BOUNDS IN POLYNOMIAL RINGS OVER ARTINIAN LOCAL RINGS

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ABSTRACT. Let R be a (mixed characteristic) Artinian local ring of length l and let X be an n-tuple of variables. This paper provides bounds over the ring $R[X]$ on the degrees of the output of several algebraic constructions in terms of l, n and the degrees of the input. For instance, if I is an ideal in $R[X]$ generated by polynomials g_i of degree at most d and if f is a polynomial of degree at most d belonging to I, then $f = q_1 f_1 + \cdots + q_s f_s$, with q_i of degree bounded in terms of d, l and n only. Similarly, the module of syzygies of I is generated by tuples all of whose entries have degree bounded in terms of d, l and n only.

1. INTRODUCTION

In [23], van den Dries proves the following faithful flatness result about certain ultraproducts (for an alternative proof, see the Appendix below; for an introduction in ultraproducts, see [7, §9.5] or [19, §2]).

1.1. **Theorem** ([12, Theorem 1.8]). *Let K_w be a collection of fields with ultraproduct K and let $X = (X_1, \dots, X_n)$ be a tuple of variables. Then the canonical embedding $K[X] \to A_\infty$ is faithfully flat, where A_∞ is the ultraproduct of the $K_w[X]$.*

Using this, Schmidt and van den Dries deduce in [12] (see also [11]) several uniform bounds in polynomial rings over fields; similar bounds will be discussed in Section 3. This was further studied in [15, 16] generalizing this to bounds on various algebraic invariants and to arbitrary affine algebras over a field. A typical application of faithful flatness is *uniform ideal membership* in the sense that, for every d, we can find an upper bound d', so that if a polynomial f_0 over some field lies in the ideal generated by polynomials $f_1, \dots, f_s$, with all the f_i of degree at most d, then this is witnessed by a linear combination $f_0 = q_1 f_1 + \cdots + q_s f_s$ with q_i polynomials of degree at most d'.

A natural question is to ask for similar bounds when the ground ring is no longer a field. In this generality (even if the base ring is $\mathbb{Z}$ or a discrete valuation ring) simply bounding the degrees will no longer suffice. For instance, over $\mathbb{Z}$, where the uniform ideal membership problem is called *Dedekind*'s problem, one also needs to bound the absolute value of the coefficients (see for instance [4]; more bounds can be found in [2], some of which reprove results in this paper by different means). Nonetheless, there is still another class of rings where bounding the degrees suffices: the class of Artinian local rings of a fixed (or bounded) length. Of course, an equicharacteristic Artinian local ring R is just a finite dimensional algebra over a field F and its length is an upperbound on the degrees of the h_i needed to write $R = F[X]/(h_1, \dots, h_s)F[X]$. Therefore, this case is already covered by Theorem 1.1. However, the mixed characteristic case does not seem to have been treated so far, and this is the aim of the present paper. Our main result therefore is the

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following (recall that a *syzygy* of an ideal $(f_1,\dots,f_s)A$, or more correctly, of a generating tuple $(f_1,\dots,f_s)$, is by definition a tuple $(a_1,\dots,a_s)$ such that $a_1f_1+\dots+a_sf_s=0$).

Main Theorem. *For each triple (d,l,n) of positive integers, we can find a positive integer $e:=e(d,l,n)$ with the following property. Let R be an Artinian local ring of length at most l. Let $X=(X_1,\dots,X_n)$ be an n-tuple of variables and let $f_0,\dots,f_s$ be polynomials in $R[X]$ of degree at most d. If f_0 lies in the ideal I generated by $f_1,\dots,f_s$, then there exist $q_i\in R[X]$ of degree at most e such that $f_0=q_1f_1+\dots+q_sf_s$.*

Moreover, the module of syzygies of I can be generated by tuples all of whose entries have degree at most e.

Note that the number s of generators f_i is bounded in terms of the other data (d,l,n) (see Lemma 2.1). As in [12] the Main Theorem follows immediately from the following result by a non-standard argument.

1.2. **Theorem.** *Fix some pair of positive integers (l,n) and an infinite index set equipped with a non-principal ultrafilter. For each index w, let R_w be an Artinian local ring of length at most l and let R be the ultraproduct of these R_w. Let A_∞ be the ultraproduct of the $R_w[X]$, where X is a fixed n-tuple of variables. There is a natural embedding $R[X]\subseteq A_\infty$ and this embedding is faithfully flat.*

Note that by [13], the ultraproduct R is in fact again an Artinian local ring of length at most l. For the reader's convenience, let me explain how the first assertion in the Main Theorem is a direct consequence of Theorem 1.2 (the second assertion is proved in a similar way; the reader can consult [12] for more details). Suppose the statement in the Main Theorem is false, so that there is no bound for some triple (d,l,m). Therefore, we can find for every w an Artinian local ring R_w of length at most l and polynomials $f_{0w},\dots,f_{sw}$ in $R_w[X]$ of degree at most d with X an n-tuple of variables, such that there is a linear combination

$$f_{0w}=q_{1w}f_{1w}+\dots+q_{sw}f_{sw}, \tag{1}$$

but in any such linear combination (1), at least one q_{iw} has degree w or higher. Note that we can take the same number s in each counterexample by Lemma 2.1 below. Let R and A_∞ denote the ultraproduct of the R_w and the $R_w[X]$ respectively. Since each f_{iw} has degree at most d, the ultraproduct f_i of the f_{iw} lies already in the subring $R[X]$ (this is because taking ultraproducts commutes with bounded sums). Set $I:=(f_1,\dots,f_s)R[X]$ (note that IA_∞ is then the ultraproduct of the ideals generated by the f_{iw}). Let $q_{i\infty}$ be the ultraproduct of the q_{iw}. This time, we can only say that $q_{i\infty}\in A_\infty$. Therefore, by Łos' Theorem, (1) yields that $f_0\in IA_\infty$. However, since $R[X]\to A_\infty$ is faithfully flat by Theorem 1.2, we get $f_0\in IA_\infty\cap R[X]=I$. Say,

$$f_0=g_1f_1+\dots+g_sf_s,$$

for some $g_i\in R[X]$. Suppose D is the maximum of the degrees of the g_i. We can choose polynomials $g_{iw}\in R_w[X]$ of degree at most D whose ultraproduct is g_i. By Łos' Theorem once more, we get

$$f_{0w}=g_{1w}f_{1w}+\dots+g_{sw}f_{sw},$$

for almost all w, contradicting our assumption for those w bigger than D. □

An immediate consequence of the Main Theorem is the following result on congruences. Note that this is an extension of [3, Proposition 4.14] to higher dimensions.

1.3. **Theorem.** *For each quadruple (d, e, n, c) of positive integers, we can find a positive integer $N := N(d, e, n, c)$ with the following property. Let $(R, \mathfrak{m})$ be a Noetherian local ring of embedding dimension at most e. Let $X = (X_1, \dots, X_n)$ be an n-tuple of variables and let $\mathfrak{p}$ be a prime ideal in $R[X]$ containing $\mathfrak{m}$. Let $f_0, \dots, f_s$ be polynomials in $R[X]$ of degree at most d. If $f_0 \in (f_1, \dots, f_s, \mathfrak{m}^c)R[X]_{\mathfrak{p}}$, then there exist $q_i \in R[X]$ of degree at most N with $q_0 \notin \mathfrak{p}$, such that*

$$q_0 f_0 \equiv q_1 f_1 + \dots + q_s f_s \mod \mathfrak{m}^c R[X].$$

In fact, if f_0 lies already in $(f_1, \dots, f_s, \mathfrak{m}^c)R[X]$, then we may take $q_0 = 1$.

Proof. Note that $R/\mathfrak{m}^c$ has length bounded in terms of e and c, since by definition of embedding dimension, $\mathfrak{m}$ is generated by at most e elements. By assumption, we can find a tuple $\mathbf{a} = (a_0, \dots, a_s)$ with entries in $R[X]$ and $a_0 \notin \mathfrak{p}$, which is a solution of the congruence

$$Y_0 f_0 \equiv Y_1 f_1 + \dots + Y_s f_s \mod \mathfrak{m}^c R[X]. \tag{2}$$

Hence, by the second part of the Main Theorem, we can find tuples $\mathbf{b}_1, \dots, \mathbf{b}_t$ of polynomials of degree at most N', for some N' only depending on (d, e, n, c), satisfying (2) and such that $\mathbf{a}$ lies in the $R[X]$-module spanned by them. In particular, since a_0 does not lie in $\mathfrak{p}$, at least one of these tuples has first entry not in $\mathfrak{p}$, and hence satisfies the requirements of the first assertion.

For the last assertion, use the first part of the Main Theorem instead. □

This has the following immediate corollary for discrete valuation rings; for an application, see [17].

1.4. **Corollary.** *For each triple (d, n, c) of positive integers, we can find a positive integer $M := M(d, n, c)$ with the following property. Let (R, π) be a discrete valuation ring. Let $X = (X_1, \dots, X_n)$ be an n-tuple of variables, let $\mathfrak{p}$ be a prime ideal of $R[X]$ containing π and let $f_1, \dots, f_s$ be polynomials in $R[X]$ of degree at most d. If $a \in R$ lies in the ideal generated by $f_1, \dots, f_s$ (respectively, this holds only locally at $\mathfrak{p}$) and a has valuation at most c, then there exist $q_i \in R[X]$ of degree at most M such that*

$$a(1 + \pi q_0) = q_1 f_1 + \dots + q_s f_s,$$

(respectively, there exist $q_i \in R[X]$ of degree at most M with $q_0 \notin \mathfrak{p}$, such that $aq_0 = q_1 f_1 + \dots + q_s f_s$).

Proof. By assumption, we can write $a = \pi^l u$, for some $l \leq c$ and some unit u in R. Put $M' := N(d, 1, n, l+1)$, where N is the bound from Theorem 1.3. It follows that we can find polynomials $q, g_i \in R[X]$ of degree at most M' with $q \notin \mathfrak{p}$ in the local case and $q = 1$ in the global case, such that

$$aq = g_1 f_1 + \dots + g_s f_s + h\pi^{l+1},$$

for some $h \in R[X]$. In other words, $aq(1 - u^{-1}\pi h)$ is equal to the sum $g_1 f_1 + \dots + g_s f_s$. However, comparing coefficients shows in the global case ($q = 1$), that h has degree at most $M' + d$, and in the local case, that $q(1 - u^{-1}\pi h)$ has degree at most $M' + d$. Therefore, $M := M' + d$ gives the required bound in either case. □

After proving the remaining statements in the next section and some further bounds in Section 3, I return to the question what happens when we work over non-Artinian rings.

However, I do not know that, if we allow in Theorem 1.2 the local rings R_w to be non-Artinian, but still impose certain bounds on the embedding dimension, whether the corresponding embedding $A \to A_\infty$ is still flat. As already explained, it is most certainly *not* faithfully flat.

2. FAITHFUL FLATNESS

2.1. **Lemma.** *For each triple (d, l, n) of positive integers, we can find a positive integer $s := s(d, l, n)$ with the following property. Let R be an Artinian local ring of length at most l. Let $X = (X_1, \dots, X_n)$ be an n-tuple of variables and let $\mathfrak{a}$ be an ideal in $R[X]$ generated by polynomials of degree at most d. Then $\mathfrak{a}$ is already generated by s polynomials of degree at most d.*

Proof. We first make the following observation. If a_m is a sequence of elements in R generating an ideal $\mathfrak{a}$ in R and $m > l$, then already l among the a_i generate $\mathfrak{a}$.

Put $A := R[X]$. Let $\mathbf{f} = (f_1, \dots, f_m)$ be a tuple of polynomials f_i of degree at most d and let I be the ideal in A they generate. Let us write $\mathcal{M}(\mathbf{f})$ for the collection of all monomials μ in the variables X of degree at most d which appear with a non-zero coefficient in some f_i and let $\delta(\mathbf{f})$ be the number of these monomials. I claim that I can be generated by at most $l\delta(\mathbf{f})$ polynomials of degree at most d. From this the statement follows easily. To prove the claim, we induct on $\delta := \delta(\mathbf{f})$. If $\delta = 0$, then all $f_i = 0$ and the assertion is clear. So let $\delta > 0$ and pick some $\mu \in \mathcal{M}(\mathbf{f})$. Let a_i be the coefficient of μ in f_i. By our first observation, we may renumber the f_i in such way that all a_i belong to $(a_1, \dots, a_l)R$. Therefore, subtracting the appropriate R-linear combination of the $f_1, \dots, f_l$ from each remaining f_i, we get a new m-tuple of the form $(f_1, \dots, f_l, \mathbf{g})$ generating I, such that $\mathcal{M}(\mathbf{g}) = \mathcal{M}(\mathbf{f}) - \{\mu\}$. In particular, since $\delta(\mathbf{g}) = \delta - 1$, our induction hypothesis yields that the ideal generated by $\mathbf{g}$ can be generated by $l(\delta - 1)$ polynomials of degree at most d. This establishes the claim and hence the lemma. □

The proof of the Lemma can easily be extended to the case where R is a local ring in which every ideal is generated by at most l elements, for some fixed l. This applies for instance, if R is a discrete valuation ring. Nonetheless, the Main Theorem is false for $R = \mathbb{Z}$ or $R = \mathbb{Z}_p$ (see [3, Section 4.7]). In particular, if R_p is equal to either $\mathbb{Z}$ or $\mathbb{Z}_p$ and if R_∞ denotes their ultraproduct, then the canonical homomorphism

$$R_\infty[X] \to A_\infty \tag{3}$$

is not faithfully flat, where A_∞ is the ultraproduct of the $R_p[X]$. In fact, inspecting the proof that Theorem 1.2 implies the Main Theorem, the above embedding cannot even be *cyclically pure*, that is to say, for some ideal I of $R_\infty[X]$ we have $I \subsetneq IA_\infty \cap R_\infty[X]$. Moreover, by Lemma 2.1, this happens already for some finitely generated ideal I.

2.2. **Corollary.** *Fix some $l, d \in \mathbb{N}$ and let X be a finite tuple of variables. Let R_w be Noetherian rings with the property that each of their ideals is generated by at most l elements. Let R_∞ and A_∞ denote the ultraproduct of the R_w and the $R_w[X]$ respectively. Let I_w be ideals in $R_w[X]$ generated by polynomials of degree at most d.*

Then there exists a finitely generated ideal I in $R_\infty[X]$, such that IA_∞ is equal to the ultraproduct of the I_w.

Proof. By Lemma 2.1 and the remark following it, there is some s, such that each I_w is of the form $(f_{1w}, \dots, f_{sw})R_w$, with all f_{iw} of degree at most d. Write each f_{iw} as a finite sum $\sum_\nu a_{i,\nu,w} X^\nu$, where X^ν runs over all monomials of total degree at most d and with each $a_{i,\nu,w} \in R_w$. Let $a_{i,\nu}$ be the ultraproduct of the $a_{i,\nu,w}$ and put $f_i := \sum_\nu a_{i,\nu} X^\nu$.

Hence each f_i lies in $R_\infty[X]$ and one checks easily that when viewed as an element in A_∞, it is the ultraproduct of the f_{iw}. Let $I := (f_1, \dots, f_s)R_\infty[X]$. Using Łos' Theorem, one now readily verifies that IA_∞ is the ultraproduct of the I_w. □

Proof of Theorem 1.2. Let $(R_w, \mathfrak{m}_w)$ and A_∞ be as in the statement of the Theorem. By [13], the ultraproduct $(R, \mathfrak{m})$ is again an Artinian local ring of length at most l. Let us put $A := R[X]$. We need to show that $A \to A_\infty$ is flat and that every proper ideal I of A remains proper in A_∞. To prove the last assertion, observe that any maximal ideal $\mathfrak{M}$ of A contains the nilpotent ideal $\mathfrak{m}$. By Theorem 1.1, the base change $A/\mathfrak{m}A \to A_\infty/\mathfrak{m}A_\infty$ is faithfully flat, so that in particular $\mathfrak{M}A_\infty \neq A_\infty$.

To prove flatness, we will induct on the length l of R. Note that almost all R_w then also have length l. If $l = 1$, then almost all R_w are fields and the assertion is just Theorem 1.1. For general $l > 1$, it suffices, by the Local Flatness Criterion [9, Theorem 22.3], to find a proper ideal $\mathfrak{a}$ in R such that $A/\mathfrak{a}A \to A_\infty/\mathfrak{a}A_\infty$ is flat and $\mathfrak{a}A \otimes_A A_\infty \cong \mathfrak{a}A_\infty$. Let η be a non-zero element in the socle of R, that is to say, so that $\eta\mathfrak{m} = 0$, and put $\mathfrak{a} := \eta R$. Choose elements η_w in R_w so that their ultraproduct is η. It follows that almost all η_w are non-zero elements in the socle of R_w. In particular, $R/\eta R$ and almost all $R_w/\eta_w R_w$ have length $l - 1$. Since $R/\eta R$ is the ultraproduct of the $R_w/\eta_w R_w$, our induction hypothesis yields that the canonical map from $(R/\eta R)[X]$ to the ultraproduct of the $(R_w/\eta_w R_w)[X]$ is faithfully flat. However, the above homomorphism is simply the base change $A/\eta A \to A_\infty/\eta A_\infty$. So remains to show that $\eta A \otimes_A A_\infty \cong \eta A_\infty$. Clearly $\eta A \cong A/\mathfrak{m}A$. Therefore, we will have shown the required isomorphism if we can show that $\operatorname{Ann}_{A_\infty}(\eta) = \mathfrak{m}A_\infty$. One inclusion is obvious, so let $a_\infty \in A_\infty$ be an element annihilating η. Choose $a_w \in R_w[X]$ with ultraproduct equal to a_∞. By Łos' Theorem, almost all $\eta_w a_w = 0$. Since $\operatorname{Ann}_{R_w}(\eta_w) = \mathfrak{m}_w$, almost each a_w lies in $\mathfrak{m}_w R_w[X]$. This in turn shows that $a_\infty \in \mathfrak{m}A_\infty$, as required. □

3. Further bounds

Throughout this section, X will always denote a tuple of variables and n will always denote the length of this tuple.

3.1. **Corollary.** *For each triple (d, l, n) of positive integers, we can find a positive integer $\alpha := \alpha(d, l, n)$ with the following property. Let $(R, \mathfrak{m})$ be an Artinian local ring of length at most l. Let I be an ideal in $R[X]$ generated by polynomials of degree at most d. Let $\mathfrak{p}$ be a minimal prime of I. If we set $S := (R[X]/I)_\mathfrak{p}$, then S has length at most α.*

Proof. We will show the existence of such a bound $\alpha(d, l, n)$ by induction on l. If $l = 1$, so that R is a field, then the existence of the uniform bound $\alpha(d, 1, n)$ follows from [15, Theorem 2.4]. For $l > 1$, let η be a non-zero element in the socle of R. That is to say, $\operatorname{Ann}_R(\eta) = \mathfrak{m}$. Hence $\operatorname{Ann}_S(\eta S)$ contains $\mathfrak{m}S$. Let us write $\ell(N)$ to denote the length of a module N. Since $\eta S \cong S/\operatorname{Ann}_S(\eta S)$, we get

$$\ell(\eta S) \leq \ell(S/\mathfrak{m}S) \leq \alpha(d, 1, n).$$

On the other hand, from the exact sequence

$$0 \to \eta S \to S \to S/\eta S \to 0$$

we get $\ell(S) = \ell(\eta S) + \ell(S/\eta S)$. Since $R/\eta R$ has length $l - 1$, our induction hypothesis yields that $\ell(S/\eta S)$ is at most $\alpha(d, l - 1, n)$. Therefore, the bound $\alpha(d, l, n) := \alpha(d, 1, n) + \alpha(d, l - 1, n)$ satisfies the requirements of the statement. □

In fact, the above argument shows that we can use $l \cdot \alpha(d, 1, n)$ as a bound.

3.2. **Corollary.** *For each triple (d,l,n) of positive integers, we can find a positive integer $\beta := \beta(d,l,n)$ with the following property. Let R be an Artinian local ring of length at most l. If I,J are ideals generated by polynomials in $R[X]$ of degree at most d, then the intersection $I \cap J$ and the colon ideal $(I : J)$ are both generated by polynomials of degree at most β.*

Proof. Suppose the assertion is false for some triple (d,l,n) and for one of these operations. Hence there are Artinian local rings $(R_w, \mathfrak{m}_w)$ of length at most l, and ideals I_w, J_w generated by polynomials in $R_w[X]$ of degree at most d, such that $I_w \cap J_w$ (respectively, $(I_w : J_w)$) cannot be generated by polynomials of degree at most w. Let R and A_∞ denote the ultraproduct of R_w and $R_w[X]$ respectively and put $A := R[X]$. Taking generators of the I_w and J_w of degree at most d and taking the ultraproducts of these generators yield ideals I,J in A such that $IA_\infty = I_\infty$ and $JA_\infty = J_\infty$ are the respective ultraproducts of the I_w and J_w. By Łos' Theorem, the ultraproduct of the ideals $I_w \cap J_w$ and $(I_w : J_w)$ are the respective ideals $I_\infty \cap J_\infty$ and $(I_\infty : J_\infty)$. By Theorem 1.2, the canonical embedding $A \to A_\infty$ is faithfully flat, so that by [9, Theorem 7.4], we get

$$
\begin{aligned}
I_\infty \cap J_\infty &= IA_\infty \cap JA_\infty = (I \cap J)A_\infty \\
(I_\infty : J_\infty) &= (IA_\infty : JA_\infty) = (I : J)A_\infty.
\end{aligned} \tag{4}
$$

Let h be the maximum degree of a set of generators of $I \cap J$ or $(I : J)$. By the above equality of ideals together with Łos' Theorem, almost all $I_w \cap J_w$ or $(I_w : J_w)$ are generated by polynomials of degree at most h, contradicting our assumption. □

3.3. **Theorem.** *For each triple (d,l,n) of positive integers, we can find a positive integer $\gamma := \gamma(d,l,n)$ with the following properties. Let R be an Artinian local ring of length at most l. Let I be an ideal in $R[X]$ generated by polynomials of degree at most d.*

- (3.3.1) *If $fg \notin I$, for all $f,g \notin I$ of degree at most γ, then I is a prime ideal.*
- (3.3.2) *If $\mathfrak{p}$ is an associated prime ideal of I, then $\mathfrak{p}$ is generated by polynomials of degree at most γ.*
- (3.3.3) *The number of associated primes of I is at most γ.*

Proof. Suppose that either one of these assertions is false for some triple (d,l,n). Therefore, we can find for each w an Artinian local ring $(R_w, \mathfrak{m}_w)$ of length at most l and an ideal $I_w = (f_{1w}, \dots, f_{sw})R_w[X]$, where f_{iw} have degree at most d and where X is an n-tuple of variables, such that in case (3.3.1), the ideal I_w is not prime but $ab \notin I_w$ for all $a,b \notin I_w$ of degree at most w; in case (3.3.2), there is an associated prime $\mathfrak{p}_w$ of I_w which cannot be generated by polynomials of degree at most w; and in case (3.3.3), the ideal I_w has at least w different associated primes.

Let $(R,\mathfrak{m})$, A_∞ and f_i denote the respective ultraproducts of the $(R_w,\mathfrak{m}_w)$, the $R_w[X]$ and the f_{iw}. It follows, as before, that R is an Artinian local ring of length at most l and that all f_i lie already in the subring $A := R[X]$. Note that the ultraproduct of the I_w is equal to IA_∞, where $I := (f_1, \dots, f_s)A$. By Theorem 1.2, the homomorphism $A \to A_\infty$ is faithfully flat. Let us first prove the following property of the homomorphism $A \to A_\infty$.

(3.3.4) I is a prime ideal if and only if IA_∞ is.

Indeed, the if part is obvious since $I = IA_\infty \cap A$ by faithful flatness. So assume that I is prime. Since $\mathfrak{m}$ is nilpotent, $\mathfrak{m} \subseteq I$. Hence A/I and A_∞/IA_∞ are in fact k-algebras, where $k := R/\mathfrak{m}$ is the residue field of R. Since A/I is a domain over the field k, the results in [12] yield that A_∞/IA_∞ is a domain, thus proving the assertion (see for instance [19, Corollary 4.4]).

Let us now return to the proof of the theorem. In case (3.3.1), Łos' Theorem yields that I_∞ is not a prime ideal, whence neither is I by (3.3.4). So there are some $a, b \notin I$, such that $ab \in I$. Let h be the maximum of their degrees and choose $a_w, b_w \in R_w[X]$ of degree at most h such that their ultraproducts are a and b respectively. By Łos' Theorem, almost all $a_w, b_w \notin I_w$ but $a_w b_w \in I_w$, contradicting our assumption.

For the remaining cases, consider a prime filtration of A/I. To be more precise, choose prime ideals $\mathfrak{q}_i$ and an increasing chain of ideals $I = I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t$ with the last ideal $I_t = \mathfrak{q}_t$ also prime, such that all

$$0 \to A/\mathfrak{q}_i \to A/I_i \to A/I_{i+1} \to 0$$

are exact, for $i = 1, \dots, t-1$. In other words, $\mathfrak{q}_i = (I_i : a_i)$ and $I_{i+1} = I_i + a_i A$, for some $a_i \in A$. Moreover, all associated primes of I are among the $\mathfrak{q}_i$ (see for instance [6, Proposition 3.7] and the discussion following it). Each ideal $\mathfrak{q}_i A_\infty$ is prime, by (3.3.4), and equal to $(I_i A_\infty : a_i)$, by (4). Moreover, we have short exact sequences

$$0 \to A_\infty/\mathfrak{q}_i A_\infty \to A_\infty/I_i A_\infty \to A_\infty/I_{i+1} A_\infty \to 0.$$

Hence, for the same reason as before, all the associated primes of IA_∞ are among the $\mathfrak{q}_i A_\infty$. In summary, we showed that any associated prime of IA_∞ is an extension $\mathfrak{q}A_\infty$ of an associated prime $\mathfrak{q}$ of I.

Assume now that we are in case (3.3.2). Since $\mathfrak{p}_w$ is an associated prime of I_w, we can find a_w such that $\mathfrak{p}_w = (I_w : a_w)$. By Łos' Theorem, the ultraproduct $\mathfrak{p}_\infty$ of the $\mathfrak{p}_w$ is equal to $(I_\infty : a_\infty)$, where a_∞ is the ultraproduct of the a_w, showing that $\mathfrak{p}_\infty$ is an associated prime of $I_\infty = IA_\infty$. By what we just said, there is an associated prime $\mathfrak{p}$ of I, such that $\mathfrak{p}_\infty = \mathfrak{p}A_\infty$. Let h be the maximum of the degrees of a set of generators of $\mathfrak{p}$. Using Łos' Theorem once more, we conclude that almost all $\mathfrak{p}_w$ are generated by polynomials of degree at most h, thus contradicting our assumption. If we are in case (3.3.3), then the above argument would yield infinitely many different associated primes of I_∞, contradicting our earlier observation that there are at most t. □

3.4. **Corollary.** *For each triple (d, l, n) of positive integers, we can find a positive integer $\delta := \delta(d, l, n)$ with the following properties. Let R be an Artinian local ring of length at most l. Let I be an ideal in $R[X]$ generated by polynomials of degree at most d. If $\mathfrak{a}$ is the radical of I, then $\mathfrak{a}$ is generated by polynomials of degree at most δ and $\mathfrak{a}^\delta \subseteq I$.*

Proof. Observe that $\mathfrak{a}$ is the intersection of the minimal primes of I. By Theorem 3.3, there are at most γ different minimal primes, each generated by polynomials of degree at most γ, for γ only depending on the triple (d, l, n). Therefore, the existence of a uniform bound on the degrees of the generators of $\mathfrak{a}$ follows from Corollary 3.2.

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be all the minimal primes of I, so that in particular, $t \le \gamma$. By Corollary 3.1, there is a bound α only depending on the triple (d, l, n), such that $(R[X]/I)_{\mathfrak{p}_i}$ has length at most α, for $i = 1, \dots, t$. Hence, for some $s_i \notin \mathfrak{p}_i$, we have

$$s_i \mathfrak{p}_i^\alpha \subseteq I. \tag{5}$$

Multiplying s_i with a suitable element not in $\mathfrak{p}_i$, we may assume that s_i is contained in the α-th power of each associated prime of I other than $\mathfrak{p}_i$. It follows that the sum $s := s_1 + \cdots + s_t$ does not belong to any associated prime of I, that is to say, that s is not a zero-divisor on $R[X]/I$. On the other hand, by (5), we get

$$s(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t)^\alpha \subseteq I.$$

Since $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$ and since s is not a zero-divisor modulo I, we get $\mathfrak{a}^\alpha \subseteq I$, as required. □

As in [15], one can also study bounds for finitely generated modules. Note that over a Noetherian ring A, any finitely generated A-module M can be realized as the cokernel of a matrix. We can now show that in case $A = R[X]$ with R Artinian, many algebraic constructions performed on the category of finitely generated A-modules can be bounded in terms of the degrees of the entries of the matrices defining the modules. The reader is invited to carry out this program, following the lines of [15]. Let me only prove one such result, since it will be used in [17]; at the same time, we put ourselves in a more general context, where the base ring is no longer a polynomial ring over an Artinian local ring R, but any finitely generated R-algebra. The previous results can easily be extended to this setup as well, using a similar argument as in the proof below.

3.5. **Theorem.** *For each triple (d, l, n) of positive integers, we can find a positive integer $\epsilon := \epsilon(d, l, n)$ with the following properties. Let R be an Artinian local ring of length at most l and X a tuple of variables of length n. Let A be of the form $R[X]/(g_1, \dots, g_s)R[X]$, with each g_i of degree at most d, and let M be a finitely generated A-module realized as the cokernel of a matrix (a_{ij}) of size at most d, with each a_{ij} (the image of) a polynomial of degree at most d.*

If $\mathfrak{a}$ is the annihilator of M, then there exist $f_i \in R[X]$ of degree at most d, such that $\mathfrak{a} = (f_1, \dots, f_s)A$.

Proof. Suppose $\mathbb{A} := (a_{ij})$ has dimensions $(a \times b)$, where by assumption $a, b \leq d$, so that $\mathbb{A}\colon A^b \to A^a$ and $M \cong A^a / \operatorname{Im} \mathbb{A}$. Therefore, an element $\theta \in R[X]$ lies in $\mathfrak{a}$ if and only if

$$\theta A^a \subseteq \operatorname{Im} \mathbb{A}. \tag{6}$$

All we need to do is to write this as a system of equations over $R[X]$, in which all coefficients have bounded degree. Namely, (6) holds, if for each i and k, the system of equations

$$\sum_j a_{ij}\xi_{jk} = \theta \boldsymbol{\delta}_{i,k} + \sum_j g_j \chi_{ijk} \tag{7}$$

in the unknowns ξ_{jk} and χ_{ijk}, is solvable in $R[X]$, where $\boldsymbol{\delta}_{i,k}$ is Kronecker's delta, that is to say, equal to one if $i = k$ and zero otherwise. Viewing (7) as a homogeneous system of equations in the unknowns ξ_{jk}, θ and χ_{ijk} the second assertion in the Main Theorem gives a bound d' on the degrees of a complete set of solutions

$$(\xi_{jk}^{(l)}, \theta^{(l)}, \chi_{ijk}^{(l)}) \quad \text{for } l = 1, \dots, t$$

Hence we can take $\epsilon = d'$, since by our previous argument, $\mathfrak{a}$ is generated by the $\theta^{(l)}$. □

4. Cyclic purity above the infinitesimals

In the general case, we have the following vanishing of *Tor*s.

4.1. **Proposition.** *For some fixed $h, e \in \mathbb{N}$, let $(R_w, \mathfrak{m}_w)$ be h-dimensional local Cohen-Macaulay rings of multiplicity at most e. Let $(R_\infty, \mathfrak{m}_\infty)$ and A_∞ denote the respective ultraproducts of R_w and $R_w[X]$, where X is a fixed n-tuple of variables and let $A := R_\infty[X]$. For each A-module N annihilated by some power of $\mathfrak{m}_\infty$, we have*

$$\operatorname{Tor}_i^A(A_\infty, N) = 0$$

for all $i \geq 1$.

Proof. From the argument in [14, Theorem 3.1], it follows that there exists an R_∞-regular sequence $(x_1, \dots, x_h)$ of length h generating an $\mathfrak{m}_\infty$-primary ideal. Note that $(x_1, \dots, x_h)$ is then also A-regular, since $R_\infty \to A$ is flat. Choose $x_{iw} \in R_w$ with ultraproduct equal to x_i. By Łos' Theorem, almost all sequences $(x_{1w},\dots,x_{hw})$ are R_w-regular. Hence, almost all of these sequences are also $R_w[X]$-regular and therefore by Łos' Theorem again, $(x_1,\dots,x_h)$ is A_∞-regular. By [9, Theorem 16.1], all this remains true upon replacing $(x_1,\dots,x_h)$ by $(x_1^m,\dots,x_h^m)$, for any $m \geq 1$. Choose m big enough so that $\mathfrak{n} := (x_1^m,\dots,x_h^m)R_\infty$ annihilates N. It follows that

$$\operatorname{Tor}_i^A(A_\infty, N) \cong \operatorname{Tor}_i^{A/\mathfrak{n}A}(A_\infty/\mathfrak{n}A_\infty, N). \tag{8}$$

On the other hand, by the Main Theorem, $A/\mathfrak{n}A \to A_\infty/\mathfrak{n}A_\infty$ is faithfully flat, so that the required vanishing follows from (8). □

Recall that a homomorphism $A \to B$ is called *cyclically pure*, if $IB \cap A = I$, for every ideal I of A. Faithfully flat homomorphisms are clearly cyclically pure. Inspecting the proof that Theorem 1.2 implies the Main Theorem, it is clear that instead of faithful flatness, cyclical purity would have sufficed. In [21, Theorem 2.2], the following criterion of cyclical purity is proven in the Noetherian case; the general case uses the same argument, which I have repeated here for the reader's convenience.

4.2. **Theorem.** *Let $(A,\mathfrak{p})$ be a local ring with residue field k and let B be an arbitrary A-algebra. If $\operatorname{Tor}_1^A(B,k)=0$ and $\mathfrak{p}B \neq B$, then $I = IB\cap A$, for all ideals I of A which are closed in the $\mathfrak{p}$-adic topology (this includes all $\mathfrak{p}$-primary ideals). In particular, if A is Noetherian, then $A \to B$ is cyclically pure.*

Proof. Assume first that I is a $\mathfrak{p}$-primary ideal. Since $\operatorname{Tor}_1^A(B,k)$ vanishes, so does $\operatorname{Tor}_1^{A/I}(B/IB,k)$ (see for instance [21, Lemma 2.1]). Since A/I is Artinian, the Local Flatness Criterion (see for instance [9, Theorem 22.3]) yields that $A/I \to B/IB$ is flat. Since $\mathfrak{p}B \neq B$, the latter homomorphism is even faithfully flat and whence in particular, $IB \cap A = I$.

For I arbitrary, observe that I is closed in the adic topology if and only if it is an intersection of $\mathfrak{p}$-primary ideals. By our previous argument, each of these $\mathfrak{p}$-primary ideals is contracted from B, and therefore, so is their intersection I. As any ideal in a Noetherian local ring is closed by Krull's Intersection Theorem, the last assertion is also clear. □

The *ideal of infinitesimals* of a (not necessarily Noetherian) local ring $(A,\mathfrak{p})$ is by definition the intersection of all powers $\mathfrak{p}^n$. For our last result, we will use some results from [17, 18] on dimension theory for ultraproducts of Noetherian local rings (this will be studied in far greater detail in [22]). We will tacitly assume that the underlying ultrafilter is countably incomplete (this can always be arranged).

4.3. **Theorem.** *For some fixed $h,e \in \mathbb{N}$, let $(R_w,\mathfrak{m}_w)$ be h-dimensional local Cohen-Macaulay rings of multiplicity at most e and let $(R_\infty,\mathfrak{m}_\infty)$ be their ultraproduct. Let ϖ be the ideal of infinitesimals of R_∞. Let A_∞ be the ultraproduct of the $R_w[X]$, where X is a fixed n-tuple of variables and let $A := R_\infty[X]$. Then the localization of the base change $A/\varpi A \to A_\infty/\varpi A_\infty$ at any maximal ideal of A containing $\mathfrak{m}_\infty$ is cyclically pure.*

Proof. Let us first show that $\tilde{R} := R_\infty/\varpi R_\infty$ is a Noetherian local ring (in fact Cohen-Macaulay of multiplicity at most e). By [1], the $\mathfrak{m}_w$ are generated by at most $h+e-1$ elements, whence so are $\mathfrak{m}_\infty$ and $\mathfrak{m}_\infty\tilde{R}$ by Łos' Theorem. Without proof, we state that $\tilde{R}$ is complete (this is where we need the assumption that the ultrafilter is countably incomplete), and therefore Noetherian by [9, Theorem 29.4] (for details see [10, Theorem 2.4] for the case of an ultrapower, and [22, Lemma 5.1] for the general case).

Hence also $A/\varpi A \cong \tilde{R}[X]$ is Noetherian. Let $(S,\mathfrak{n})$ denote the localization of $A/\varpi A$ at a maximal ideal $\mathfrak{n}$ of A containing $\mathfrak{m}_\infty$. Let $S_\infty := A_\infty \otimes_A S$. We need to show that $S \to$

S_∞ is cyclically pure. Since $A/\mathfrak{m}_\infty A \to A_\infty/\mathfrak{m}_\infty A_\infty$ is faithfully flat by Theorem 1.1, it is clear that $\mathfrak{n}S_\infty \neq S_\infty$. Let k be the residue field of S. In view of Theorem 4.2, it remains to show that $\operatorname{Tor}_1^S(S_\infty, k) = 0$. However, this is clear by Proposition 4.1. □

APPENDIX A. AN ALTERNATIVE PROOF OF THEOREM 1.1

Because of the central role played by Theorem 1.1 in these results, we will provide an alternative, self-contained proof. We need a lemma ([5, Lemma 4.8]), the easy proof of which we will repeat here.

A.1. **Lemma.** *Let R be a Noetherian local ring and let M be a big Cohen-Macaulay module over R. If any permutation of an M-regular sequence is again M-regular, then M is a balanced big Cohen-Macaulay module.*

Proof. We induct on the dimension d of R, where there is nothing to show if $d = 0$. So assume $d > 0$. By assumption, there exists a system of parameters $(x_1, \dots, x_d)$ of R which is an M-regular sequence. Let $(y_1, \dots, y_d)$ be an arbitrary system of parameters. By prime avoidance, we can find $z \in R$, such that $(x_1, \dots, x_{d-1}, z)$ and $(y_1, \dots, y_{d-1}, z)$ are both systems of parameters. Since a power of x_n is a multiple of z modulo $(x_1, \dots, x_{d-1})$, the sequence $(x_1, \dots, x_{d-1}, z)$ is M-regular as well. Therefore, the permuted sequence $(z, x_1, \dots, x_{d-1})$ is also M-regular. In particular, $(x_1, \dots, x_{d-1})$ is M/zM-regular, showing that M/zM is a big Cohen-Macaulay over R/zR. Moreover, the property that a permutation of an M-regular sequence is again M-regular passes to the quotient M/zM. Our induction hypothesis therefore shows that $(y_1, \dots, y_{d-1})$, being a system of parameters in R/zR, is M/zM-regular. Hence $(z, y_1, \dots, y_{d-1})$ is M-regular and therefore so is $(y_1, \dots, y_{d-1}, z)$. As some power of z is a multiple of y_d modulo $(y_1, \dots, y_{d-1})$, the element y_d is $M/(y_1, \dots, y_{d-1})M$-regular, showing that $(y_1, \dots, y_d)$ is M-regular, as required. □

Proof of Theorem 1.1. Let F_w be an algebraic closure of K_w and let F be the ultraproduct of the F_w. (Although F is algebraically closed, it is in general larger than the algebraic closure of K). If we showed that the natural map from $F[X]$ to the ultraproduct of the $F_w[X]$ is faithfully flat, then so is $K[X] \to A_\infty$ by faithfully flat descent. Therefore, we may assume from the start that each K_w is algebraically closed, and hence so is K.

Let $A := K[X]$ and $A_w := K_w[X]$. By the Nullstellensatz, any maximal ideal $\mathfrak{M}$ of A is of the form $(X_1 - a_1, \dots, X_n - a_n)A$ with $a_i \in K$. Choose $a_{iw} \in K_w$ with ultraproduct equal to a_i and let $\mathfrak{M}_w$ be the maximal ideal $(X_1 - a_{1w}, \dots, X_n - a_{nw})A_w$. It follows that $\mathfrak{M}A_\infty$ is the ultraproduct of the $\mathfrak{M}_w$, whence in particular a proper ideal. So remains to show that $A \to A_\infty$ is flat. Since this is a local property, we may localize at a maximal ideal $\mathfrak{M}$, which after a translation, we may take to be $(X_1, \dots, X_n)A$. It is easy to see that $(X_1, \dots, X_n)$ is A_∞-regular (since it is A_w-regular). In particular, $(A_\infty)_{\mathfrak{M}A_\infty}$ is a big Cohen-Macaulay module over $A_{\mathfrak{M}}$. Any permutation of an $(A_\infty)_{\mathfrak{M}A_\infty}$-regular sequence is again $(A_\infty)_{\mathfrak{M}A_\infty}$-regular, since this is true in each Noetherian local ring $(A_w)_{(X_1,\dots,X_n)A_w}$ by [9, Theorem 16.3]. Hence, $(A_\infty)_{\mathfrak{M}A_\infty}$ is a balanced big Cohen-Macaulay $A_{\mathfrak{M}}$-module by Lemma A.1. However, a balanced big Cohen-Macaulay module over a regular local ring is automatically flat, by [20, Theorem 3.5] (see also the proof of [8, Theorem 9.1]). □

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