affine algebras R_w of uniformly bounded \mathfrak{D}_w -complexity is pseudo-regular and isodimensional. Indeed, we already observed that then R is isodimensional, and therefore by Theorem 6.5, pseudo-regular. For a homological characterization of pseudo-regularity, see Corollary 11.5 below.

6.7. Example. If R denotes the localization of $\mathfrak{D}[X, Y]/(X^2 + Y^3 + \pi)$ at the maximal ideal generated by X, Y and π , then R is pseudo-regular (namely X and Y generate the maximal ideal, so $\epsilon = 2$, and since $R/\pi R$ has dimension one, $d = 2$ as well). Note though that $R/\pi R$ is not regular.

6.8. Corollary. *Let \mathfrak{D} be an ultra-DVR and let R be an isodimensional local \mathfrak{D} -affine algebra. If R is pseudo-regular, then it is pseudo-Cohen-Macaulay.*

Proof. Let R_w be an \mathfrak{D} -approximation of R . By Theorem 6.5, almost all R_w are regular whence Cohen-Macaulay. This in turn implies that R is pseudo-Cohen-Macaulay by Theorem 6.3. \square

Without the isodimensionality assumption, the result is false. For instance, let a be a non-zero element in the ideal of infinitesimals of \mathfrak{D} and put $R := \mathfrak{D}/a\mathfrak{D}$. It follows that R has geometric dimension one, whence is pseudo-regular, but its depth is zero.

6.9. Transfer. Let me now elaborate on why the results in this section are instances of transfer between positive and mixed characteristic. Suppose $\tilde{\mathfrak{D}}$ is a second ultra-DVR, realized as the ultraproduct of discrete valuation rings $\tilde{\mathfrak{D}}_w$ and suppose $\mathfrak{D} \cong \tilde{\mathfrak{D}}$. Note that this does *not* imply that \mathfrak{D}_w and $\tilde{\mathfrak{D}}_w$ are almost all pair-wise isomorphic. In fact, in the next sections, one set of discrete valuation rings will be of mixed characteristic and the other set of prime characteristic. Let R be a local \mathfrak{D} -affine algebra. Since R is then also local $\tilde{\mathfrak{D}}$ -affine, it admits a non-standard $\tilde{\mathfrak{D}}$ -hull and $\tilde{\mathfrak{D}}$ -approximations with respect to this second set of discrete valuation rings; let us denote them by $\mathfrak{L}_{\tilde{\mathfrak{D}}}(R)$ and \tilde{R}_w respectively. Suppose \mathfrak{D}_w and $\tilde{\mathfrak{D}}_w$ have pair-wise isomorphic residue fields (as will be the case below). Since the $R_w/\pi_w R_w$ are an approximation of the κ -algebra $R/\pi R$ (in the sense of [39]) and, mutatis mutandis, so are the $\tilde{R}_w/\tilde{\pi}_w \tilde{R}_w$, where $\tilde{\pi}_w$ is a uniformizing parameter of $\tilde{\mathfrak{D}}_w$, we get from [39, 3.2.3] that almost all $R_w/\pi_w R_w$ are isomorphic to $\tilde{R}_w/\tilde{\pi}_w \tilde{R}_w$. Therefore, if we assume that there is no torsion, then R_w and \tilde{R}_w have the same dimension, and one set consists of almost all Cohen-Macaulay local rings if and only if the other set does (note that this argument does not yet use the above *pseudo* notions). However, this argument breaks down in the presence of torsion, or, when we want to transfer the regularity property. This can be overcome by using the notions defined in this section, provided we have a uniform upper bound on the parameter degree.

Suppose, for some $d, e \in \mathbb{N}$, that almost all R_w have dimension d and parameter degree at most e . Note that in view of Corollary 5.11 this last condition is automatically satisfied if almost all R_w are torsion-free over \mathfrak{D}_w ; and that it is implied by the assumption that almost all R_w have uniformly bounded homological multiplicity (see [43, Corollary 4.6]). Applying Proposition 5.10 twice gives first that R is isodimensional, with geometric dimension d , and then that almost all \tilde{R}_w have dimension d and uniformly bounded parameter degree. Now, Theorems 6.3 and 6.5 tell us that almost all R_w are respectively Cohen-Macaulay or regular, if and only if almost all \tilde{R}_w are.

7. BIG COHEN-MACAULAY ALGEBRAS

In [3, 41], ultraproducts of absolute integral closures in characteristic p were used to define big Cohen-Macaulay algebras in equicharacteristic zero. This same process can be used in the current mixed characteristic setting. Recall that for an arbitrary domain B , we define its *absolute integral closure* as the integral closure of B in some algebraic closure of its field of fractions and denote it B^+ . This is uniquely defined up to B -algebra isomorphism.

For each prime number p , let $\mathfrak{D}_p^{\text{mix}}$ be a mixed characteristic complete discrete valuation ring with uniformizing parameter π_p and residue field κ_p of characteristic p , and let \mathfrak{D} , π and κ be their respective ultraproducts. Put $\mathfrak{D}_p^{\text{eq}} := \kappa_p[[t]]$, for t a single indeterminate. By Theorem 2.3, the Ax-Kochen-Ershov Theorem, \mathfrak{D} is isomorphic to the ultraproduct of the $\mathfrak{D}_p^{\text{eq}}$. As before, $\mathfrak{I}_{\mathfrak{D}}$ denotes the ideal of infinitesimals of \mathfrak{D} . Put $A := \mathfrak{D}[X]$, for a fixed tuple of indeterminates X , and let $\mathfrak{L}_{\mathfrak{D}}^{\text{eq}}(A)$ and $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(A)$ be its respective equicharacteristic and mixed characteristic non-standard \mathfrak{D} -hull, that is to say, the ultraproduct of respectively the $A_p^{\text{eq}} := \mathfrak{D}_p^{\text{eq}}[X]$ and the $A_p^{\text{mix}} := \mathfrak{D}_p^{\text{mix}}[X]$.

Throughout, R will be a local \mathfrak{D} -affine domain with R_p^{eq} and $\mathfrak{L}_{\mathfrak{D}}^{\text{eq}}(R)$ respectively an equicharacteristic \mathfrak{D} -approximation and the equicharacteristic non-standard \mathfrak{D} -hull of R (so that $\mathfrak{L}_{\mathfrak{D}}^{\text{eq}}(R)$ is the ultraproduct of the R_p^{eq}). By Theorem 4.4, almost all R_p^{eq} are local domains.

7.1. Definition. Define $\mathcal{B}(R)$ as the ultraproduct of the $(R_p^{\text{eq}})^+$.

Since $(R_p^{\text{eq}})^+$ is well-defined up to R_p^{eq} -algebra isomorphism, we have that $\mathcal{B}(R)$ is well-defined up to R -algebra isomorphism. Moreover, this construction is weakly functorial in the following sense. Let $R \rightarrow S$ be an \mathfrak{D} -algebra homomorphism between local \mathfrak{D} -affine domains. This induces $\mathfrak{D}_p^{\text{eq}}$ -algebra homomorphisms $R_p^{\text{eq}} \rightarrow S_p^{\text{eq}}$ of the corresponding equicharacteristic \mathfrak{D} -approximations. These in turn yield homomorphisms $(R_p^{\text{eq}})^+ \rightarrow (S_p^{\text{eq}})^+$ between the absolute integral closures. Taking ultraproducts, we get an \mathfrak{D} -algebra homomorphism $\mathcal{B}(R) \rightarrow \mathcal{B}(S)$ and a commutative diagram

$$(6) \quad \begin{array}{ccc} R & \xrightarrow{\quad} & S \\ \downarrow & & \downarrow \\ \mathcal{B}(R) & \xrightarrow{\quad} & \mathcal{B}(S). \end{array}$$

7.2. Theorem. *If R is a local \mathfrak{D} -affine domain, then any generic sequence in R is $\mathcal{B}(R)$ -regular.*

Proof. Let $\mathfrak{L}_{\mathfrak{D}}^{\text{eq}}(R)$ and R_p^{eq} be respectively, the equicharacteristic non-standard \mathfrak{D} -hull and an equicharacteristic \mathfrak{D} -approximation of R . Let \mathbf{x} be a generic sequence, and let \mathbf{x}_p be an \mathfrak{D} -approximation of \mathbf{x} . By Corollary 5.8, almost each \mathbf{x}_p is a system of parameters in R_p^{eq} , whence is $(R_p^{\text{eq}})^+$ -regular by [19]. By Łos' Theorem, \mathbf{x} is $\mathcal{B}(R)$ -regular. \square

8. IMPROVED NEW INTERSECTION THEOREM

The remaining sections will establish various asymptotic versions in mixed characteristic of the Homological Conjectures listed in the abstract. We start with discussing Intersection Theorems. By [30], we now know that the New Intersection Theorem holds for all Noetherian local rings. However, this is not yet known for the Improved New Intersection Theorem. We need some terminology and notation (all taken from [7]).

Let C be an arbitrary Noetherian local ring and $\varphi: C^a \rightarrow C^b$ a linear map between finite free C -modules. We will always think of φ as an $(a \times b)$ -matrix over C . For $r > 0$, recall that the r -th Fitting ideal of φ , denoted $I_r(\varphi)$, is the ideal in C generated by all $(r \times r)$ minors of φ ; if r exceeds the size of the matrix, we put $I_r(\varphi) := (0)$.

By a *finite free complex* over C we mean a complex

$$(F_{\bullet}) \quad 0 \rightarrow C^{a_s} \xrightarrow{\varphi_s} C^{a_{s-1}} \xrightarrow{\varphi_{s-1}} \dots \xrightarrow{\varphi_2} C^{a_1} \xrightarrow{\varphi_1} C^{a_0} \rightarrow 0.$$

We call s the *length* of the complex, and for each i , we define

$$r_i := \sum_{j=i}^s (-1)^{j-i} a_j.$$

We will refer to r_i as the *expected rank* of φ_i . We will call the residue ring $C/I_{r_i}(\varphi_i)$ the i -th Fitting ring of F_{\bullet} and we will denote it $\mathfrak{R}_i(F_{\bullet})$.

The i -th *homology* of F_{\bullet} is by definition the quotient module

$$H_i(F_{\bullet}) := \text{Ker}(\varphi_i) / \text{Im}(\varphi_{i+1}).$$

We call F_{\bullet} *acyclic*, if all $H_i(F_{\bullet}) = 0$ for $i > 0$. In that case, F_{\bullet} yields a *finite free resolution* of $H_0(F_{\bullet})$.

In case C is a Z -affine algebra with Z a local ring, we say that F_\bullet has Z -complexity at most c , if its length s is at most c , if all $a_i \leq c$, and if every entry of each φ_i has Z -complexity at most c . Below we will say that an element τ in a homology module $H_i(F_\bullet)$ has Z -complexity at most c , if it is the image of a tuple in $\text{Ker}(\varphi_i)$ of Z -complexity at most c (for more details, see §11 below).

8.1. Theorem (Asymptotic Improved New Intersection Theorem). *For each c , there exists a bound $\text{INIT}(c)$ with the following property. Let V be a mixed characteristic discrete valuation ring and let (C, \mathfrak{m}) be a local V -affine domain. Let F_\bullet be a finite free complex over C . Assume $H_0(F_\bullet)$ has a minimal generator τ , such that $C\tau$ has finite length and assume that c simultaneously bounds the V -complexity of C , τ and F_\bullet , the parameter degree of each Fitting ring $\mathfrak{R}_i(F_\bullet)$, and the length of $C\tau$.*

If $\mathfrak{R}_i(F_\bullet)$ has dimension at most $\dim C - i$, for $i = 1, \dots, s$, then the dimension of C is at most the length of the complex F_\bullet , provided the characteristic of the residue field of \mathfrak{D} is bigger than $\text{INIT}(c)$.

Proof. If $\pi C = 0$, then C contains the residue field of V and in that case the Theorem is known (see for instance [7, Theorem 9.4.1] or [11, 16]). So we may moreover assume that C is flat over V . By faithful flat descent, we may replace V and C by \widehat{V} and a suitable localization of $\widehat{V} \otimes_V C$ respectively, where \widehat{V} is the completion of V . In other words, we only need to prove the result for a torsion-free local domain over a complete discrete valuation ring of mixed characteristic. Suppose this last assertion is false for some c , so that there exists an infinite set U of prime numbers and for each $p \in U$ a counterexample consisting of the following data:

- a mixed characteristic complete discrete valuation ring $\mathfrak{D}_p^{\text{mix}}$ with uniformizing parameter π_p , whose residue field has characteristic p ;
- a local $\mathfrak{D}_p^{\text{mix}}$ -affine domain R_p^{mix} of $\mathfrak{D}_p^{\text{mix}}$ -complexity at most c ;
- a finite free complex

$$(F_{p_\bullet}^{\text{mix}}) \quad 0 \rightarrow (R_p^{\text{mix}})^{a_s} \xrightarrow{\varphi_{s,p}} (R_p^{\text{mix}})^{a_{s-1}} \xrightarrow{\varphi_{s-1,p}} \dots \xrightarrow{\varphi_{2,p}} (R_p^{\text{mix}})^{a_1} \xrightarrow{\varphi_{1,p}} (R_p^{\text{mix}})^{a_0} \rightarrow 0$$

of length s and of $\mathfrak{D}_p^{\text{mix}}$ -complexity at most c , such that the i -th Fitting ring $\mathfrak{R}_i(F_{p_\bullet}^{\text{mix}})$ has dimension at most $d - i$ and parameter degree at most c ;

- a minimal generator τ_p of $H_0(F_{p_\bullet}^{\text{mix}})$ of $\mathfrak{D}_p^{\text{mix}}$ -complexity at most c , generating a module of length at most c ,

but such that s is strictly less than the dimension of R_p^{mix} . Choose some non-principal ultrafilter on the set of prime numbers which contains U . In particular, we have a counterexample with the above properties for almost all p . Without loss of generality, we may assume that the dimension of each R_p^{mix} and that the ranks of each $F_{p_\bullet}^{\text{mix}}$ are independent from p , since there are only finitely many possibilities, and hence precisely one such possibility almost always holds. In particular, the expected ranks do not depend on p .

Let \mathfrak{D} and π be the respective ultraproduct of the $\mathfrak{D}_p^{\text{mix}}$ and the π_p . Let R and $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)$ be the respective restricted ultraproduct and ultraproduct of the R_p^{mix} . It follows from Theorem 4.4, that R is a local \mathfrak{D} -affine domain, and from Theorem 4.2, that $R \rightarrow \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)$ is faithfully flat. Let d be the geometric dimension of R , so that almost all R_p^{mix} have dimension d by Theorem 5.4. Let φ_i be the ultraproduct of the $\varphi_{i,p}$. It follows from

Lemma 2.1 that each φ_i is already defined over R . Hence by Łos' Theorem

$$(F_\bullet) \quad 0 \rightarrow R^{a_s} \xrightarrow{\varphi_s} R^{a_{s-1}} \xrightarrow{\varphi_{s-1}} \dots \xrightarrow{\varphi_2} R^{a_1} \xrightarrow{\varphi_1} R^{a_0} \rightarrow 0$$

is a finite free complex. Let M denote its zero-th homology and fix some i . By Łos' Theorem, $I_{r_i}(\varphi_{i,p})$ is an \mathfrak{D} -approximation of $I_{r_i}(\varphi_i)$. By the uniform boundedness of the parameter degrees, $\mathfrak{R}_i(F_\bullet)$ is isodimensional by Proposition 5.10. If d_i is the geometric dimension of $\mathfrak{R}_i(F_\bullet)$, then $d - d_i$ is equal to the height of almost all $I_{r_i}(\varphi_{i,p})$ and to the geometric height of $I_{r_i}(\varphi_i)$, by Theorem 5.13. In particular, by assumption, $i \leq d - d_i$, and therefore, by definition of geometric height, we can find a generic sequence \mathbf{x}_i in R whose first i entries belong to $I_{r_i}(\varphi_i)$.

Let $B := \mathcal{B}(R)$. Since \mathbf{x}_i is B -regular by Theorem 7.2, the grade of $I_{r_i}(\varphi_i)B$ is at least i . Since this holds for all i , the Buchsbaum-Eisenbud-Northcott Acyclicity Theorem ([7, Theorem 9.1.6]) proves that $F_\bullet \otimes_R B$ is acyclic. Since B has depth at least d , it follows from [7, Theorem 9.1.2] that the zero-th homology of $F_\bullet \otimes_R B$, that is to say, $M \otimes_R B$, has depth at least $d - s$.

Let τ be the ultraproduct of the τ_p . Note that each τ_p is by assumption the image of a tuple in $(R_p^{\text{mix}})^{a_0}$ of $\mathfrak{D}_p^{\text{mix}}$ -complexity at most c , so that τ is already defined over R by Lemma 2.1. By Łos' Theorem, τ is a minimal generator of

$$H_0(F_\bullet \otimes \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)) = M \otimes \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R),$$

and by [34, Proposition 1.1] or [25, Proposition 9.1], the length of $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)\tau$ is at most c . By faithful flatness, $\tau \in M - \mathfrak{m}M$, where \mathfrak{m} is the maximal ideal of R , and $R\tau$ has length at most c . In particular, the image of $\tau \otimes 1$ in $M/\mathfrak{m}M \otimes B/\mathfrak{m}B$ is non-zero, and therefore $\tau \otimes 1$ itself is a non-zero element of $M \otimes B$. Since \mathfrak{m}^c annihilates $\tau \otimes 1$, we get that $M \otimes B$ has depth zero. Together with the conclusion from the previous paragraph, we get that $d \leq s$, contradiction. \square

This type of argument *ex absurdum*, to obtain uniform bounds via ultraproducts, is very common and will be used constantly in the sequel. We will shorten the argument by saying from the start that by way of contradiction, we may assume that for some c , there exist for almost each p a counterexample with such and such properties.

9. MONOMIAL AND DIRECT SUMMAND CONJECTURES

We keep notation as in the previous section, so that in particular \mathfrak{D} will denote the ultraproduct of mixed characteristic complete discrete valuation rings $\mathfrak{D}_p^{\text{mix}}$. In order to formulate a non-standard version of the Monomial Conjecture, we need some terminology. Let \mathbb{N}_∞ be the ultrapower of \mathbb{N} . Let C_w be rings, $X := (X_1, \dots, X_d)$ indeterminates and A_∞ the ultraproduct of the $C_w[X]$. Although each $C_w[X]$ is \mathbb{N} -graded, it is not true that A_∞ is \mathbb{N}_∞ -graded, since we might have infinite sums of monomials in A_∞ . Nonetheless, for each $\nu_\infty \in (\mathbb{N}_\infty)^d$, the element X^{ν_∞} is well-defined, namely, if ν_∞ is the ultraproduct of elements $\nu_w \in \mathbb{N}$, then

$$X^{\nu_\infty} := \text{ulim}_{w \rightarrow \infty} X^{\nu_w}.$$

In particular, if B_∞ is an arbitrary ultraproduct of rings B_w and if \mathbf{x} is a d -tuple in B_∞ , then \mathbf{x}^{ν_∞} is a well-defined element of B_∞ .

By a *cone* H in a semi-group Γ (e.g., $\Gamma = \mathbb{N}^d$ or $\Gamma = \mathbb{N}_\infty^d$), we mean a subset H of Γ such that $\nu + \Gamma \subseteq H$, for every $\nu \in H$, where $\nu + \Gamma$ stands for the collection of all

$\nu + \gamma$ with $\gamma \in \Gamma$. A cone H is *finitely generated*, if there exist $\nu_1, \dots, \nu_s \in H$, called *generators* of the cone, such that

$$H = \bigcup_i \nu_i + \Gamma.$$

If H is a cone in \mathbb{N}^d , we let J_H be the monomial ideal in $\mathbb{Z}[Y]$ generated by all Y^ν with $\nu \in H$, where Y is a d -tuple of indeterminates. If H is generated by ν_1, \dots, ν_s , then J_H is generated by $X^{\nu_1}, \dots, X^{\nu_s}$. Conversely, if J is a monomial ideal in $\mathbb{Z}[Y]$, then the collection of all ν for which $Y^\nu \in J$, is a cone in \mathbb{N}^d . Since $\mathbb{Z}[Y]$ is Noetherian, every cone in \mathbb{N}^d is finitely generated. This is no longer true for a cone in \mathbb{N}_∞^d .

Let B be an arbitrary ring. We will use the following well-known fact about regular sequences. If \mathbf{x} is a B -regular sequence (in fact, it suffices that \mathbf{x} is quasi-regular), H a cone in \mathbb{N}^d and $\nu \notin H$, then \mathbf{x}^ν does not lie in the ideal $J_H(\mathbf{x})$ generated by all \mathbf{x}^θ with $\theta \in H$.

9.1. Corollary. *Let R be a local \mathfrak{D} -affine domain with equicharacteristic non-standard \mathfrak{D} -hull $\mathfrak{L}_\mathfrak{D}^{\text{eq}}(R)$. Let \mathbf{x} be a generic sequence in R , let H be a cone in \mathbb{N}_∞^d and let $\nu \in \mathbb{N}_\infty^d$. If $\nu \notin H$, then*

$$(7) \quad \mathbf{x}^\nu \notin (\mathbf{x}^\mu \mid \mu \in H) \mathfrak{L}_\mathfrak{D}^{\text{eq}}(R).$$

Proof. Suppose (7) is false for some choice of cone H of \mathbb{N}_∞^d and some $\nu_0 \notin H$. In other words, we can find f_{i_∞} in $\mathfrak{L}_\mathfrak{D}^{\text{eq}}(R)$ and tuples ν_i in H , such that

$$(8) \quad \mathbf{x}^{\nu_0} = f_{1_\infty} \mathbf{x}^{\nu_1} + \dots + f_{s_\infty} \mathbf{x}^{\nu_s}.$$

Since $R \rightarrow \mathcal{B}(R)$ factors through $\mathfrak{L}_\mathfrak{D}^{\text{eq}}(R)$, we can view (8) as a relation in $\mathcal{B}(R)$, and we want to show that that is impossible. Let R_p^{eq} be an equicharacteristic \mathfrak{D} -approximation of R , so that $\mathcal{B}(R)$ is the ultraproduct of the $(R_p^{\text{eq}})^+$. Choose tuples $\nu_{ip} \in \mathbb{N}$, elements $f_{ip} \in (R_p^{\text{eq}})^+$ and tuples \mathbf{x}_p in R_p^{eq} whose respective ultraproducts are ν_i , f_{i_∞} and \mathbf{x} . By Łos' Theorem, we get that

$$(9) \quad \mathbf{x}_p^{\nu_{0p}} = f_{1p} \mathbf{x}_p^{\nu_{1p}} + \dots + f_{sp} \mathbf{x}_p^{\nu_{sp}}$$

in $(R_p^{\text{eq}})^+$, for almost all p . Łos' Theorem also yields that ν_{0p} does not lie in the cone of \mathbb{N}^d generated by $\nu_{1p}, \dots, \nu_{sp}$, for almost all p . However, \mathbf{x} is $\mathcal{B}(R)$ -regular by Theorem 7.2, whence, almost all \mathbf{x}_p are $(R_p^{\text{eq}})^+$ -regular by Łos' Theorem. By our above discussion on regular sequences, (9) cannot hold for those p . \square

9.2. Theorem (Asymptotic Monomial Conjecture I). *For each c , there exists a bound $MC(c)$ with the following property. Let Y be a tuple of indeterminates and J a monomial ideal in $\mathbb{Z}[Y]$. Let V be a mixed characteristic discrete valuation ring and let C be a local V -affine domain. Let \mathbf{y} be a system of parameters in C and let $J(\mathbf{y})C$ denote the ideal in C obtained from J by the substitution $Y \mapsto \mathbf{y}$. Assume $JV[Y]$, C and \mathbf{y} have V -complexity at most c and $\pi^c \in \mathbf{y}C$.*

If Y^ν is a monomial of degree at most c not belonging to J , then $\mathbf{y}^\nu \notin J(\mathbf{y})C$, provided the characteristic of the residue field of V is bigger than $MC(c)$.

Proof. Note that since C has V -complexity at most c , its dimension d is at most c . By faithful flat descent, we may reduce to the case that V is complete. Suppose the result is false for some c , so that we can find for almost each prime number p ,

- a mixed characteristic complete discrete valuation ring $\mathfrak{D}_p^{\text{mix}}$ with uniformizing parameter π_p , whose residue field has characteristic p ,
- a local $\mathfrak{D}_p^{\text{mix}}$ -affine domain R_p^{mix} of $\mathfrak{D}_p^{\text{mix}}$ -complexity at most c ,

- multi-indices $\nu_{0p}, \dots, \nu_{tp}$ such that $|\nu_{ip}| \leq c$ and ν_{0p} is not in the cone generated by the remaining tuples,
- a system of parameters \mathbf{y}_p of $\mathfrak{D}_p^{\text{mix}}$ -complexity at most c generating an ideal containing $(\pi_p)^c$,

such that

$$(10) \quad \mathbf{y}_p^{\nu_{0p}} \in (\mathbf{y}_p^{\nu_{1p}}, \dots, \mathbf{y}_p^{\nu_{tp}})R_p^{\text{mix}}.$$

Note that the possible number t of tuples ν_{ip} is bounded in terms of c and hence can be taken to be independent of p . Let \mathfrak{D} be the ultraproduct of the $\mathfrak{D}_p^{\text{mix}}$ and let R and $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)$ be the respective restricted ultraproduct and ultraproduct of the R_p^{mix} . Since R is then a domain, it is isodimensional. Let \mathbf{y} and ν_i be the respective ultraproducts of \mathbf{y}_p and ν_{ip} . In particular, $|\nu_i| \leq c$, so that $\nu_i \in \mathbb{N}^d$. Let H be the cone in \mathbb{N}^d generated by ν_1, \dots, ν_t . By Łos' Theorem, $\nu_0 \notin H$. The sequence \mathbf{y} is defined over R , by Lemma 2.1, and is generic in R , by Corollary 5.8. By an application of Łos' Theorem to (10) together with Theorem 4.2, we get

$$\mathbf{y}^{\nu_0} \in (\mathbf{y}^{\nu_1}, \dots, \mathbf{y}^{\nu_t})R.$$

However, this contradicts Corollary 9.1 for the cone H . \square

9.3. *Remark.* In [38, Theorem 1.1], this result was stated erroneously without imposing a bound on the degrees of the monomials. I can only prove this more general result in the special case given by Corollary 9.5 below.

Using some results from [46], we can remove the restriction on C to be a domain. Namely, by the usual argument, we reduce to the domain case by killing a minimal prime \mathfrak{p} of C of maximal dimension (that is to say, so that $\dim C = \dim C/\mathfrak{p}$). However, in order to apply the theorem to the domain C/\mathfrak{p} , we must be guaranteed that its V -complexity is at most c' , for some c' only depending on c . Such a bound does indeed exist by [46, Theorems 9.2 and 9.12].

9.4. **Theorem** (Asymptotic Direct Summand Conjecture). *For each c , we can find a bound $DS(c)$ with the following property. Let V be a mixed characteristic discrete valuation ring and let $C \rightarrow D$ be a finite, injective local V -algebra homomorphism of V -complexity at most c .*

If C is regular, then C is a direct summand of D (as a C -module), provided the characteristic of the residue field of V is bigger than the bound $DS(c)$.

Proof. If $\pi C = 0$, we are in the equicharacteristic case and the result is well-known. So we may assume that $V \subseteq C$. We leave it to the reader to make the reduction to the case that V is complete and D is torsion-free over V . Towards a contradiction, suppose for some c and almost each p , we have a mixed characteristic complete discrete valuation ring $\mathfrak{D}_p^{\text{mix}}$ with residue field of characteristic p , and a finite, injective local $\mathfrak{D}_p^{\text{mix}}$ -algebra homomorphism $R_p^{\text{mix}} \rightarrow S_p^{\text{mix}}$ of $\mathfrak{D}_p^{\text{mix}}$ -complexity at most c , such that R_p^{mix} is regular but not a direct summand of S_p^{mix} .

By the transfer described in §6.9, these data in mixed characteristic yield corresponding data in equal characteristic. In particular, we have for almost each p , an equicharacteristic p complete discrete valuation ring $\mathfrak{D}_p^{\text{eq}}$, and a finite, injective local $\mathfrak{D}_p^{\text{eq}}$ -algebra homomorphism $R_p^{\text{eq}} \rightarrow S_p^{\text{eq}}$ of $\mathfrak{D}_p^{\text{eq}}$ -complexity at most c , such that R_p^{eq} is regular. Although, we did not discuss transfer of homomorphisms and their properties, it is not hard to see, using faithfully flat descent, that almost no R_p^{eq} is a direct summand of S_p^{eq} . However, this is in violation of the Direct Summand theorem in equicharacteristic. \square

9.5. Corollary (Asymptotic Monomial Conjecture II). *For each c , we can find a bound $MC'(c)$ with the following property. Let V be a mixed characteristic discrete valuation ring, let D be a local V -affine algebra and let (x_1, \dots, x_d) be a system of parameters in D .*

If there exists a finite, injective local V -algebra homomorphism $C \subseteq D$ of V -complexity at most c , such that the x_i belong to C and generate its maximal ideal, then $(x_1 \cdots x_d)^t$ does not belong to $(x_1^{t+1}, \dots, x_d^{t+1})D$, for all $t \geq 0$, provided the residue field of V is bigger than $MC'(c)$.

Proof. We may take $MC'(c)$ equal to the bound $DS(c)$ from Theorem 9.4. Indeed, since D has dimension d , so does C , showing that C is regular. Hence C is a direct summand of D by Theorem 9.4, so that we are done by [7, Lemma 9.2.2]. \square

Note that the bounds provided by Theorem 9.2 for the problem at hand depend a priori also on the exponent t , so that the corollary gives a stronger result. Interestingly, by Cohen's Structure Theorem, any system of parameters in a complete local V -affine domain arises as the image of a regular system of parameters under a finite extension. However, since we are forced to work with non-complete V -affine algebras, it is not clear yet to which extent the above theorem applies.

10. PURE SUBRINGS OF REGULAR RINGS

We keep notation as in the previous section, so that in particular \mathfrak{D} will denote the ultraproduct of mixed characteristic complete discrete valuation rings $\mathfrak{D}_p^{\text{mix}}$. Our goal is to show an asymptotic version of the Hochster-Roberts Theorem in [22]. Recall that a ring homomorphism $C \rightarrow D$ is called *cyclically pure* if every ideal I in C is extended from D , that is to say, if $I = ID \cap C$.

10.1. Theorem. *If R is a pseudo-regular isodimensional local \mathfrak{D} -affine algebra, then $R \rightarrow \mathcal{B}(R)$ is faithfully flat.*

Proof. Let L be a linear form in a finite number of indeterminates Y with coefficients in R and let \mathfrak{b} be a solution in $B := \mathcal{B}(R)$ of $L = 0$. Let R_p^{eq} , L_p^{eq} and $\mathfrak{b}_p^{\text{eq}}$ be equicharacteristic \mathfrak{D} -approximations of R , L and \mathfrak{b} respectively. By Łos' Theorem, $\mathfrak{b}_p^{\text{eq}}$ is a solution in $(R_p^{\text{eq}})^+$ of the linear equation $L_p^{\text{eq}} = 0$. By [2, Corollary 4.27], we can find tuples $\mathfrak{a}_{1p}^{\text{eq}}, \dots, \mathfrak{a}_{sp}^{\text{eq}}$ over R_p^{eq} generating the module of solutions of $L_p^{\text{eq}} = 0$, all of $\mathfrak{D}_p^{\text{eq}}$ -complexity at most c , for some c independent from p and s . Let $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ be the respective ultraproducts, which are then defined over R by Lemma 2.1. By Łos' Theorem, $L(\mathfrak{a}_i) = 0$, for each i . On the other hand, almost all R_p^{eq} are regular, by Theorem 6.5. Therefore, $R_p^{\text{eq}} \rightarrow (R_p^{\text{eq}})^+$ is flat by [24, Theorem 9.1]. Hence we can write $\mathfrak{b}_p^{\text{eq}}$ as a linear combination over $(R_p^{\text{eq}})^+$ of the $\mathfrak{a}_{ip}^{\text{eq}}$. By Łos' Theorem, \mathfrak{b} is a B -linear combination of the solutions \mathfrak{a}_i , showing that $R \rightarrow B$ is flat whence faithfully flat. \square

10.2. Proposition. *Let $R \rightarrow S$ be an injective homomorphism of local isodimensional \mathfrak{D} -affine algebras. If $R/\pi R \rightarrow S/\pi S$ is cyclically pure and S is a pseudo-regular local ring, then R is pseudo-Cohen-Macaulay.*

Proof. Since S is a domain by Corollary 6.6, so is R . If $\pi R = 0$, we are in an equicharacteristic Noetherian situation and the statement becomes the Hochster-Roberts Theorem [22]. Therefore, we may assume π is R -regular, so that we can choose a generic sequence $\mathbf{x} := (x_1, \dots, x_d)$ in R with $x_1 = \pi$. For each $n \leq d$, let $I_n := (x_1, \dots, x_n)R$. Suppose $rx_{n+1} \in I_n$, for some $r \in R$. By Theorem 7.2, the sequence \mathbf{x} is a $\mathcal{B}(R)$ -regular.

Therefore, $r \in I_n \mathcal{B}(R)$. Since the homomorphism $R \rightarrow S$ induces a homomorphism $\mathcal{B}(R) \rightarrow \mathcal{B}(S)$, we get $r \in I_n \mathcal{B}(S)$. By Theorem 10.1, we have $I_n \mathcal{B}(S) \cap S = I_n S$, so that $r \in I_n S$. Using finally that $R/\pi R \rightarrow S/\pi S$ is cyclically pure and $\pi \in I_n$, we get $r \in I_n$. This shows that \mathfrak{x} is R -regular, so that R has depth at least d and hence is pseudo-Cohen-Macaulay. \square

10.3. Theorem (Asymptotic Hochster-Roberts Theorem). *For each c , we can find a bound $HR(c)$ with the following property. Let V be a mixed characteristic discrete valuation ring and let $C \rightarrow D$ be a local V -algebra homomorphism of V -complexity at most c .*

If $C \rightarrow D$ is cyclically pure and D is regular, then C is Cohen-Macaulay, provided the characteristic of the residue field of V is at least $HR(c)$.

Proof. As before, we may reduce to the case that V is complete and that $V \subseteq C$. Suppose this assertion is then false for some c , so that we can find for almost each prime number p , a mixed characteristic complete discrete valuation ring $\mathfrak{O}_p^{\text{mix}}$ with residue field of characteristic p and a cyclically pure $\mathfrak{O}_p^{\text{mix}}$ -algebra homomorphism $R_p^{\text{mix}} \rightarrow S_p^{\text{mix}}$ of $\mathfrak{O}_p^{\text{mix}}$ -complexity at most c , such that S_p^{mix} is regular but R_p^{mix} is not Cohen-Macaulay. Let $R \rightarrow S$ and $\mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R) \rightarrow \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(S)$ be respectively the restricted ultraproduct and the ultraproduct of the $R_p^{\text{mix}} \rightarrow S_p^{\text{mix}}$. Theorem 6.3 implies that R is not pseudo-Cohen-Macaulay, and Theorem 6.5, that S is pseudo-regular. I claim that $R/\pi R \rightarrow S/\pi S$ is cyclically pure. Assuming this claim, we get from Proposition 10.2 that R is pseudo-Cohen-Macaulay, contradiction.

To prove the claim, let I be an arbitrary ideal in R containing π . Let $r \in IS \cap R$, so that we need to show that $r \in I$. Note that I is finitely generated, as $R/\pi R$ is Noetherian. Let I_p^{mix} and r_p^{mix} be mixed characteristic \mathfrak{O} -approximations in R_p^{mix} of I and r respectively. By Łos' Theorem, almost all r_p^{mix} lie in $I_p^{\text{mix}} S_p^{\text{mix}} \cap R_p^{\text{mix}}$, whence in I_p^{mix} by cyclical purity. By Łos' Theorem, $r \in I \mathfrak{L}_{\mathfrak{D}}^{\text{mix}}(R)$, so that $r \in I$ by faithful flatness, as we needed to prove. \square

11. ASYMPTOTIC VANISHING FOR MAPS OF TOR

11.1. Proposition. *If $R \rightarrow S$ is an integral extension of local \mathfrak{D} -affine domains, then $\mathcal{B}(R) = \mathcal{B}(S)$.*

Proof. Since any integral extension is a direct limit of finite extensions, we may assume that $R \rightarrow S$ is finite. Choose an equicharacteristic \mathfrak{D} -approximation $R_p^{\text{eq}} \rightarrow S_p^{\text{eq}}$ of $R \rightarrow S$. By Theorem 4.4 and Łos' Theorem, almost all R_p^{eq} and S_p^{eq} are domains and the extension $R_p^{\text{eq}} \rightarrow S_p^{\text{eq}}$ is finite. Therefore, $(R_p^{\text{eq}})^+ = (S_p^{\text{eq}})^+$, so that in the ultraproduct, we get $\mathcal{B}(R) = \mathcal{B}(S)$. \square

11.2. Theorem. *Let $R \rightarrow S \rightarrow T$ be local \mathfrak{D} -algebra homomorphisms between local \mathfrak{D} -affine domains. Assume that R and T are pseudo-regular and that $R \rightarrow S$ is integral and injective. For every R -module M , the induced map $\text{Tor}_i^R(S, M) \rightarrow \text{Tor}_i^R(T, M)$ is zero, for all $i \geq 1$.*

Proof. Since $R \rightarrow S$ is integral, we have that $\mathcal{B}(R) = \mathcal{B}(S)$ by Proposition 11.1. Therefore, $\text{Tor}_i^R(\mathcal{B}(S), M) = 0$, for all $i \geq 1$, by Theorem 10.1. By weak functoriality, we

have, for each $i \geq 1$, a commutative diagram

$$(11) \quad \begin{array}{ccc} \mathrm{Tor}_i^R(S, M) & \longrightarrow & \mathrm{Tor}_i^R(T, M) \\ \downarrow & & \downarrow \\ 0 = \mathrm{Tor}_i^R(\mathcal{B}(S), M) & \longrightarrow & \mathrm{Tor}_i^R(\mathcal{B}(T), M). \end{array}$$

In particular, the composite map in this diagram is zero, so that the statement follows once we have shown that the last vertical map is injective. However, this is clear, since $T \rightarrow \mathcal{B}(T)$ is faithfully flat by Theorem 10.1. \square

To make use of this theorem, we need to incorporate modules in our present setup. I will not provide full details, since many results are completely analogous to the case where we work over a field, and this has been treated in detail in [35]. Of course, we do not have the full equivalent of Theorem 2.2 to our disposal, but for most purposes, the flatness result in Theorem 4.2 suffices.

Let C be an arbitrary Noetherian local ring and M a finitely generated module over C . We say that a finite free complex F_\bullet is a *finite free resolution of M up to level n* , if $H_0(F_\bullet) = M$ and all $H_j(F_\bullet) = 0$, for $j = 1, \dots, n$. Hence, if n is strictly larger than the length of F_\bullet , then this just means that F_\bullet is a finite free resolution of M (compare with the terminology introduced in the beginning of §8).

Suppose moreover that Z is a Noetherian local ring and C is a local Z -affine algebra. We say that M has *Z -complexity at most c* , if C has Z -complexity at most c and if M can be realized as the cokernel of a matrix of Z -complexity at most c (meaning that its size is at most c and all its entries have Z -complexity at most c).

11.3. Proposition. *For each pair (c, n) , there exist bounds $RES(c, n)$ and $HOM(c)$ with the following property. Let V be a mixed characteristic discrete valuation ring and let C be a local V -affine algebra of V -complexity at most c .*

- Any finitely generated C -module of V -complexity at most c , admits a (minimal) finite free resolution up to level n of V -complexity at most $RES(c, n)$.
- Any finite free complex over C of V -complexity at most c , has homology modules of V -complexity at most $HOM(c)$.

Proof. The first assertion follows by induction from the already quoted [2, Corollary 4.27] on bounds of syzygies (compare with the proof of [35, Theorem 4.3]). It is also clear that we may take this resolution to be minimal (=every tuple in one of the kernels has its entries in the maximal ideal), if we choose to do so. The second assertion is derived from the flatness of the non-standard \mathfrak{D} -hull in exactly the same manner as the corresponding result for fields was obtained in [35, Lemma 4.2 and Theorem 4.3]. \square

Recall that the *weak global dimension* of a ring C is by definition the supremum (possibly infinite) of the weak homological dimensions (=flat dimensions) of all C -modules, that is to say, the supremum of all n for which $\mathrm{Tor}_n^C(\cdot, \cdot)$ is not identically zero.

11.4. Corollary. *A pseudo-regular local \mathfrak{D} -affine domain has finite weak global dimension.*

Proof. Let R be a pseudo-regular local \mathfrak{D} -affine domain. Given an arbitrary R -module M , we have to show that M has finite flat dimension, that is to say, admits a finite flat resolution. Assume first that M is finitely presented. Hence we can realize M as the cokernel of some matrix Γ . Let $\mathfrak{L}(R)$ be the non-standard \mathfrak{D} -hull of R and let R_w and Γ_w be \mathfrak{D} -approximations of R and Γ respectively. Let M_w be the cokernel of Γ_w . Let d be the geometric dimension of R . By Proposition 11.3, we can find a finite free resolution $F_{w\bullet}$ up to level d of each M_w , of \mathfrak{D}_w -complexity at most c , for some c depending only on Γ , whence independent from w . Since almost each R_w is regular by Theorem 6.5 and has dimension d by Theorem 5.4, almost each M_w has projective dimension at most d , so that we can even assume that $F_{w\bullet}$ is a finite free resolution of M_w . Let F_\bullet be the restricted ultraproduct of the $F_{w\bullet}$ (that is to say, the finite free complex over R given by the restricted ultraproduct of the matrices in $F_{w\bullet}$). By Łos' Theorem, $F_\bullet \otimes_R \mathfrak{L}(R)$ is a free resolution of $M \otimes_R \mathfrak{L}(R)$, and therefore by faithful flat descent, F_\bullet is a free resolution of M , proving that M has projective dimension at most d .

Assume now that M is arbitrary. By what we just proved, we have for every finitely generated ideal I of R that $\mathrm{Tor}_{d+1}^R(M, R/I)$ vanishes. Hence, if H is a d -th syzygy of M , then $\mathrm{Tor}_1^R(H, R/I) = 0$. Since this holds for every finitely generated ideal of R , we get from [27, Theorem 7.7] that H is flat over R . Hence M has finite flat dimension (at most d). \square

By [26], any flat R -module has projective dimension less than the finitistic global dimension of R (the supremum of all projective dimensions of modules of finite projective dimension). Therefore, if, moreover, the finitistic global dimension of R is finite, then so is its global dimension. For a Noetherian local ring, its global dimension is finite if and only if its residue field has finite projective dimension (if and only if it is regular). The following is the pseudo analogue of this.

11.5. Corollary. *A local \mathfrak{D} -affine domain is pseudo-regular if and only if it is a coherent regular ring in the sense of [6], if and only if its residue field has finite projective dimension.*

Proof. In [6] or [12, §5], a local ring R is called a *coherent regular ring*, if every finitely generated ideal of R has finite projective dimension. If R is a pseudo-regular local \mathfrak{D} -affine domain, then this property was established in the course of the proof of Corollary 11.4. Conversely, suppose R is a local \mathfrak{D} -affine domain in which every finitely generated ideal has finite projective dimension. In particular, its residue field k admits a finite projective resolution, say of length n . Let R_w and k_w be \mathfrak{D} -approximations of R and k respectively. Since the k_w have uniformly bounded \mathfrak{D}_w -complexity, Proposition 11.3 allows us to take a minimal finite free resolution $F_{w\bullet}$ of k_w up to level n , with the property that each $F_{w\bullet}$ has \mathfrak{D}_w -complexity at most c , for some c independent from w . Let F_\bullet be the restricted ultraproduct of these resolutions. By Łos' Theorem and faithfully flat descent, F_\bullet is a minimal finite free resolution of k up to level n . Since F_\bullet is minimal and since k has by assumption projective dimension n , it follows that the final morphism (that is to say, the left most arrow) in F_\bullet is injective. By Łos' Theorem, so are almost all final morphisms in $F_{w\bullet}$, showing that almost all k_w have finite projective dimension. By Serre's characterization of regular local rings, we conclude that almost all R_w are regular. Theorem 6.5 then yields that R is pseudo-regular, as we wanted to show. \square

Closer inspection of the above argument shows that the residue field of a pseudo-regular local \mathfrak{D} -affine domain R has projective dimension equal to the geometric dimension of R . In particular, the weak global dimension of R is equal to its geometric dimension.

11.6. Theorem (Asymptotic Vanishing for Maps of Tors). *For each c , we can find a bound $VT(c)$ with the following property. Let V be a mixed characteristic discrete valuation ring, let $C \rightarrow D \rightarrow E$ be local V -algebra homomorphisms of local V -affine domains and let M be a finitely generated R -module, all of V -complexity at most c .*

If C and E are regular and $C \rightarrow D$ is finite and injective, then the natural map $\mathrm{Tor}_n^C(D, M) \rightarrow \mathrm{Tor}_n^C(E, M)$ is zero, for all $n \geq 1$, provided the characteristic of the residue field of V is at least $VT(c)$.

Proof. Note that C has dimension at most c and therefore $\mathrm{Tor}_n^C(\cdot, \cdot)$ vanishes identically for all $n > c$ and the assertion trivially holds for these values of n . If $\pi^C = 0$, we are in the equicharacteristic case, for which the result is known ([24, Theorem 9.7]). Hence we may assume that all rings are torsion-free over V . Moreover, without loss of generality, we may assume that V is complete. Suppose even in this restricted setting, there is no such bound for c and some $1 \leq n \leq c$. Hence, for almost each prime number p , we can find a counterexample consisting of the following data:

- a mixed characteristic complete discrete valuation ring $\mathfrak{D}_p^{\mathrm{mix}}$ of residual characteristic p ;
- local R_p^{mix} -algebra homomorphisms $R_p^{\mathrm{mix}} \rightarrow S_p^{\mathrm{mix}} \rightarrow T_p^{\mathrm{mix}}$ of $\mathfrak{D}_p^{\mathrm{mix}}$ -complexity at most c between torsion-free local domains, with R_p^{mix} and T_p^{mix} regular and $R_p^{\mathrm{mix}} \rightarrow S_p^{\mathrm{mix}}$ finite and injective;
- a finitely generated R_p^{mix} -module M_p^{mix} of $\mathfrak{D}_p^{\mathrm{mix}}$ -complexity at most c ;

such that

$$\mathrm{Tor}_n^{R_p^{\mathrm{mix}}}(S_p^{\mathrm{mix}}, M_p^{\mathrm{mix}}) \rightarrow \mathrm{Tor}_n^{R_p^{\mathrm{mix}}}(T_p^{\mathrm{mix}}, M_p^{\mathrm{mix}})$$

is non-zero.

Let \mathfrak{D} be the ultraproduct of the $\mathfrak{D}_p^{\mathrm{mix}}$ and let M be the restricted ultraproduct of the M_p^{mix} (that is to say, M is the cokernel of the restricted ultraproduct of matrices whose cokernel is M_p^{mix}). Let $R \rightarrow S \rightarrow T$ and $\mathfrak{L}_{\mathfrak{D}}^{\mathrm{mix}}(R) \rightarrow \mathfrak{L}_{\mathfrak{D}}^{\mathrm{mix}}(S) \rightarrow \mathfrak{L}_{\mathfrak{D}}^{\mathrm{mix}}(T)$ be the respective restricted ultraproduct and mixed characteristic ultraproduct of the homomorphisms $R_p^{\mathrm{mix}} \rightarrow S_p^{\mathrm{mix}} \rightarrow T_p^{\mathrm{mix}}$. It follows from Corollary 6.6 and Theorems 4.4 and 6.5, that R , S and T are local \mathfrak{D} -affine domains with R and T pseudo-regular. By Łos' Theorem, using that the $R_p^{\mathrm{mix}} \rightarrow S_p^{\mathrm{mix}}$ have bounded $\mathfrak{D}_p^{\mathrm{mix}}$ -complexity, $\mathfrak{L}_{\mathfrak{D}}^{\mathrm{mix}}(R) \rightarrow \mathfrak{L}_{\mathfrak{D}}^{\mathrm{mix}}(S)$ is finite, whence so is $R \rightarrow S$ by faithful flat descent. By Theorem 11.2, the natural homomorphism $\mathrm{Tor}_n^R(S, M) \rightarrow \mathrm{Tor}_n^R(T, M)$ is therefore zero.

By Proposition 11.3, we can find a finite free resolution $F_{p_\bullet}^{\mathrm{mix}}$ of M_p^{mix} up to level n , of $\mathfrak{D}_p^{\mathrm{mix}}$ complexity at most c' , for some c' only depending on c (note that $n \leq c$). By definition of Tor, we have isomorphisms

$$\begin{aligned} \mathrm{Tor}_n^{R_p^{\mathrm{mix}}}(S_p^{\mathrm{mix}}, M_p^{\mathrm{mix}}) &\cong H_n(F_{p_\bullet}^{\mathrm{mix}} \otimes_{R_p^{\mathrm{mix}}} S_p^{\mathrm{mix}}) \\ \mathrm{Tor}_n^{R_p^{\mathrm{mix}}}(T_p^{\mathrm{mix}}, M_p^{\mathrm{mix}}) &\cong H_n(F_{p_\bullet}^{\mathrm{mix}} \otimes_{R_p^{\mathrm{mix}}} T_p^{\mathrm{mix}}) \end{aligned}$$

In particular, by Proposition 11.3, both modules have $\mathfrak{D}_p^{\mathrm{mix}}$ -complexity at most c' , for some c' only depending on c , whence only on c . Let H_S and H_T be their respective restricted ultraproduct, so that by Łos' Theorem and our assumptions, $H_S \rightarrow H_T$ is non-zero. Let F_\bullet be the restricted ultraproduct of the $F_{p_\bullet}^{\mathrm{mix}}$. By Łos' Theorem and faithful flatness, H_S and H_T are isomorphic to $H_n(F_\bullet \otimes_R S)$ and $H_n(F_\bullet \otimes_R T)$ respectively. Since F_\bullet is a finite free resolution of M up to level n by another application of Łos' Theorem and faithful flatness, these two modules are also isomorphic to $\mathrm{Tor}_n^R(S, M)$ and

$\mathrm{Tor}_n^R(T, M)$ respectively. Hence the natural map between these two modules is non-zero, contradiction. \square

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