BOUNDS IN COHOMOLOGY

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Abstract. We introduce a measure of complexity for affine algebras and their finitely generated modules, in terms of the degrees of the polynomials used in their description. We then study how various cohomological operations and numerical invariants are uniformly bounded with respect to these complexities. We apply this to give first order characterisations of certain algebraic-geometric properties. This enables us to apply the Lefschetz Principle to transfer properties between various characteristics. As an application, we obtain the following version of the Zariski-Lipman Conjecture in positive characteristic: let R be the local ring of a point P on a hypersurface over an algebraically closed field K such that the module of K-invariant derivatives on R is free, then P is a non-singular point, provided the characteristic is larger than some bound only depending on the degree of the hypersurface.

1. Introduction

1.1. Transfer Principles. Often a remarkable dichotomy in algebraic-geometric results between zero and positive characteristic can be observed both in their statements and their proofs. Thus it appears that problems involving singularities are harder to prove in positive characteristic than in zero characteristic (e.g., resolution of singularities), or are even false in positive characteristic (Zariski-Lipman Conjecture, see below). In contrast, homological questions tend to be easier to prove in positive characteristic in view of the presence of the Frobenius morphism; here the Bass Conjecture (see below) is a good example. However, in some cases the validity of a result in one case of the characteristic can be inferred from its validity in the other case. The most naive way to do this is by reduction: for a simple example take the Diophantine equation $X^2+Y^2 = 3Z^2$; it has no solutions in positive integers for it has no solutions modulo 3. A much more sophisticated transfer principle is the Lefschetz Principle as formulated by Weil. Unfortunately, the latter principle is metamathematical in nature.[1] We propose to use in this paper a first order version of this principle, which we will continue to call the *Lefschetz Principle*, for sake of simplicity. Most succinctly stated, it is the following isomorphism of fields

$$\mathbb{C} \cong \prod_{\mathcal{U}} \mathbb{F}_p^{\mathrm{alg}}, \tag{1}$$

where $\mathbb{F}_p^{\mathrm{alg}}$ is the algebraic closure of the p-element field $\mathbb{F}_p$ (p prime) and $\mathcal{U}$ is a non-principal ultrafilter on the set of primes.

Key words and phrases. Cohomology, Betti numbers, Zariski-Lipman Conjecture, Bass Conjecture, Lefschetz Principle.

[1] Perhaps the best attempts to formulate this principle in a formal, model-theoretic language are [**Ek**] and [**BE**] ; for a more general version than (1) below, see [**FJ**, Theorem 8.3] .

Before explaining the bearing of this isomorphism, let us first indicate some of the results in this paper obtained through an application of the Lefschetz Principle. In Section 5 we will show that the validity of both the Bass Conjecture and the New Intersection Theorem for affine local rings over an algebraically closed field of characteristic zero can be derived from its validity in case the characteristic is positive. In the original proofs, a similar transfer is shown, albeit by ad hoc means. However, we do reach some new conclusions in Section 6, where we show some *low degree* version of the Zariski-Lipman Conjecture in positive characteristic. More precisely, we show that if the local ring R of a point x on a scheme X of finite type over an algebraically closed field K of characteristic p, admits a free module of K-derivations $\mathrm{Der}_K(R)$, then X is normal at the point x, provided the polynomials defining X and the prime ideal of x have small degree with respect to the characteristic p. Moreover, if X is a hypersurface, then under the same assumptions, x is a non-singular point on X. Note that both results are false without any restriction on the degree. We derive these results using the Lefschetz Principle in conjunction with known cases ([**Lip**] and [**SS**]) of the Zariski-Lipman Conjecture in characteristic zero. One could also prove, using the Lefschetz Principle in the same way, a *low degree* version of Embedded Resolution of Singularities, as already observed by Eklof in [**Ek 2**] . However, it is important to note that, whereas the latter property can also be derived from a direct investigation of the proof in characteristic zero (for instance, using Bierstone and Milman's explicit proof [**BM**]), this is no longer true for the former properties. Indeed, the proofs to both the New Intersection Theorem and the Bass Conjecture use an application of the Frobenius functor and whence are not available in characteristic zero and something like Hochster's Finiteness Theorem (see [**Ho**]) or our method is needed. Likewise, the aforesaid results [**Lip**] and [**SS**] on the Zariski-Lipman Conjecture, use both analytic methods in their proof and whence cannot be transferred to positive characteristic. Therefore, to my knowledge, our (indirect) proof is the only one available at present.

In the rest of this introduction, we will sketch the technique behind this transfer principle. A *definable* set U of K^n, for some $n = 0, 1, \dots$, is an element of the smallest Boolean algebra $\mathcal{D}(K)$ closed under the following two rules:

(D1) any constructible set $V \subset K^m$ lies in $\mathcal{D}(K)$;
(D2) if $U \subset K^{n+1}$ lies in $\mathcal{D}(K)$ and $\pi\colon K^{n+1} \to K^n$ is the projection on the first n coordinates, then also $\pi(U)$ lies in $\mathcal{D}(K)$.

Recall that a *constructible* set is a finite union of locally closed sets $F \setminus Z$, with F and Z zero sets of finitely many polynomials over K. Let R be a subring of K. If we allow in (D1) only constructible sets in which all equations have coefficients from R, then we obtain the subclass $\mathcal{D}_R(K)$ of *R-definable* subsets. In particular $\mathcal{D}_K(K) = \mathcal{D}(K)$. If R is the prime subfield of K, then we write $\mathcal{D}_\emptyset(K)$ for $\mathcal{D}_R(K)$ and call its elements *$\emptyset$-definable* (read *zero*-definable). It follows from Chevalley's theorem (or from algebraic quantifier elimination) that $\mathcal{D}(K)$ with K algebraically closed coincides with the collection of all constructible sets of K.

A *first order formula* (or, simply *formula*) is a functorial way of assigning to an algebraically closed field K a $\emptyset$-definable subset of K. More precisely, a formula in the free variables $\boldsymbol{x} = (x_1, \dots, x_n)$, is an expression

$$\varphi = (\exists \boldsymbol{y}_0)(\forall \boldsymbol{y}_1)\dots(\exists \boldsymbol{y}_{s-1})(\forall \boldsymbol{y}_s) \bigvee_{i<m} \bigwedge_{j<n} p_{ij}(\boldsymbol{x}, \boldsymbol{y}) = 0 \wedge q_{ij}(\boldsymbol{x}, \boldsymbol{y}) \neq 0$$

with p_{ij} and q_{ij} polynomials over $\mathbb{Z}$ and $\boldsymbol{y}_j$ (possibly empty) tuples of variables. Such a formula φ defines for an arbitrary field K a $\emptyset$-definable subset $|\varphi|_K$ as follows. An n-tuple $\boldsymbol{x} \in K^n$ belongs to $|\varphi|_K$, if and only if, there exists a tuple $\boldsymbol{y}_0$ over K, such that for all tuples $\boldsymbol{y}_1$ over K, ..., such that there exists a tuple $\boldsymbol{y}_{s-1}$ over K, such that for all tuples $\boldsymbol{y}_s$ over K, we have for some $i < m$ and all $j < n$ that $p_{ij}(\boldsymbol{x}, \boldsymbol{y}_0, \dots, \boldsymbol{y}_s) = 0$ and $q_{ij}(\boldsymbol{x}, \boldsymbol{y}_0, \dots, \boldsymbol{y}_s) \neq 0$. The reader easily checks that $|\varphi|_K$ belongs to $\mathcal{D}_\emptyset(K)$. Note that it is not true in general that if $K \subset L$ is an extension of fields then $|\varphi|_K$ maps into $|\varphi|_L$ under the inclusion $K^n \subset L^n$. However, if K and L are algebraically closed, then this is true, again by quantifier elimination. In other words, a formula φ can be viewed as a functor from the full subcategory of algebraically closed fields to the category of sets.

We will sometimes make use of the following version of the Compactness Theorem for first order logic.

1.2. First Order Compactness. *Let φ_i be a sequence of first order formulae in n free variables. If for every field K, we have that*

$$|\varphi_0|_K = \bigcup_{i \geq 1} |\varphi_i|_K$$

(as subsets of K^n), then there is some i_0, such that, for each field K, we have that

$$|\varphi_0|_K = \bigcup_{i=1}^{i_0} |\varphi_i|_K .$$

We say that a formula φ in the free variables $(x_1, \dots, x_n)$ is *true* in K, if $|\varphi|_K = K^n$ (i.e., every n-tuple over K *satisfies* the formula φ). The Lefschetz Principle (1) is now equivalent with the following.

1.3. Lefschetz Principle. *Let φ be a formula. Then φ is true in $\mathbb{C}$, if and only if, it is true in infinitely many $\mathbb{F}_p^{\mathrm{alg}}$. Moreover, in this case, there is an $N = N(\varphi) \in \mathbb{N}$, such that φ holds in any algebraically closed field of characteristic bigger than N or equal to zero.*[2]

Note that not any functor which assigns a $\emptyset$-definable subset to an algebraically closed field has this property; the notion of a formula is not a moot one.

1.4. Definability. With this setup, let us now look at an example in which we want to apply this principle. Let K be an algebraically closed field. With an *affine local ring* over K, we mean a local ring $(R, \mathfrak{m})$ which is essentially of finite type over K, i.e., a localisation of a finitely generated K-algebra. The (affine version of the) Bass conjecture for K states that any affine local ring over K admitting a finitely generated module of finite injective dimension must be Cohen-Macaulay (the converse is also true). As such, this does not appear to be a statement which can be translated into a first order formula. For one thing, we quantify over *all* affine local rings of K. Moreover, it is not clear how to encode in a first order way properties as *finite injective dimension* and *Cohen-Macaulayness*.

[2]If one wishes, the dichotomy mentioned above becomes in the first order case a dichotomy between *all* and *almost all* characteristics.

To bypass the first problem we will only make statements about certain subclasses $\underline{\mathrm{AffLoc}}_d(K)$ of affine local rings over K (for $d = 0, 1, \dots$). We want of course that the union of all the $\underline{\mathrm{AffLoc}}_d(K)$ gives the class of all (isomorphism classes of) affine local rings $\underline{\mathrm{AffLoc}}(K)$ over K and also that each class $\underline{\mathrm{AffLoc}}_d(K)$ can be identified with a $\emptyset$-definable subset $L_{d,K}$ of K, in some affine space $K^{N(d)}$ where $N(d)$ depends on d. For convenience's sake, we will also require that $\underline{\mathrm{AffLoc}}_d(K) \subset \underline{\mathrm{AffLoc}}_{d+1}(K)$ and the corresponding $\emptyset$-definable subsets also satisfy $L_{d,K} \subset L_{d+1,K}$. Here we fix an embedding of K^n in K^{n+m} via $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0)$ and assume that always $N(d) \leq N(d+1)$ (by adding zeros if necessary). However, since we want to use the Lefschetz Principle, we should make the construction functorial in K. In other words, for each d, we want to construct a formula $\mathtt{AffLoc}_d$ (in the free variables $(x_1, \dots, x_{N(d)})$), such that $|\mathtt{AffLoc}_d|_K = L_{d,K}$. Concretely, let $\underline{\mathrm{AffLoc}}_d(K)$ be the class of all affine local rings of the form

$$R = \left(\frac{K[X_1, \dots, X_n]}{(f_1, \dots, f_s)} \right)_{(g_1, \dots, g_t)} \tag{2}$$

where $(g_1, \dots, g_t)$ is a prime ideal, $n, s, t \leq d$ and each f_i, g_j has degree at most d. Adding zeros if necessary, we even may take $n = s = t = d$. If the above bounds hold for R, then we say that its *complexity* is at most d. Each affine local ring R as in (2) of complexity at most d is now encoded by giving the tuple $\boldsymbol{a}$ of all coefficients of the f_i and the g_j, listed in a fixed order. We'll write this as $R = \mathcal{R}(\boldsymbol{a})$. Note that each polynomial f over K in the variables $(X_1, \dots, X_d)$ of degree at most d is of the form

$$f = \sum_{|\nu| \leq d} a_\nu X^\nu$$

where $a_\nu \in K$ and $\nu = (\nu_1, \dots, \nu_d)$ is a tuple of indices with $|\nu| = \nu_1 + \dots + \nu_d \leq d$. Hence the the polynomial f is determined by the tuple (a_ν). In other words, when $R = \mathcal{R}(\boldsymbol{a})$, then $\boldsymbol{a}$ consists of $2d$ such tuples (determining the f_i and the g_j), subject to the condition that the last d tuples determine polynomials which generate a prime ideal. We will show in **(2.3)** below that there exists a formula $\mathtt{AffLoc}_d$ which precisely expresses this fact. Summarizing, we found, for each d, a formula $\mathtt{AffLoc}_d$, such that there is a surjective map $\boldsymbol{a} \mapsto \mathcal{R}(\boldsymbol{a})$ from $|\mathtt{AffLoc}_d|_K$ onto $\underline{\mathrm{AffLoc}}_d(K)$. In fact, if we view $\underline{\mathrm{AffLoc}}_d$ as a functor from the category of fields to the category of sets, sending a field K to the collection of affine local K-algebras $\underline{\mathrm{AffLoc}}_d(K)$ of complexity at most d, and, if we view $\mathtt{AffLoc}_d$ as a functor as well as explained above, then we have a surjective natural transformation

$$\mathtt{AffLoc}_d \to \underline{\mathrm{AffLoc}}_d. \tag{3}$$

Here we call a natural transformation $\eta\colon F \to G$ of functors *surjective*, if for each object M, we have that $\eta(M)\colon F(M) \to G(M)$ is surjective.

Remark. A surjective natural transformation is in general the best one can hope for. The reason for this is that as of now, I do not know whether one can express by a first order formula in the codes $\boldsymbol{a}$ and $\boldsymbol{b}$ that $\mathcal{R}(\boldsymbol{a}) \cong \mathcal{R}(\boldsymbol{b})$. In general, the isomorphism problem is a subtle matter and we intend to return to this question in a future paper. Let us just indicate the main problem. Fix some bound d and let

$\boldsymbol{a}, \boldsymbol{b}$ be codes for two affine local K-algebras $R = \mathcal{R}(\boldsymbol{a})$ and $S = \mathcal{R}(\boldsymbol{b})$ of complexity at most d. For each e, one can write down a formula $\mathtt{Iso}_e$ expressing that R and S are isomorphic via an isomorphism ϕ, so that both ϕ and its inverse ϕ^{-1} are defined by polynomials of degree at most e (i.e., ϕ and ϕ^{-1} have *complexity* at most e, in the terminology of **(2.5)** below). However, if $R \cong S$, both of complexity at most d, then there is no a priori bound on the least e for which such an isomorphism of complexity at most e exists. In other words, $\boldsymbol{a}$ and $\boldsymbol{b}$ define isomorphic affine local rings, if and only if, $(\boldsymbol{a}, \boldsymbol{b})$ belongs to

$$\bigcup_e |\mathtt{Iso}_e|_K .$$

However, the latter is not in general a definable set (the collection of definable sets is not closed under countable unions, only under finite unions). It would be, though, if we could find an a priori bound on e in terms of d. However, the present techniques are insufficient to obtain such bound, as we have to find solutions to a quadratic system of equations. Theorem **(2.2)** below only allows one to deduce bounds in case the system of equations is linear. What is needed is therefore a non-linear version of **(2.2)**. As such, this is impossible, but using a more local notion of complexity, *etale complexity*, we will show in a future paper how to find bounds to non-linear systems of equations.

If one were to find a first order formula expressing that $\mathcal{R}(\boldsymbol{a}) \cong \mathcal{R}(\boldsymbol{b})$, then by a technique called *Elimination of Imaginaries*, one could replace in (3) the formula $\mathtt{AffLoc}_d$ so that the resulting natural transformation is bijective. However, for our present purposes, such a refinement will not be necessary.

We will write code(R) for the collection of all tuples $\boldsymbol{a}$ belonging to $|\mathtt{AffLoc}_d|_K$ for which $\mathcal{R}(\boldsymbol{a}) \cong R$. We call such a tuple $\boldsymbol{a} \in \text{code}(R)$ simply a *code* for R and it will not matter which code we pick. The reader should keep in mind that in general code(R) is not a definable set.

Next we want to encode finitely generated modules. Fix some affine local K-algebra R. Note that every element of R is of the form f/g, with f and g polynomials over K. Any finitely generated R-module M admits a representation

$$R^s \xrightarrow{\Gamma} R^t \to M \to 0 \tag{4}$$

with Γ an $s \times t$-matrix with entries in R. We will say that M has *complexity*[3] at most d, if we can find a representation (4) with $s, t \leq d$ and each entry of Γ is a quotient of two polynomials of degree at most d. Let $\boldsymbol{m}$ be the tuple of coefficients of all polynomials involved in describing Γ. We will indicate this by writing $M = \mathcal{M}(\boldsymbol{m})$ and we let code(M) be the collection of all $\boldsymbol{m}$ for which $\mathcal{M}(\boldsymbol{m}) \cong M$. The reason that we can not apply Elimination of Imaginaries here either is because again, we do not know how to express by aid of a formula that two tuples $\boldsymbol{m}$ and $\boldsymbol{n}$ yield isomorphic modules $\mathcal{M}(\boldsymbol{m}) \cong \mathcal{M}(\boldsymbol{n})$.

Again we should make this construction functorial in the base field K. For this purpose, we should vary R as well. In other words, we should consider the functor <u>Mod</u> from the category of algebraically closed fields to sets, which assigns to some field K the set <u>Mod</u>(K) of all pairs (R, M) with R an affine local K-algebra and

[3] This notion of complexity should not be confused with a notion from commutative algebra used to measure the asymptotic behaviour of Betti numbers.

M a finitely generated R-module. We let $\underline{\mathrm{Mod}}_d(K)$ be the collection of all pairs (R, M) with both R and M of complexity at most d. From the above discussion it then follows that there exists for each d a formula $\mathtt{Mod}_d$ and a surjective natural transformation

$$\eta\colon \mathtt{Mod}_d \to \underline{\mathrm{Mod}}_d.$$

We will let code(R, M) be the collection of all pairs $(\boldsymbol{a}, \boldsymbol{m})$ for which $\mathcal{R}(\boldsymbol{a}) \cong R$ and $\mathcal{M}(\boldsymbol{m}) \cong M$.

Returning to the example of the Bass Conjecture, we would like to find a formula $\mathtt{CM}_d$, such that for each algebraically closed field K and each tuple $\boldsymbol{a}$ in $|\mathtt{AffLoc}_d|_K$, we have that $\boldsymbol{a}$ belongs to $|\mathtt{CM}_d|_K$, if and only if, $\mathcal{R}(\boldsymbol{a})$ is Cohen-Macaulay. Similarly, we want a formula $\mathtt{FinInj}_d$, such that for each algebraically closed field K and each tuple $(\boldsymbol{a}, \boldsymbol{m})$ in $|\mathtt{Mod}_d|_K$, we have that $(\boldsymbol{a}, \boldsymbol{m})$ belongs to $|\mathtt{FinInj}_d|_K$, if and only if, the module $\mathcal{M}(\boldsymbol{m})$ has finite injective dimension over $\mathcal{R}(\boldsymbol{a})$. Assume the existence of such formulae and let $\mathtt{Bass}_{d,e}$ be the formula

$$\mathtt{FinInj}_e(\boldsymbol{a}, \boldsymbol{m}) \to \mathtt{CM}_d(\boldsymbol{a}), \tag{5}$$

for $d \leq e$ (where in $\mathtt{FinInj}_e(\boldsymbol{a}, \boldsymbol{m})$, we might have added some zeros to $\boldsymbol{a}$ to make it of the right length; such considerations on the length of a tuple will not be made explicit, as it will be clear from the context what the appropriate length should be). We can now express the validity of the Bass Conjecture over some algebraically closed field K by stating that all $\mathtt{Bass}_{d,e}$ are true. Putting (5) together with the Lefschetz Principle **(1.3)**, we can conclude that if the Bass Conjecture is true over every (or over infinitely many) $\mathbb{F}_p^{\mathrm{alg}}$, then it is true for every algebraically closed field of characteristic zero. As a matter of fact, Peskine and Szpiro first proved the Conjecture in positive characteristic and then used an ad hoc technique to lift it to zero characteristic. The main purpose of this paper is to show how this and similar liftings can be made via first order definability. In doing so, we will provide a general framework in which many other problems can be formulated in a similar first order way and whence become available for an application of the Lefschetz Principle. The basic tool to obtain formulae such as $\mathtt{CM}_d$ and $\mathtt{FinInj}_d$ is first to give a cohomological characterisation of the properties they seek to encode. To finish we then will show how various bounds in cohomology, depending merely on the complexity of the initial data, exist. That such bounds are necessary, follows from the fact that a (first order) formula should only contain a finite number of variables, disjunctions and conjunctions. Therefore the apparent infinite numbers of variables or conjuncts/disjuncts required, can be reduced to finitely many.

1.5. Geometric Point of View. For the reader who does not feel too confident with model theoretic terminology, we propose the following alternative reading of this paper. Assume first of all that the field K is algebraically closed; this will be his only concession to an easier reading for only in that case definable sets (in the sense of **(1.1)**) are constructible. In this paper, we study various geometric or algebraic objects M defined (in the non-technical sense of the word) over K, which are described by certain (tuples of) parameters $\mathbf{a}$ (called *codes* above). Let us make more explicit in which way this dependence on parameters is to be understood. Let $\mathcal{L} \to \mathcal{A}$ be a finitely generated $\mathbb{Z}$-algebra morphism between $\mathbb{Z}$-algebras of finite type. We think of $\operatorname{Spec} \mathcal{L}$ as the *parameter space* for, typically, $\mathcal{L}$ will just be a polynomial ring over $\mathbb{Z}$. Fix an algebraically closed field K. Let $\mathfrak{U}$ denote the collection of (isomorphism classes of) affine coordinate rings of closed fibres $\pi^{-1}(u)$ of the map

$$\pi\colon \operatorname{Spec}(\mathcal{A} \otimes_{\mathbb{Z}} K) \to \operatorname{Spec}(\mathcal{L} \otimes_{\mathbb{Z}} K)$$

where u runs over all closed points of $\operatorname{Spec}(\mathcal{L} \otimes_{\mathbb{Z}} K)$. In other words, $\mathfrak{U}$ is the collection of (isomorphism classes of) quotient rings $(\mathcal{A} \otimes_{\mathbb{Z}} K)/\mathfrak{m}(\mathcal{A} \otimes_{\mathbb{Z}} K)$, where $\mathfrak{m}$ runs over the maximal ideals of $\mathcal{L} \otimes_{\mathbb{Z}} K$. Such a collection of finitely generated K-algebras will be called a *bounded family.* For each $A \in \mathfrak{U}$, let $\operatorname{code} A$ be the collection of all closed points $u \in \operatorname{Spec}(\mathcal{L} \otimes_{\mathbb{Z}} K)$, for which A is isomorphic to the coordinate ring of $\pi^{-1}(u)$. We call any u in $\operatorname{code} A$ a *parameter* (or, *code*) for A.[4]

It is almost immediate that there exists, for each bounded family $\mathfrak{U}$, a bound $D_{\mathfrak{U}}$, such that each K-algebra in $\mathfrak{U}$ has complexity at most $D_{\mathfrak{U}}$ (see **(2.1)** for definitions). Conversely, the collection of all affine K-algebras of complexity at most d, is a bounded family. Namely, let $\mathcal{L}$ be the polynomial ring $\mathbb{Z}[\xi]$, where $\xi = (\xi_{\nu,i})$, for $i = 1, \dots, N$ and $\nu \in \mathbb{N}^d$ with $|\nu| \leq d$, is a collection of variables, and where $\mathcal{A}$ is the ring

$$\frac{\mathbb{Z}[\xi, X]}{(\sum_{|\nu|\leq d} \xi_{\nu,1} X^\nu, \dots, \sum_{|\nu|\leq d} \xi_{\nu,N} X^\nu)}$$

where $X = (X_1, \dots, X_d)$. Here $N = \binom{2d}{d}$; see **(2.1)** for details. As in **(2.3)** below, it follows from **(2.2)** that there exists a constructible set Π in $\operatorname{Spec} \mathcal{L}$, such that, for each algebraically closed field K, the fibre $\pi^{-1}(u)$ is irreducible, if and only if, u belongs to $\Pi \times_{\mathbb{Z}} K \subset \operatorname{Spec}(\mathcal{L} \otimes_{\mathbb{Z}} K)$.

Similarly, with $\mathcal{L}$ and $\mathcal{A}$ as before, let $\mathcal{M}$ be a finitely generated $\mathcal{A}$-module, with representation

$$\mathcal{A}^s \xrightarrow{\mathcal{G}\times} \mathcal{A}^t \to \mathcal{M} \to 0$$

where $\mathcal{G}$ is a $(s \times t)$-matrix over $\mathcal{A}$. Let $A \in \mathfrak{U}$ and consider the collection of all modules $(\mathcal{M} \otimes_{\mathbb{Z}} K)/\mathfrak{m}_u(\mathcal{M} \otimes_{\mathbb{Z}} K)$, where u runs over all possible parameters for A and where $\mathfrak{m}_u$ is the maximal ideal in $\mathcal{L} \otimes_{\mathbb{Z}} K$ corresponding to the closed point u. Call this collection of A-modules again a *bounded family.* From the above exact sequence it follows that this family is obtained as all possible cokernels of specialisations of the matrix $\mathcal{G}$ to closed points $u \in \operatorname{code} A$. Or, alternatively, if $\mathcal{F}$ is the coherent $\mathcal{O}_{\operatorname{Spec} \mathcal{A}}$-sheaf corresponding to $\mathcal{M}$, then we are looking at the collection of all restrictions of $\mathcal{F} \times_{\mathbb{Z}} K$ to closed fibres $\pi^{-1}(u)$, where u runs over all parameters of A. Hence in the terminology of **(1.3)**, there is a bound on the complexity of each member in the family. Again, by choosing $\mathcal{G}$ to be a generic $\mathcal{A}$-matrix, one shows that the collection of all A-modules of complexity at most d is a bounded family.

By taking products

$$\pi \times \psi\colon \operatorname{Spec} \mathcal{A} \times \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} \mathcal{L}$$

one could in the same way get a bounded family consisting of pairs, triples, etc., of K-algebras, modules, etc. In particular, using the constructible set Π of above, one can construct a constructible set Π' in $\operatorname{Spec} \mathcal{L}$, such that for each closed point u in

[4] To be entirely precise, one should call u an π-parameter for A.

$\Pi' \times_{\mathbb{Z}} K$, the closed fibre $\pi^{-1}(u)$ has coordinate ring A and the closed fibre $\psi^{-1}(u)$ has coordinate ring $A/\mathfrak{p}$, with $\mathfrak{p}$ a prime ideal of A. Let $R = A_{\mathfrak{p}}$, then we call u a *parameter* for R and the collection of all these affine local K-algebras forms a *bounded family*. Once more does any bounded family of affine local K-algebras have bounded complexity and, conversely, the collection of all local affine K-algebras of complexity at most d forms a bounded family. We leave it up to the reader to perform a similar analysis for the other algebraic-geometric objects M (schemes, cycles, etc.) appearing in this paper and to give a precise meaning of the notion of a *parameter* for M.

Now, let $(M_1, \dots, M_s)$ be an s-tuple of algebraic-geometric objects and let $\operatorname{Spec} \mathcal{L}$ be a parameter space for s-tuples of such objects. Whenever in the text we say that

> *such-and-such property of the objects M_i can be expressed by a first order formula in their codes $\mathbf{a}_i$,*

replace this by

> *there is a constructible set $\Gamma \subset \operatorname{Spec} \mathcal{L}$ so that property such-and-such holds for the objects M_i, if and only if, there exists a parameter for their tuple $(M_1, \dots, M_s)$ in $\Gamma \times_{\mathbb{Z}} K$.*

In other words, if the property is *uniformly constructible in the parameter space*.

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2. Complexity and Bounds

2.1. Definition. In the previous section we introduced the notion of complexity without providing much details. In this and subsequent sections we will introduce more systematically various complexities and study their properties. We call a ring A *affine*, if it is a finitely generated algebra over some field K. In other words, A is of the form $K[X]/(f_1,\dots,f_s)$, with $X=(X_1,\dots,X_n)$. We say that an ideal $\mathfrak{a}=(f_1,\dots,f_s)$ of $K[X_1,\dots,X_n]$ has *degree type* at most d, if $n\leq d$ and $\mathfrak{a}$ can be generated by polynomials f_i of degree at most d. We will denote this by $\operatorname{deg.type}\mathfrak{a}\leq d$. Note that there is no need to also bound the number s of generators of the ideal $\mathfrak{a}=(f_1,\dots,f_s)$. Indeed, recall that $\binom{k+n-1}{n-1}$ equals the number of monomials of degree k in n variables. Therefore, the number of monomials of degree at most d in n variables is given by

$$(*)\qquad \sum_{k=0}^{d}\binom{k+n-1}{n-1}=\binom{n+d}{n}.$$

Now, if $\operatorname{deg.type}\mathfrak{a}\leq d$, then we can write $\mathfrak{a}=(f_1,\dots,f_s)$ with all $f_i\in K[X]$ of degree at most d in $X=(X_1,\dots,X_n)$ with $n\leq d$. Take some monomial ordering on the variables $X=(X_1,\dots,X_n)$ and normalize each f_i so that its leading term with respect to this ordering has coefficient one. If two of the f_i have the same leading term, then subtracting one from the other gives a pair of generators with different leading terms. Therefore, we can arrange for all generators to have different leading term and hence the maximal number of generators needed to generate $\mathfrak{a}$ is at most the number of monomials of degree at most d, that is to say, is at most $\binom{2d}{d}$, since $n\leq d$.

We say that an affine algebra A has *complexity* at most d, if $A=K[X]/\mathfrak{a}$ with $\operatorname{deg.type}\mathfrak{a}\leq d$. If $\mathfrak{a}$ is an ideal of A, then we say that $\mathfrak{a}$ has *degree type* at most d and write $\operatorname{deg.type}\mathfrak{a}\leq d$ if A has complexity at most d and there exists an ideal $\mathfrak{A}$ in $K[X]$ of degree at most d, such that $\mathfrak{a}=\mathfrak{A}A$. In fact, the degree function on $K[X]$ induces a degree function on A by calling $f\in A$ of *degree* at most d if it admits some lifting $F\in K[X]$ of degree at most d. Therefore, we will often say that an ideal is generated by polynomials of degree at most d, even if the affine ring we work in is not a polynomial ring. In particular, if $\operatorname{deg.type}\mathfrak{a}\leq d$, then $A/\mathfrak{a}$ has complexity at most d. We call a ring R an *affine local* ring, if it is the localisation of an affine ring A with respect to a prime ideal $\mathfrak{p}$. We say that its *complexity* is at most d, if $\operatorname{deg.type}\mathfrak{p}\leq d$ (note that by definition this includes that also A has complexity at most d).

Caution. The notion of complexity depends on the field K over which we work. For instance, if $K\subset L$ is a finite extension of fields of degree d, then L has complexity d over K (to wit, $L=K[X]/(f)$ with f an irreducible polynomial of degree d), but has complexity 1 over itself (to wit, $L=L[X]/(X)$). This ambiguity is resolved over algebraically closed fields. Nonetheless, even if the base field is not algebraically closed, we will not make reference to it when discussing complexity; it should be clear from the context what is meant.

Since there is a bound $N = \binom{2d}{d}$ on the number of generators for an ideal $\mathfrak{a}$ with $\operatorname{deg.type}\mathfrak{a} \leq d$, we can describe it by giving a tuple $\boldsymbol{a}$ in the base field K of length N^2 listing all coefficients of a generating set for $\mathfrak{a}$. We will indicate this by writing $\mathcal{I}(\boldsymbol{a}) = \mathfrak{a}$ and we let $\operatorname{code}(\mathfrak{a})$ be the collection of all tuples $\boldsymbol{a}$ for which $\mathfrak{a} = \mathcal{I}(\boldsymbol{a})$. Similarly any affine ring A can be described by means of a tuple $\boldsymbol{a}$ over K (of length N^2) as $A = K[X]/\mathcal{I}(\boldsymbol{a})$. We will indicate this by writing $A = \mathcal{A}(\boldsymbol{a})$ and we let $\operatorname{code}(A)$ be the collection of all $\boldsymbol{a}$ for which $\mathcal{A}(\boldsymbol{a}) \cong A$. In the previous section we claimed that a similar result holds for affine local rings, but for this we need already some results on bounds in polynomial rings. The following Theorem is a compilation of several results obtained in [**SvdD**] . These bounds will serve as the cornerstone for our own results on bounds in cohomology. Some of these results were already known for long time, but the authors introduced to the subject a novel technique: they obtained these bounds via some model theoretic non-standard arguments. In this paper we will not need this technique, only the results obtained from it and we refer the reader for further details to loc. cit.

2.2. Theorem [VAN DEN DRIES-SCHMIDT]**.** *For each d, there exists a bound D with the following properties. Let K be a field. Let $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$ be variables. Let A be an affine K-algebra of complexity at most d and let $\mathfrak{a}$ be an ideal of A of degree type at most d.*

(i) *Let $t \in \mathbb{N}$ and let f_i, f_{ij}, for $i = 1, \dots, t$ and $j = 1, \dots, d$, be polynomials over K in the variables X of degree at most d. Suppose the linear system of equations*

$$\begin{aligned} f_1 &= f_{11}Y_1 + \dots f_{1d}Y_d \\ f_2 &= f_{21}Y_1 + \dots f_{2d}Y_d \\ &\vdots \\ f_t &= f_{t1}Y_1 + \dots f_{td}Y_d \end{aligned}$$

has a solution for the Y-variables over $K[X]$, then it has already a solution $(q_1, \dots, q_d)$ with all $q_i \in K[X]$ of degree at most D. Moreover, if all the $f_i = 0$, then any solution of this (homogeneous) system of equations is a linear combination with coefficients in $K[X]$ of solutions of degree at most D.

(ii) *Let*

$$\mathfrak{a} = \mathfrak{g}_1 \cap \dots \cap \mathfrak{g}_s$$

be a minimal primary decomposition of $\mathfrak{a}$. Then $s \leq D$ and each $\mathfrak{g}_i$ has degree type at most D. The radical $\operatorname{rad}\mathfrak{a}$ of $\mathfrak{a}$ and the radicals $\mathfrak{p}_i$ of $\mathfrak{g}_i$ (so that the $\mathfrak{p}_i$ are the associated primes of $\mathfrak{a}$) all have degree type at most D. Moreover, $\mathfrak{p}_i^D$ is contained in $\mathfrak{g}_i$ and, similarly, $(\operatorname{rad}\mathfrak{a})^D$ lies in $\mathfrak{a}$.

(iii) *If for each $f, g \in A$ of degree at most D, we have that $fg \in \mathfrak{a}$ implies that f or g belongs to $\mathfrak{a}$, then $\mathfrak{a}$ is a prime ideal.*

(iv) *If A is moreover a domain and any monic equation*

$$T^s + f_{s-1}T^{s-1} + \dots + f_0 = 0$$

with $s \leq D$ and $f_i \in A$ of degree at most D, has no solutions in the fraction field of A which are not already in A, then A is normal.

2.3. Remark. Before we derive some further bounds from these results, let us pause to show in some detail how we can find a formula $\texttt{AffLoc}_d$ as proclaimed in **(1.4)**. Let $\mathfrak{a}$ be an ideal of degree type at most d and let $\boldsymbol{a} \in \operatorname{code}(\mathfrak{a})$. Using (iii) and (i), we will show how to write down a formula $\texttt{Prime}_d$, such that $\boldsymbol{a}$ belongs to $|\texttt{Prime}_d|_K$, if and only if, $\mathfrak{a} = \mathcal{I}(\boldsymbol{a})$ is a prime ideal; from this the construction of $\texttt{AffLoc}_d$ is immediate. To express that $\mathfrak{a}$ is prime, we have to express by (iii) that for any two polynomials f and g of degree at most D, we have that $fg \in \mathfrak{a}$ implies that one of them lies in $\mathfrak{a}$. Now, applying (i) to the bound D, we can find a bound D', depending only on D and whence only on d, such that $fg \in \mathfrak{a}$, if and only if, there exist polynomials q_i of degree at most D' expressing fg as a linear combination of the generators of $\mathfrak{a}$. The same bound can be used to express that $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$. We can now finish the construction of the wanted formula $\texttt{Prime}_d$, by observing that the tuple of coefficients of a sum or a product of two polynomials f and g is easily expressed in terms of the coefficients of f and g. In summary, the main point is that we only need a finite and fixed number of coefficients to describe the contents of (iii). This number does depend on d, but on nothing else and so does our formula only depend on d.

In the above discussion, we actually showed that there are formulae $\texttt{IdMem}_d$ in the code $\boldsymbol{a}$ of an affine K-algebra A, in the code $\boldsymbol{i}$ of an ideal $\mathfrak{a}$ of A with $\operatorname{deg.type} \mathfrak{a} \leq d$ and in the code $\boldsymbol{u}$ of a polynomial f over K of degree at most d, such that $f \in \mathfrak{a}$, if and only if, $(\boldsymbol{a}, \boldsymbol{u}, \boldsymbol{i})$ belongs to $|\texttt{IdMem}_d|_K$. Let us show that this can even be extended to the case of an affine local ring R. Suppose R is of the form $(K[X]/I)_{\mathfrak{p}}$, where $\operatorname{deg.type} I \leq d$ and $\mathfrak{p}$ a prime ideal containing I with $\operatorname{deg.type} \mathfrak{p} \leq d$. Let $\mathfrak{a}$ be an ideal in $K[X]$ with $\operatorname{deg.type} \mathfrak{a} \leq d$ and f a polynomial of degree at most d. Then $f \in \mathfrak{a}R$, if there exists some $q \notin \mathfrak{p}$, such that $qf \in I + \mathfrak{a}$. In other words, if we have that

$$(\mathfrak{a} + I : f) \not\subset \mathfrak{p}.$$

Hence we have translated the ideal membership $f \in \mathfrak{a}R$ in the local ring R into an ideal containment problem between ideals of bounded degree type in a polynomial ring, which is first order definable by what we said above.

In the sequel we will not always give the details for writing down a formula, but content ourselves with merely giving the bounds necessary for doing so and leave the actual construction of the formula to the diligent reader.

2.4. Theorem. *For each d, there exists a bound D with the following properties. Let K be a field and let A be an affine (local) K-algebra of complexity at most d. Let $\mathfrak{a}$ be an ideal of A of degree type at most d and let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{a}$.*

- (v) *The exponent of the Artinian local ring $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$ is at most D, i.e., $\mathfrak{p}^D$ is zero in $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$.*
- (vi) *The length $\ell(A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}})$ of $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$ is at most D. In particular, if $A/\mathfrak{a}$ has finite length, then this length is at most D.*

Proof. Note that by [**Eis**, Corollary 2.19] the ring $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$ is indeed Artinian, i.e., of finite length. By (ii), there exists a D depending only on d, such that $\operatorname{deg.type} \mathfrak{p} \leq D$. Put $\mathfrak{g} = \mathfrak{a}A_{\mathfrak{p}} \cap A$. This is the $\mathfrak{p}$-primary component of $\mathfrak{a}$, and hence its degree type is at most D by (ii). Moreover, let $f \in A$ be such that $f \neq 0$ but $f^n = 0$ in $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$, for some $n \in \mathbb{N}$. This means that $f \notin \mathfrak{g}$ but $f^n \in \mathfrak{g}$. By (ii), already $f^D \in \mathfrak{g}$, i.e., $f^D = 0$ in $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$. This proves (v).

As $\mathfrak{p}$ is generated by at most D_2 elements, where D_2 only depends on d, it follows that the embedding dimension of $A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p}$ is also bounded by D_2. Recall that the *embedding dimension* of a Noetherian local ring $(R,\mathfrak{m})$, is the minimal number of generators of $\mathfrak{m}$. By NAKAYAMA's Lemma, the minimal number of generators of any finite R-module M is equal to the dimension of $M/\mathfrak{m}M$ over the residue field k of R, see for instance [**Mats**, Theorem 2.3] . Hence, in particular, the embedding dimension of R is equal to the dimension of $\mathfrak{m}/\mathfrak{m}^2$. If R is moreover Artinian, then the length $\ell(R)$ of R equals the sum of the lengths of all $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. Since the length of $\mathfrak{m}^i/\mathfrak{m}^{i+1}$, equals the number of generators of $\mathfrak{m}^i$, we have an estimate

$$\ell(\mathfrak{m}^i/\mathfrak{m}^{i+1}) \leq \binom{i+r-1}{r-1},$$

(note that the latter number equals the number of monomials of degree i in the r generators of $\mathfrak{m}$). Moreover, $\mathfrak{m}^i/\mathfrak{m}^{i+1} = 0$, for i bigger than the exponent e of R. Putting all this together while using $(*)$, we see that

$$\ell(R) \leq \binom{e+r}{r}, \tag{6}$$

where e is the exponent and r is the embedding dimension of R. Therefore, using the bounds $e \leq D_1$ and $r \leq D_2$, the first part of (vi) follows from the estimate (6) applied to the Artinian local ring $R = A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p}$.

If $A/\mathfrak{a}$ has finite length, use the above together with the equality

$$\ell_A(A/\mathfrak{a}) = \sum_{\mathfrak{p}\in\operatorname{Spec} A} \ell_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p}),$$

where the only non-zero contributions come from the minimal primes of $\mathfrak{a}$, and hence only D terms are non-zero by (ii). ■

Remark. The converse does not hold in general: to bound the length does not imply to bound the complexity. An easy counterexample is given by taking $K = \mathbb{F}_p$. Then every finite field $\mathbb{F}$ of characteristic p is a quotient of $\mathbb{F}_p[X]$ of complexity equal to the degree $(\mathbb{F} : \mathbb{F}_p)$ but obviously has length one as an Artinian ring. If K is algebraically closed, this is no longer an obstruction and one can easily show that any ideal $\mathfrak{a}$ of $K[X]$ has degree type at most d, if $K[X]/\mathfrak{a}$ has length at most $d+1$. Indeed, for then there is some tuple $\boldsymbol{a} = (a_1,\dots,a_d)$ in K, such that

$$(X_1 - a_1,\dots,X_d - a_d)^{d+1} \subset \mathfrak{a},$$

so that we can choose generators for $\mathfrak{a}$ of degree at most d.

2.5. Definition. Let K be a field and A and B affine K-algebras. Let $\phi\colon A \to B$ be a K-algebra morphism. We say that ϕ has *complexity* at most d, if A and B admit representations $A = K[X]/I$ and $B = K[Y]/J$, with $\operatorname{deg.type} I \leq d$ and $\operatorname{deg.type} J \leq d$, and, if there exist $F_i \in K[Y]$ of degree at most d, such that the K-algebra morphism

$$K[X] \to K[Y]\colon X_i \mapsto F_i(Y) \tag{7}$$

induces the morphism ϕ. If A and B are affine local K-algebras, then we allow in (7) that X_i is sent to a fraction F_i/G_i with both F_i and G_i of degree at most d.

Next we will show a uniform version of Elimination Theory. Its short proof uses the following fact on Gröbner bases. Let $X = (X_1, \dots, X_s)$ and $Y = (Y_1, \dots, Y_t)$ be variables and let M be a submodule of $K[X,Y]^n$. Suppose $\mu_1, \dots, \mu_k$ is a Gröbner basis of M with respect to an elimination ordering (see [**Eis**, Proposition 15.29 and Exercise 15.37] for details). If $\mu_1, \dots, \mu_l$ are those μ_i whose entries do not depend on the Y-variables, then $\mu_1, \dots, \mu_l$ is a Gröbner basis of $M \cap (K[X])^n$.

In fact, many of the bounds in this paper could equally well be obtained from the theory of Gröbner bases. See for instance [**Eis**, 15.10] for an alternative approach to uniformity in commutative algebra via Gröbner bases (see also [**Vas**]).

2.6. Theorem. *For each d, there exists a bound D with the following property. Let K be a field and let $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$ be variables. If $\mathfrak{b}$ is an ideal in $K[X,Y]$ of degree type at most d, then $\mathfrak{b} \cap K[X]$ has degree type at most D.*

Proof. Let $g_1, \dots, g_s$ be a Gröbner basis for $\mathfrak{b}$ (with respect to the lexicographical order on (X,Y)). As observed in **(2.5)** above, those g_i which do not involve the variables Y form a generating set of $\mathfrak{b} \cap K[X]$. It follows from [**Eis**, 15.9] that the degree of a Gröbner basis of $\mathfrak{b}$ is bounded in terms of $d = \operatorname{deg.type} \mathfrak{b}$. In fact, Möller and Mora (see [**Eis**, loc. cit.]) show that one can take

$$D = (d^2 + 2d + 2)^{2^{d+1}(d+1)},$$

when the lexicographic ordering is used. ■

2.7. Theorem. *For each d, there exists a bound D with the following property. Let K be a field and let A and B be affine (local) K-algebras of complexity at most d. Let $\phi\colon A \to B$ be a K-algebra morphism of complexity at most d. If $\mathfrak{b}$ is an ideal of degree type at most d, then $\mathfrak{b} \cap A = \phi^{-1}(\mathfrak{b})$ has degree type at most D.*

Proof. Let A and B have representations $A = K[X]/I$ and $B = K[Y]/J$, with $\operatorname{deg.type} I \le d$ and $\operatorname{deg.type} J \le d$ (we leave the local case to the reader). Let $F_i \in K[Y]$ be of degree at most d, such that the K-algebra morphism

$$K[X] \to K[Y]\colon X_i \mapsto F_i$$

induces the morphism ϕ. Suppose $\mathfrak{b}$ is generated by the images in B of polynomials $G_j \in K[Y]$ of degree at most d. The image in A of a polynomial $f \in K[X]$ belongs to $\mathfrak{b} \cap A$, if and only if, f belongs to the ideal

$$J + (G_1, \dots, G_s) + (X_1 - F_1, \dots, X_d - F_d)$$

in $K[X,Y]$. Note that $X_i = F_i(Y)$ are the equations defining the graph of ϕ. The result now follows from **(2.6)**. ■

3. Complexity of Modules

3.1. Definition. In Section 1 we also introduced the notion of complexity for a finitely generated module. However, in this section it will be more convenient to work with a slightly different notion, which we call *deg-complexity.* In **(3.8)** below, we then will show how both notions are connected to each other and that one can use either one for constructing formulae. Let A be an affine ring (or perhaps an affine local ring) and let M be a finitely generated A-module. Extending the definition of the degree type of an ideal, we will say that a submodule M of A^d has *degree type* at most d and we write $\operatorname{deg.type} M \leq d$, if A has complexity at most d and M is generated by d-tuples of *degree* at most d, i.e., by tuples in A^d with entries of degree at most d. In case A is an affine local ring of complexity at most d, then we require that each entry is of the form p/q with both p and q of degree at most d. In the sequel we will only continue to treat the global affine case and leave the details for the local case to the reader.

Note that we bound simultaneously the length of the tuples and their degrees as well as the complexity of the base ring. By the same argument as in the ideal case, there exists a bound D (depending only on d), such that a submodule M of degree type at most d, can be generated by at most D elements. For an arbitrary finitely generated A-module M, we say that its *deg-complexity* is at most d, if there exists submodules $N_1 \subset N_2 \subset A^d$, both of degree type at most d, such that $M \cong N_1/N_2$. (Hence implicit is also that A itself has complexity at most d). Clearly also the minimal number of generators of a module of deg-complexity at most d is bounded in terms of d. Let M be as above of deg-complexity at most d. We can encode M by a tuple $\boldsymbol{m} = (\boldsymbol{n}_1, \boldsymbol{n}_2)$ over K, where $\boldsymbol{n}_i$ is an enumeration of all coefficients of the generators of $N_i \subset A^d$. We will indicate this by $M \cong \mathcal{M}(\boldsymbol{m})$ and we let $\operatorname{code}(M)$ be the collection of all tuples $\boldsymbol{m}$ for which $\mathcal{M}(\boldsymbol{m}) \cong M$. This is consistent with our notation from Section 1 where N_1 was taken to be the whole A^d (and whence $\boldsymbol{n}_1$ just lists the d generators $(1,0,\dots,0) \dots (0,0,\dots,1)$). More generally, the set $|\mathtt{Mod}_d|_K$, which is functorially defined by the formula $\mathtt{Mod}_d$, consists of tuples $(\boldsymbol{a}, \boldsymbol{m})$ such that $M = \mathcal{M}(\boldsymbol{m})$ is a finitely generated module over $A = \mathcal{A}(\boldsymbol{a})$ of complexity at most d. We will write $\operatorname{code}(A, M)$ for the collection of all such tuples $(\boldsymbol{a}, \boldsymbol{m})$. Since each finite A-module M is a quotient A^r/N of a free A-module, we see that each finite A-module has some finite deg-complexity, namely at most the maximum of $\operatorname{deg.type} N$ and r. Moreover, since the complexity of M was defined to be at most $\operatorname{deg.type} N$ in **(1.4)**, it follows that the deg-complexity of M is always smaller than or equal to its complexity.

Caution. Let M be a submodule of A^d, then its degree type is bigger than or equal to its deg-complexity, since it is the quotient of itself by the zero-module. However, the opposite inequality does not hold. The easiest example is the ideal fA. Its degree type is equal to the degree of f, but its deg-complexity is one, since it is isomorphic with A.

3.2. Lemma. *For each $d \in \mathbb{N}$, there is a bound D with the following properties. Let A be an affine (local) ring of complexity at most d. Let M and M' be submodules of A^d of degree type at most d and let $\mathfrak{a}$ be an ideal of A of degree type at most d. Then the degree type of the following submodules are all bounded by D.*

(vii) $\mathfrak{a}M$;

(viii) $M + M'$;
(ix) $(M :_A M') = \{ a \in A \mid aM' \subset M \}$;
(x) $M \cap M'$.

Proof. For (viii) the bound d suffices and for (vii) the bound $2d$. To prove (ix), let $\mu_1, \dots, \mu_s$ be a set of generators of M, where each $\mu_i = (a_{i1}, \dots, a_{id})$ with $a_{ij} \in A$ of degree at most d and let $\mu'_1, \dots, \mu'_s$ be a set of generators of M', where each $\mu'_i = (a'_{i1}, \dots, a'_{id})$ with $a'_{ij} \in A$ of degree at most d.[5] Then a polynomial $f \in A$ belongs to $(M :_A M')$, if and only if, there exist $r_{ik} \in A$, such that

$$f a'_{ij} = r_{i1} a_{1j} + \cdots + r_{is} a_{sj}, \tag{8}$$

for all $i = 1, \dots, s$ and $j = 1, \dots, d$. View this as a linear homogeneous system of equations in the unknowns f and r_{ik} with coefficients a_{ij} and a'_{ij}. By (i) of the Schmidt-van den Dries Theorem **(2.2)**, the set of solutions (f, r_{ik}) of (8) is generated by solutions of degree bounded by some D. In particular, there exist $f_1, \dots, f_t$ of degree at most D, such that each f as above is a linear combination of these f_i. In other words, we have that $(M :_A M') = (f_1, \dots, f_t)$ has degree type at most D.

To prove (x), a similar argument applies. With notation as before, a d-tuple $\nu = (b_1, \dots, b_d)$ belongs to $M \cap M'$, if and only if, there exist $r_i, r'_i \in A$, such that

$$b_i = r_1 a_{1i} + \cdots + r_s a_{si} = r'_1 a'_{1i} + \cdots + r'_s a'_{si},$$

for all $i = 1, \dots, d$. As a homogeneous linear system of equations in the unknowns b_i, r_i and r'_i, with coefficients a_{ij} and a'_{ij}, this has again a generating set of solutions of degree bounded by D, as above. ■

3.3. Corollary. *For each $d \in \mathbb{N}$, there is a bound D with the following properties. If M is a finitely generated A-module of deg-complexity at most d, where A is an affine (local) ring, and $\mathfrak{a}$ is an ideal of A of degree type at most d, then the modules $M/\mathfrak{a}M$ and $\operatorname{Ann}_M(\mathfrak{a}) = \{ \mu \in M \mid \mu\mathfrak{a} = 0 \}$ have deg-complexity at most D. Moreover, $\operatorname{Ann}_A(M)$ has degree type at most D.*

If, moreover, Γ is a $(d \times d)$-matrix over A of degree at most d (i.e., all its entries have degree at most d), then the deg-complexity of Z and C is at most D, where Z (respectively, C) is the kernel (respectively, cokernel) of the morphism $\Gamma^\times$ induced by Γ, that is to say, Z and C are given by the exact sequence

$$0 \to Z \to M^d \xrightarrow{\Gamma^\times} M^d \to C \to 0.$$

Proof. Let $N_2 \subset N_1 \subset A^d$ be submodules of degree type at most d, such that $M \cong N_1/N_2$. We have that $M/\mathfrak{a}M \cong N_1/(\mathfrak{a}N_1 + N_2)$. By **(3.2)**, we have that $\mathfrak{a}N_1 + N_2$ has degree type at most D, where this bound only depends on d. Therefore, the deg-complexity of $M/\mathfrak{a}M$ is at most D.

Similarly, $\operatorname{Ann}_M(\mathfrak{a})$ is isomorphic to the module $((N_2 :_A \mathfrak{a}A^d) \cap N_1)/N_2$ and the ideal $\operatorname{Ann}_A(M)$ is equal to $(N_1 :_A N_2)$. Using **(3.2)** once more, we again conclude

[5] Note that the number s can be bounded in terms of d only, and hence there is no harm in taken the same number everywhere. In the sequel, we will frequently do so without further warning.

that the deg-complexity of $\operatorname{Ann}_M(\mathfrak{a})$ (respectively, the degree type of $\operatorname{Ann}_A(M)$) is bounded in terms of d. To prove the last statement, use (ii) of **(2.2)** and the fact that $Z \cong (\Gamma^\times_{N_1})^{-1}(N_2^d)/N_2^d$ and $C \cong N_1^d/(\operatorname{Im}\Gamma^\times_{N_1} + N_2^d)$, where $\Gamma^\times_{N_1}$ denotes the morphism

$$N_1^d \xrightarrow{\Gamma^\times} N_1^d.$$

■

Remark. Note that if M has deg-complexity at most d, then any associated prime $\mathfrak{p}$ of M has uniformly bounded degree type. Indeed, $\mathfrak{p}$ is then an associated prime of $\operatorname{Ann}_A(M)$ and the result follows from **(3.3)** and (ii) of **(2.2)**.

3.4. Corollary. *For each $d \in \mathbb{N}$, there is a bound D with the following properties. Suppose K is a field and A is an affine (local) K-algebra of complexity at most d. Let M be a finitely generated A-module of deg-complexity at most d. If M has finite length, then its length $\ell_A(M)$ is at most D. More generally, if $\mathfrak{p}$ is a prime ideal of A such that $M_\mathfrak{p}$ has finite length as an $A_\mathfrak{p}$-module, then $\ell_{A_\mathfrak{p}}(M_\mathfrak{p}) \leq D$.*

Proof. Let $\mathfrak{a} = \operatorname{Ann}_A(M)$ and let r be the minimal number of generators of M. Note that r is uniformly bounded in terms of d. If M has finite length then its dimension is zero. Since the dimension of M is by definition the dimension of $A/\mathfrak{a}$, it follows that also $A/\mathfrak{a}$ has finite length (see [**Eis**, Corollary 2.17]). Moreover, since there is a surjective morphism $(A/\mathfrak{a})^r \twoheadrightarrow M$, we see that $\ell(M) \leq r\ell(A/\mathfrak{a})$. By **(3.3)** there is some bound D depending only on d, such that $\operatorname{deg.type}\mathfrak{a} \leq D$. By **(2.4)** it follows that also $\ell(A/\mathfrak{a})$ is uniformly bounded and whence also $\ell(M)$, as required.

A similar argument works for the general case. Firstly, observe that if $M_\mathfrak{p}$ has finite length as an $A_\mathfrak{p}$-module, then $\mathfrak{p}$ must be a minimal prime ideal of $\mathfrak{a} = \operatorname{Ann}_A(M)$, see for instance [**Eis**, Corollary 2.18] . Since $\operatorname{deg.type}\mathfrak{a} \leq D$, it follows then from **(2.4)** that there is also a uniform bound for the length of $A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p}$. As above, we have an estimate for the length of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module by $r\ell(A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p})$. This yields the wanted uniform bound. ■

Remark. Again the converse is not true in general, neither in the local nor in the global affine case, albeit for different reasons. The obstruction in the former case comes from the fact that the residue field is in general transcendental over K and in the latter case the obstruction comes from finite field extensions of K. Of course, by taking K algebraically closed, we overcome the latter obstruction and one can show that if M is a module of finite length at most d over an affine K-algebra A of complexity at most d, with K algebraically closed, then its complexity is at most D, for some bound D depending only on d. However, this assumption is not sufficient for the local case, as the following example shows. Let $R = K(X)$, X a single variable, so that according to our definitions, R has complexity at most 1 (over K). Let $S_n = K(X)[Y]/(Y^2 + X^n + 1)$, which is then a finite R-module of length 2, whereas its complexity is n. However, if we view S_n as an affine $K(X)$-algebra, then its complexity (over the field $K(X)$) is just 2. One can show in more generality that if R is an affine local ring (over an arbitrary field) of complexity at most d and M an R-module of length at most d, then taking complexities with respect to the residue field k of R (rather than with respect to K), we do have that M has complexity at most D, for some bound D depending only on d. This follows quite easily from the fact that $S = R/\operatorname{Ann}_R(M)$ has length at most d^2 (embed R in d copies of M via a set of d generators of M) and hence $S \cong K[Z]/I$, where I is an ideal containing $(Z_1, \dots, Z_s)^{d^2}$. One calculates that then M has complexity at most d^3 viewed as an S-module. We then conclude with an application of **(3.9)** below.

3.5. Proposition. *For each $d \in \mathbb{N}$, there is a bound D with the following property. Suppose A is an affine (local) ring of complexity at most d and M a finitely generated A-module of deg-complexity at most d. If M has finite length, then its* socle $\operatorname{Soc} M$ *(i.e., the sum of all simple submodules of M), has deg-complexity at most D.*

Moreover, for each $d, l \in \mathbb{N}$, there exists a formula $(\texttt{Len=}l)_d$ with the following property. Suppose K is a field and A an affine (local) K-algebra of complexity at most d. Let M be a finitely generated A-module of deg-complexity at most d. Take $\boldsymbol{a} \in \operatorname{code}(A)$ and $\boldsymbol{m} \in \operatorname{code}(M)$. Then $(\boldsymbol{a}, \boldsymbol{m})$ belongs to $|(\texttt{Len=}l)_d|_K$, if and only if, M has length l as an A-module.

Proof. Let K be a field and $(\boldsymbol{a}, \boldsymbol{m})$ a tuple belonging to $|\texttt{Mod}_d|_K$. Put $A = \mathcal{A}(\boldsymbol{a})$ and $M = \mathcal{M}(\boldsymbol{m})$. If M has finite length then all prime ideals containing its annihilator are maximal and there are only finitely many such primes, see [**Eis**, Corollary 2.17] . Since any simple module is of the form $A/\mathfrak{m}$, with $\mathfrak{m}$ a maximal ideal, one calculates that

$$\operatorname{Soc} M = \sum_{\operatorname{Ann}_A(M) \subset \mathfrak{m}} \operatorname{Ann}_M(\mathfrak{m}).$$

Recall that $\operatorname{Ann}_M(\mathfrak{m})$ is the set of all $\mu \in M$, such that $\mu\mathfrak{m} = 0$. By (`ii`) of **(2.2)** and **(3.3)**, it follows that $\operatorname{Soc} M$ has deg-complexity at most D, where D is some bound depending only on d.

We will construct the formulae $(\texttt{Len=}l)_d$ by induction on $l \in \mathbb{N}$. For $l = 0$, we have to express by means of the code $\boldsymbol{m} = (\boldsymbol{n}_1, \boldsymbol{n}_2)$ of M that M is the zero module. By assumption, M is of the form N_1/N_2, for some submodules $N_2 \subset N_1 \subset A^d$ of degree type at most d and we just need to express that $N_1 = N_2$, i.e., that the generators of N_1 encoded by $\boldsymbol{n}_1$ are all linear combinations of the generators of N_2 encoded by $\boldsymbol{n}_2$. Another application of (`i`) of **(2.2)** finishes then the construction of $(\texttt{Len=}0)_d$.

For arbitrary $l > 0$, we will define $(\texttt{Len=}l)_d$ as the formula expressing that there exists a non-zero $\mu \in \operatorname{Soc} M$ such that $M/A\mu$ has length $l - 1$. Here is in some detail the construction. From the proof of the first part, it follows that there exists some $\mu \in N_1 \subset A^d$ of degree at most D, such that its image in M is a non-zero element of $\operatorname{Soc} M$. Therefore, we must claim that there exists a tuple $\boldsymbol{b}$ encoding a d-tuple of polynomials μ of degree at most D, such that μ viewed as an element of A^d belongs to N_1 but not to N_2 and $\mathfrak{p}\mu$ is contained in N_2, for some minimal prime $\mathfrak{p}$ of M. The latter can be expressed also by a formula, as we know from (`ii`) that any minimal prime of M will have degree type at most D. Hence the image of such a μ in M is indeed a non-zero element of the socle. Finally, assume $\boldsymbol{v}$ is the code for the quotient module $M/A\mu$ (which can easily be derived from the codes $\boldsymbol{m} = (\boldsymbol{n}_1, \boldsymbol{n}_2)$ and $\boldsymbol{b}$), then we require that $(\boldsymbol{a}, \boldsymbol{v})$ belongs to $|(\texttt{Len=}l - 1)_d|_K$. This finishes the construction of $(\texttt{Len=}l)_d$ and it is now immediate by induction that this formula encodes that M has length l. ∎

3.6. Corollary. *For each $d \in \mathbb{N}$, there exists a formula $\mathtt{FinLen}_d$ with the following property. Suppose K is a field and A an affine (local) K-algebra of complexity at most d. Let M be a finitely generated A-module of deg-complexity at most d. Take $\boldsymbol{a} \in \operatorname{code}(A)$ and $\boldsymbol{m} \in \operatorname{code}(M)$. Then $(\boldsymbol{a}, \boldsymbol{m})$ belongs to $|\mathtt{FinLen}_d|_K$, if and only if, M has finite length as an A-module.*

Proof. By **(3.4)** there is a bound D depending only on d, such that if M has finite length, then this length is at most D. Hence the formula

$$\bigvee_{i \leq D} (\mathtt{Len=}i)_d$$

expresses that M has finite length. ■

3.7. Proposition. *For each $d \in \mathbb{N}$, there exists a bound D with the following property. If A is an affine (local) ring of complexity at most d and M a finitely generated A-module of deg-complexity at most d, then M has a syzygy of deg-complexity at most D, i.e., there exists a short exact sequence*

$$0 \to Z \longrightarrow A^s \longrightarrow M \to 0, \tag{9}$$

with $s \leq D$ and $\operatorname{deg.type} Z \leq D$.

Proof. Let $M \cong N_1/N_2$, with $N_2 \subset N_1 \subset A^d$ submodules of degree type at most d. Let $\mu_1, \dots, \mu_s \in A^d$ be tuples of degree at most d generating N_1. Let Γ be the $(s \times d)$-matrix with rows the μ_i. Viewing Γ as a morphism

$$\Gamma^{\times} \colon A^s \to A^d \colon \alpha \mapsto \alpha\Gamma$$

we see that $N_1 = \operatorname{Im} \Gamma^{\times}$. Therefore, if $Z = (\Gamma^{\times})^{-1}(N_2)$, then the sequence (9) is indeed exact, where $A^s \to M$ is the composed morphism $A^s \twoheadrightarrow N_1 \twoheadrightarrow M$. Now, an s-tuple $\alpha \in A^s$ belongs to Z, if and only if, $\alpha\Gamma \in N_2$. Writing this out in a set of generators of N_2 of bounded degree, we obtain once more a homogeneous linear system of equations with all coefficients of degree at most d. Another application of **(2.2)** yields a bound D, such that Z is generated by tuples each of which have degree at most D. ■

3.8. Remark. In the course of the above proof we obtained the following. For any A-module M of complexity at most d, we can find an $(s \times s)$-matrix Ψ with entries of degree at most D and $s \leq D$, such that the sequence

$$A^s \xrightarrow{\Psi^{\times}} A^s \to M \to 0$$

is exact.

In other words, if M has deg-complexity at most d, then its complexity, as defined in **(1.4)**, is at most D. We have already observed that the deg-complexity is always at most the complexity, so one complexity notion can be bounded in terms of the other. We will express this by saying that both complexity notions are *mutually bounded*. If two complexity notions are mutually bounded then one can translate bounds for one notion in terms of the other. Therefore, in the sequel, deg-complexity will only play a minor role and we will from now on look for bounds in terms of the complexity of a module.

3.9. Theorem. *For each $d \in \mathbb{N}$, there exists a bound D with the following property. Suppose A and B are affine (local) rings of complexity at most d and $\phi\colon A \to B$ a K-algebra morphism of complexity at most d. Let Γ be a $(d \times d)$-matrix over B with cokernel M, so that we have an exact sequence*

$$B^d \xrightarrow{\Gamma^\times} B^d \xrightarrow{\pi} M \to 0.$$

Suppose that M is also generated over A by the $\pi(e_i)$, where $\{e_1, \dots, e_d\}$ is the standard basis of B^d. Then M has complexity at most D viewed as an A-module.

Proof. Write $A = K[X]/I$ and $B = K[Y]/J$ with $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$, and where $\operatorname{deg.type} I \leq D$ and $\operatorname{deg.type} J \leq d$. Let C denote the image of $\Gamma^\times$ and let H be the submodule of A^d of all d-tuples $\alpha = (a_1, \dots, a_d)$ for which $\phi(\alpha) = (\phi(a_1), \dots, \phi(a_d))$ lies in C. We then have an exact sequence

$$0 \to H \longrightarrow A^d \xrightarrow{\pi} M \to 0, \tag{10}$$

where we still have written π for its restriction to A^d. By our assumption the latter morphism is surjective and it is now easy to verify that (10) is indeed exact. Hence the statement is proven once we showed that $\operatorname{deg.type} H \leq D$, for some bound D depending only on d. To prove that $\operatorname{deg.type} H \leq D$, take a Gröbner basis for H as explained in **(2.5)** and use the same argument as in the proof of **(2.7)**; we leave the details for the reader. ■

4. Bounds in Cohomology

A highly recommendable general reference for the material in this Section is Appendix 3 in [**Eis**] .

4.1. Definition. Let A be an affine (local) ring. Let $\mathfrak{F}$ be a functor (covariant or contravariant) on the category of finitely generated A-modules. We say that $\mathfrak{F}$ is *bounded*, if for each d, there exists a bound D, such that the complexity of $\mathfrak{F}(M)$ is at most D, for any finitely generated A-module M of complexity at most d.

We say that the functor $\mathfrak{F}$ is *linear*, if it is additive and if for any finitely generated A-module M and any $a \in A$, we have that multiplication by a on M is send under $\mathfrak{F}$ to multiplication by a (or, perhaps by $-a$) on $\mathfrak{F}(M)$, i.e., that $\mathfrak{F}(a^\times) = \pm a^\times$. Let us write $\gamma_{i,M}\colon M \hookrightarrow M^d$ (respectively, $\pi_{i,M}\colon M^d \twoheadrightarrow M$) for the embedding in (respectively, projection onto) the i-th coordinate. Then any $(d \times d)$-matrix $\Gamma = (a_{ij})$ induces a morphism $\Gamma^\times\colon M^d \to M^d$ and we have that

$$\Gamma^\times = \sum_{1 \leq i,j \leq d} \gamma_{i,M} \circ a_{ij}^\times \circ \pi_{j,M}. \tag{11}$$

Applying $\mathfrak{F}$ to both sides of (11) and using linearity, we conclude that also $\mathfrak{F}(\Gamma^\times) = \Gamma^\times$ in the covariant case and $\mathfrak{F}(\Gamma^\times) = ({}^{\mathrm{tr}}\Gamma)^\times$ in the contravariant case, where ${}^{\mathrm{tr}}\Gamma$ denotes the transposed of Γ.

Again, what we really want is to make this definition independent of the affine algebra as well as of the base field. Unfortunately, category theory does not have the right formalism for this, so that we will have to make do with the following vague statement. Suppose that $\mathfrak{F}$ is a rule which assigns to an arbitrary affine ring A over some field K in some *consistent way* a functor (respectively, a bi-functor) $\mathfrak{F}_A$ on the category of finitely generated A-modules. We call $\mathfrak{F}$ bounded, if for each $d \in \mathbb{N}$ there exists a bound $D = D(d)$, such that for any affine ring A of complexity at most d and any finitely generated A-module M (respectively, any two finitely generated A-modules M and N) of complexity at most d, we have that $\mathfrak{F}_A(M)$ (respectively, $\mathfrak{F}_A(M,N)$) has complexity at most D. In other words, the $\mathfrak{F}_A$ are bounded (bi)functors and their bounding functions only depend on the complexity of A. Of course the vague term here is *in a consistent way* and we can only say that all families of functors that will be considered in this paper fall in that class. For example, the bi-functors $\cdot \otimes_A \cdot$ and $\operatorname{Hom}_A(\cdot,\cdot)$ will be considered consistent families. Henceforth, we will adopt the following strategy to deal with this vagueness. We will prove the boundedness of a functor and check that the bounding function $D(d)$ only depends on the complexity of the base ring. This then allows us to view the functor as a member of a consistent family. A similar approach will be made when constructing formulae.

The following lemma is an easy consequence of **(2.2)**, but it is crucial for obtaining bounds in cohomology.

4.2. Lemma. *For each $d \in \mathbb{N}$, there exists a bound D with the following property. Suppose A is an affine (local) ring of complexity at most d and M a finitely generated A-module of complexity at most d. Let Γ and Δ be $(d \times d)$-matrices of degree at most d. If $\Gamma\Delta = 0$, then the homology module $H = \operatorname{Ker}\Delta^\times / \operatorname{Im}\Gamma^\times$ of the complex*

$$M^d \xrightarrow{\Gamma^\times} M^d \xrightarrow{\Delta^\times} M^d$$

has complexity at most D.

Proof. We will reason on the deg-complexity of M, which also is bounded by d. So, there exist submodules $N_2 \subset N_1 \subset A^d$ of degree type at most d, such that $M \cong N_1/N_2$. One calculates that H is then isomorphic with the quotient of $H_1 = \left\{ v \in N_1^d \mid v\Delta \in N_2^d \right\}$ by its submodule $H_2 = \Gamma^\times(N_1^d) + N_2^d$. Using (`i`) of **(2.2)**, it follows that there exists a bound D on the degree type of both H_1 and H_2 and whence on the deg-complexity of H. We then finish off by an application of **(3.8)**. ∎

4.3. Theorem. *Let A be an affine (local) ring. Let $\mathfrak{F}$ be a linear functor (covariant or contravariant) on the category of finitely generated A-modules. Assume that either $\mathfrak{F}$ is covariant and right exact or contravariant and left exact and let $\mathrm{S}^i\,\mathfrak{F}$ denote its right derived functors in the former and its left derived functors in the latter case. Then each $\mathrm{S}^i\,\mathfrak{F}$ is bounded, for $i = 0, 1, \dots$ (and hence in particular $\mathfrak{F}$ itself is bounded).*

More precisely, there exist, for each $d, i \in \mathbb{N}$, a first order formula i-`Deriv`$[\mathfrak{F}]_d$, only depending on d, i and the code of A, with the following properties. For each tuple $\boldsymbol{m}$, we can find a tuple $\boldsymbol{v}$, such that $(\boldsymbol{m},\boldsymbol{v})$ belongs to $|i$-`Deriv`$[\mathfrak{F}]_d|_K$. Moreover, for any tuple $(\boldsymbol{m},\boldsymbol{v})$ belonging to $|i$-`Deriv`$[\mathfrak{F}]_d|_K$, we have that $V \cong \mathrm{S}^i\,\mathfrak{F}(M)$, where $M = \mathcal{M}(\boldsymbol{m})$ and $V = \mathcal{M}(\boldsymbol{v})$.

Proof. Let us just treat the covariant case; for the contravariant case one only needs to reverse the arrows. Let M be a finitely generated A-module of complexity at most d. So we have an exact sequence

$$A^d \xrightarrow{\Gamma^\times} A^d \to M \to 0$$

with Γ of degree at most d. Let M_1 be the image of $\Gamma^\times$ so that its degree type is at most d. By **(3.7)**, there exists an exact sequence

$$0 \to M_2 \longrightarrow A^{D_1} \longrightarrow M_1 \to 0$$

with M_2 of complexity at most D_1, where D_1 only depends on d and on the complexity of A. Repeating this process, we can construct an exact sequence $F_\bullet$ given by

$$F_{i+1} \xrightarrow{\Gamma^\times_{i+1}} F_i \xrightarrow{\Gamma^\times_i} \dots \xrightarrow{\Gamma^\times_1} F_0 \to M \to 0 \tag{12}$$

with each Γ_j a matrix of degree at most D_i (depending only on d, i and the complexity of A), and each F_j a finite free A-module of rank at most D_i, for all $j \leq i$. Using linearity, the exact sequence $F_\bullet$ transforms into a complex $\mathfrak{F}(F_\bullet)$ where the morphisms are still given by the matrices Γ_j. Therefore, by an application of **(4.2)**, the latter complex has homology of complexity at most D'_i, for some bound $D'_i \geq D_i$ depending only on d and the complexity of A. The proof of the first statement follows in view of the identity $\mathrm{H}_i(\mathfrak{F}(F_\bullet)) = \mathrm{S}^i\,\mathfrak{F}(M)$.

Moreover, the matrices in the exact sequence (12) have all bounded degree at most D_i. To write down the formula $i\text{-}\mathtt{Deriv}[\mathfrak{F}]_d$ in the code $\boldsymbol{v}$ of some module we do the following. We proclaim the existence of a free complex $F_\bullet$ of length $i+1$ as in (12) with all matrices Γ_j, for $j \leq i+1$, of degree at most D_i, such that $F_\bullet$ is free and $\boldsymbol{v}$ is a code for $\mathrm{H}_i(\mathfrak{F}(F_\bullet))$. ∎

Remark. If $\mathfrak{F}_A$ is a consistent family of functors, then the formula $i\text{-}\mathtt{Deriv}[\mathfrak{F}_A]_d$ can be refined to a formula expressing for a tuple $(\boldsymbol{a}, \boldsymbol{m}, \boldsymbol{v})$ that

$$\mathcal{M}(\boldsymbol{v}) \cong \mathrm{S}^i\,\mathfrak{F}_{\mathcal{A}(\boldsymbol{a})}(\mathcal{M}(\boldsymbol{m})).$$

Let us just work this out for the bi-functors $\otimes$ and *Hom*.

4.4. Corollary. *The bi-functors* $\mathrm{Tor}_i^A(\cdot,\cdot)$ *and* $\mathrm{Ext}_A^i(\cdot,\cdot)$ *form bounded consistent families of functors. There are formulae* $(\mathtt{Tor}_i)_d$ *and* $(\mathtt{Ext}^i)_d$ *with the following properties. Let K be a field. If a tuple* $(\boldsymbol{a}, \boldsymbol{m}, \boldsymbol{n}, \boldsymbol{v})$ *belongs to* $\left|(\mathtt{Tor}_i)_d\right|_K$ *(respectively, to* $\left|(\mathtt{Ext}^i)_d\right|_K$*), then* $\mathcal{M}(\boldsymbol{v})$ *is isomorphic with*

$$\mathrm{Tor}_i^{\mathcal{A}(\boldsymbol{a})}(\mathcal{M}(\boldsymbol{m}), \mathcal{M}(\boldsymbol{n})) \quad \textit{respectively,} \quad \mathrm{Ext}^i_{\mathcal{A}(\boldsymbol{a})}(\mathcal{M}(\boldsymbol{m}), \mathcal{M}(\boldsymbol{n})).$$

Moreover, for each triple $(\boldsymbol{a}, \boldsymbol{m}, \boldsymbol{n})$ *we can find at least one tuple* $\boldsymbol{v}$*, such that* $(\boldsymbol{a}, \boldsymbol{m}, \boldsymbol{n}, \boldsymbol{v})$ *belongs to* $\left|(\mathtt{Tor}_i)_d\right|_K$ *(respectively, to* $\left|(\mathtt{Ext}^i)_d\right|_K$*).*

Proof. The proof is similar to the one for **(4.3)**. Let us just treat the case of the *Tor*-functor. Let A be an affine (local) ring of complexity at most d and let M and N be finitely generated A-modules of complexity at most d. Put $\mathfrak{F} = \cdot \otimes_A N$. The exact sequence $F_\bullet$ of (12) transforms into a complex $\mathfrak{F}(F_\bullet)$. Each module in this complex is a Cartesian product of N, except for the last one, which is $M \otimes N$.

Therefore, there is some bound D_i only depending on d, such that the complexity of the complex $\mathfrak{F}(F_\bullet)$ is at most D_i. Hence its homology has also bounded complexity by **(4.2)**. The formula $(\mathtt{Tor}_i)_d$ is now obtained by proclaiming the existence of matrices Γ_i making the sequence $F_\bullet$ exact. This will involve the codes $\boldsymbol{a} \in \operatorname{code}(A)$, $\boldsymbol{m} \in \operatorname{code}(M)$ and $\boldsymbol{n} \in \operatorname{code}(N)$ and a bounded number of new variables describing the coefficients of these matrices. We then can construct the code $\boldsymbol{v}$ for the i-th homology module of $\mathfrak{F}(F_\bullet)$ using some $\boldsymbol{m} \in \operatorname{code}(M)$. We leave the details to the reader. ■

4.5. Theorem. *For each $d, i \in \mathbb{N}$, there exists a bound D_i with the following property. Suppose A is an affine (local) ring of complexity at most d. If M and N are finitely generated A-modules of complexity at most d for which $M \otimes_A N$ (respectively, $\operatorname{Hom}_A(M, N)$) has finite length, then their i-th* Betti number *$\beta_i^A(M, N)$, defined as the length of $\operatorname{Tor}_i^A(M, N)$ (respectively, their i-th* Bass number *$\mu_A^i(M, N)$, defined as the length of $\operatorname{Ext}_A^i(M, N)$) is bounded by D_i.*

More generally, if $\mathfrak{p}$ is a minimal prime ideal of $M \otimes_A N$, then $\beta_i^{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \leq D_i$ (respectively, $\mu_{A_\mathfrak{p}}^i(M_\mathfrak{p}, N_\mathfrak{p}) \leq D_i$).

Proof. Immediately from **(4.4)** and **(3.4)**. For the second statement, observe that deg. type $\mathfrak{p}$ is also uniformly bounded, by the remark following **(3.3)**. ■

4.6. Corollary. *For each $d, i \in \mathbb{N}$, there exists a bound D_i with the following property. Suppose A is an affine (local) ring of complexity at most d. If M is a finitely generated A-module of complexity at most d and $\mathfrak{p}$ a prime ideal of degree type at most d, then the length $\beta_i^{A_\mathfrak{p}}(M_\mathfrak{p})$ of $\operatorname{Tor}_i^{A_\mathfrak{p}}(M_\mathfrak{p}, k(\mathfrak{p}))$ is at most D_i. Here $k(\mathfrak{p})$ denotes the residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ of $\mathfrak{p}$.*

Proof. Let $H = \operatorname{Tor}_i^A(M, A/\mathfrak{p})$. By **(4.4)** its complexity is uniformly bounded in terms of d. However, $\beta_i^{A_\mathfrak{p}}(M_\mathfrak{p})$ is just the length of $H_\mathfrak{p}$ and $\mathfrak{p}$ is a minimal prime of H, so that we are done by an application of **(3.4)**. ■

In fact, using **(3.5)** we can write down formulae expressing in the code of the modules and the base ring (having complexity at most d) that their i-th Betti or Bass number equals a prescribed value l. See the proof of **(5.1)** below for some more details. We finish this section with one more cohomological bound.

4.7. Theorem. *For each $d \in \mathbb{N}$, there exists a bound D with the following property. Let A be an affine (local) ring of complexity at most d and M a finitely generated A-module of complexity at most d. If $\mathfrak{a}$ is an ideal of degree type at most d, then the Koszul homology group $\mathrm{H}_i(\mathfrak{a}, M)$ has complexity at most D, for each i.*

Proof. Let $\mathfrak{a} = (f_1, \dots, f_s)$ with each f_i of degree at most d. The n-th term in the Koszul complex $\mathrm{K}_\bullet(f)$ of $f = (f_1, \dots, f_s)$ is given by the morphism $\mathrm{d}\colon K_n \to K_{n-1}$ defined as

$$\mathrm{d}(e_{i_1 \dots i_n}) = \sum_{k=1}^{n} (-1)^{k-1} f_{i_k} e_{i_1 \dots \widehat{\imath_k} \dots i_n}$$

where K_n is the free A-module with basis $e_{i_1 \dots i_n}$. In particular, all the matrices in the Koszul complex $\mathrm{K}_\bullet(f)$ have degree at most d, since their entries are $\pm f_i$. By **(4.2)** the homology $\mathrm{H}_i(\mathfrak{a}, M)$ of the complex $\mathrm{K}_\bullet(f) \otimes_A M$ has therefore complexity at most D, for some bound D depending only on d. ■

5. First Order Definable Properties

In this section we will use the bounds obtained in the previous section to show how various properties can be made first order definable. We first need a result on the definability of the height and the depth of an ideal. Depth has an easy cohomological interpretation, but height does not. The next best thing would be to define height via the Hilbert polynomial. Nonetheless, the following argument is more elementary, as it only uses KRULL's Principal Ideal Theorem, and so we prefer to present it here. We will return to the definability of the Hilbert polynomial in a future paper [**Sch 2**] .

5.1. Proposition. *For each $d, h \in \mathbb{N}$, there exists a formula $(\texttt{Height} = h)_d$ with the following property. Let K be a field, let A be an affine (local) K-algebra of complexity at most d and take $\boldsymbol{a} \in \operatorname{code}(A)$. Let $\mathfrak{a}$ be an ideal in A of complexity at most d and take $\boldsymbol{i} \in \operatorname{code}(\mathfrak{a})$. Then the tuple $(\boldsymbol{a}, \boldsymbol{i})$ belongs to $|(\texttt{Height} = h)_d|_K$, if and only if, $\mathfrak{a}$ has height h.*

Similarly, there exists a formula $(\texttt{Depth} = h)_d$, so that the tuple $(\boldsymbol{a}, \boldsymbol{i})$ belongs to $|(\texttt{Depth} = h)_d|_K$, if and only if, $\mathfrak{a}$ has depth h in A.

Proof. We will construct the formulae $(\texttt{Height} = h)_d$ by induction on h. If $h = 0$, then we seek to express in a first order way that there exists an associated prime $\mathfrak{p}$ of $\mathfrak{a}$, such that $\mathfrak{p}$ is a minimal prime of A. This is indeed equivalent with $\mathfrak{a}$ having height zero (note that every associated prime ideal of an ideal I is in particular a minimal prime ideal of I). By **(2.2)**, there exists a bound D, such that for any associated prime $\mathfrak{p}$ of $\mathfrak{a}$ and for any associated prime $\mathfrak{g}$ of A, we have that $\mathfrak{p}$ and $\mathfrak{g}$ have degree type at most D. Moreover, by **(3.3)**, we can enlarge D so that the annihilator ideals $\operatorname{Ann}_A(\mathfrak{g})$ and $(\mathfrak{a} : \mathfrak{p})$ also have degree type at most D. Therefore, we will claim the existence of a tuple $\boldsymbol{p}$ belonging to $|\texttt{Prime}_D|_K$ (so that $\mathfrak{p} = \mathcal{I}(\boldsymbol{p})$ is a prime ideal; see **(2.3)**), with the properties that $\mathfrak{a} \subset \mathfrak{p}$ and $(\mathfrak{a} : \mathfrak{p}) \neq 0$, and, moreover, that for any other tuple $\boldsymbol{q}$ belonging to $|\texttt{Prime}_D|_K$, if $\mathcal{I}(\boldsymbol{q}) \subset \mathfrak{p}$, then in fact $\mathcal{I}(\boldsymbol{q}) = \mathfrak{p}$. The latter condition means that $\mathfrak{p}$ is a minimal prime (of A), whereas the former means that it is an associated prime of $\mathfrak{a}$. It should now be clear how to write down the formula $(\texttt{Height} = 0)_d$.

For general $h > 0$, we do the following. Let $\mathfrak{a} = (f_1, \dots, f_s)$ where the f_i have degree at most d. Let $\mathfrak{a}_j$ be the ideal generated by $f_1, \dots, f_j$, where we put $\mathfrak{a}_0 = (0)$. Let t be the maximal value of j less than s, for which $\mathfrak{a}_j$ has height strictly smaller than the height of $\mathfrak{a}$. By KRULL's Principal Ideal Theorem the height of $\mathfrak{a}_t$ is then exactly one less than the height of $\mathfrak{a}$. Therefore, the formula $(\texttt{Height} = h)_d$ is defined as follows. A tuple $(\boldsymbol{a}, \boldsymbol{i})$ belongs to $|(\texttt{Height} = h)_d|_K$, if and only if, it does not belong to any of the $|(\texttt{Height} = i)_d|_K$, for $i < h$, and, for some $t < s$, we have that $\mathfrak{a}_t$ has height $h - 1$, whereas $\mathfrak{a}(A/\mathfrak{a}_{t+1})$ has height zero. Indeed, if this holds, then clearly the height of $\mathfrak{a}$ is at least h by the former condition. As $\mathfrak{a}(A/\mathfrak{a}_{t+1})$ has height zero, it follows that $\mathfrak{a}$ and $\mathfrak{a}_{t+1}$ have the same height. As $\mathfrak{a}_{t+1}$ has height at most the height of $\mathfrak{a}_t$ plus one, that is to say, at most $(h-1)+1 = h$, it follows that $\mathfrak{a}$ has height at most h. It should also be clear that the above statement can be translated in a first order formula using induction. Note that $\operatorname{deg.type} \mathfrak{a}_j \leq d$ and all $A/\mathfrak{a}_j$ have complexity at most d.

For the construction of $(\texttt{Depth} = h)_d$ we will use the characterisation [**Mats**, Theorem 16.7] that $\mathfrak{a}$ has depth h, if and only if, all $\operatorname{Ext}^i_A(A/\mathfrak{a}, A)$ vanish for

$i < h$ whereas $\operatorname{Ext}^h_A(A/\mathfrak{a}, A) \neq 0$. Using **(4.4)** we can make first order statements about the modules $\operatorname{Ext}^i_A(A/\mathfrak{a}, A)$. In particular, using **(3.5)** we can express their vanishing in the tuples $\boldsymbol{a}$ and $\boldsymbol{i}$ by first order formulae: simply express that the length of the module is zero. ■

5.2. Corollary. *For each d, there exists a bound D with the following property. Let A be an affine (local) ring of complexity at most d and $\mathfrak{a}$ an ideal in A of degree type at most d. Then there exist $x_1, \dots, x_s \in \mathfrak{a}$ of degree at most D, such that $(x_1, \dots, x_s)$ is a maximal A-regular sequence in $\mathfrak{a}$ (where s equals the depth of $\mathfrak{a}$).*

Proof. It suffices to show that for fixed d and s, we can find a bound on the degrees of an A-regular sequence in $\mathfrak{a}$, provided that the depth of $\mathfrak{a}$ is at least s. Fix some $d, s, m \in \mathbb{N}$. Let $\mathtt{RS}_{d,s,m}$ be the formula expressing the following two facts in the codes of A and $\mathfrak{a}$ and in the codes of s polynomials f_i of degree at most m.

- The polynomials $f_1, \dots, f_s$ lie in the ideal $\mathfrak{a}$.
- For each $i = 1, \dots, s$, if g is a polynomial of degree at most m, such that $f_i g$ lies in the ideal I_{i-1} generated by $f_1, \dots, f_{i-1}$, then already $g \in I_{i-1}$.

Using the fact that ideal membership can be expressed in a first order way by **(2.3)**, it follows that such a first order formula $\mathtt{RS}_{d,s,m}$ does indeed exist. Moreover, if $\mathtt{RegSeq}_{d,s,m}$ is the formula expressing in the code of A and $\mathfrak{a}$ that there exists an s-tuple $(f_1, \dots, f_s)$ in $\mathfrak{a}$ of degree at most m, such that $\mathtt{RS}_{d,s,m}$ holds, then $\mathfrak{a}$ has depth at least s, if and only if, one of the $\mathtt{RegSeq}_{d,s,m}$ holds.

In other words, for fixed s and d and for every field K we have that

$$|(\mathtt{Depth} = s)_d|_K = \bigcup_{m<\omega} \left|\mathtt{RegSeq}_{d,s,m}\right|_K.$$

By first order compactness **(1.2)**, there exists an m_0, such that

$$(\mathtt{Depth} = s)_d \leftrightarrow \mathtt{RegSeq}_{d,s,m_0}$$

is true in any field K. This means precisely that there exists a regular sequence of length s in $\mathfrak{a}$ of degree at most this m_0, as required. ■

5.3. Theorem. *For each $d \in \mathbb{N}$ and each property $\underline{\mathrm{P}}$ of local rings listed below, there exists a corresponding formula $\mathtt{P}_d$ with the following property. Let K be a field and R an affine local K-algebra of complexity d. Then R has the property $\underline{\mathrm{P}}$, if and only if, there is a tuple $\boldsymbol{a} \in \operatorname{code}(R)$ which belongs to $|\mathtt{P}_d|_K$. Here $\underline{\mathrm{P}}$ is either one of the following properties:*

(xi) *regular;*
(xii) *complete intersection;*
(xiii) *Gorenstein;*
(xiv) *Cohen-Macaulay.*

Proof. Let K be an arbitrary field and let R be an affine local K-algebra of complexity at most d. Let $\boldsymbol{a} \in \operatorname{code}(R)$. Let k be the residue field of R, which has then also complexity at most d.

To prove (xii), use the criterion [**BH**] that R is a complete intersection, if and only if,

$$\beta_2^R(k) = \binom{\beta_1^R(k)}{2} + \beta_1^R(k) - h \tag{13}$$

where h is the dimension of R. By **(4.6)**, each quantity in (13) is bounded by some D depending only on d. Let us show in some detail how to write down the required formula. Let W be the collection of all triples $(b_1, b_2, h) \in \mathbb{N}^3$ with $b_1, b_2, h \leq D$ and such that

$$b_2 = \binom{b_1}{2} + b_1 - h.$$

The formula $\mathtt{CI}_d$ has then the following form

$$\bigvee_{(b_1,b_2,h)\in W} b_1 = \beta_1^R(k) \wedge b_2 = \beta_2^R(k) \wedge h = \dim R.$$

As written down, this is not yet a first order formula, but it can easily be turned into a genuine one. For the statement $h = \dim R$, this is easily done by **(5.1)**. Just observe that $R = A_{\mathfrak{p}}$ for some affine algebra A and some prime ideal $\mathfrak{p}$ of height h with $\operatorname{deg.type} \mathfrak{p} \leq d$. Let us show how to express that $b_1 = \beta_1^R(k)$ (see also remark following **(4.6)**), the other cases are similarly dealt with. Let $\boldsymbol{p} \in \operatorname{code}(k)$, where we view k as an R-module, i.e., $\mathcal{M}(\boldsymbol{p}) = k$. This tuple is easily derived from $\boldsymbol{a}$ and so we assume that $\boldsymbol{p}$ below is given in terms of $\boldsymbol{a}$. Then $b_1 = \beta_1^R(k)$ is encoded by

$$(\exists \boldsymbol{v})[(\mathtt{Tor}_1)_d(\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{p}, \boldsymbol{v}) \wedge (\mathtt{Len}{=}b_1)_D(\boldsymbol{a}, \boldsymbol{v})]$$

where we used the formulae defined in **(4.4)** and **(3.5)**.

For (`xi`), observe that by Serre's criterion R is regular, if and only if, its residue field k has finite projective dimension. As this projective dimension is then at most the dimension of R and hence at most d, the regularity of R is equivalent with the vanishing of $\operatorname{Tor}^R_{d+1}(k, k)$ (see [**Mats**, §19] for details). As the latter module has complexity at most D, for some bound D depending only on d, we can express the vanishing of $\operatorname{Tor}^R_{d+1}(k, k)$ by means of a first order formula $\mathtt{Reg}_d$, using **(3.5)** and **(4.4)** as above.

For (`xiii`), we use the criterion [**Mats**, Theorem 18.1.(3)] that R is Gorenstein, if and only if, $\operatorname{Ext}^i_R(k, R)$ vanishes for some $i > h$ and hence, if and only if, $\operatorname{Ext}^{d+1}_R(k, R)$ vanishes, as $h \leq d$. For (`xiv`), we let $\mathtt{CM}_d$ express that the height of $R = A_{\mathfrak{p}}$ equals its depth and this is easy by means of **(5.1)**. ■

5.4. Proposition. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and let M be a finite R-module. Then M has finite injective dimension, if and only if, all $\operatorname{Ext}^{t+1+i}_R(k, M)$ vanish, for $i = 0, \dots, h$, where t is the depth of R and h its dimension.*

Proof. Assume first that M has finite injective dimension. By [**BH**, Theorem 3.1.17] its injective dimension is then equal to t and in view of [**BH**, Proposition 3.1.14] it follows that all $\operatorname{Ext}^{t+1+i}_R(k, M)$ vanish.

Conversely, applying [**BH**, Proposition 3.1.13] to $\operatorname{Ext}^{t+i+1}_R(k, M) = 0$, we conclude that $\operatorname{Ext}^{t+i}_R(R/\mathfrak{p}, M)$ vanishes, for each prime ideal $\mathfrak{p}$ of height $h-1$ and each $i < h$. Successively applying this trick shows that $\operatorname{Ext}^{t+1}_R(R/\mathfrak{p}, M) = 0$, for all prime ideals $\mathfrak{p}$ of R and whence by [**BH**, Corollary 3.1.12] that M has injective dimension at most t. ■

5.5. Corollary. *For each d, there exists a first order formula* $\mathtt{FinInj}_d$ *with the following property. Let K be a field and R an affine local K-algebra of complexity d. Let M be a finitely generated R-module of complexity at most d. Let $\boldsymbol{a} \in \mathrm{code}(R)$ and $\boldsymbol{m} \in \mathrm{code}(M)$. Then $(\boldsymbol{a}, \boldsymbol{m})$ belongs to $|\mathtt{FinInj}_d|_K$, if and only if, M has finite injective dimension as an R-module.*

Proof. By **(5.1)**, both the depth and the dimension of R are expressible by means of a formula. Now use the criterion **(5.4)** together with **(3.5)** and **(4.4)** in the same way as they were used in the proof of **(5.3)**. ■

5.6. Theorem. *The validity of the Bass Conjecture for local rings essentially of finite type over an algebraically closed field of positive characteristic, implies the validity of the conjecture for local rings essentially of finite type over an algebraically closed field of characteristic zero.*

Remark. The Bass Conjecture states that if a Noetherian local ring admits a finitely generated module of finite injective dimension, then it is Cohen-Macaulay. In fact the converse also holds and its proof is rather easy (see remark below). This conjecture is now proven in full generality by ROBERTS' New Intersection Theorem, but was originally proven by SZPIRO and PESKINE using their New Intersection Theorem in positive characteristic (see also **(5.8)** below). The same authors then derived the zero characteristic case using an ad hoc lifting procedure, which we have replaced here, at least for the affine case, by the Lefschetz Principle.

Proof. Fix some $d, e \in \mathbb{N}$. Let K be an arbitrary field and R an affine local K-algebra of complexity at most d. Let $\boldsymbol{a} \in \mathrm{code}(R)$. Using **(5.5)**, we can write down a first order formula $\mathtt{FinInj}_{d,e}$ expressing that R admits a finitely generated module of complexity at most e and of finite injective dimension. Let $\mathtt{Bass}_{d,e}$ be the formula

$$\mathtt{FinInj}_e(\boldsymbol{a}, \boldsymbol{m}) \to \mathtt{CM}_d(\boldsymbol{a}).$$

Hence the Bass Conjecture for affine K-algebras simply states that all $\mathtt{Bass}_{d,e}$ are true in K and our claim then follows by the Lefschetz Principle, as explained in Section 1. ■

Remark. A dual version of the Bass Conjecture is the statement that any local ring admitting a module of finite length and finite projective dimension, is Cohen-Macaulay (see for instance [**Str**, Proposition 9.1.10]). We can construct formulae $\mathtt{Bss}_d$ only depending on d, such that this dual version is equivalent with all $\mathtt{Bss}_d$ being true. Indeed, if R is Cohen-Macaulay, then a module of finite length and finite projective dimension can be constructed in the following way: let $(x_1, \dots, x_h)$ be a maximal regular sequence in R and let $B = R/(x_1, \dots, x_h)$. Then B is zero dimensional and whence has finite length and, as an R-module, its projective dimension is d. It follows from **(5.2)** that we can find some D_1, only depending on d, such that R admits a maximal regular sequence $(x_1, \dots, x_h)$ of degree at most D_1. Hence from **(2.4)**, there exists D, only depending on d, such that $B = R/(x_1, \dots, x_h)$ has length at most D. Therefore, we can take for $\mathtt{Bss}_d$ the formulae

$$(\exists \boldsymbol{m})[\mathtt{FinProj}_D(\boldsymbol{a}, \boldsymbol{m}) \wedge \mathtt{FinLen}_D(\boldsymbol{a}, \boldsymbol{m})] \to \mathtt{CM}_d(\boldsymbol{a}),$$

where $\mathtt{FinProj}_e$ is a formula expressing that a module of complexity at most e has finite projective dimension (see **(6.3)** below for how to write down such a formula, taking into account that projective dimension never exceeds Krull dimension) and where $\mathtt{FinLen}_e$ is the formula from **(3.6)** expressing that the module has finite length.

5.7. Definition. Let A be an affine (local) ring. A complex $F_\bullet$ of the form

$$(F_\bullet) \qquad 0 \to F_s \xrightarrow{\Gamma_s^\times} F_{s-1} \xrightarrow{\Gamma_{s-1}^\times} \dots \xrightarrow{\Gamma_2^\times} F_1 \xrightarrow{\Gamma_1^\times} F_0 \to 0$$

is called a *finite free complex* of length s, if all F_i are finitely generated free A-modules. We say that $F_\bullet$ has complexity at most d, if A has complexity at most d, every matrix Γ_i has degree type (and dimensions) at most d and $s \leq d$. Put

$$r_i = \sum_{i=j}^{s} (-1)^{j-i} \operatorname{rank} F_j,$$

called the *expected rank* of $\Gamma_i^\times$ and let $\mathfrak{W}_i(F_\bullet)$ be the r_i-th Fitting ideal $\mathfrak{W}_{r_i}(\Gamma_i^\times)$ of the matrix Γ_i (see **(6.1)** below for the definition of Fitting ideal). The Buchsbaum-Eisenbud Acyclicity Theorem (see [**BH**, Theorem 1.4.12]) states that $F_\bullet$ is acyclic, if and only if, $\mathfrak{W}_i(F_\bullet)$ has depth at least i, for $i = 1, \dots, s$. Recall that $F_\bullet$ is called *acyclic*, if

$$0 \to F_s \xrightarrow{\Gamma_s^\times} F_{s-1} \xrightarrow{\Gamma_{s-1}^\times} \dots \xrightarrow{\Gamma_2^\times} F_1 \xrightarrow{\Gamma_1^\times} F_0$$

is exact, i.e., if $F_\bullet$ is a free resolution of the cokernel of $\Gamma_1^\times$.

In case $(R, \mathfrak{m})$ is a Noetherian local ring of positive characteristic p, another acyclicity theorem is the *New Intersection Theorem* by SZPIRO and PESKINE. It simply states that if the homology groups of $F_\bullet$ have finite length and s is strictly smaller than the dimension of R, then $F_\bullet$ is exact. The authors then lift this result to the case of a local ring containing a field of zero characteristic by means of Artin Approximation. For the next theorem we assume the validity of the New Intersection Theorem in positive characteristic and we show how to lift this via first order definability to zero characteristic, at least in the affine case.

5.8. New Intersection Theorem. *Let R be an affine local ring and $F_\bullet$ a finite free complex of length s over R. If s is strictly smaller than the dimension of R and each homology group of $F_\bullet$ has finite length, then $F_\bullet$ is exact.*

Proof. Let R be an affine local ring of complexity at most d. Let h be its dimension. Let $F_\bullet$ be a finite free complex over R of length $s < h$ and assume that $F_\bullet$ has complexity at most d. It then follows from **(4.2)** that its homology groups $\mathrm{H}_i(F_\bullet)$ all have complexity at most D_1, where D_1 only depends on d. Moreover, by **(3.6)**, there exists a first order formula $\mathtt{FinLenH}_d$ expressing in the code for R and the complex $F_\bullet$ that all $\mathrm{H}_i(F_\bullet)$ have finite length. Similarly, there exists a first order formula $\mathtt{Exact}_d$, expressing that the complex $F_\bullet$ is exact. Let $\mathtt{NIT}_d$ be the formula expressing that for an affine local ring R of complexity at most d and a finite free complex $F_\bullet$ over R of complexity at most d and of length s strictly smaller than the dimension h of R, we have that

$$\mathtt{FinLenH}_d \to \mathtt{Exact}_d.$$

See the proof of **(5.3)** how to express that $s < h$ by means of a first order formula. The content of the New Intersection Theorem for affine algebras is then simply that $\mathtt{NIT}_d$ holds for every d. By the Peskine-Szpiro result in positive characteristic (see for instance [**BH**, Theorem 8.2.6]), the $\mathtt{NIT}_d$ all hold in positive characteristic and whence in all characteristics by the Lefschetz Principle (see Section 1). ■

6. Zariski-Lipman Conjecture

6.1. Definition. Let M be a finitely generated module over A, where A, as always, denotes an affine (local) ring. Let

$$A^n \xrightarrow{\Gamma^\times} A^n \to M \to 0$$

be a representation of M. The *i-th Fitting ideal* $\mathfrak{W}_i(M)$ is defined to be the ideal of A generated by the $(n-i) \times (n-i)$ subdeterminants of Γ, for $i < n$. For $i \geq n$, we put $\mathfrak{W}_i(M) = A$. These ideals are independent from the particular representation of the module M; see [**Eis**, Section 20.2] for more details. By expanding determinants along a row one verifies that the Fitting ideals form an ascending chain

$$0 \subset \mathfrak{W}_0(M) \subset \mathfrak{W}_1(M) \subset \cdots \subset \mathfrak{W}_{n-1}(M) \subset A.$$

The *lower rank* $\underline{r}(M)$ of M is defined as the minimal index such that $\mathfrak{W}_i(M) \neq 0$ and the *upper rank* $\overline{r}(M)$ as the minimal index for which $\mathfrak{W}_i(M) = A$. One checks that the lower rank equals the minimum of the minimal number of generators of $M_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all prime ideals of A and the upper rank is the maximum of those same numbers.

In particular, suppose $A = K[X]/I$ with K a field, $X = (X_1, \dots, X_n)$. Let $I = (f_1, \dots, f_m)$. Then the module of *K-differentials* $\Omega_K(A)$ admits a representation

$$A^m \xrightarrow{\mathrm{Jac}(f)^\times} A^n \to \Omega_K(A) \to 0,$$

where $\mathrm{Jac}(f)$ is the Jacobian matrix of partial derivatives $(\partial f_i/\partial X_j)$. It follows that the rank of $\mathrm{Jac}(f)$ is $n - \underline{r}$ where $\underline{r} = \underline{r}(\Omega_K(A))$. The corresponding non-zero ideal $\mathfrak{W}_{\underline{r}}(\Omega_K(A))$ is called the *Jacobian ideal* of A and will be denoted by $\mathfrak{J}_K(A)$.

Let $\mathfrak{p}$ be a prime ideal of A. We have that $A_{\mathfrak{p}}$ is *geometrically regular*, or *$\mathfrak{p}$-smooth* over K, if and only if, $\mathfrak{J}_K(A) \not\subset \mathfrak{p}$ (see [**Ohm**, Remark (a) p. 103] and [**Mats**, Theorem 28.7]).

6.2. Corollary. *For each $d \in \mathbb{N}$, there exists a bound D with the following property. Let K be a field and A an affine (local) K-algebra of complexity at most d. Let M be a finitely generated A-module of complexity at most d. Then each Fitting ideal $\mathfrak{W}_i(M)$ has degree type at most D.*

Moreover, the module of differentials $\Omega_K(A)$ has complexity at most D and the Jacobian ideal $\mathfrak{J}_K(A)$ has degree type at most D.

Proof. Immediate from the definitions. Note that a Fitting ideal can only be proper for $i \leq d$. ■

Remark. Similarly, one can find formulae expressing that a module of complexity at most d has a prescribed lower or upper rank.

6.3. Proposition. *For each $d, q \in \mathbb{N}$, there exists a first order formula* $(\mathtt{pd} = q)_d$ *with the following property. Suppose K is a field and R is an affine local K-algebra of complexity at most d. Let M be a finitely generated A-module of complexity at most d. Take $\boldsymbol{a} \in \operatorname{code}(R)$ and $\boldsymbol{m} \in \operatorname{code}(M)$. Then $(\boldsymbol{a}, \boldsymbol{m})$ belongs to $|(\mathtt{pd} = q)_d|_K$, if and only if, M has projective dimension q as an R-module.*

Proof. Let R and M be as in the statement and let k be the residue field of R. It follows from **(4.4)**, that the module $\operatorname{Tor}^R_{q+1}(M, k)$ has complexity at most D, for some bound depending only on d. Hence if we express that the latter module is zero (by requiring that its length is zero using **(3.5)**), then by [**Mats**, Lemma 1 p. 154] , we have expressed that M has projective dimension at most q. From this the required formula is easy to construct. ■

Remark. By a Theorem of FERRAND and VASCONCELOS (see for instance [**Ohm**, Theorem 35.3]), an affine reduced local K-algebra R is a complete intersection, if and only if, $\Omega_K(R)$ has projective dimension at most 1. This gives, using by **(6.2)** and **(6.3)**, an alternative construction for the first order formula $\mathtt{CI}_d$ of **(5.3)** expressing that R is a complete intersection, at least in the reduced case.

6.4. Theorem. *There are first order sentences (=formulae without free variables)* $\mathtt{ZarLip}_d$*, such that* $\mathtt{ZarLip}_d$ *holds in an algebraically closed field K, for all d, if and only if, the Zariski-Lipman Conjecture is true for K.*

Therefore, if the Zariski-Lipman Conjecture holds for some algebraically closed field of characteristic zero, then it holds for any algebraically closed field of characteristic zero.

Proof. The content of the Zariski-Lipman Conjecture is the following. Let R be a reduced affine local K-algebra, with K an algebraically closed field. The Conjecture claims that if $\operatorname{Der}_K(R)$ is free, then R is regular.

Recall that $\operatorname{Der}_K(R)$ is the module of *K-invariant derivations* on R, i.e., the set of K-linear endomorphisms δ on R which satisfy the *Leibnitz rule* $\delta(ab) = a\delta(b) + b\delta(a)$. Moreover, we have an isomorphism (see for instance [**Mats**, page 192])

$$\operatorname{Der}_K(R) = \operatorname{Hom}_R(\Omega_K(R), R).$$

Therefore, by **(6.2)** and **(4.4)**, there is a bound D (only depending on d) on the complexity of $\operatorname{Der}_K(R)$. Moreover, by **(6.3)**, there is a first order formula $\mathtt{DerFree}_d$ in the code $\boldsymbol{a} \in \operatorname{code}(R)$ expressing that $\operatorname{Der}_K(R)$ has projective dimension 0, i.e., that $\operatorname{Der}_K(R)$ is free. Using **(2.2)**, one can easily write down a formula $\mathtt{Red}_d$ expressing in the code $\boldsymbol{a}$ of R that $R = \mathcal{R}(\boldsymbol{a})$ is a reduced local ring: namely, express that the zero ideal is radical. Hence the Zariski-Lipman Conjecture for an algebraically closed field K states that

$$|\mathtt{Red}_d|_K \cap |\mathtt{DerFree}_d|_K \subset |\mathtt{Reg}_d|_K\,,$$

for all d. In other words, we can take for $\mathtt{ZarLip}_d$ the sentence

$$\mathtt{Red}_d(\boldsymbol{a}) \wedge \mathtt{DerFree}_d(\boldsymbol{a}) \to \mathtt{Reg}_d(\boldsymbol{a}).$$

The last statement then follows from the Lefschetz Principle. ■

6.5. Corollary. *For each $d \in \mathbb{N}$, there exists $N(d) \in \mathbb{N}$ with the following property. Let K be an algebraically closed field of characteristic at least $N(d)$ and let R be a reduced affine local K-algebra R of complexity at most d. If the module $\operatorname{Der}_K(R)$ of K-derivations is free, then R is normal.*

Proof. LIPMAN shows in [**Lip**] that if we replace the condition for R to be regular by the weaker condition that R be normal in the Zariski-Lipman Conjecture, then this weaker statement holds for any algebraically closed field of zero characteristic. It follows from (iv) in **(2.2)**, that the condition for R to be normal, can be expressed by a first order formula $\mathtt{Normal}_d$ in the code $\boldsymbol{a}$ of R. Fix some d and let $\mathtt{weakZL}_d$ be the formula

$$\mathtt{Red}_d(\boldsymbol{a}) \wedge \mathtt{DerFree}_d(\boldsymbol{a}) \to \mathtt{Normal}_d(\boldsymbol{a}).$$

Hence $\mathtt{weakZL}_d$ is true in $\mathbb{C}$. Therefore it is true in almost all characteristics by the Lefschetz Principle (see Section 1), which is precisely what the statement claims. ■

6.6. Corollary. *For each $d \in \mathbb{N}$, there exists $N(d) \in \mathbb{N}$ with the following property. Let K be an algebraically closed field of characteristic at least $N(d)$ and let R be the local ring of a point P on a reduced hypersurface V over K of degree at most d. If the module $\operatorname{Der}_K(R)$ of K-derivations is free, then P is a non-singular point on V.*

Proof. SCHEJA-STORCH show in [**SS**] that the Zariski-Lipman Conjecture holds for any local ring R of a (reduced) *hypersurface* over an algebraically closed field K of zero characteristic. I.e, R is the localisation of a K-algebra of the form $K[X]/f$, where f is a single square-free polynomial and $X = (X_1, \dots, X_d)$. Let $\mathtt{Hyp}_d$ be a formula in the code of R expressing this fact and set $\mathtt{HyperZL}_d$ equal to

$$\mathtt{Hyp}_d(\boldsymbol{a}) \wedge \mathtt{DerFree}_d(\boldsymbol{a}) \to \mathtt{Reg}_d(\boldsymbol{a}),$$

As $\mathtt{HyperZL}_d$ is true in $\mathbb{C}$, it is true in almost all characteristics by the Lefschetz Principle (see Section 1), which is precisely what the statement claims. ■

Remark. In [**Lip**] , the author shows by some easy examples that the conjecture in general is false in characteristic $p > 0$, e.g., take the singular curve defined by $X^p = Y^{p+1}$. Since normality is the same as regularity for curves, it follows that $N(d)$ in **(6.5)** and **(6.6)** will never be one.

Nonetheless, this suggests a possible strategy to the original conjecture using positive characteristic methods. Namely, show that for the degree d small with respect to the characteristic p, the conjecture holds, i.e., the module $\operatorname{Der}_K(R)$ of K-derivations being free implies that R is regular (with R as before of complexity at most d). Indeed, for then (19), for a fixed d, holds in algebraically closed fields of sufficiently large characteristic and whence, by the Lefschetz principle, in any algebraically closed field of characteristic zero.

Keeping d small with respect to p avoids the pathology coming from non-constant polynomials with zero derivative, but we would still have 'typical' positive characteristic tools at hand, such as the Frobenius automorphism, to prove the result. On the other hand, any proof of the Zariski-Lipman Conjecture would yield its validity in positive characteristic for 'small' degree, again by an application of the Lefschetz Principle to (19), so that the above sketched heuristic is perfectly defensible.

7. Bounds in Intersection Theory

7.1. Definition. In this section we will extend some of the previous results to coherent sheaves on schemes of finite type over a field. In fact, **(7.5)** below was my original motivation for starting the present research. To start, we will have to define a notion of complexity for schemes of finite type over a field. Firstly, let us say that a Zariski closed subset V of $\operatorname{Spec} A$ has complexity at most d, if A has complexity at most d and there exists an ideal $\mathfrak{a}$ of degree type at most d in A, such that V is the the Zariski closed set defined by $\mathfrak{a}$. Note that in view of (`ii`) of **(2.2)** we might as well have defined this by requiring that the radical ideal defining V has complexity at most d; both notions of complexity are then mutually bounded (see **(3.8)**). However, if we are interested in the subscheme structure of V given by a particular ideal $\mathfrak{a}$, then its complexity is defined as the degree type of that ideal. In summary, if V is a closed subscheme of $\operatorname{Spec} A$ defined by an ideal $\mathfrak{a}$, then its complexity c_{Zar} as a Zariski closed subset is given as the minimum of $\operatorname{deg.type} I$ where I runs through all ideals with the same radical as $\mathfrak{a}$, whereas its complexity c_{Sch} as a subscheme is given by $\operatorname{deg.type} \mathfrak{a}$. Therefore $c_{\mathrm{Zar}} \leq c_{\mathrm{Sch}}$, and conversely c_{Sch} is uniformly bounded by c_{Zar}, in view of (`ii`). Similarly, we say that a Zariski open subset has complexity at most d, if its complement is a Zariski closed subset of complexity at most d.

We will say that a scheme X of finite type over some field K has *complexity* at most d, if the following holds. There exists a finite affine covering $X = U_1 \cup \dots \cup U_s$ with $s \leq d$ and each $U_i = \operatorname{Spec} A_i$ with A_i an affine ring of complexity at most d. Moreover, each $U_{ij} = U_i \cap U_j$ viewed as a Zariski open subset of U_i has complexity at most d and the patching isomorphisms $U_{ij} \to U_{ji}$ have complexity at most d. The above definition can be extended without any difficulty to schemes which are essentially (=locally) of finite type over some field. Similarly, we extend the notions of complexity for closed subschemes and Zariski closed or open sets from affine to arbitrary schemes of finite type. If X is a scheme of finite type of complexity at most d, then a coherent $\mathcal{O}_X$ module $\mathcal{F}$ is said to have complexity at most d, if there exist finitely generated A_i-modules M_i of complexity at most d, such that $\mathcal{F}(U_i)$ is the sheaf associated to the module M_i, for some affine covering $X = U_1 \cup \dots \cup U_s$ as above with transition functions of complexity at most d. In particular, we say that a point x of X has complexity at most d, if its local ring has complexity at most d.

Of course, most of the theorems so far proven in the affine (local) case go through for arbitrary schemes (essentially) of finite type over some field. So far we have used the existence of uniform bounds to obtain definability. In the following result on the openness of certain loci, the argument gets reverted.

7.2. Theorem. *For each d, there exists a bound D with the following properties. Let X be a scheme of finite type over some algebraically closed field K. If X has complexity at most d, then the locus of all points where X is geometrically regular, regular, complete intersection, Gorenstein or Cohen-Macaulay is a Zariski open of complexity at most D.*

Proof. For a definition of geometrically regular see **(6.1)**. Now use **(6.2)** which says that the complexity of the Jacobian matrix is uniformly bounded; we leave the details for the reader. For the remaining properties we will use **(5.3)**. Let

$P(X)$ stand for any of the loci of points where X is regular, complete intersection, Gorenstein or Cohen-Macaulay. Since the problem is local, we may assume that $X = \operatorname{Spec} A$ is affine, with A an affine K-algebra of complexity at most d.

Let $X_{\max}$ be the collection of all maximal ideals of A (i.e., closed points of X), which by the Nullstellensatz can be viewed as a subset of affine K-space K^d. That is to say, every maximal ideal $\mathfrak{m}$ is of the form $(X_1 - x_1, \dots, X_d - x_d)$, with $\boldsymbol{x} = (x_1, \dots, x_d) \in K^d$. Therefore, $\boldsymbol{x}$ is the unique code for $\mathfrak{m}$ and we denote this by $\mathfrak{m} = \mathfrak{m}_{\boldsymbol{x}}$.

By **(5.3)**, we can find a formula $\mathtt{P}$, such that $(\boldsymbol{a}, \boldsymbol{x})$ belongs to $|\mathtt{P}|_K$, if and only if, the maximal ideal $\mathfrak{m}_{\boldsymbol{x}}$ belongs to the locus $P(X) \cap X_{\max}$, where $X = \operatorname{Spec} \mathcal{A}(\boldsymbol{a})$. Fix some affine K-algebra A, let $X = \operatorname{Spec} A$ and let $\boldsymbol{a} \in \operatorname{code}(A)$. Let $\mathtt{P}_A$ be the formula $\mathtt{P}$ where the code $\boldsymbol{a}$ of A has been fixed. By quantifier elimination (see Section 1), we know that $|\mathtt{P}|_K$ is a constructible set and therefore so is $|\mathtt{P}_A|_K$. Moreover, from our above discussion, we have that $P(X) \cap X_{\max} = |\mathtt{P}_A|_K$. Let Σ be the constructible subset of X given by $\mathtt{P}_A$, i.e., let Σ be given by the equations and inequalities of $\mathtt{P}_A$. It is a well known fact (see for instance [**Mats**, §24] , especially Exercises 24.2 and 24.3) that $P(X)$ is an open set in all four cases. As $P(X)$ and Σ are two constructible sets with the same closed points, they are actually equal. This follows from the fact that an affine algebra is a *Jacobson ring*, that is to say, that every radical ideal is the intersection of all the maximal ideals containing it, see [**Mats**, Theorem 5.5] . Let D be the maximum of the degrees of all polynomials occurring in the formula $\mathtt{P}$. It follows that $\Sigma = P(X)$ has complexity at most D. ■

Remark. It follows also from **(5.3)** that we can write down formulae expressing that a point $x \in X$ of complexity at most d belongs to one of these loci.

The fact that the above loci are open could also be proved using the present definability results; we intend to return to the study of these and similar questions in a future paper [**Sch 2**] . There we will also show how to extend these results to arbitrary, not necessarily algebraically closed fields.

7.3. Definition. Let X be a scheme of finite type over some field K. Recall that $Z^k(X)$ denotes the free abelian group on closed reduced irreducible subschemes (*subvarieties*, for short) of X of codimension k. An element of $Z^k(X)$ is called a *k-cycle* on X. The direct sum of these $Z^k(X)$ is denoted by $Z^*(X)$ and its elements are referred to as *(algebraic) cycles* on X. We say that a cycle α has complexity at most d, if X itself has complexity at most d and α is of the form

$$\alpha = \sum_{i=1}^{s} n_i Y_i$$

with $s \leq d$ and $|n_i| \leq d$ and each Y_i a closed subvariety of complexity at most d.

Since the Zariski topology on X is Noetherian, we can write X uniquely as $X_1 \cup \dots \cup X_s$, where the X_i are subvarieties of X with $X_i \not\subset X_j$. These subvarieties X_i are called the *irreducible components* of X. The *cycle associated to* X is by definition the cycle $\sum n_i X_i$, where n_i is the length of $\mathcal{O}_{X,\eta_i}$ and where η_i is the generic point of X_i, for $i = 1, \dots, s$. In particular, if $X = \operatorname{Spec} A$ is affine, then X_i is the closed subset defined by a minimal prime $\mathfrak{g}_i$ of A and n_i is the length of the Artinian local ring $A_{\mathfrak{g}_i}$.

Assume now that X is moreover regular. Let us briefly review some intersection theory for closed subschemes on a regular scheme of finite type over a field. Let Y_1 and Y_2 be two closed subschemes of X. Their intersection $Y_1 \cap Y_2$ is by definition the scheme $Y_1 \times_X Y_2$, which is a closed subscheme of X. We say that Y_1 and Y_2 *intersect properly*, if the codimension of each irreducible component F of $Y_1 \cap Y_2$ equals $\operatorname{codim} Y_1 + \operatorname{codim} Y_2$. If this is the case, let η be the generic point of such an irreducible component F. We define, following SERRE in [**Ser**] , the *local intersection number* by

$$i(\eta; Y_1, Y_2) = \sum_{n=0}^{\infty} (-1)^n \beta_n^{\mathcal{O}_{X,\eta}}(\mathcal{O}_{Y_1,\eta}, \mathcal{O}_{Y_2,\eta}),$$

where the β_n are the Betti numbers as defined in **(4.5)**. Note that this sum is finite. Indeed, since X is regular, every $\mathcal{O}_X$-module has finite projective dimension by [**Mats**, Theorem 19.2] and therefore $\beta_n^{\mathcal{O}_{X,\eta}}(\mathcal{O}_{Y_1,\eta}, \mathcal{O}_{Y_2,\eta}) = 0$ for n strictly bigger than the dimension of X. The *intersection cycle* of Y_1 and Y_2 is then defined as the element in $Z^*(X)$ given by

$$Y_1 \cdot Y_2 = \sum_F i(\eta_F; Y_1, Y_2) F, \tag{16}$$

where the sum runs over all irreducible components F of $Y_1 \cap Y_2$ and η_F denotes the generic point of F. If Y_1 and Y_2 do not intersect properly, then a more complicated definition is required, using CHOW's Moving Lemma. (We will not treat this case here.) Finally, the intersection of two cycles which intersect properly (meaning that each subvariety in the support of one cycle intersects properly every subvariety in the support of the other cycle), is defined by extending (16) by linearity.

7.4. Theorem. *For each $d \in \mathbb{N}$, there exists a bound D with the following property. Let X be a scheme of finite type over a field K, of complexity at most d. Let Y be a closed subscheme of X of complexity at most d. Then the cycle δ_Y associated to Y has complexity at most D.*

Proof. By (ii) of **(2.2)**, the number s of generic points of Y and the complexity of each irreducible component Y_i of Y is uniformly bounded and by **(2.4)**, so is each n_i. ■

7.5. Theorem. *For each $d \in \mathbb{N}$, there exists a bound D with the following property. Let X be a scheme of finite type of complexity at most d. Let α_1 and α_2 be two cycles of X of complexity at most d. If α_1 and α_2 intersect properly, then their intersection cycle $\alpha_1 \cdot \alpha_2$ has complexity at most D.*

Proof. Without loss of generality we may reduce to the case that $X = \operatorname{Spec} A$ is affine and that $\alpha_i = Y_i$ is a closed subscheme. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be the ideals of A defining Y_1 and Y_2 respectively. Let η be a generic point of $Y_1 \cap Y_2$. To η corresponds a minimal prime ideal $\mathfrak{p}$ of $\mathfrak{a}_1 + \mathfrak{a}_2$. By (ii) of **(2.2)**, the degree type of $\mathfrak{p}$ is at most D, where D only depends on d. The local intersection number

$$i(\eta; Y_1, Y_2) = \sum_{n=0}^{\infty} (-1)^n \beta_n^{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{a}_1 A_\mathfrak{p}, A_\mathfrak{p}/\mathfrak{a}_2 A_\mathfrak{p})$$

is uniformly bounded by **(4.5)** and the corollary follows. (Note that X can have dimension at most d, as its complexity is at most d). ■

Remark 1. By now the reader will have no problem in writing down a formula in the codes of three cycles expressing that the last one is the intersection of the two first ones, provided they intersect properly.

Remark 2. On $Z^*(X)$ several equivalence relations (rational equivalence, numerical equivalence,...) are defined. MACINTYRE has assured me that HRUSHOVSKI has a counterexample to the first order definability of rational equivalence. However, MACINTYRE himself argues in [**Mac**] that it would follow from the Standard Conjectures of GROTHENDIECK that numerical equivalence is first order definable.

7.6. Proposition. *For each d, there exists a bound D with the following property. Let K be an arbitrary field and A and B two affine K-algebras of complexity at most d. Let $\varphi\colon A \to B$ be a morphism of complexity at most d. Then for every finitely generated A-module M of complexity at most d, the B-module $M \otimes_A B$ has complexity at most D.*

Proof. Choose a $(d \times d)$-matrix Γ with entries of degree at most d, such that the following sequence is exact

$$A^d \xrightarrow{\Gamma^\times} A^d \to M \to 0.$$

Tensoring over B yields an exact sequence

$$B^d \xrightarrow{\Gamma_B^\times} B^d \to M \otimes_A B \to 0.$$

Here Γ_B is the image of the matrix Γ in B. As φ is given by polynomials of degree at most d, Γ_B has all its entries of degree at most d^2. Using **(3.3)**, the statement now follows readily. ■

Let us just give one application of this observation.

7.7. Theorem. *For each d, there exists a bound D with the following property. Let K be an arbitrary field and let $f: Y \to X$ be a morphism of complexity at most d between schemes of finite type over K. Let Z be a closed subscheme of Y of complexity at most d. For any closed point $x \in X$, the intersection cycle*

$$f^{-1}(x) \cdot Z$$

has complexity at most D, whenever the intersection is proper.

Proof. Without loss of generality, we may take $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ to be affine. A closed point x of X then corresponds to a maximal ideal $\mathfrak{m}$ of A. Assume first that x is K-rational, so that K is the residue field of x and hence has complexity at most d, since $\mathfrak{m}$ is generated by polynomials of degree one. Therefore, by **(7.6)**, there is a bound D, only depending on d, with the property that the coordinate ring $B/\mathfrak{m}B$ of the closed fibre $f^{-1}(x)$ has complexity at most D. For a general closed point, let L denote its residue field. Let $A_1 = A \otimes_K L$ and $B_1 = B \otimes_K L$. Let $\mathfrak{m}_1$ be any maximal ideal of A_1 lying above $\mathfrak{m}$ and $x_1 \in X_1 = \operatorname{Spec} A_1$ the corresponding closed point. In particular, we have an isomorphism of coordinate rings

$$B/\mathfrak{m}B \cong B_1/\mathfrak{m}_1 B_1 \tag{17}$$

of the closed fibres of x and x_1 respectively. We can now apply the above argument, since x_1 is L-rational. In particular, since D does not depend on the field K, the complexity of the affine coordinate ring of the fibre $f^{-1}(x)$ is therefore also bounded by D, in view of (17).

So in either case, the fibre ring has complexity at most D and we now finish with an application of **(7.5)**. ■

Remark. More generally, the same proof shows that for any closed subvariety F of X of complexity at most d, the complexity of the intersection cycle $f^{-1}(F) \cdot Z$ is bounded by D, whenever the intersection is proper.

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