MIXED CHARACTERISTIC HOMOLOGICAL THEOREMS IN LOW DEGREES

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ABSTRACT. Let R be a locally finitely generated algebra over a discrete valuation ring V of mixed characteristic. For any of the homological properties, the Direct Summand Theorem, the Monomial Theorem, the Improved New Intersection Theorem, the Vanishing of Maps of Tors and the Hochster-Roberts Theorem, we show that it holds for R and possibly some other data defined over R, provided the residual characteristic of V is sufficiently large in terms of the complexity of the data, where the complexity is primarily given in terms of the degrees of the polynomials over V that define the data, but possibly also by some additional invariants.

1. THE RESULTS

Let V be a mixed characteristic discrete valuation ring with uniformizing parameter π and residue field κ of characteristic p. We say that R is a *local V-affine algebra* of *V-complexity* at most c, if it is of the form $(V[X]/I)_{\mathfrak{m}}$, with X a tuple of at most c variables, I and $\mathfrak{m}$ ideals generated by polynomials of degree at most c, and $\mathfrak{m}$ a prime ideal containing I and π. Similarly, we say that an element a in R (respectively, a tuple $\mathbf{x}$ in R; a matrix Γ defined over R; an ideal I in R; a finitely generated R-module M; or, an R-algebra S) has V-complexity at most c, if R has V-complexity at most c and a is the image in R of a fraction f/g with f and g polynomials of degree at most c and $g \notin \mathfrak{m}$ (respectively, the length of $\mathbf{x}$ is at most c and each of its entries has V-complexity at most c; the dimensions of Γ are at most c and each of its entries has V-complexity at most c; the ideal I is generated by elements of V-complexity at most c; the module M can be realized as the cokernel of a matrix of V-complexity at most c; and, the R-algebra S has V-complexity at most c).

Theorem 1.1 (Asymptotic Homological $\mathcal{P}$-Theorem)**.** *Let $\mathcal{P}$ be one of the homological properties listed below. For each $c \in \mathbb{N}$, there exists a bound $c' \in \mathbb{N}$, such that if V is a mixed characteristic discrete valuation ring, R a local V-affine algebra, and ϖ some other data defined over R, all of V-complexity at most c (and possibly with some additional constraints in terms of c indicated below), and if the residual characteristic of V is at least c', then property $\mathcal{P}$ holds for ϖ.*

- **Direct Summand Theorem.:** *Given a module-finite ring extension $R \subset S$, if R is regular, then $R \subset S$ splits as an R-module morphism.*
- **Monomial Theorem.:** *Given at most c monomials Y^{μ_i} in at most c variables Y and given a system of parameters $\mathbf{x}$ of R, such that $\mathbf{x}R \cap V$ has V-adic valuation at most c, if Y^{μ_0} does not belong to the ideal in $\mathbb{Z}[Y]$ generated by the remaining*

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monomials Y^{μ_i}, then $\mathbf{x}^{\mu_0}$ does not belong to the ideal in R generated by the remaining $\mathbf{x}^{\mu_i}$.

Improved New Intersection Theorem.: *Given a finite free complex*

$$(F_\bullet) \qquad 0 \to R^{a_s} \xrightarrow{\Gamma_s} R^{a_{s-1}} \xrightarrow{\Gamma_{s-1}} \dots \xrightarrow{\Gamma_2} R^{a_1} \xrightarrow{\Gamma_1} R^{a_0} \to 0$$

over R with $s, a_i \leq c$ and a minimal generator τ of $H_0(F_\bullet)$ generating a module of length at most c, if each $R/I_{r_i}(\Gamma_i)$ has dimension at most $d-i$ and parameter degree[1] *at most c, where*

$$r_i := \sum_{j=i}^{s} (-1)^{j-i} a_j,$$

and d is the dimension of R, then $F_\bullet$ has length at least d. Here we write $I_n(\Gamma)$ for the ideal generated by all $n \times n$-minors of a matrix Γ.

Vanishing for Maps of Tors.: *Given V-algebra homomorphisms $R \to S \to T$ and a finitely generated R-module M, if R and T are regular and if $R \to S$ is integral and injective, then the natural map*

$$\operatorname{Tor}_n^R(S, M) \to \operatorname{Tor}_n^R(T, M)$$

is zero.

Hochster-Roberts Theorem.: *Given a cyclically pure*[2] *homomorphism $R \to S$ of V-algebras, if S is regular, then R is Cohen-Macaulay.*

2. The method

If V is equicharacteristic, then each of these homological properties holds unconditionally, that is to say, without any bound on the complexity ([3, 8, 19]). We will use the Ax-Kochen-Ershov Principle to deduce Theorem 1.1 from this. Let me sketch the idea before I give more details. After a faithfully flat extension, we may assume that V is moreover complete. Towards a contradiction, suppose for some c, no such bound exists. This means that for each p, we can find a complete discrete valuation ring V_p of characteristic zero and residual characteristic p, and some data ϖ_p of V_p-complexity at most c for which $\mathcal{P}$ fails. Let κ_p be the residue field of V_p. Define $V_p^{\text{eq}} := \kappa_p[[t]]$, for t a single variable. Using the Ax-Kochen-Ershov Principle, we can construct for each p, similar data ϖ_p^{eq} defined over the discrete valuation rings V_p^{eq}, so that for infinitely many p, property $\mathcal{P}$ does not hold for ϖ_p^{eq}, leading to the desired contradiction.

I will now explain this in more detail. The relation between the discrete valuation rings V_p and V_p^{eq} is given by the following result due to Ax-Kochen [2] and Ershov [4, 5].

Theorem 2.1 (Ax-Kochen-Ershov). *For a fixed choice of a non-principal ultrafilter on the set of prime numbers, the ultraproduct of all V_p is isomorphic to the ultraproduct of all V_p^{eq}.*

For a quick review on ultraproducts, including Łos' Theorem, see [16, §2]; for a more detailed treatment, see [11]. Fix a non-principal ultrafilter on the set of prime numbers. Identify both ultraproducts via a fixed isomorphism and denote the common object

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[1] The *parameter degree* of a Noetherian local ring S is defined as the minimal possible length of a residue ring $S/\mathbf{x}S$, where $\mathbf{x}$ runs over all systems of parameters of S (note that homological multiplicity is an upper bound for parameter degree by [15, §4]).

[2] A homomorphism $R \to S$ is *cyclically pure* if $I = IS \cap R$, for every ideal I in R.

by $\mathfrak{O}$. By Łos' Theorem, $\mathfrak{O}$ is an equicharacteristic zero Henselian (non-discrete, non-Noetherian) valuation ring with maximal ideal generated by a single element π. Fix a tuple of variables X. It is no longer true that the ultraproduct A_∞^{mix} of the $V_p[X]$ is isomorphic to the ultraproduct A_∞^{eq} of the $V_p^{\text{eq}}[X]$. Nonetheless, both ultraproducts contain $\mathfrak{O}[X]$ as a subring. More precisely, if $f_p \in V_p[X]$ have degree at most c, for some c independent from p, then their ultraproduct f_∞ in A_∞^{mix} is an element of the subring $\mathfrak{O}[X]$, and every element in $\mathfrak{O}[X]$ is realized in this manner. In particular, f_∞ can also be viewed as an element in A_∞^{eq}, that is to say, as the ultraproduct of elements $f_p^{\text{eq}} \in V_p^{\text{eq}}[X]$. In this way, we can associate to a sequence of elements f_p of uniformly bounded V_p-complexity, a sequence of elements f_p^{eq} of uniformly bounded V_p^{eq}-complexity. Although this assignment is not unique, any two choices will be the same almost everywhere (in the sense of the ultrafilter). Similarly, we can associate to a sequence of local V_p-affine algebras R_p of uniformly bounded complexity (or any other object defined in finite terms over V_p), a sequence of local V_p^{eq}-affine algebras R_p^{eq}; the latter are called an equicharacteristic *approximation* of the former.

Let R_∞^{mix} and R_∞^{eq} be the respective ultraproducts of R_p and R_p^{eq}. These rings have a common local subring $(\Re, \mathfrak{m})$, consisting precisely of ultraproducts of elements of uniformly bounded complexity. Using for instance the result in [1] regarding uniform bounds on the complexity of modules of syzygies, one shows that both extensions $\Re \to R_\infty^{\text{mix}}$ and $\Re \to R_\infty^{\text{eq}}$ are faithfully flat. Moreover, using results from [12, 13, 14], every finitely generated prime ideal of $\Re$ remains prime when extended to either R_∞^{mix} or R_∞^{eq}. It follows that almost all R_p are domains if, and only if, $\Re$ is a domain if, and only if, almost all R_p^{eq} are domains.

The idea is to view $\Re$ as an equicharacteristic zero version of the R_p (or, for that matter, of the R_p^{eq}), so that we are lead to prove an analogue of the homological property $\mathcal{P}$ for $\Re$ (and whatever other data required, arising in a similar fashion from data of uniformly bounded V_p-complexity). However, in carrying out this project, we are faced with a serious obstruction: $\Re$ is in general not Noetherian. This prompts for a non-Noetherian version of the local algebra required for discussing homological properties. To this end, we define the *pseudo-dimension* of $\Re$ to be the smallest length of a tuple generating an $\mathfrak{m}$-primary ideal (note that the Krull dimension is infinite and hence of no use). We say that $\Re$ is *pseudo-regular* if its pseudo-dimension equals its embedding dimension (=the minimal number of generators of $\mathfrak{m}$), and *pseudo-Cohen-Macaulay*, if its pseudo-dimension is equal to its depth (in the sense of [6]). To derive for instance the asymptotic Hochster-Roberts Theorem, we can now use the fact that almost all R_p are regular (respectively, Cohen-Macaulay) if, and only if, $\Re$ is pseudo-regular (respectively, pseudo-Cohen-Macaulay) if, and only if, almost all R_p^{eq} are regular (respectively, Cohen-Macaulay).

The main tool in establishing a variant of each $\mathcal{P}$ over $\mathfrak{O}$ is via an $\mathfrak{O}$-analogue of a big Cohen-Macaulay algebra. Hochster has demonstrated (see for instance [7, 8]) how efficiently big Cohen-Macaulay modules can be used to prove homological theorems. More recently, Hochster and Huneke have given various strengthenings and generalizations using big Cohen-Macaulay *algebras*. Big Cohen-Macaulay algebras in equicharacteristic zero are obtained by reduction from their existence in characteristic p via absolute integral closures ([9, 10]). In [18], I gave an alternative construction for local $\mathbb{C}$-affine domains, using ultraproducts, and it is this approach we will adopt here. Namely, for $\Re$ a local $\mathfrak{O}$-affine domain, let $\mathcal{B}(\Re)$ be the ultraproduct of the absolute integral closures $(R_p^{\text{eq}})^+$.

Theorem 2.2 (Big Cohen-Macaulay Algebra)**.** *Let* $(\mathfrak{R}, \mathfrak{m})$ *be a local* $\mathfrak{O}$*-affine domain. Every tuple in* $\mathfrak{R}$ *of length equal to the pseudo-dimension of* $\mathfrak{R}$ *and generating an* $\mathfrak{m}$*-primary ideal, is* $\mathcal{B}(\mathfrak{R})$*-regular.*

Proof. Let $\mathbf{x}$ be a tuple of length equal to the pseudo-dimension d of $\mathfrak{R}$ so that $\mathbf{x}\mathfrak{R}$ is $\mathfrak{m}$-primary. Choose d-tuples $\mathbf{x}_p^{\text{eq}}$ in R_p^{eq} whose ultraproduct is $\mathbf{x}$. One can show that almost all R_p^{eq} have dimension d. By Łos' Theorem, almost all $\mathbf{x}_p^{\text{eq}} R_p^{\text{eq}}$ are primary to the maximal ideal. Hence almost all $\mathbf{x}_p^{\text{eq}}$ are systems of parameters, whence $(R_p^{\text{eq}})^+$-regular by [9]. By another application of Łos' Theorem, $\mathbf{x}$ is $\mathcal{B}(\mathfrak{R})$-regular. □

Details can be found in the forthcoming [17].

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