For a Noetherian local ring R, if $R/\mathfrak{a}$ is Cohen-Macaulay, then the ideal $\mathfrak{a}$ can be generated by at most $(e-2)(\nu-d-1)+2$ elements, where ν is the embedding dimension of R and where d and $e \geq 3$ are the dimension and the multiplicity of $R/\mathfrak{a}$ respectively. This bound is in general much sharper than the bounds given by Sally or Boratyński-Eisenbud-Rees in case $\mathfrak{a}$ has height bigger than 2. Moreover, no Cohen-Macaulay assumption on R is required.

Number of generators of a Cohen-Macaulay ideal

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1. INTRODUCTION

The program of bounding (in either direction) the minimal number of generators $\mu_R(\mathfrak{a})$ of an ideal $\mathfrak{a}$ in a Noetherian ring R is an ambitious one, and the list of papers on the subject is impressive (below I only will mention very few). According to one's taste, either absolute bounds (that is to say, independent of the ideal), or bounds in terms of other invariants associated to R and $\mathfrak{a}$, are given. Examples of the former can be found for instance in [4, 9, 10] or in some recent generalizations due to the author in [11]; see below for a further discussion. In this paper, upper bounds are given in terms of the embedding dimension of R and the multiplicity of the residue ring $R/\mathfrak{a}$, under the additional assumption that $\mathfrak{a}$ is a *Cohen-Macaulay ideal* (that is to say, such that $R/\mathfrak{a}$ is Cohen-Macaulay; SALLY has already argued in [9, p. 81] that at least *some vestige of the Cohen-Macaulay hypothesis must remain*). The principal result of this paper is the following (see Theorem 2.3).

MAIN THEOREM. *Let R be a Noetherian local ring of embedding dimension ν. If $\mathfrak{a}$ is a Cohen-Macaulay ideal of R, such that $R/\mathfrak{a}$ has dimension d and multiplicity $e \geq 3$, then $\mathfrak{a}$ can be generated by at most $(e-2)(\nu-d-1)+2$ elements.*

In case $e \leq 2$, at most $\nu - d + 1$ generators suffice.

Using the FORSTER-SWAN Theorem (Theorem 3.1), similar bounds can be obtained in the non-local case: at most d more generators than in the local case are required. As a corollary, we obtain the following result (this is a special case of the last statement in Theorem 3.4, with A the coordinate ring of Y, so that $\nu \leq n$).

COROLLARY A. *Let X be an affine smooth variety. If Y is a subscheme of affine n-space containing X, then X is the (ideal-theoretic) intersection of Y and $n+1$ hypersurfaces.*

I will now briefly compare these results with others from the literature and say something about the proof of the Main Theorem. In the remainder of this introduction, R will denote a Noetherian local ring with maximal ideal $\mathfrak{m}$. Recall that then $\mu(M) = \mu_R(M)$ is simply the vector space dimension of $M/\mathfrak{m}M$, by Nakayama's Lemma. Already interesting is the case $\mathfrak{a} = \mathfrak{m}$, so that $\mu(\mathfrak{m})$ is the *embedding dimension* $\operatorname{embdim}(R)$ of R. In [1], ABHYANKAR proves the following inequality under the additional assumption that R is Cohen-Macaulay,

$$\operatorname{embdim}(R) \leq \dim(R) + \operatorname{mult}(R) - 1 \qquad (\texttt{[Abh]})$$

where $\dim(R)$ is the Krull dimension of R and $\operatorname{mult}(R)$ the multiplicity of R, that is to say, the multiplicity $\operatorname{mult}(\mathfrak{m})$ of $\mathfrak{m}$ (on R). This was then generalized to arbitrary primary ideals $\mathfrak{a}$ (so that $R/\mathfrak{a}$ has finite length) by SALLY in [7, 8, 9] to

$$\mu(\mathfrak{a}) \leq \dim(R) + \operatorname{nildeg}(R/\mathfrak{a})^{\dim(R)-1} \operatorname{mult}(R) - 1 \qquad (1)$$

where $\operatorname{nildeg}(R/\mathfrak{a})$ denotes the *nilpotency degree* of $R/\mathfrak{a}$, that is to say, the smallest number t such that $\mathfrak{m}^t \subset \mathfrak{a}$.

To obtain results for non-primary Cohen-Macaulay ideals, SALLY reduces to the primary case using superficial elements. In [9, Chapter 5, Theorem 2.3], she shows the existence of the bound

$$\mu(\mathfrak{a}) \leq \operatorname{ht}(\mathfrak{a}) + (\operatorname{mult}(R/\mathfrak{a}))^{\operatorname{ht}(\mathfrak{a})-1} \operatorname{mult}(R) - 1, \qquad (\texttt{[Sal]})$$

where $\operatorname{ht}(\mathfrak{a})$ is the height of the ideal $\mathfrak{a}$. Unfortunately, the exponent $\operatorname{ht}(\mathfrak{a}) - 1$ will often make the bound too large in case $\operatorname{ht}(\mathfrak{a}) > 2$. Moreover, ABHYANKAR's bound (`[Abh]`) is only attained in special situations. Since each of SALLY's bounds specializes to ABHYANKAR's bound (`[Abh]`) when we put $\mathfrak{a} = \mathfrak{m}$, her bounds will in general be too crude. Moreover, they require the Cohen-Macaulay assumption not only on $R/\mathfrak{a}$ but also on R, since ABHYANKAR gives in [1, (5.1)] plenty of examples of non Cohen-Macaulay Noetherian local rings for which (`[Abh]`) fails.

Another upper bound can be found in [4, Theorem 5 and Lemma 6] for R a δ-dimensional local Cohen-Macaulay ring and $\mathfrak{a}$ a primary ideal,

$$\mu(\mathfrak{a}) \leq (\delta! \operatorname{length}(R/\mathfrak{a}))^{1-\frac{1}{\delta}} \operatorname{mult}(R) + \delta - 1. \qquad (\texttt{[BER]})$$

They also point out that this bound is derived essentially from a similar point of view as in SALLY's work.

In contrast, without any Cohen-Macaulay assumption on the ring R, I depart from the following bound (see Theorem 2.2), for $\mathfrak{a}$ a primary ideal,

$$\mu(\mathfrak{a}) \leq (\operatorname{length}(R/\mathfrak{a}) - 2)(\operatorname{embdim}(R) - 1) + 2,$$

provided $\operatorname{length}(R/\mathfrak{a}) > 2$. Moreover, $\mu(\mathfrak{a}) \leq \operatorname{embdim}(R) + 1$ whenever $\operatorname{length}(R/\mathfrak{a})$ is at most 2. The proof is an easy homological argument, using the fact that the first Betti number $\dim_k(\operatorname{Tor}_1^R(R/\mathfrak{a}, k))$ is equal to $\mu(\mathfrak{a})$, where k denotes the residue field of R. The bound in the Main Theorem then follows essentially by the same reduction argument as in the other quoted papers (albeit in the present paper phrased in terms of sufficiently general systems of parameters).

Absolute Bounds versus Relative Bounds

There seems to be a substantial difference in the kind of upper bounds one can expect as the height of the ideal goes up. Height one Cohen-Macaulay ideals are absolutely bounded by [11, Theorem 2.3])–in case R is either Cohen-Macaulay ([3]) or contains a field ([11, Corollary 4.2]), the multiplicity of R serves as an absolute bound. Unfortunately, this phenomenon is not reflected in the bound of the Main Theorem. For height two Cohen-Macaulay ideals absolute bounds still exist under some additional Gorenstein assumption (like fixing the type of the residue ring); see again [11]. The family of MACAULAY space curves with unbounded numbers of defining equations (see for instance [2]), shows that some additional control on the singularities is required. In the following crude hierarchy of local singularities

regular $\implies$ complete intersection $\implies$ Gorenstein $\implies$ Cohen-Macaulay

only the first two admit absolute bounds regardless of the height of the ideal (for complete intersections use Corollary 3.2).[1] Therefore, for arbitrary height two Cohen-Macaulay ideals, some other invariants of the ideal, or, preferably, of the residue ring $R/\mathfrak{a}$ will enter; this is what is meant here with a *relative bound*. If R is Cohen-Macaulay with regularity defect ρ (that is to say, $\rho := \operatorname{embdim}(R) - \dim(R)$), then SALLY's bound (`[Sal]`) gives an estimate of $e \cdot \operatorname{mult}(R) + 1$ on the number of generators of a height two ideal $\mathfrak{a}$ with $e := \operatorname{mult}(R/\mathfrak{a})$, whereas the Main Theorem gives $(e-2)(\rho+1)+2$ (whenever $e > 3$, otherwise we can take $\rho + 1$ as an upper bound). In view of ABHYANKAR's inequality (`[Abh]`), we have that $\rho+1 \leq \operatorname{mult}(R)$, so that the present bound is always as sharp as SALLY's, and in fact, by [9, p. 81, Remark (2)], optimal when R is regular ($\rho = 0$).

The bound in the Main Theorem will in general be much sharper than SALLY's bound (`[Sal]`) when the height is at least three, since it remains linear in the multiplicity of the residue ring. The bound (`[BER]`), albeit only valid for primary ideals in Cohen-Macaulay local rings, is sharper when the length of $R/\mathfrak{a}$ grows big (for R fixed), as it is more sensitive to the growth of the minimal number of generators of powers of ideals.

I would like to thank VASCONCELOS for his valuable comments and especially for drawing my attention to the fact that the dimension of the

[1] I do not know of any natural class of ideals of height three other than the class of complete intersections, which still admits an absolute bound.

residue ring should enter the estimate in the Main Theorem in the way it is now stated.

2. BOUNDS ON COHEN-MACAULAY IDEALS

For the proof of our first estimate, it is more convenient to use the following alternative description of the minimal number of generators in case $(R, \mathfrak{m})$ is a Noetherain local ring with residue field k. Let I be an arbitrary ideal of R. Tensoring the exact sequence

$$0 \to I \to R \to R/I \to 0$$

with k, yields $\mathrm{Tor}_1^R(R/I, k) \cong I/\mathfrak{m}I$. Therefore, by Nakayama's Lemma, the dimension of $\mathrm{Tor}_1^R(R/I, k)$ (the so-called *first Betti number* of R/I) is equal to the minimal number of generators of I.

The following bound is sharp for $\mathfrak{a} = \mathfrak{m}$ and therefore seems preferable to depart from in lieu of ABHYANKAR's bound [Abh].

2.1. LEMMA. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. If $\mathfrak{a}$ is a primary ideal of R, then*

$$\mu(\mathfrak{a}) \leq \mathrm{length}(R/\mathfrak{a}) \cdot (\mathrm{embdim}(R) - 1) + 1.$$

Proof. We will induct on the length l of $R/\mathfrak{a}$. If $l = 1$, then $\mathfrak{a} = \mathfrak{m}$, and the bound follows from the definition of embedding dimension as the minimal number of generators of the maximal ideal. Therefore, assume $l > 1$. Let $a \in R$ be such that its image in $R/\mathfrak{a}$ is a non-zero element of the socle of $R/\mathfrak{a}$. In other words, $a(R/\mathfrak{a}) \cong k$, where k denotes the residue field of R. Tensoring the exact sequence

$$0 \to a(R/\mathfrak{a}) \to R/\mathfrak{a} \to R/(\mathfrak{a} + aR) \to 0$$

with k and using the isomorphism $a(R/\mathfrak{a}) \cong k$, the last six terms of the long exact Tor-sequence are

$$\begin{aligned}\mathrm{Tor}_1^R(k, k) \to \mathrm{Tor}_1^R(R/\mathfrak{a}, k) \to \mathrm{Tor}_1^R(R/(\mathfrak{a} + aR), k) \to \\ k \to k \to k \to 0.\end{aligned}$$

Since the penultimate arrow is an isomorphism, we have in fact an exact sequence

$$\mathrm{Tor}_1^R(k, k) \to \mathrm{Tor}_1^R(R/\mathfrak{a}, k) \to \mathrm{Tor}_1^R(R/(\mathfrak{a} + aR), k) \to k \to 0.$$

Therefore, the dimension of the second vector space is at most the sum of the dimensions of the first and the third vector space minus one. Using the correspondence between first Betti numbers and the minimal numbers of generators, we see that the first vector space has dimension equal to

$\nu := \text{embdim}(R)$, whereas the third vector space has dimension equal to $\mu(\mathfrak{a} + aR)$. Since $R/(\mathfrak{a} + aR)$ has length $l - 1$, our induction hypothesis yields that $\mu(\mathfrak{a} + aR) \le (l - 1)(\nu - 1) + 1$. Therefore, $\mathfrak{a}$ can be generated by at most $(l - 1)(\nu - 1) + 1 + \nu - 1 = l(\nu - 1) + 1$ elements. ∎

Making the observation that $\mu(\mathfrak{a}) \le \text{embdim}(R) + 1$ whenever $R/\mathfrak{a}$ has length at most 3, we obtain the following improvement.

2.2. THEOREM. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. If $\mathfrak{a}$ is a primary ideal of R, then*

$$\mu(\mathfrak{a}) \le (\text{length}(R/\mathfrak{a}) - 2)(\text{embdim}(R) - 1) + 2.$$

provided $\text{length}(R/\mathfrak{a}) > 3$. *In the remaining case, $\mathfrak{a}$ can always be generated by at most* $\text{embdim}(R) + 1$ *elements.*

Proof. Put $\nu := \text{embdim}(R)$ and $l := \text{length}(R/\mathfrak{a})$. I claim that it suffices to prove the last claim. Indeed, suppose we showed that $\mu(\mathfrak{a}) \le \nu+1$ whenever $l \le 3$ (note that both estimates agree when $l = 3$). In the induction in the proof of Lemma 2.1, we basically showed that $\mu(\mathfrak{a})$ is generated by at most $\nu - 1$ more elements than $\mathfrak{a} + aR$, where (the image of) a is a non-zero element of the socle of $R/\mathfrak{a}$. Therefore, if we start our induction hypothesis from $l = 3$, we obtain that $\mu(\mathfrak{a}) \le (l-3)(\nu-1)+\nu+1 = (l - 2)(\nu - 1) + 2$.

So remains to prove the last statement. Suppose $l = 3$ (the case $l = 2$ is even simpler). Let $x_1, \dots, x_\nu$ be a minimal set of generators of $\mathfrak{m}$. Since not all x_i belong to $\mathfrak{a}$, we may assume that $x_1 \notin \mathfrak{a}$. Suppose first that $\mathfrak{a}+x_1R \neq \mathfrak{m}$, so that after renumbering, we may assume that $x_2 \notin \mathfrak{a}+x_1R$. It follows that the following chain of ideals is strict and maximal, that is to say, no ideal can be properly inserted further

$$\mathfrak{a} \subsetneq \mathfrak{a} + x_1R \subsetneq \mathfrak{a} + x_1R + x_2R = \mathfrak{m}. \tag{2}$$

In particular, it follows that for each $i = 3, \dots, \nu$, we can find a linear combination y_i of x_1 and x_2, such that $x_i + y_i \in \mathfrak{a}$. Since $x_1, x_2, x_3 + y_3, \dots, x_\nu + y_\nu$ are also a minimal set of generators of $\mathfrak{m}$, we may replace each x_i by $x_i + y_i$ and assume from the start that $x_3, \dots, x_\nu \in \mathfrak{a}$. Since by Nakayama's Lemma, they are then necessarily part of a minimal system of generators of $\mathfrak{a}$, we showed that $\mu_S(\mathfrak{a}S) = \mu_R(\mathfrak{a}) - \nu + 2$, where $S := R/(x_3, \dots, x_\nu)R$. Therefore, it suffices to prove that $\mu_S(\mathfrak{a}S) \le 3$. However, $\text{embdim}(S) = 2$ and the image of the chain (2) in S is still strict. In particular, since $\mathfrak{a}S + x_1^2S$ is contained in $\mathfrak{a}S + x_1S$, we see that it must be equal to $\mathfrak{a}$. In other words, $x_1^2 \in \mathfrak{a}S$. The same argument shows that $x_1x_2 \in \mathfrak{a}S$,. Moreover, since the chain (2) is also strict if we interchange x_1 and x_2, the same argument also shows that $x_2^2 \in \mathfrak{a}S$. In conclusion, $\mathfrak{m}^2S \subset \mathfrak{a}S$ and since $\text{length}(S/\mathfrak{m}^2S) = l = 3$, this must even be an equality, showing our claim.

In the remaining case that $x_2, \dots, x_\nu$ all belong already to $\mathfrak{a}$, it follows that $\mu(\mathfrak{a}) - \nu + 1 = \mu(\mathfrak{a}R/(x_2, \dots, x_\nu)R = 1$, since $R/(x_2, \dots, x_\nu)R$ has embedding dimension one, so that every ideal is principal. ∎

2.3. THEOREM. *For R a Noetherian local ring and $\mathfrak{a}$ a Cohen-Macaulay ideal in R, we have that*

$$\mu(\mathfrak{a}) \leq (\operatorname{mult}(R/\mathfrak{a}) - 2)(\operatorname{embdim}(R) - \dim(R/\mathfrak{a}) - 1) + 2$$

provided $\operatorname{mult}(R/\mathfrak{a}) > 2$. *In the remaining case, we have that* $\mu(\mathfrak{a}) \leq \operatorname{embdim}(R) - \dim(R/\mathfrak{a}) + 1$.

Proof. Let ν be the embedding dimension of R. Let d and e be respectively the dimension and the multiplicity of the Cohen-Macaulay local ring $R/\mathfrak{a}$. We seek to show that $\mathfrak{a}$ can be generated by at most $(e-2)(\nu-d-1)+2$ elements, provided $e > 2$, and by $\nu-d+1$ elements if $e \leq 2$. For simplicity's sake, I will only deal with the case $e > 2$; the (easier) case $e \leq 2$ follows by the same argument by substituting at the right place the appropriate bound from Theorem 2.2.

We will induct on d. If $d = 0$, then the bound follows from Lemma 2.1, since in this case e is equal to the length of $R/\mathfrak{a}$. Therefore, assume $d > 0$. Since $R/\mathfrak{a}$ is Cohen-Macaulay, we can find an $(R/\mathfrak{a})$-regular element x. For sake of simplicity, I will assume that the residue field k of R is infinite (by some base change $R \to R(T)$ we may reduce to this case). Using [6, Theorem 14.14], any sufficiently general choice of d elements in $\mathfrak{m}R/\mathfrak{a}$, generates a parameter ideal I of $R/\mathfrak{a}$ such that I is a reduction of $\mathfrak{m}$. Note that by [6, Theorem 14.13], e is equal to the multiplicity of the ideal I. Without loss of generality, we may assume that x is one of these sufficiently general elements generating I and, moreover, that $x \notin \mathfrak{m}^2$. It follows from [6, Theorem 14.11], that e is equal to the multiplicity of the ideal $IR/(\mathfrak{a} + xR)$. Let us denote R/xR by $\bar{R}$, so that $\bar{R}$ is again Cohen-Macaulay. Since $I\bar{R}/\mathfrak{a}\bar{R}$ is also a reduction of $\mathfrak{m}\bar{R}/\mathfrak{a}\bar{R}$, we conclude by another application of [6, Theorem 14.13] that $\bar{R}/\mathfrak{a}\bar{R}$ also has multiplicity e. Moreover, $\bar{R}/\mathfrak{a}\bar{R}$ has dimension $d-1$. Since x does not lie in $\mathfrak{m}^2$, it is part of a minimal system of generators of $\mathfrak{m}$, so that $\bar{R}$ has embedding dimension $\nu - 1$. By our induction hypothesis, it follows that $\mathfrak{a}\bar{R}$ is generated by at most $(e-2)(\nu-1-(d-1)-1)+2 = (e-2)(\nu-d-1)+2$ elements. Since x is a non-zero divisor on $R/\mathfrak{a}$ so that $\operatorname{Tor}_1^R(\bar{R}, R/\mathfrak{a}) = 0$, we get from the exact sequence

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0,$$

after tensoring with $\bar{R}$, an exact sequence

$$0 \to \mathfrak{a}/x\mathfrak{a} \to \bar{R} \to \bar{R}/\mathfrak{a}\bar{R} \to 0$$

showing that $\mathfrak{a}/x\mathfrak{a}$ can be identified with $\mathfrak{a}\bar{R}$. Therefore, $\mathfrak{a}/x\mathfrak{a}$ is generated by at most $(e-2)(\nu-d-1)+2$ elements. Finally, Nakayama's Lemma then yields that $\mathfrak{a}$ itself is generated by at most that many elements. ∎

3. THE GLOBAL CASE

Recall the FORSTER-SWAN Theorem proven in [5] (see also [6, Theorem 5.7]).

3.1. THEOREM (FORSTER-SWAN). *Let A be a Noetherian ring and M a finitely generated A-module. For each prime ideal $\mathfrak{p}$ of A, let $f(\mathfrak{p}, M)$ denote the sum of* $\dim(A/\mathfrak{p})$ *and* $\mu_{A_\mathfrak{p}}(M_\mathfrak{p})$. *If N is the maximum of all $f(\mathfrak{p}, M)$ for $\mathfrak{p}$ running over all prime ideals in the support of M, then M can be generated by at most N elements.*

3.2. COROLLARY. *Let A be a δ-dimensional Noetherian ring and $\mathfrak{a}$ an ideal of A. Let N be a bound on the number of generators of each $\mathfrak{a}A_\mathfrak{m}$, where $\mathfrak{m}$ runs over all maximal ideals of A. Then $\mathfrak{a}$ can be generated by at most* $\max\{\delta + 1, N + \dim(A/\mathfrak{a})\}$ *elements.*

Proof. Let $\mathfrak{p}$ be an arbitrary prime ideal of A. If $\mathfrak{a}$ is not contained in $\mathfrak{p}$, then $\mathfrak{a}A_\mathfrak{p} = A_\mathfrak{p}$ is generated by a single element, so that $f(\mathfrak{p}, \mathfrak{a}) = \dim(A/\mathfrak{p}) + 1 \leq \delta + 1$. If $\mathfrak{a} \subset \mathfrak{p}$, then $\dim(A/\mathfrak{p}) \leq \dim(A/\mathfrak{a})$. Choose a maximal ideal $\mathfrak{m}$ of A, containing $\mathfrak{p}$. Since $\mathfrak{a}A_\mathfrak{p}$ is a localization of $\mathfrak{a}A_\mathfrak{m}$, it is generated by at most N elements. The assertion now follows from Theorem 3.1. ∎

3.3. DEFINITION. Let A be a Noetherian ring. We call the *geometric embedding dimension* of A the maximum of the embedding dimensions of the $A_\mathfrak{m}$, where $\mathfrak{m}$ runs over all prime ideals of A and denote it by $\operatorname{embdim}(A)$. Similarly, we define the *geometric multiplicity* of A as the maximum of the multiplicities of each $A_\mathfrak{m}$, where $\mathfrak{m}$ runs over all maximal ideals of A, and we denote it by $\operatorname{mult}(A)$.

Of course, the geometric embedding dimension or the geometric multiplicity may be infinite, but is always finite for finitely generated algebras over a field.

3.4. THEOREM. *Let A be a Noetherian ring. If $\mathfrak{a}$ is a Cohen-Macaulay ideal of A, then*

$$\mu(\mathfrak{a}) \leq (\operatorname{mult}(A/\mathfrak{a}) - 2)(\operatorname{embdim}(A) - \dim(A/\mathfrak{a}) - 1) + \dim(A/\mathfrak{a}) + 2,$$

provided $\operatorname{mult}(A/\mathfrak{a}) > 2$. *In the remaining case,* $\mu(\mathfrak{a}) \leq \operatorname{embdim}(A) + 1$.

Proof. Immediate from Theorem 2.3 and Corollary 3.2. Just observe that each of these bounds is at least $\dim(A) + 1$. ∎

The following special case ($\operatorname{embdim}(A) = 3$ and $\dim(A/\mathfrak{a}) = 1$) deserves separate mentioning; an affine space curve is a pure 1-dimensional subscheme of affine 3-space (whence in particular is Cohen-Macaulay).

3.5. COROLLARY. *Any affine space curve C of multiplicity $e \geq 3$ requires at most $e + 1$ defining equations (ideal-theoretically).*

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