UNIFORM BOUNDS IN ALGEBRAIC GEOMETRY AND COMMUTATIVE ALGEBRA

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Abstract. In this survey article, we will introduce various measures of complexity for algebraic constructions in polynomial rings over fields and show how they are often uniformly bounded by the complexity of the starting data. In problems which have a linear nature, the degree of the polynomials provide a sufficient notion of complexity. However, in the non-linear case, the more sophisticated measure of etale complexity is needed.

These bounds lead often to the constructible nature of geometric problems, where in the non-linear case, one should work in the etale site rather than in the Zariski site. As another application of the existence of these bounds we mention the possibility of transferring results from one characteristic to another by means of the Lefschetz Principle. We will give some examples of new results as well as some new proofs to old results.

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1. Introduction

Constructive Algebra is mostly seen as the theoretical counterpart of Computational Algebra: whereas the latter seeks to describe explicit methods and algorithms, the former merely concentrates on the effective nature of many constructions to be found in algebra. This dichotomy is best understood by an example. In [6], Hermann showed the following.

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1.1. Theorem. *Let K be a field and ξ a finite set of variables. If a polynomial f in $K[\xi]$ of degree d is a linear combination of some other polynomials $g_1, \dots, g_s$ of degree d, then one can find polynomials $q_i \in K[\xi]$ of degree at most d', such that*

$$f = q_1 g_1 + \dots + q_s g_s, \tag{1}$$

where d' does only depend on d and on the number of variables ξ, and not on the particular polynomials f and g_i.

This *effective* result belongs to the field of Constructive Algebra. Using Groebner bases, Buchberger obtained in [2] the same result, but by an explicit description of an algorithm that calculates the polynomials q_i in (1). This lead to a direct implementation into various algebraic software programs, which was not practically feasible in the case of Hermann's explicit proof using elimination theory, in view of the exponential growth of degrees of polynomials involved in this elimination process. Not surprisingly, also (model-theoretic) non-standard proofs for this result have been put forward; we will discuss below the proof of Schmidt and van den Dries in [10]. Such proofs lack even more practical implementation, but they provide sometimes extra information. In the particular case of our example, it comes immediate that the bound d' is even independent of the field. This is not hard by the other methods either, but for more complicated examples, this is no longer that evident.

In this survey paper, we will set ourselves yet another goal. We seek to derive theoretical results from the existence of certain effective results. In other words, we want to derive *pure* results in Algebraic Geometry and Commutative Algebra, that is to say, results which do not refer to any model theoretic, constructive or computational concept. An example is perhaps in order.

1.2. Example. Given a scheme of finite type X over an algebraically closed field K, then there exists for each i a finite constructible partition

$$X = F_1^{(i)} \cup \dots \cup F_s^{(i)} \tag{2}$$

such that on each $F_j^{(i)}$, the i-th Betti number (see (3) below) is constant.

Sometimes, our results will only be pure to a certain extent as shown by the next example.

1.3. Example (Zariski-Lipman Conjecture for Hypersurfaces)**.** For each d, there is a finite set of exceptional characteristics, such that if K is an algebraically closed field of characteristic not in the exceptional set, and such that if x is a closed point on a degree d hypersurface X over K with the property that the module of K-derivations $\mathrm{Der}_K(\mathcal{O}_{X,x})$ is free (where $\mathcal{O}_{X,x}$ is the local ring of x at X), then x is a regular point of X.

Note that we prefer to use the term "scheme" over the more common term "variety", for it emphasizes the fact that we look at *all* points, not just the K-rational (or, *closed*) points–therefore including for instance *generic* points as well. So will $\mathbb{A}^n_B$ denote the affine space over the ring B, that is to say $\operatorname{Spec} B[\xi_1, \dots, \xi_n]$, whereas B^n simply denotes the collection of all n-tuples with entries from B. For $B = K$ a field, K^n is precisely the collection of all K-rational points of $\mathbb{A}^n_K$. More generally, if X is a scheme, then $X(K)$ will denote the collection of K-rational points of X.

Transferring Properties. One of the main tools to derive a pure result such as 1.3 from effective results is the Lefschetz Principle. It enables one to carry over results in positive characteristic to similar results in zero characteristic, and vice versa, at least for problems over algebraically closed fields which are *first order definable*. Let us explain this a little more. The model theoretic Lefschetz Principle states the following.

1.4. Theorem. *If σ is a sentence in the language of rings, then σ is true in any algebraically closed field K of characteristic zero, if and only if, there are infinitely many primes p and, for each such prime p, an algebraically closed field K_p of that characteristic, such that σ holds in K_p.*

In fact, in the above statement, we may replace the condition *infinitely many primes* by *all but finitely many primes*.

In stead of giving a formal definition of a (first order) *sentence* (in the language of rings), let us just give here a geometric interpretation. Let F be a constructible subset (in the Zariski sense) of affine n-space $\mathbb{A}^n_{\mathbb{Z}}$ over the integers. A sentence σ is then a statement (or its negation), of the form $F(K)$ *is non-empty*, where K is an algebraically closed field and $F(K)$ is the subset of K^n of all K-rational points in F (or rather, of the base change of F to K; we will not make this distinction). This is a constructible subset of K^n for the induced Zariski topology. Or, by taking some Boolean operations, a sentence could be a statement of the form *the K-rational points of F and G are the same* (i.e., $F(K) = G(K)$), where G is another constructible subset of $\mathbb{A}^n_{\mathbb{Z}}$. If one allows more general fields than just algebraically closed ones, a more complicated definition is required, by lack of Quantifier Elimination. The geometric interpretation of Quantifier Elimination, is Chevalley's Theorem (*the image of a constructible set under a map of finite type is again constructible*), together with the Nullstellensatz (*any closed point is K-rational*).

Let us explain by means of Example 1.3 how the Lefschetz Principle ties in with the existence of uniform bounds, in order to obtain such a *pure* result. For sake of exposition, let us assume that X from the example is in fact a hypersurface of $\mathbb{A}^n_K$ of degree d. Hence X is given as the zero locus of a single polynomial $f = \sum a_i\xi^i$ of degree d. Let N be the number of monomials of degree at most d in the n variables $\xi = (\xi_1, \dots, \xi_n)$. Hence f is given by listing the N-tuple of its coefficients $a = (a_i)$. Let us denote this by writing $X = X(a)$. We will show that there exists a constructible subset F_d of $\mathbb{A}^{n+N}_{\mathbb{Z}}$, with the property that a K-rational point (x, a) belongs to $F_d(K)$, if and only if, x is a K-rational point on the hypersurface $X = X(a)$ and the module of K-derivations of $\mathcal{O}_{X,x}$ is free. Similarly, there exists a constructible subset G_d of $\mathbb{A}^{n+N}_{\mathbb{Z}}$, such that $(x, a) \in G_d(K)$, if and only if, x is a regular K-rational point on the hypersurface $X = X(a)$. The main point here is that F_d and G_d do not depend on X nor on the field K, but only on the bound d. Let σ_d be the sentence expressing that for an algebraically closed field K, the K-rational points of F_d and G_d are the same. In other words, if each σ_d holds in some algebraically closed field K, then the conclusion in 1.3 is true for the field K. Now, it has been proved by Scheja and Storch in [9], that this is true for K of characteristic zero. Therefore, by an application of the Lefschetz Principle, we obtain our result. Note that the set of exceptional characteristics depends on d, as we must apply Theorem 1.4 to each sentence σ_d separately.

So, how do we prove the existence of these constructible sets F_d and G_d? The conditions that the module of derivations of the local ring $\mathcal{O}_{X,x}$ is free, or that $\mathcal{O}_{X,x}$ is regular, are *algebraic* in nature. For instance, for the second condition, one needs to express the existence of an $(n-1)$-tuple of elements in $\mathcal{O}_{X,x}$ generating its maximal ideal. Since $\mathcal{O}_{X,x}$ is a localization of the coordinate ring $K[\xi]/(f)$, such a tuple can be taken as a tuple of polynomials in $K[\xi]$. However, we cannot just assert in a first order way that an $(n-1)$-tuple of polynomials exists; we need to know what their possible degree is. Remember that a polynomial is encoded as its tuple of coefficients; the length of this tuple being completely determined by the degree. Therefore, we also need that the degrees of the entries of this $(n-1)$-tuple are uniformly bounded in terms of d alone.

To summarize, the existence of uniform bounds for some problem from Algebraic Geometry or Commutative Algebra, entails the definability (i.e., the constructive nature) of the problem and hence allows one to apply the Lefschetz Principle to it, in order to transfer results from one characteristic to another. In Section 7, we will explain how many of these uniform bounds can be derived from Theorem 1.1. The idea is that many constructions in Algebraic Geometry or Commutative Algebra are not only algebraic but even linear. Since the obstruction of extending linear algebra over a field to linear algebra over a ring is measured by the *Ext* and *Tor* cohomology groups, we need to study uniform bounds in that context.

Constructible Sets. Let us now turn to Example 1.2, which is yet another theoretical application of the existence of bounds. Here we use the Compactness Theorem and/or Noetherianity instead of the Lefschetz Principle, to obtain some further uniformity results. Let us explain the problematic and techniques again by means of the example. In a similar fashion as in the previous example, one obtains, for each d and each i and β, a constructible subset $F_d^{(i,\beta)}$ of $\mathbb{A}_{\mathbb{Z}}^{n+N}$, with the property that if K is an algebraically closed field and (x,a) is a K-rational point, then $(x,a) \in F_d^{(i,\beta)}(K)$, if and only if, a is a list of coefficients of a tuple of polynomials in $K[\xi]$ of degree at most d, defining a closed subscheme $X = X(a)$ of $\mathbb{A}_K^n$, and x is a K-rational point of X for which the i-th Betti number equals β. With the *i-th Betti number* of an (arbitrary) point $z \in X$, we mean the dimension of the i-th *Tor* module

$$\beta = \dim \operatorname{Tor}_i^{\mathcal{O}_{X,z}}(k(z), k(z)) \tag{3}$$

where $k(z)$ is the residue field of the point z. As we already said, we will denote the collection of K-rational points of X by $X(K)$. In the terminology of the geometer, these are precisely the closed points of X, and these then correspond by the Nullstellensatz to the maximal ideals of the coordinate ring of X. Therefore, for a fixed i, we get a partition of $X(K)$ in constructible sets

$$\Sigma^{(i,\beta)} := \left\{ x \in X(K) \mid F_d^{(i,\beta)}(x,a) \text{ holds in } K \right\}, \tag{4}$$

where β runs over all natural numbers. Let us denote by $X^{(i,\beta)}$ the constructible subset of X given by the equations and inequalities defining $\Sigma^{(i,\beta)}$. We would like to show that

i. There are only finitely many non-empty strata $\Sigma^{(i,\beta)}$.
ii. We get a (finite) partition by constructible sets $X^{(i,\beta)}$ of the whole scheme X, not just of its K-rational points.

The difference between the topologies of $X(K)$ and X is reflected by the following proposition. As an immediate consequence of it, we obtain a positive answer to Question (`i`), as soon as we have a positive answer to Question (`ii`).

1.5. Proposition. *Let X be a scheme of finite type over a field K and let $\mathcal{S}$ be a partition of X (as a scheme) by constructible sets. Then $\mathcal{S}$ is finite.*

Moreover, we have the following effective version. For each d, there exists a bound D, such that if X has degree complexity at most d and each constructible set $F \in \mathcal{S}$ has degree complexity at most d, then $\mathcal{S}$ has cardinality at most D.

The degree complexity of certain algebraic objects will be defined below in Section 2. For now suffices to give the following intuitive version. An algebraic object has *degree complexity* at most d, if it can be described algebraically using polynomials of degree at most d in at most d variables. For instance, a closed subscheme of $\mathbb{A}^d_K$ given as the zero locus of polynomials of degree at most d, has degree complexity at most d.

Proposition 1.5 is certainly false for $X(K)$, for the partition by all singletons is constructible but not finite. Geometrically, Proposition 1.5 says that a scheme of finite type is not only compact in the Zariski topology, but even in the constructible topology. Model-theoretically, for $X = \mathbb{A}^n_K$ and K algebraically closed, the content of Proposition 1.5 is that X is the space of n-types, which is a Stone space. Moreover, the Stone topology coincides with the constructible topology, in view of Quantifier Elimination. The model theoretic version of Proposition 1.5 is the compactness of this Stone space:

1.6. Theorem (First Order Compactness)**.** *Let φ_i be a sequence of first order formulae in n free variables. If for some (resp. every) algebraically closed field K, we have that*

$$K^n = \bigcup_{i\geq 1} |\varphi_i|^K \tag{5}$$

(as subsets of K^n), then there is some i_0, such that,

$$K^n = \bigcup_{i=1}^{i_0} |\varphi_i|^K. \tag{6}$$

For the general definition of a formula, see page 10 below. With $|\varphi|^K$ we mean all tuples in K which satisfy the formula φ. The following corollary is a scheme theoretic generalization of an observation of van den Dries in [15].

1.7. Corollary. *Let X be a scheme (of finite type over an algebraically closed field K) and let W be a subset of X. Suppose there exist constructible sets F_n and G_n, such that*

$$\bigcup_{n=0}^{\infty} F_n = W = \bigcap_{n=0}^{\infty} G_n. \tag{7}$$

Then W is constructible.

Let us now turn to Question (`ii`). To obtain a partition of the non-K-rational points, we will not work with the type space, but we will use purely algebraic tools. Of particular importance will be the notion of saturatedness: a subset $F \subset K^n$ is called *saturated*, if for every point $x \in F$, we can find a K-rational point y in F

lying in the Zariski closure of x (i.e., y is a *specialization* of x). Any constructible set has this property.

We show that, in the notation introduced just before Question (ii), each $X^{(i,\beta)}$ is saturated. By a general result, see 11.2, the $X^{(i,\beta)}$ then necessarily cover the whole space X. This will be explained more in Section 11, but since there is no model theory used in this part of our work, we will not provide details. They can be found in the paper [13].

Non-linear Problems. The problems so far discussed were all *linear* problems. However, not every algebraic construction is linear. Let us give a simple example: in order to express that two schemes of finite type over an algebraically closed field are isomorphic, we need to give a map together with its inverse. Expressing that two maps are each others inverse, reduces to expressing that the composition of two polynomials yields the identity map modulo some other equations (defining the scheme). This clearly is no longer a linear problem. It is even not immediately clear that it can be expressed by means of polynomial equations. In fact one needs polynomial equations with constraints to express this; see below for more details. In summary, it is not clear whether the following question posed by EKLOF in [5] has an affirmative answer.

1.8. Question. *Does there exists for each d a bound d' with the property that, if X and Y are two isomorphic schemes of degree complexity at most d over an algebraically closed field K, then an isomorphism can already be given using polynomials of degree at most d'.*

A first approach, therefore, would be to try to extend the results of Theorem 1.1 for non-linear equations, i.e., replacing Equation (1) by an equation of the form

$$\sum_{\nu} g_\nu X^\nu = 0 \tag{8}$$

to be solved in the $X = (X_1, \dots, X_m)$-variables by polynomials $q_1, \dots, q_m \in K[\xi]$,. Here ν runs over a finite set of multi-indices and $g_\nu \in K[\xi]$. However, in Section 9 we will present an example due to SCHMIDT and VAN DEN DRIES, showing that no such uniform bounds can exist. One might though hope that uniform bounds exist when working in overrings of $K[\xi]$. Since we want to remain algebraic, the following ring seems a plausible candidate. Let $K[[\xi]]^{\text{alg}}$ be the ring of all formal power series f which are algebraic over $K[\xi]$, i.e., $f \in K[[\xi]]$ satisfying a non-trivial equation

$$a_d f^d + \dots a_1 f + a_0 = 0 \tag{9}$$

with $a_i \in K[\xi]$ and $a_d \neq 0$. Unfortunately, Equation (9) does not uniquely define the algebraic power series f, as several power series might be a solution to it. To avoid this ambiguity in an algebraic characterization of f, we use the following alternative description. The ring $K[[\xi]]^{\text{alg}}$ is the henselization of $K[\xi]$ at the maximal ideal $(\xi_1, \dots, \xi_n)$. This means that $K[[\xi]]^{\text{alg}}$ is the direct limit of all local etale extensions of $K[\xi]_{(\xi_1,\dots,\xi_n)}$. Without giving the full definition, this means in the simplest case that in Equation (9) we may furthermore assume that $f(0)$ is a simple root of the polynomial

$$a_d(0)T^d + \dots + a_1(0) + a_0(0) = 0. \tag{10}$$

One verifies that then f can be uniquely recovered from Equation (9) and from its value $f(0)$ at $\xi = 0$. The minimal degree d for which this can be achieved is then called the *etale complexity* of f. A note of caution: not always does a single equation with a simple root at $\xi = 0$ suffice, but in general a system of equations with invertible Jacobian matrix at $\xi = 0$ is needed.

The key result on uniform bounds in the non-linear case is then the following theorem.

1.9. Theorem. *For each d, there exists a d', with the following property. Let $F_1(\xi, X), \dots, F_s(\xi, X)$ be polynomials over a field K of total degree at most d and suppose that there exists a tuple $f = (f_1, \dots, f_m)$ over $K[[\xi]]$, such that*

$$F_1(\xi, f(\xi)) = \cdots = F_s(\xi, f(\xi)) = 0. \tag{11}$$

Then there already exists such a tuple f over $K[[\xi]]^{alg}$ with each f_i of etale complexity at most d'.

Let me briefly comment on its proof. In Section 2, we will explain how Theorem 1.1 follows from non-standard methods and from the algebraic fact that the extension

$$K^*[\xi] \subset K[\xi]^* \tag{12}$$

is faithfully flat. Here, at the left, we have the polynomial ring over the ultraproduct K^* (with respect to some fixed non-principal ultrafilter) of fields $K^{[\nu]}$, and at the right, we have the ultraproduct of the polynomial rings $K^{[\nu]}[\xi]$. Faithful flatness is tightly connected to solvability of linear equations. In order to deal with a non-linear situation, we need to replace faithfully flatness by its non-linear counterpart. This turns out to be Artin Approximation, which says that the extension

$$K[[\xi]]^{\mathrm{alg}} \subset K[[\xi]] \tag{13}$$

is *existentially closed*. This model-theoretic concept translates, in this particular case, into the algebraic fact that a system of polynomial equations with coefficients in $K[[\xi]]^{\mathrm{alg}}$ which is solvable in $K[[\xi]]$, is already so in $K[[\xi]]^{\mathrm{alg}}$. (In fact, we need Artin Approximation with constraints, and of the latter, its non-standard version; for some details see Section 9 below).

Geometrically, the effect of working with etale complexity rather than with degree complexity, amounts to abandoning the Zariski site in favor of the etale site. Therefore, the original Question 1.8 has to be modified: the notion of isomorphism has to be taken in the sense of etale topology, not in the sense of Zariski topology. This amounts to a much more local (infinitesimal) notion of isomorphism. However, with this adaptation, we can answer Question 1.8 in the affirmative, see Theorem 10.3 below.

Complete Intersections. Finally, we want to mention an open problem on uniform bounds, which would solve an outstanding open problem on space curves over the complex numbers, as already observed by VAN DEN DRIES in [15].

1.10. Question. *Does there exist for each d a bound d' with the following property? If $\mathfrak{a}$ is a radical ideal generated by polynomials of degree at most d over some field K in $n \leq d$ variables ξ, and if there exist polynomials $f_1, \dots, f_s$, with $s \leq d$, such that the radical of the ideal generated by the f_i equals $\mathfrak{a}$, then we can already find such polynomials f_i of degree at most d'.*

Apply this to a height one prime ideal $\mathfrak{p}$ with $n = 3$ and $s = 2$. This prime ideal $\mathfrak{p}$ then defines a (reduced and irreducible) curve in $\mathbb{A}^3_K$. The condition on the radical means that this curve can be realized, set theoretically, as the intersection of two surfaces. It is not known whether every curve in $\mathbb{A}^3_{\mathbb{C}}$ is, as a set, the intersection of two surfaces, but it is known over algebraically closed fields of positive characteristic, by a theorem in [3] of Cowsik and Nori. Therefore, a positive solution to Question 1.10, would imply the definability of the collection of all curves of bounded degree complexity which are set-theoretically the intersection of two surfaces and whence by the Lefschetz Principle (as explained above), one would obtain that every curve in $\mathbb{A}^3_{\mathbb{C}}$ is, as a set, the intersection of two surfaces. In view of the non-linear nature of the problem (since we have to deal with radicals), I would propose the following weakening of the conjecture.

1.11. Question. *Is every curve singularity formally a set-theoretical complete intersection? In particular, is every curve isomorphic in the etale topology with a set-theoretical intersection of two surfaces?*

The term *formally* refers here to the completion of the local ring of the singularity.

2. Asymptotically Definable Functors

In this section we give some more rigorous definitions for what we will understand as *first order definability*.

2.1. Definition. Fix some field K and let Ω be some set. We seek to encode Ω in K. This can be done most easily if we assume that K is also algebraically closed. Moreover, we want this construction also to be functorial in the field. This amounts in letting Ω be a functor rather than a set. More precisely, let $\underline{ACF}$ denote the category of algebraically closed fields and let Ω be a functor from $\underline{ACF}$ to the category of sets $\underline{Set}$. If Θ is another functor from $\underline{ACF}$ to $\underline{Set}$ and $\eta\colon \Omega \to \Theta$ a natural transformation, then we call η *injective* (respectively, *surjective*), if $\eta_d(K)$ is, for any algebraically closed field K (and in the injective case, we call Ω a *subfunctor* of Θ and denote this by $\Omega \subset \Theta$). We call an ascending chain of subfunctors

$$\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega \tag{14}$$

a *filtration*, if $\Omega(K)$ equals the union of the $\Omega_d(K)$, for any algebraically closed field K. Let us call Ω together with a filtration (14) *asymptotically definable*, if for each d, there exists a constructible set $V_d \subset \mathbb{A}^{n_d}_{\mathbb{Z}}$ and a surjective natural transformation $\eta_d\colon V_d \to \Omega_d$. Here an arbitrary constructible set $W \subset \mathbb{A}^n_{\mathbb{Z}}$ is viewed as a functor from $\underline{ACF}$ to $\underline{Set}$, by associating to an algebraically closed field K the K-rational points $W(K)$ of W. Before introducing more terminology, let us pause to give some examples.

2.2. Example. Let Ω and Θ be asymptotically definable functors. The *product functor* $\Omega \times \Theta$ sends an algebraically closed field K to the product $\Omega(K) \times \Theta(K)$. It is again an asymptotically definable functor.

2.3. Example. Let Ω be the functor assigning to an algebraically closed field K the polynomial ring $K[\xi]$, where $\xi = (\xi_1, \dots, \xi_n)$ are a fixed set of variables. Let $\Omega_d(K)$ be the collection of all polynomials f of degree at most d. Hence there exist

$a_\nu \in K$, such that

$$f = \sum_{|\nu| \le d} a_\nu \xi^\nu. \tag{15}$$

Let V_d be the whole affine space $\mathbb{A}_{\mathbb{Z}}^{n(d)}$, where $n(d)$ is the number of distinct monomials of degree at most d in the variables ξ and let $\eta_d(K)$ be the map associating to an $n(d)$-tuple $a = (a_\nu)_\nu$ over K the polynomial (15). It is clear that this yields an asymptotic definition of Ω. In other words, polynomials f are encoded by their coefficient tuples a.

2.4. Example. More generally, let Ω be the functor associating to an algebraically closed field K the collection of all ideals of the polynomial ring $K[\xi]$ and let $\Omega_d(K)$ be the subcollection of all ideals $\mathfrak{a}$ generated by polynomials of degree at most d. We will simply paraphrase the latter condition by saying that $\mathfrak{a}$ has *degree complexity* at most d. To show that this filtration by degree complexity yields an asymptotic definition, we simply need to take for V_d again an affine space, this time of dimension $n(d) \times n(d)$, and let η_d be the $n(d)$-fold product of the map defined in Example 2.3, composed with the map sending $n(d)$ polynomials to the ideal they generate. In other words, ideals are now encoded by a bunch of tuples $\{a_1, \dots, a_{n(d)}\}$, where each tuple a_i encodes a generator of the ideal.

2.5. Example. Continuing in this way, we might take for Ω the functor associating to an algebraically closed field K the collection of all finitely generated $K[\xi]$-modules and let $\Omega_d(K)$ be the subcollection of all finite $K[\xi]$-modules M which admit a representation

$$K[\xi]^s \xrightarrow{\mathbb{A}\times} K[\xi]^s \to M \to 0 \tag{16}$$

where $s \le d$ and $\mathbb{A}$ is an $(s \times s)$-matrix over $K[\xi]$, with all its entries of degree at most d. This time η_d will be the $(d \times d)$-fold product of Example 2.3, followed by the map sending $d \times d$ polynomials to the cokernel of the matrix given by these polynomials.

2.6. Definition. From the above examples we see that giving a filtration (15) is equivalent with giving a notion of *complexity* on Ω. That is to say, any natural transformation $c\colon \Omega \to \mathbb{N}$ (where $\mathbb{N}$ is viewed as the constant functor $K \mapsto \mathbb{N}$) yields a filtration $\Omega_d(K) = c(K)^{-1}[0, d]$ and conversely, every filtration determines a complexity degree c by declaring $c(K)(x) \le d$, if $x \in \Omega_d(K)$. Therefore encodings often come to us via some complexity degree.

The filtrations in examples 2.2–2.5 might seem randomly chosen and indeed other (natural or not) choices can be made. However, what will matter in the sequel is that once a complexity fixed, it will be important to study in how far the further structure on the $\Omega_d(K)$ is compatible with this filtration. In this light, the two following notions are essential.

Let Θ be a subfunctor of Ω. We say that Θ is an *asymptotically definable subfunctor*, if there exist, for each d, a constructible subset W_d of V_d, such that

$$\eta_d(K)\big(W_d(K)\big) = \Theta(K) \cap \Omega_d(K), \tag{17}$$

for any algebraically closed field K. Let Ω and Θ be functors with a filtration and let $\eta\colon \Omega \to \Theta$ be a natural transformation. We call η *bounded*, if, for any d, there exists d', such that $\eta(K)$ maps $\Omega_d(K)$ inside $\Theta_{d'}(K)$, for any algebraically closed field K. For a trivial example, let Ω be as in Example 2.3, then the functor assigning to K the collection of all irreducible polynomials is an asymptotically definable subfunctor and any partial derivative gives a bounded transformation.

In the sequel, we might sometimes ignore the functorial character, and simply say that the polynomial ring is an asymptotically definable *set* and the set of irreducible polynomials is an asymptotically definable *subset*.

A less trivial example is the following.

2.7. Example. Let Π be the functor which assigns to an algebraically closed field K the set $\operatorname{Spec} K[\xi]$ of all prime ideals of $K[\xi]$. This is an asymptotically definable subfunctor of the functor in Example 2.4.

In order to prove this, one needs the following two facts taken from [10].

2.8. Theorem. *For each d, there exists d', with the following properties. Let K be an arbitrary field and let $\xi = (\xi_1, \dots, \xi_n)$ be a tuple of variables with $n \leq d$.*

iii. *If $f_i \in K[\xi]$, for $i < s$, have degree at most d and f_0 belongs to the ideal generated by the f_i, then there exist $q_i \in K[\xi]$ of degree at most d', such that*

$$f_0 = q_1 f_1 + \dots + q_s f_s. \tag{18}$$

iv. *An ideal $\mathfrak{p}$ of $K[\xi]$ of degree complexity at most d (in the sense of 2.4), is prime, provided for any $f, g \in K[\xi]$ of degree at most d' with $f, g \notin \mathfrak{p}$, implies $fg \notin \mathfrak{p}$.*

Formulae. Before we turn to explaining how Theorem 2.8 yields asymptotic definability of the set (functor) Π of prime ideals defined in Example 2.7, we introduce some more terminology. The reason we need to introduce formulae, is that the collection of constructible sets over a non-algebraically closed field behaves badly with respect to projection (due to the absence of Chevalley's Theorem). For instance, the parabola in $\mathbb{R}^2$ given by $x = y^2$ projects onto the interval $[0, +\infty)$ in the x-axis, which is clearly not a constructible set.

A constructible set V in K^n, where K is an algebraically closed field, is a finite union of locally closed subsets V_i, each of which is given by some equations $f_1^{(i)} = \dots = f_s^{(i)} = 0$ and an inequality $f_0^{(i)} \neq 0$. In other words, for a K-rational point $x \in K^n$, to lie in V is equivalent with

$$\bigvee_i (f_1^{(i)}(x) = \dots = f_s^{(i)}(x) = 0 \wedge f_0^{(i)}(x) \neq 0). \tag{19}$$

An expression (19) is an example of a *formula*. To be more precise, it is a formula with *parameters* (namely the coefficients of the $f_j^{(i)}$) from K in the (free) variables $x = (x_1, \dots, x_n)$. If the coefficients come from a smaller field K_0, then we say that the formula has parameters from K_0. In particular, if K_0 is the prime field, or, equivalently, if the coefficients come from $\mathbb{Z}$, then we express this by saying that (19) is a formula *without parameters*. The latter collection of formulae will be of main interest to us and therefore we adopt the convention that *formula* will mean *formula without parameters*, unless we explicitly mention its parameters.

However, there exist formulae more complex than (19). Namely, we can add some quantifiers to the formula. So will a formula

$$(\exists y)[\bigvee_i (f_1^{(i)}(x,y) = \dots = f_s^{(i)}(x,y) = 0 \wedge f_0^{(i)}(x,y) \neq 0)] \tag{20}$$

also be a statement about tuples x. If we interpret (20) over an algebraically closed field K, then its solution set (i.e., all the tuples $x \in K^n$ satisfying the formula), corresponds to the projection of the locally closed set in K^{n+m} given by

$$\bigvee_i (f(i)_1(x,y) = \cdots = f(i)_s(x,y) = 0 \wedge f(i)_0(x,y) \neq 0). \tag{21}$$

By Chevalley's Theorem this is again a constructible set and whence given by a formula as (19), i.e., without quantifiers. However, if K is not algebraically closed then this is no longer true as the above example over the reals shows. Therefore formulae as (19) are called *quantifier free*. A general formula $\varphi(x)$ in the variables x is then of the form: a quantifier free formula in variables x, y preceded by a bunch of quantifiers $(\exists y_i)$ or $(\forall y_i)$.

Formulae, unfortunately, are no longer functors as schemes are, at least not if we allow arbitrary morphisms. However, we have still a base change property in the following sense. Let A be a ring, then we can interpret (19) as a set of equations and inequalities over A and thus it defines a subset of A^n. For an arbitrary formula φ in n free variables, we have a map from the category of rings to the category of sets by assigning to a ring A the *solution set* $|\varphi|^A$ of φ in A^n given by all tuples $a \in A^n$ satisfying the formula φ. This means that whenever we have a quantifier $(\exists y)$ or $(\forall y)$ occurring in φ, then we interpret this as meaning $(\exists y \in A)$ and $(\forall y \in A)$ respectively (which are just formal renderings of the more informal *there exists some y in A* and *for all y in A*, respectively). Two formulae φ and ψ in n free variables are said to be equivalent over a ring A, if they define the same subset in A^n, i.e., if $|\varphi|^A = |\psi|^A$. Note that this might depend on the ring: for instance the formulae $\varphi(x) =: (\exists y) x = 1 \wedge y^2 + 1 = 0$ and $\psi(x) =: (x = x + 1)$ both define the empty set over $\mathbb{R}$ but the former defines the singleton $\{1\}$ over $\mathbb{C}$ whereas the latter still defines the empty set.

The reason why we cannot call a formula a functor, is that it is not compatible with arbitrary morphisms. For instance, the formula $\varphi(x)$ given as

$$(\forall y)[x + y^2 \neq 1] \tag{22}$$

defines the interval $1 < x$ in $\mathbb{R}$, but defines the empty set over $\mathbb{C}$, so that the embedding $\mathbb{R} \subset \mathbb{C}$ does not give a map $|\varphi|^{\mathbb{R}} \to |\varphi|^{\mathbb{C}}$. However, one can make a formula into a functor by allowing fewer morphisms. One should take the category of rings with homomorphisms the *elementary embeddings*. These are precisely the morphism $A \to B$ sending $|\varphi|^A$ into $|\varphi|^B$, for all formulae φ. Unfortunately, there tend to be in general very few elementary embeddings. It is a happy fact that any embedding of algebraically closed fields is elementary. Of course, on the category $\underline{ACF}$ any formula is representable by a constructible set as already explained. The only other elementary embeddings we will encounter here are the diagonal maps $A \to A^*$ of a ring into its ultrapower, to be defined below.

A very special instance of a formula is one where there are no free variables. Such a formula φ is called a *sentence* by the model theorists. According to our setup, this defines, for a ring A, a subset in $A^0 = 0$. In other words, either $|\varphi|^A$ is empty, in which case we say that φ *is false* or *does not hold* in A, or $|\varphi|^A$ is the singleton $\{0\}$, in which case we say that φ *is true* or *holds* in A. Sentences will play a crucial role in transferring properties from one ring to another.

Proof of the asymptotic definability of Π. Firstly, let us show that the subset of pairs $(f, \mathfrak{a})$ with $f \in \mathfrak{a}$ is an asymptotically definable subset of the product of $K[\xi]$ with the collection of all ideals of $K[\xi]$. Here we will suppress any reference to the field and simply pretend that we work over a fixed field K, the reader should check that all constructions are indeed functorial. Let $f_0 \in (f_1, \dots, f_s)$, where $f_i \in K[\xi]$ have degree at most d. Let a_i be the $n(d)$-tuple of coefficients giving f_i. In other words, $\eta_d(a_i) = f_i$, where we took η_d from Example 2.3. The condition that $f_0 \in (f_1, \dots, f_s)$ can now be expressed by the formula

$$(\exists y_i)\eta_d(a_o) = \sum_{i=1}^{s} \eta_{d'}(y_i)\eta_d(a_i). \tag{23}$$

Strictly speaking this is not yet a formula, as the $\eta_d(x)$ denote polynomials. However, addition and multiplication of polynomials are easily expressible in terms of their coefficients and this is how we will interpret Formula (23). This loose way of writing down formulae is common practice and we will happily adopt this convention. Even more, we could altogether suppress the coefficient map η_d and simply restate Formula (23) by

(24) *there exist polynomials q_i of degree at most d', such that $f_0 = \sum q_i f_i$,*

and we could then leave it up to the reader to verify that this can indeed be expressed by a formula. At any rate, as the Formula (23) defines a constructible subset (in the a_i), our claim follows.

To accomplish the task of writing down a formula (and whence giving a constructible subset) for expressing that an ideal $\mathfrak{a}$ of degree complexity at most d is prime, we simply translate Condition (`iv`) in a formula

(25) *if two polynomials f, g of degree at most d' do not belong to $\mathfrak{p}$, then neither does their product fg,*

where of course ideal membership and multiplication of polynomials should be replaced by their respective defining formulae in the tuples defining f, g and $\mathfrak{p}$. ■

In the sequel we will often just provide the necessary bounds as in Theorem 2.8 which are essential to write formulae such as (23) and (25), and leave to the diligent reader the details of the corresponding proof.

3. Ultraproducts

In this section we will describe an algebraic construction which turns out to be very useful in transferring results from one ring to another ring. Strangely enough this construction is not well known among algebraists.

3.1. Definition. A *filter* $\mathcal{U}$ on $\mathbb{N}$ is a collection of non-empty subsets of $\mathbb{N}$, closed under intersection and oversets. In other words, $\mathcal{U}$ is a filter if, for $I, J \subset \mathbb{N}$, we have that

`v.` $\emptyset \notin \mathcal{U}$;
`vi.` $I, J \in \mathcal{U}$ then also $I \cap J \in \mathcal{U}$;
`vii.` $I \subset J$ and $I \in \mathcal{U}$, then also $J \in \mathcal{U}$.

A filter $\mathcal{U}$ is called an *ultrafilter*, if it is maximal among all filters (with respect to inclusion). Alternatively, if the converse of (`vi`) holds, i.e., `viii`. $I \cap J \in \mathcal{U}$, if and only if, $I, J \in \mathcal{U}$.

This is also equivalent with the condition that for every subset I of $\mathbb{N}$, either I or its complement belongs to $\mathcal{U}$. An ultrafilter is called *non-principal*, if it does not contain a singleton. This turns out to be equivalent with $\mathcal{U}$ containing all co-finite subsets.

In the sequel, fix once and for all a non-principal ultrafilter $\mathcal{U}$. With aid of this ultrafilter, we will construct a new ring A^* from any sequence of rings $A^{[m]}$, where $m \in \mathbb{N}$, in such a way that the new ring A^* *shares many of the properties which most of the rings $A^{[m]}$ have in common*. We will make this statement more precise in a minute, but let us first give the construction. A crude way of encoding all rings $A^{[m]}$ is just by taking their direct product $\prod_m A^{[m]}$. However, if all $A^{[m]}$ are fields, then this property is definitely lost by taking their direct product, whereas this is surely one of the properties which we would very much like to preserve. Therefore we define an equivalence relation on $\prod A^{[m]}$ by calling two sequences $(a^{[m]})_m$ and $(b^{[m]})_m$ equivalent modulo $\mathcal{U}$, where $a^{[m]}, b^{[m]} \in A^{[m]}$, if the set of indices on which they agree belongs to $\mathcal{U}$. Facts such as this, namely that

$$\left\{ m \in \mathbb{N} \mid a^{[m]} = b^{[m]} \right\} \in \mathcal{U} \tag{26}$$

will be loosely expressed by saying that *for almost all m*, we have that $a^{[m]} = b^{[m]}$. It follows from (`viii`) that if two properties each separately hold for almost all m, then they also hold jointly for almost all m and conversely. This justifies our informal terminology. We let now A^* be the collection of equivalence classes $[a^{[m]}]$ of sequences $(a^{[m]})_m$ in $\prod A^{[m]}$ and call A^* the *ultraproduct* of the $A^{[m]}$. The (pointwise) addition and multiplication in $\prod A^{[m]}$ is compatible with this equivalence relation and whence A^* is again a ring. In fact, if all (or almost all) rings $A^{[m]}$ are fields (respectively, domains, local rings) then so is A^*. This follows from the following theorem on preservation of properties in an ultraproduct, for which the model theoretic formalism proves once more to be very useful.

3.2. Theorem (Łos). *Let φ be a sentence and let $A^{[m]}$ be a sequence of rings. Let A^* denote their ultraproduct (relative to some non-principal ultrafilter). Then φ holds in A^*, if and only if, φ holds in $A^{[m]}$, for almost all m.*

The following corollary of Łos's Theorem will be be extremely important for the present work. It is generally called the Lefschetz Principle (although some authors have an even stronger principle in mind when they refer to the Lefschetz Principle). Of course, it is just Theorem 1.4 reformulated in the language of ultraproducts.

3.3. Theorem (Lefschetz Principle). *Let $\mathbb{F}_p^{alg}$ denote the algebraic closure of the p element field $\mathbb{F}_p$. Let $\mathcal{U}$ be a non-principal ultrafilter on the set of primes (enumerate the primes to get an ultrafilter on $\mathbb{N}$). Then the ultraproduct of the $\mathbb{F}_p^{alg}$ is isomorphic to the field of the complex numbers.*

Proof. Let $\mathbb{F}^*$ denote the ultraproduct of the $\mathbb{F}_p^{\text{alg}}$. By Theorem 3.2 this must be a field. Moreover, since any prime l is non-zero in almost all $\mathbb{F}_p^{\text{alg}}$, it follows that $\mathbb{F}^*$ must have characteristic zero. It is an easy exercise to express by means of a sentence $\texttt{Root}_d$ that any polynomial of degree d admits a root. As $\texttt{Root}_d$ holds in any $\mathbb{F}_p^{\text{alg}}$, it must also hold in $\mathbb{F}^*$, again by Theorem 3.2. Therefore $\mathbb{F}^*$ is algebraically closed. A cardinality argument shows that $\mathbb{F}^*$ has the cardinality of the continuum. The statement now follows from STEINITZ' Theorem that any two algebraically closed fields of the same characteristic and the same uncountable cardinality must be isomorphic. ■

It is an easy exercise to write down the appropriate sentences expressing that a ring A is a field, a domain or a local ring. For instance for the latter property, one needs to express that the sum of any two non-units is again a non-unit. Hence the sentence

$$\texttt{Local} = (\forall x, y)[(\forall z)[xz \neq 1 \wedge yz \neq 1] \to (\forall z)[(x+y)z \neq 1]], \tag{27}$$

will do. In particular, the ultraproduct of local rings is again local. More generally, suppose $\mathfrak{a}^{[m]}$ is an ideal in $A^{[m]}$, for each m, then the collection of $[a^{[m]}]$ with $a^{[m]} \in \mathfrak{a}^{[m]}$, for almost all m, is an ideal in A^* and will denoted by $\mathfrak{a}^*$. In fact, one can take ultraproducts not only of rings, but of any sequence of first order structures. In particular, $\mathfrak{a}^*$ is then nothing else than the ultraproduct of the ideals $\mathfrak{a}^{[m]}$. Using Theorem 3.2, the reader can check that almost all $\mathfrak{a}^{[m]}$ are prime (respectively, radical or maximal), if and only if, $\mathfrak{a}^*$ is.

A special but non-trivial instance of the construction of an ultraproduct is when all rings $A^{[m]}$ are the same, say, $A^{[m]} = A$. The corresponding ultraproduct A^* is then called an *ultrapower* and it is a field, domain, etc., if and only if, A is. There is an obvious (diagonal) embedding $A \hookrightarrow A^*$ by identifying $a \in A$ with the image of the constant sequence $a^{[m]} = a$ in A^*. This map is never an isomorphism, as A^* will be much larger than A. However, it follows from Theorem 3.2 that this embedding is elementary.

An example is in order to show the limitations of Theorem 3.2. Let $K^{[m]}$ be a sequence of fields and fix some set of variables $\xi = (\xi_1, \dots, \xi_n)$. Let $K^{[m]}[\xi]$ be the corresponding sequence of polynomial rings and let $K[\xi]^*$ be their ultraproduct. There is no way that we can express by means of a sentence that a ring is a polynomial ring over some field and hence $K[\xi]^*$ will not be a polynomial ring. In fact, in general ultraproducts tend to be very complicated, almost always non-Noetherian and often intractable rings; the field case is a fortunate exception. How then can they be put to use? For our purposes two descent techniques will be crucial. The second one will be discussed in Section 9. The first one was observed by van den Dries (see for instance [15]) and used in [10] to obtain the bounds we already mentioned. Keeping notation as before, let K^* be the ultraproduct of the fields $K^{[m]}$. Then inside $K[\xi]^*$ lives K^* and also the elements ξ_i. Therefore, even the polynomial ring $(K^*)[\xi]$ lies inside $K[\xi]^*$. The crucial algebraic fact proved by van den Dries about this embedding is the following result.

3.4. Theorem ([15])**.** *Let $K^{[m]}$ be a sequence of fields with ultraproduct K^*. Let $K[\xi]^*$ be the ultraproduct of the sequence $K^{[m]}[\xi]$, where $\xi = (\xi_1, \dots, \xi_n)$. Then the extension of rings*

$$K^*[\xi] \subset K[\xi]^* \tag{28}$$

is faithfully flat.

This result, which a priori seems a result on ultraproducts, translates into the following result on bounds in polynomial rings.

3.5. Theorem. *For each d, there exists a bound d' with the following property. Let K be a field and let $\xi = (\xi_1, \dots, \xi_n)$ be variables with $n \leq d$. Let f_i, f_{ij} be* *polynomials over K in the variables ξ and let $Y = (Y_1, \dots, Y_s)$ be an extra set of variables. If the linear system of equations*

$$\begin{cases} f_1 = f_{11}Y_1 + \dots f_{1s}Y_s \\ f_2 = f_{21}Y_1 + \dots f_{2s}Y_s \\ \quad\vdots \\ f_t = f_{t1}Y_1 + \dots f_{ts}Y_s \end{cases} \tag{29}$$

has a solution $q = (q_1, \dots, q_s)$ over $K[\xi]$, then it has already a solution q with all q_i of degree at most d'. Moreover, if all the f_i are identically zero, then any solution of this (homogeneous) system of equations is a linear combination of solutions of degree at most d'.

4. Modules and Schemes

4.1. Definition. In this section we will study the effective nature of the module theory over (a quotient of) a polynomial ring. The following terminology will be in order for the rest of this paper. We fix an algebraically closed field K (of arbitrary characteristic). An *affine (K-)algebra* will be a quotient of the polynomial ring $K[\xi]$, where $\xi = (\xi_1, \dots, \xi_n)$ will be a fixed set of variables.

We say that an affine algebra A has *degree complexity* at most d, if A can be written as the quotient of a polynomial ring $K[\xi]$ in at most d variables modulo an ideal generated by polynomials of degree at most d, i.e.,

$$A = \frac{K[\xi]}{(f_1, \dots, f_s)} \tag{30}$$

with $f_i \in K[\xi]$ of degree at most d (and the number of variables is also at most d). This notion of complexity gives at once the following.

4.2. Example. The functor which assigns to an algebraically closed field K the the collection of all affine K-algebras, is asymptotically definable.

An ideal $\mathfrak{a}$ in A is said to have *degree complexity* at most d, if A has degree complexity at most d and for the representation (30) exhibiting this fact, we can find $F_i \in K[\xi]$ of degree at most d, such that $\mathfrak{a} = (F_1, \dots, F_t)A$. Extending Example 2.4 we get.

4.3. Example. The functor which assigns to an algebraically closed field K the the collection of all pairs $(A, \mathfrak{a})$, where A is an affine K-algebra and $\mathfrak{a}$ an ideal in A, is asymptotically definable.

With an *affine local (K-)algebra*, we mean a localization of an affine algebra with respect to a prime ideal. We say that its *degree complexity* is at most d, if the prime ideal has degree complexity at most d. From Example (2.7), we obtain.

4.4. Example. The functor which assigns to an algebraically closed field K the the collection of all affine local K-algebras, is asymptotically definable. In fact, it is an asymptotically definable subfunctor of the functor from Example 4.3, after identifying a pair $(A, \mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal, with the localization $A_{\mathfrak{p}}$.

A K-algebra morphism $\phi\colon A \to B$ of affine K-algebras has *degree complexity* at most d, if we can find a representation (30) for A and a similar one

$$B = \frac{K[\zeta_1, \dots, \zeta_m]}{(g_1, \dots, g_s)} \tag{31}$$

for B, such that ϕ is given by $\xi_i \mapsto H_i(\zeta)$ and such that $n, m \leq d$ and all polynomials f_i, g_i and H_i have degree at most d. (In particular A and B have degree complexity at most d). We obtain:

4.5. Example. The functor which assigns to an algebraically closed field K the the collection of all triples (A, B, ϕ), where A and B are affine K-algebras and ϕ a K-algebra morphism between them, is asymptotically definable.

We have already discussed finitely generated modules over $K[\xi]$ in Example (2.5). Without any effort, one can extend this to a finitely generated module M over an arbitrary affine (local) algebra A as follows. The *degree complexity* of M is at most d, if A has degree complexity at most d and M is isomorphic to the cokernel of a matrix $\mathbb{A}$ over A of dimensions at most d and with entries (images of) polynomials of degree at most d.

4.6. Example. The functor which assigns to an algebraically closed field K the the collection of all pairs (A, M), where A is an affine K-algebra and M a finitely generated A-module, is asymptotically definable. Similarly, we could take pairs (A, M) where this time A is an affine local algebra. We will make no distinction in notation between these two functors.

Schemes. Below, we want to translate our results in a more geometric setting. In particular, we will make use of the language of schemes. With a *scheme* we will always mean a scheme of finite type over an algebraically closed field. Locally, such a scheme X looks like an *affine scheme* $\operatorname{Spec} A$, with A an affine algebra. We will say that the affine scheme $\operatorname{Spec} A$ has *degree complexity* at most d, if A has degree complexity at most d. Note that by Example 2.7, each affine scheme is an asymptotically definable set. If V is a closed subscheme of $\operatorname{Spec} A$, then we say that its *degree complexity* is at most d, if the ideal $\mathfrak{a}$ of A defining V (we denote this by $V = \mathrm{V}(\mathfrak{a})$), has degree complexity at most d. Note that V itself is an affine scheme $\operatorname{Spec} A/\mathfrak{a}$ and that the two notions of degree complexity, namely as a subscheme or as an affine scheme, coincide. The underlying set $|V|$ of a closed subscheme is called a *(Zariski) closed subset*. One can put different scheme structures on such a closed subset $|V|$ where $V = \mathrm{V}(\mathfrak{a})$, corresponding to various ideals $\mathfrak{b}$ having the same radical as $\mathfrak{a}$. In particular, the *induced reduced subscheme structure* is given by taking as defining ideal the radical $\operatorname{rad}\mathfrak{a}$ of $\mathfrak{a}$. In [10] it is shown that the degree complexity of the radical is uniformly bounded in terms of the degree complexity of the ideal.

We say that a Zariski open U of $\operatorname{Spec} A$ has degree complexity at most d, if its complement, with the induced reduced subscheme structure, is a closed subscheme of degree complexity at most d. There is a small ambiguity here: if U itself is an affine scheme, then both notions of degree complexity might not coincide, but at least one is uniformly bounded in terms of the other. This is an immediate consequence of our observation on the degree complexity of the radical in the previous paragraph. Such ambiguities might arise at other occasions as well, but as long as we have mutual uniform boundedness, we do not care. A map $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$

has *degree complexity* at most d, if the corresponding K-algebra morphism $A \to B$ has degree complexity at most d.

Now, a general scheme X (remember, always of finite type), is completely determined by giving data (U_i, U_{ij}, f_{ij}), where $1 \leq i \neq j \leq t$, with $U_i = \operatorname{Spec} A_i$ an affine scheme, $U_{ij} = \operatorname{Spec} A_{ij}$ an open affine subscheme of U_i and $f_{ij} \colon U_{ij} \to U_{ji}$ an isomorphism with inverse f_{ji}. Namely, X is obtained by *gluing* together the affine schemes U_i via their isomorphic open subsets U_{ij}. We say that X has *degree complexity* at most d, if $t \leq d$ and if all affine schemes U_i and U_{ij}, and all maps f_{ij} have degree complexity at most d.

A *coherent* $\mathcal{O}_X$*-module* $\mathcal{F}$ is then determined by giving for each i a finitely generated A_i-module M_i, such that

$$M_i \otimes_{A_i} A_{ij} \cong M_j \otimes_{A_j} A_{ji} \tag{32}$$

where we identify A_{ij} with A_{ji} via the isomorphism induced by f_{ij}. If, moreover, all M_i have degree complexity at most d, then we say that $\mathcal{F}$ has *degree complexity* at most d. In summary, we get the following examples.

4.7. Example. The functor which assigns to an algebraically closed field K the collection of all schemes (over K) and the functor which assigns to K the collection of pairs $(X, \mathcal{F})$ with $\mathcal{F}$ a coherent $\mathcal{O}_X$-module over the scheme X, are asymptotically definable functors.

5. Intersection Theory

Recall that an *(algebraic) cycle* on a scheme X is a formal sum

$$\alpha = \sum_{i=1}^{t} n_i Y_i \tag{33}$$

where $n_i \in \mathbb{Z}$ and Y_i are closed reduced irreducible subschemes (*subvarieties*, for short) of X. We denote the group of all cycles on X by $Z^*(X)$. We say that α has *degree complexity* at most d, if $t \leq d$ and all $|n_i| \leq d$ and if all subschemes Y_i have degree complexity at most d (whenever Y_i belongs to the *support* of α, i.e., $n_i \neq 0$).

5.1. Example. The functor which assigns to an algebraically closed field K the collection of all pairs (X, α) with X a scheme and α a cycle on X, is asymptotically definable.

Since the Zariski topology on X is Noetherian, we can write X uniquely as

$$X = X_1 \cup \cdots \cup X_s, \tag{34}$$

where the X_i are subvarieties of X with $X_i \not\subset X_j$ for $i \neq j$. These subvarieties X_i are called the *irreducible components* of X. The *cycle associated to* X is by definition the cycle $\sum n_i X_i$, where n_i is the length of $\mathcal{O}_{X,\eta_i}$ and where η_i is the generic point of X_i, for $i = 1, \dots, s$. In particular, if $X = \operatorname{Spec} A$ is affine, then X_i is the closed subset defined by a minimal prime $\mathfrak{g}_i$ of A and n_i is the length of the Artinian local ring $A_{\mathfrak{g}_i}$.

Assume now that X is moreover regular. Let us briefly review some intersection theory for closed subschemes on a regular scheme of finite type over a field. Let Y_1 and Y_2 be two closed subschemes of X. Their intersection $Y_1 \cap Y_2$ is by definition the scheme $Y_1 \times_X Y_2$, which is a closed subscheme of X. We say that Y_1 and Y_2 *intersect properly*, if the codimension of each irreducible component F of $Y_1 \cap Y_2$

equals $\operatorname{codim} Y_1 + \operatorname{codim} Y_2$. If this is the case, let η be the generic point of such an irreducible component F. We define, following SERRE in [14], the *local intersection number* by

$$i(\eta; Y_1, Y_2) = \sum_{n=0}^{\infty} (-1)^n \ell(\operatorname{Tor}_n^{\mathcal{O}_{X,\eta}}(\mathcal{O}_{Y_1,\eta}, \mathcal{O}_{Y_2,\eta})), \tag{35}$$

where ℓ denotes the length of a module (see Definition 7.1 below for more details). Note that this sum is finite. Indeed, since X is regular, every $\mathcal{O}_X$-module has finite projective dimension by [7, Theorem 19.2] and therefore

$$\operatorname{Tor}_n^{\mathcal{O}_{X,\eta}}(\mathcal{O}_{Y_1,\eta}, \mathcal{O}_{Y_2,\eta}) = 0 \tag{36}$$

for n strictly bigger than the dimension of X. The *intersection cycle* of Y_1 and Y_2 is then defined as the element in $Z^*(X)$ given by

$$Y_1 \cdot Y_2 = \sum_F i(\eta_F; Y_1, Y_2) F, \tag{37}$$

where the sum runs over all irreducible components F of $Y_1 \cap Y_2$ and η_F denotes the generic point of F.

If Y_1 and Y_2 do not intersect properly, then a more complicated definition is required, using CHOW's Moving Lemma. (We will not treat this case here.) Finally, the intersection of two cycles which intersect properly (meaning that each subvariety in the support of one cycle intersects properly every subvariety in the support of the other cycle), is defined by extending formula (37) by linearity.

In Section 7, we will investigate asymptotically definable subfunctors and bounded natural transformations of the examples of asymptotically definable functors from the last two sections. For instance, we will show that the intersection product is asymptotically definable. Namely, the functor Ω which assigns to an algebraically closed field K, the collection of all triples (X, α, β) with X a scheme and α and β algebraic cycles on X which intersect properly, is an asymptotically definable subfunctor of the functor

$$K \mapsto \{ (X, \alpha, \beta) \mid X \text{ is a scheme over } K \text{ and } \alpha, \beta \in Z^*(X) \}. \tag{38}$$

Moreover, the intersection product defines a natural transformation from Ω to the functor from Example 5.1. We will show that this natural transformation is bounded, i.e., the intersection product has degree complexity uniformly bounded in terms of the degree complexity of the factors.

6. Asymptotically Definable Structures

One could generalize the above definitions to a more abstract model-theoretic setting. In this short section, we give a brief outline how this can be done. For the duration of this section, let T be a theory in some language $\mathcal{L}$. We denote by $\underline{Mod}_T$ the category with objects the models of T and morphisms elementary embeddings. In particular, if T is the empty theory, then $\underline{Mod}_{\mathcal{L}}$ denotes the category of all $\mathcal{L}$-structures with elementary embeddings as morphisms.

Let $\mathcal{K}$ be another language. For simplicity, we will assume that $\mathcal{K}$ has only relation symbols. (The general case can easily be reduced to this). Let Ω be a functor from $\underline{Mod}_T$ to $\underline{Mod}_{\mathcal{K}}$. In other words, $\Omega(M)$ is a $\mathcal{K}$-structure, for every model M of T. We say that Ω is *asymptotically definable* if the following holds. There exists a filtration

$$\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega \tag{39}$$

(i.e., $\Omega_d(M)$ forms a filtration of $\Omega(M)$, for every model M of T), and there exist $\mathcal{L}$-formulae φ_d and surjective natural transformations

$$\eta_d \colon \varphi_d \to \Omega_d \tag{40}$$

with the following property. (Note that an $\mathcal{L}$-formula is a functor on $\underline{Mod}_T$ by sending a model M to its solution set $|\varphi|^M$). Let R be a unary relation symbol of the language $\mathcal{K}$. Then there exist $\mathcal{L}$-formulae ψ_d^R, such that T proves $\psi_d^R \to \varphi_d$ and such that

$$\eta_d(M)(\left|\psi_d^R\right|^M) = |R|^M \cap \Omega_d(M). \tag{41}$$

In other words, the relation R is an asymptotically definable subfunctor. For arbitrary s-ary relations, we have the same property, but now we have to take the s-fold product of the map η_d.

For instance, if $\mathcal{L}$ is the language of rings and T is the theory of algebraically closed fields and $\mathcal{K}$ is the empty language, we retrieve our previous definition (a structure in the empty language is just a set). In fact, if we take for $\mathcal{K}$ also the language of rings, then the functor of Example 2.3 is asymptotically definable in this new sense, since addition, subtraction and multiplication are clearly definable operations.

7. Uniform Bounds in Cohomology

7.1. Definition. To show that a certain functor Θ is an asymptotically definable subfunctor of an asymptotically definable functor Ω, means that we have to find formulae ψ_d (or, equivalently, constructible subsets G_d of $\mathbb{A}^N_{\mathbb{Z}}$) such that

$$\eta_d(K)(|\psi_d|^K) = \Theta(K) \cap \Omega_d(K) \tag{42}$$

where $\Omega_d(K)$ denotes all the objects in $\Omega(K)$ of complexity at most d and where η_d is the coding map from a constructible set $F_d(K) \subset K^N$ (for some N) to $\Omega_d(K)$, with F_d a constructible subset of $\mathbb{A}^N_{\mathbb{Z}}$ containing G_d.

We wrote out the above in its full gruesome details to convince the reader that a more flexible and legible notation is in order. Let Ω be an asymptotically definable functor and keep our notation from above. Let x be an object of $\Omega(K)$. Hence x could stand here for an affine algebra A, or for a pair (X, α), with X a scheme and $\alpha \in Z^*(X)$, etc. If x has degree complexity at most d, i.e., $x \in \Omega_d(K)$, then x can be written as $x = \eta_d(a)$, for some tuple a in K^N belonging to $F_d(K)$. In other words, a is a *code* for x. Let φ_d be the formula which defines F_d, so that $\varphi_d(a)$ holds in K. We then simply say that

$$x \textit{ satisfies formula } \varphi_d \textit{ over } K \tag{43}$$

In other words, we will confuse an object with its code. This leaves us with the ambiguity that several tuples might define the same code, and in fact, very often the thus defined equivalence relation on codes ($a \equiv b$, if $\eta_d(a) = \eta_d(b)$) is **not** definable. Therefore, a statement of the form (43) should be interpreted as

$$\textit{there is some code } a \textit{ of } x \textit{ which satisfies } \varphi_d \textit{ over } K. \tag{44}$$

In this section, we will be mainly concerned with subfunctors of the asymptotically definable functor of Example 4.6. Our first result is on the definable nature of the length of a module. Recall that an A-module M has finite *length* l, written $\ell_A(M) = l$, if M admits a decomposition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M \tag{45}$$

with each M_i a submodule of M, such that the subsequent quotients M_{i+1}/M_i are non-zero *simple* A-modules, i.e., with no non-trivial submodules.

7.2. Proposition ([12, Proposition 3.5])**.** *For each $d, l \in \mathbb{N}$, there exists a formula $(\texttt{Len} = l)_d$ with the following property. Let K be an algebraically closed field. Let A be an affine (local) K-algebra of degree complexity at most d and M a finitely generated A-module of degree complexity at most d. Then the pair (A, M) satisfies the formula $(\texttt{Len} = l)_d$ over K, if and only if, M has length l as an A-module.*

As an immediate corollary we get that the functor which assigns to an algebraically closed field the collection of pairs $(A, 0)$, with A an affine (local) algebra and 0 the zero A-module, is an asymptotically definable subfunctor of the asymptotically definable functor from Example 4.6. To define just a singleton within a whole class (i.e., the zero module among all finitely generated modules) seems an utter triviality, but it is not! Keep in mind that there is in general no formula which expresses that two tuples encode the same object. However, here we do have such a formula when the tuples encode the zero module. In general, for finitely generated modules over an affine local ring, it is a highly non-trivial fact that such a formula exists. This will be discussed in Section 9. Another corollary is:

7.3. Corollary. *For each $d \in \mathbb{N}$, there exists a formula $\texttt{FinLen}_d$ with the following property. Suppose K is an algebraically closed field and A an affine (local) K-algebra. Let M be a finitely generated A-module of degree complexity at most d. Then the pair (A, M) satisfies the formula $\texttt{FinLen}_d$ over K, if and only if, M has finite length as an A-module.*

Proof. Implicit in Proposition 7.2 is the fact that there is a bound d' depending only on d, such that if M has finite length, then this length is at most d'. Hence the formula

$$\bigvee_{i \leq d'} (\texttt{Len} = i)_d$$

expresses that M has finite length. ∎

Using Theorem 3.5, one can shown that basically all module operations are bounded:

7.4. Proposition. *For each d, there exists a bound d' with the property that if $N, N' \subset M$ are finitely generated A-modules of degree complexity at most d, then so are $N + N'$, $N \cap N'$ and M/N. Moreover, the ideal $(N : N')$ of all $a \in A$ such that $aN' \subset N$, has degree complexity at most d.*

We refer to [12] for more details. In loc. cit., we use yet another measure of complexity, called *deg-complexity*, defined for submodules of a free A-module, as an intermediate tool to obtain the proposition (see [12, Remark 3.8] where it is explained how both measures of complexity are mutually bounded).

Our next goal is to develop in a constructive manner some homological algebra. The following result will prove to be crucial.

7.5. Theorem. *For each $d \in \mathbb{N}$, there is a bound d' with the following properties. Let A be an affine (local) algebra and M a finitely generated A-module of degree complexity at most d. Let $\mathbb{G}$ be a $(d \times d)$-matrix over A of degree at most d (i.e., all its entries have degree at most d), then the degree complexity of Z and C are at most d', where Z (respectively, C) is the kernel (respectively, cokernel) of the morphism $\mathbb{G}^{\times}$ induced by $\mathbb{G}$, that is to say, Z and C are given by the exact sequence*

$$0 \to Z \to M^d \xrightarrow{\mathbb{G}^{\times}} M^d \to C \to 0. \tag{46}$$

Homological algebra studies derived functors on abelian categories. These functors should not be confused with the asymptotically definable functors previously introduced. To avoid too much clash of terminology, we will work in a less abstract setting than in the paper [12]. The functorial operations that matter to us, are taking *Hom* and tensor products. Let us denote by Ω_2 the asymptotically definable functor which assigns to an algebraically closed field K, the set of triples (A, M, N), where A is an affine (local) K-algebra and M and N are finitely generated A-modules. Tensor product then yields a natural transformation $\otimes$ from Ω_2 to the functor of Example 4.6, by sending a triple (A, M, N) to the pair $(A, M \otimes_A N)$. Similarly, we get a natural transformation *Hom* sending a triple (A, M, N) to the pair $(A, \operatorname{Hom}_A(M, N))$.

7.6. Proposition. *The natural transformations $\otimes$ and* Hom *are bounded.*

Note that this means that the degree complexities of $M \otimes_A N$ and $\operatorname{Hom}_A(M, N)$ are uniformly bounded in terms of the degree complexity of M and N. Taking tensor products with respect to a fixed A-module M is not an exact functor on the category of all A-modules, but only right exact. Its derived functors are known as the *Tor* functors. The derived functors of *Hom* are called the *Ext* functors. Our main result is then:

7.7. Theorem. *For each d and i, there exist formulae $(\mathtt{Tor}_i)_d$ and $(\mathtt{Ext}^i)_d$, such that if K is an algebraically closed field and if M, N and T are finitely generated A-modules of degree complexity at most d over the affine (local) K-algebra A, then the quadruple (A, M, N, T) satisfies the formula $(\mathtt{Tor}_i)_d$ (respectively, the formula $(\mathtt{Ext}^i)_d$), if and only if, we have that*

$$T \cong \operatorname{Tor}_i^A(M, N) \qquad \text{(respectively,} \quad T \cong \operatorname{Ext}_A^i(M, N)) \tag{47}$$

Implicit in the existence of these formulae is that the natural transformations Tor_i and Ext^i are bounded. The main use of this theorem is in combination with Proposition 7.2, enabling us to express that a certain i-th *Tor* or *Ext* module vanishes. Before we derive some corollaries, let us first give an indication of the proof.

One calculates a derived functor such as *Tor* as follows. Let M and N be finitely generated A-modules. Choose any *projective resolution* of M, that is to say, an exact sequence

$$P_{i+1} \xrightarrow{\mathbb{A}_i^{\times}} P_i \xrightarrow{\mathbb{A}_{i-1}^{\times}} \dots \xrightarrow{\mathbb{A}_0^{\times}} P_0 \to M \to 0 \tag{48}$$

with the P_j finite free A-modules. It follows from Theorem 7.5, that we can uniformly bound the ranks of the P_j as well as the degrees of the entries of the matrices $\mathbb{A}_j$ in terms of d and i only.

Tensoring this sequence with N no longer gives an exact sequence, but merely a complex

$$P_{i+1} \otimes_A N \xrightarrow{\mathbb{A}_i^\times} P_i \otimes_A N \xrightarrow{\mathbb{A}_{i-1}^\times} \dots \xrightarrow{\mathbb{A}_0^\times} P_0 \otimes_A N \to M \otimes_A N \to 0 \tag{49}$$

which is only exact at $M \otimes_A N$. The *homology* modules of this complex measure precisely the extent to which the complex is not exact. In this case, they are denoted by $\mathrm{Tor}_i^A(M,N)$. In particular, $\mathrm{Tor}_1^A(M,N)$ is the quotient of the kernel of the morphism $\mathbb{A}_0 \colon P_1 \otimes_A N \to P_0 \otimes_A N$ by the image of the morphism $\mathbb{A}_1 \colon P_2 \otimes_A N \to P_1 \otimes_A N$. By Theorem 7.5 both this kernel and this image have uniformly bounded degree complexity in terms of the degree complexity of M and N, and hence so does their quotient, by Proposition 7.4.

From Theorem 7.7 together with Formula (35), the following is now straightforward.

7.8. Theorem. *The intersection of algebraic cycles which intersect properly is a bounded and asymptotically definable operation.*

8. Bounds in Commutative Algebra

In this section, we will give some examples of effective results in Commutative Algebra. Homological Algebra has proven to be an indispensable tool for ring and module theorists. Therefore, it should not come as a surprise that our effective results come from the uniformity of the *Tor* and *Ext* cohomology groups discussed in the previous section. Nonetheless, perhaps the most important invariant is not homological in nature: the Krull dimension of a ring. Recall that the *(Krull) dimension* of an affine K-algebra A is the maximal possible length of a strict chain of prime ideals in A. If A is a domain, its dimension equals the transcendence degree of its fraction field over K. The *dimension* of a finitely generated A-module M is by definition the dimension of the ring $A/\operatorname{Ann}_A(M)$. In spite of its non-homological nature, dimension also behaves uniformly:

8.1. Theorem ([12, Proposition 5.1])**.** *For each d and h, there exists a formula $(\mathtt{dim} = h)_d$, such that a pair (A,M) satisfies the formula over K, where K is an algebraically closed field, A an affine (local) K-algebra and M a finitely generated A-module of degree complexity at most d, if and only if, M has dimension h (as an A-module).*

Note that a finitely generated A-module is zero dimensional, if and only if, it has finite length, so that by Corollary 7.3, the case $h = 0$ is clear. For the general case, one uses Krull's Principal Ideal Theorem.

8.2. Theorem ([12, Theorem 5.3])**.** *The subfunctors of the asymptotically definable functor from Example 4.4 given by assigning to an algebraically closed field the set of all regular (respectively, Gorenstein, Cohen-Macaulay, or complete intersection) local affine K-algebras, is asymptotically definable.*

Let us just explain this for the regular case. It follows from Serre's Theorem, that a local affine K-algebra R is regular, if and only if, its residue field k has finite projective dimension. The latter condition is equivalent with $\mathrm{Tor}_i^R(k,k) = 0$, for some i strictly bigger than the dimension of R. Now, the dimension of R can never exceed the number of variables in the polynomial ring of which R is the localization of some quotient. Hence if R has degree complexity at most d, then its dimension is at most d. So we may take $i = d+1$ in the criterion and now the rest is easy by an application of Theorem 7.7 and Proposition 7.2. (Note that the degree complexity of the residue field k is also at most d by definition).

The other conditions can equally well be translated in to the vanishing of some *Tor* or *Ext* groups derived from R. For some of them, we might need also Theorem 8.1.

A module variant is the following:

8.3. Theorem ([12, Corollary 5.5]). *The subfunctor of the asymptotically definable functor from Example 4.6 given by assigning to an algebraically closed field K the set of all pairs (A, M) with A an affine K-algebra and M a finitely generated A-module of finite injective dimension, is asymptotically definable.*

From these effective results, we can now derive our first pure, albeit not new, result on the Bass Conjecture. This conjecture, and now theorem, states the following.

8.4. Theorem (Bass Conjecture). *If a Noetherian local ring R admits a finitely generated module of finite injective dimension, then R is Cohen-Macaulay.*

In fact the converse also holds and its proof is rather easy. Moreover, this proof shows that an R-module of finite injective dimension can be found with degree complexity bounded in terms of the degree complexity of R. This conjecture is now proven in full generality by Roberts' New Intersection Theorem, but was originally proven for a smaller class of local rings by Szpiro and Peskine in [8] using their New Intersection Theorem in positive characteristic. Their proof uses the action of Frobenius on local cohomology groups and whence cannot be transferred to the zero characteristic case. To deal with the latter case, the authors introduced an ad hoc lifting procedure, which we can replace now, at least for the affine case, by the Lefschetz Principle. Indeed, our observation above on the converse, together with Theorem 8.3 and Theorem 8.2, shows that, for each d, there exist formulae $\mathtt{CM}_d$ and $\mathtt{Injdim}_d$, such that an affine local K-algebra R of degree complexity at most d satisfies these formulae over an algebraically closed field K, if and only if, R is Cohen-Macaulay (respectively, admits a finitely generated module of finite injective dimension). Hence the sentences $\mathtt{Bass}_d$, for $d = 1, 2, \dots$, expressing that $\mathtt{CM}_d$ and $\mathtt{Injdim}_d$ are equivalent formulae, holds in K, if and only if, the Bass Conjecture 8.4 holds for affine local K-algebras of degree complexity at most d. By the result of Szpiro and Peskine in positive characteristic and the Lefschetz Principle 1.4, we derive the validity of the conjecture in zero characteristic as well.

Another example, the Zariski-Lipman Conjecture, was already discussed in some length in Example 1.3.

8.5. Theorem (Zariski-Lipman Conjecture for Hypersurfaces). *For each d, there is a finite set C_d of exceptional characteristics, such that if K is an algebraically closed field of characteristic not in the exceptional set C_d, and such that if x is a K-rational point on a degree d hypersurface X over K with the property that $\mathrm{Der}_K(\mathcal{O}_{X,x})$ is free, then x is a regular point of X. Here $\mathrm{Der}_K(\mathcal{O}_{X,x})$ denotes the module of K-derivations of the local ring $\mathcal{O}_{X,x}$ of x at X.*

Again this result cannot be obtained in positive characteristic by just simply mimicking the proof in zero characteristic, as the latter uses transcendental methods. Since this time we go from zero to positive characteristic, there is a price to be paid for *going against the stream*. Namely, we are only able to say something about almost all characteristics. Moreover, the exceptional set C_d of characteristics depends heavily on the degree of the hypersurface. In fact, easy examples show that the Zariski-Lipman Conjecture is in general false in positive characteristic, but all these counterexamples are in characteristics of the same order of magnitude as the degree. We trust the reader will be able to convince himself that the argument sketched in the introduction is valid. Just note that a finitely generated module M over a local ring R is free, if and only if, $\operatorname{Tor}_1^R(M,k)=0$, where k is again the residue field of R. In [12] some further special instances are discussed where one can transfer the validity of the Zariski-Lipman Conjecture to positive characteristic.

In fact, one possibly could invert the argument. If one were able to show that the full Zariski-Lipman Conjecture holds for a fixed degree in almost all positive characteristics, then it would by the Lefschetz Principle also hold in zero characteristic. With the full Zariski-Lipman Conjecture we mean a similar statement as in Theorem 8.5, where X is now not just a hypersurface, but can have any codimension.

9. Non-Linear Case: Etale Complexity

In this section we will discuss how to deal with a situation in which the algebraic or geometric problem has no longer a linear description. We start with a counterexample to uniformity.

9.1. Theorem. *It is not true that there exists for each d a bound d' with the following property. Let K be an algebraically closed field and let $\xi=(\xi_1,\dots,\xi_n)$ and $Y=(Y_1,\dots,Y_m)$ be variables. Consider the system of equations*

$$\begin{cases} F_1(\xi,Y) &= 0\\ F_2(\xi,Y) &= 0\\ &\vdots\\ F_s(\xi,Y) &= 0. \end{cases} \tag{50}$$

over $K[\xi]$, in the unknowns Y, where the total degree of each F_i is at most d and where $n,m\leq d$. If there is a solution $f=(f_1,\dots,f_m)$ for the Y-variables with $f_i(\xi)\in K[\xi]$, then there exists already such a solution of degree at most d'.

Proof. In fact even if we fix the algebraically closed field K from the start, no such bound d' exists. The following counterexample is based on an example taken from the already quoted paper [10]. In it, the authors show that for all $t\in\mathbb{N}$, there exists an $a\in K$ different from 0 and 1, such that the coordinate ring

$$A_a=\frac{K[\xi,\zeta]}{(\zeta^2-\xi(\xi-1)(\xi-a))} \tag{51}$$

of the elliptic curve has a non-constant unit, but any non-constant unit has degree at least t. This observation follows from the fact that an elliptic curve can have torsion points of arbitrary high order.

Now, consider the following (quadratic) system of equations

$$XY = 1 + Z(\zeta^2 - \xi(\xi-1)(\xi-a)) \tag{52a}$$
$$X = P(\xi - C) + Q(\zeta - D) \tag{52b}$$
$$CC_1 = 1 \tag{52c}$$
$$DD_1 = 1 \tag{52d}$$

in the variables $(X, Y, Z, P, Q, C, C_1, D, D_1)$. We claim that this system has a solution in $K[\xi, \zeta]$, if and only if, the ring A_a has a non-constant unit. Indeed, let $(f, g, h, p, q, c, c_1, d, d_1)$ be such a solution. Then Equations (52c) and (52d) show that c and d are units in $K[\xi, \zeta]$ and whence belong to K. From Equation (52b) it then follows that (c, d) is a root of the polynomial f, so that in particular, the latter is not constant. Finally, Equation (52a) simply expresses that f is a unit in A_a. We leave it up to the reader to check that conversely, the existence of a non-constant unit in A_a provides a solution to this system of equations.

Hence if a uniform bound would exist to solve quadratic equations, then there would be a bound on the solution of minimal degree, contradicting the above. ■

As already mentioned in the introduction, to obtain uniformity results, we have to replace the polynomial ring $K[\xi]$ by the algebraic power series ring $K[[\xi]]^{\mathrm{alg}}$. The following algebraic description of an algebraic power series, enables us to introduce a new measure of complexity, turning $K[[\xi]]^{\mathrm{alg}}$ into an asymptotically definable set.

9.2. Proposition. *For each $\omega_1 \in K[[\xi]]^{alg}$, we can find a natural number N, an N-tuple ω of algebraic power series with first entry ω_1 and an N-tuple H of polynomials over K in the variables ξ and $Y = (Y_1, \dots, Y_N)$, such that $H(\xi, \omega(\xi)) = 0$ and such that the Jacobian matrix*

$$\mathrm{Jac}(H) = \begin{pmatrix} \partial H_1/\partial Y_1 & \partial H_1/\partial Y_2 & \dots & \partial H_1/\partial Y_N \\ \partial H_2/\partial Y_1 & \partial H_2/\partial Y_2 & \dots & \partial H_2/\partial Y_N \\ \vdots & \vdots & \ddots & \vdots \\ \partial H_N/\partial Y_1 & \partial H_N/\partial Y_2 & \dots & \partial H_N/\partial Y_N \end{pmatrix} \tag{53}$$

evaluated at $\xi = 0$ and $Y = u$, where $u = \omega(0)$, is invertible.

Moreover, ω is the unique solution to $H(Y) = 0$ which evaluates to u at $\xi = 0$.

A system H_i as in the proposition is called a *Hensel system* for ω at u. We say that ω_1 (or, ω) has *etale complexity* at most d, if we can find an Hensel system of length $N \leq d$ and of total degree at most d. (We always assume that the number of ξ-variables is at most d). This makes $K[[\xi]]^{\mathrm{alg}}$ into an asymptotically definable set and we now have the non-linear counterpart of Theorem 3.5.

9.3. Theorem ([11]). *For each $d \in \mathbb{N}$, there exists a $d' \in \mathbb{N}$ with the following properties. Let K be a field. Suppose the $F_i \in K[\xi, Y]$ in the system of equations* (50) *have total degree at most d and also $n, m \leq d$. If $F = 0$ has a solution for the Y-variables in $K[[\xi]]$, then it has a solution in $K[[\xi]]^{alg}$ of etale complexity at most d'.*

More generally, if we allow the F_i to be in $K[[\xi]]^{alg}[Y]$ such that, viewed as polynomials in the Y-variables, their degree is at most d and each coefficient has etale complexity at most d, then the conclusion in the above statement remains valid.

This theorem should be complemented with the following result, called *Strong Artin Approximation*, which gives us an effective way of testing whether an algebraic or formal power series solution exists.

9.4. Theorem ([1])**.** *For each $d \in \mathbb{N}$, there exists $d' \in \mathbb{N}$, such that any system of equations* (50) *has a solution ω in algebraic (or formal) power series with initial condition $\omega(0) = u$, if and only if, it has a polynomial solution f modulo $(\xi_1, \dots, \xi_n)^{d'+1}$ with initial condition $f(0) = u$.*

An algorithm to solve polynomial equations. Combining these two theorems, we can now describe briefly an algorithm which effectively computes a solution in $K[[\xi]]^{\mathrm{alg}}$ for the system (50). Effectively here has perhaps to be taken with a grain of salt, because we will assume that one has been able to calculate the bound d' in both theorems. Moreover, we will assume that the field arithmetic in an algebraically closed field K can be effectively carried out. One should bare in mind that the theory of algebraically closed fields is decidable. Here is the description of the algorithm.

Firstly, we need to check whether a solution exists at all. According to Theorem 9.4, this amounts in finding polynomials $(f_1, \dots, f_m)$ of degree at most d', such that all terms in $F_i(\xi, f(\xi))$ up to order d' vanish. Writing down the f_i as general polynomials in ξ of degree d' with unknown coefficients and expressing the above condition, we end up with a polynomial system of equations in these unknown coefficients. The difference with our original problem is now that we have to solve this system in the field K and not in the polynomial ring. As we mentioned before, we assume that this can be done in an effective way, and hence we can predict whether an algebraic power series solution exists or not. We can even include from the start some initial condition $f(0) = u$.

So, let's assume we know that an algebraic power series solution ω exists with initial condition $\omega(0) = u$. Hence by Theorem 9.3, we must be able to find a Hensel system $H(Y) = 0$ for ω at u of total degree and length N at most d'. Let us still write ω and Y for the corresponding enlarged N-tuples. By a strengthening of Proposition 9.2, we may even assume that the H_i generate a prime ideal (necessarily of height N by the Jacobian Criterion). The uniqueness part in Proposition 9.2 then implies that the (model-theoretic) type of the solution ω is completely determined by this prime ideal, so that in particular, since ω is also a solution of $F(Y) = 0$, we must have an inclusion

$$(F_1, \dots, F_s) \subset (H_1, \dots, H_N). \tag{54}$$

Now, let us write down the H_i with unknown coefficients. We then express that the H_i form a Hensel system at u; this yields some polynomial equations in the unknown coefficients. Using Theorem 3.5, we can also express in the unknown coefficients that condition (54) is satisfied. Hence a solution in K for this system in unknown coefficients gives then a complete description of a solution of system (50).

The stronger condition on the Hensel system just mentioned actually translates geometrically in the fact that the map of affine schemes

$$h\colon W = \operatorname{Spec} \frac{K[\xi, Y]}{(H_1, \dots, H_N)} \to \mathbb{A}^n_K \tag{55}$$

is an *etale* extension. In other words, we found a solution to system (50) which *lives* on W.

Any etale map is open, so that in particular, the image $\operatorname{Im} h$ is a Zariski open of $\mathbb{A}^n_K$. Moreover, it will follow from our discussion in Section 11 that the degree complexity of this Zariski open is uniformly bounded in terms of d' and whence in terms of d. Note that by construction, the origin lies in $\operatorname{Im} h$. If we would carry out the just described algorithm at every K-rational point of $\mathbb{A}^n_K$, not just at the origin, then we would have constructed an etale covering of $\mathbb{A}^n_K$, where on each piece we would have a solution to system (50). We refer to such a solution as a solution *locally in the etale topology*. Note that we can take such a covering to be finite, and, moreover, its cardinality can be taken uniformly bounded in terms of d, in view of Proposition 1.5.

10. Isomorphism Problems

As we mentioned already in the introduction, and as we explained at the end of the previous section, an application of Theorem 9.3 will often lead to results which hold locally in the etale topology. Sometimes etale descent and local/global principles enable us to obtain results for the Zariski topology as well, as the next example shows.

10.1. Theorem. *For each d, there exists a formula $\mathtt{Iso}_d$ with the following property. Let R be an affine local K-algebra with K an algebraically closed field. Let M_1 and M_2 be two finitely generated R-modules of degree complexity at most d. Then the triple (R, M_1, M_2) satisfies the formula $\mathtt{Iso}_d$ over K, if and only if, M_1 and M_2 are isomorphic as R-modules.*

Proof. Since R is a quotient of a localization of a polynomial ring, we may assume already without loss of generality that R is a localization of $K[\xi]$ with respect to a prime ideal $\mathfrak{p}$. Let $\widehat{R}$ denote its $\mathfrak{p}$-adic completion. By Cohen's Structure Theorem for complete regular local rings, we know that

$$\widehat{R} \cong K[[\zeta]], \tag{56}$$

where $\zeta = (\zeta_1, \dots, \zeta_h)$ with h the dimension of R. In fact, choosing h generators $g_1, \dots, g_h$ for the maximal ideal of R, the isomorphism (56) is given by sending g_i to the variable ζ_i. Therefore, for the sake of exposition, we may assume that $\mathfrak{p}$ is a maximal ideal, which we then, after some translation, may assume to be $(\xi_1, \dots, \xi_n)$. In particular, we may take $\widehat{R} = K[[\xi]]$.

By definition of degree complexity, we can write M_i as the cokernel of a morphism $R^d \to R^d$ given by a $(d \times d)$-matrix $\mathbb{A}_i$ with (polynomial) entries of degree at most d. Lemma 10.2 below shows that our problem can be translated in terms of (non-linear) polynomial equations. Therefore, if M_1 and M_2 are isomorphic, then system (59) has a solution in R and whence in the overring $K[[\xi]]^{\mathrm{alg}}$. By Theorem 9.3, there exists already a solution of etale complexity at most d'. Conversely, if a solution in $K[[\xi]]^{\mathrm{alg}}$ of system (59) exists, then again by the Lemma 10.2, we have that

$$M_1 \otimes K[[\xi]]^{\mathrm{alg}} \cong M_2 \otimes K[[\xi]]^{\mathrm{alg}}. \tag{57}$$

Hence in particular, we have an isomorphism

$$M_1 \otimes K[[\xi]] \cong M_2 \otimes K[[\xi]]. \tag{58}$$

However, using [4, Exercise 7.5 and Theorem 7.2], it follows that if an isomorphism (58) exists, then in fact M_1 and M_2 are isomorphic as R-modules. From these observations, the reader should now be convinced that a formula as claimed exists. ■

10.2. Lemma. *Let A be a ring and let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two $(d \times d)$-matrices with entries in A. The cokernels of these two matrices are isomorphic, if and only if, there exist $(d \times d)$-matrices $\mathbb{X}_i$, $\mathbb{U}_i$, $\mathbb{V}_i$, such that*

$$\begin{aligned} \mathbb{E} - \mathbb{X}_1\mathbb{X}_2 &= \mathbb{U}_1\mathbb{A}_1 \\ \mathbb{E} - \mathbb{X}_2\mathbb{X}_1 &= \mathbb{U}_2\mathbb{A}_2 \\ \mathbb{A}_1\mathbb{X}_1 &= \mathbb{V}_1\mathbb{A}_2 \\ \mathbb{A}_2\mathbb{X}_2 &= \mathbb{V}_2\mathbb{A}_1, \end{aligned} \tag{59}$$

where $\mathbb{E}$ denotes the $(d \times d)$-identity matrix.

For the next result, mentioned already in the introduction, we are less fortunate: the etale topology is strictly finer than the Zariski topology. For instance any two smooth (irreducible and reduced) curves are locally isomorphic for the etale topology. This is an easy consequence of Cohen's Structure Theorem together with Artin Approximation. In the presence of singularities, we obtain the following definability result. We will say that two schemes X and Y are *locally isomorphic in the etale topology*, if there exist surjective etale maps $T \twoheadrightarrow X$ and $T \twoheadrightarrow Y$. The denomination *local* stems from the fact that we do not assume T to be connected. Alternatively, X and Y are locally isomorphic in the etale topology, if each K-rational point on one scheme admits an etale neighborhood which is isomorphic to an etale neighborhood of a K-rational point on the other scheme.

10.3. Theorem. *For each d, there exists a formula $\mathtt{Etiso}_d$ with the following property. Let X and Y be schemes of degree complexity at most d over an algebraically closed field K. Then the pair (X, Y) satisfies the formula $\mathtt{Etiso}_d$ over K, if and only if, X and Y are locally isomorphic in the etale topology.*

Let us just show how one translates the isomorphism problem into a system of polynomial equations.

10.4. Lemma. *Let $A = K[\xi]/(f_1, \dots, f_s)$ and $B = K[\zeta]/(g_1, \dots, g_s)$ be two affine domains of dimension d. Then $A \cong B$ (as K-algebras), if and only if, the following system of equations*

$$\begin{cases} f_i(X) &= \sum_{j=1}^s C_{ij} g_j(\zeta) \\ g_i(Y) &= \sum_{j=1}^s D_{ij1} f_j(\xi) \\ \xi_i &= X_i + \sum_{j=1}^s D_{ij2} f_j(\xi) + D_{ij3}(\zeta_j - Y_j) \end{cases} \tag{60}$$

for $i = 1, \dots, s$, has a solution for X in $K[\zeta]$ and for the remaining variables Y, C, D in $K[\xi, \zeta]$.

Similarly, if $A = K[[\xi]]^{alg}/(f_1, \dots, f_s)$ and $B = K[[\zeta]]^{alg}/(g_1, \dots, g_s)$ are d-dimensional (local) domains and we now allow solutions in $K[[\xi, \zeta]]^{alg}$ with the solution for X still only depending on ζ, then the system (60) *has a solution, if and only if, we have an isomorphism $A \cong B$.*

The new ingredient here is that some of the solutions are subject to some constraints. It is known that Artin Approximation holds for a system of equations with a single constraint (i.e., a constraint of the form: the first k solutions depend only on the first l variables). When we use this result instead of the classical Artin

Approximation in the proof of Theorem 9.3, then we get an effective version with single constraint, which we then can apply to the system (60). Note that in general, Artin Approximation fails when we allow multiple constraints.

11. Constructible Invariants

An invariant is often thought of as an integer attached to an algebraic or geometric object, in such way that isomorphic objects get the same number assigned. The hope then is to find enough invariants, such that whenever two objects have the same value for each invariant, then they are in fact isomorphic. For instance, the dimension of a vector space completely determines its isomorphism type, or, the characteristic and the transcendence degree over the prime field, completely characterizes an algebraically closed field. In general, complete sets of invariants are far from known. Moreover, one will need also invariants which take on other kinds of values, not just integers. Below we'll see examples where the value is a certain formal power series over $\mathbb{Z}$. For our purposes, an invariant will be defined as follows.

11.1. Definition. Let $\mathbb{S}$ be a set. We call a map ω an *($\mathbb{S}$-)invariant*, if it assigns to each pair (R, M) a value in $\mathbb{S}$, where R is an affine local ring and M a finitely generated R-module. If $\mathbb{S}$ is just the natural numbers, then we call ω a *numerical invariant*.

If X is a scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, then we write

$$\omega(x, \mathcal{F}) := \omega(\mathcal{O}_{X,x}, \mathcal{F}_x) \tag{61}$$

for x an arbitrary point of X. This defines a partition on X, by the *level sets*

$$\omega_{\mathcal{F}}^{-1}(s) := \{\, x \in X \mid \omega(x, \mathcal{F}) = s \,\} \tag{62}$$

where s runs over all possible values in $\mathbb{S}$. Here we are a bit sloppy in our usage of the word *partition*, namely we call a collection $\{X_i\}$ of subsets of X a partition, if they are mutually disjoint and cover the whole set X; so we do not always require the X_i to be non-empty. When we then say that a partition is *finite*, we mean that all but finitely many X_i are empty.

Our main goal is now to find necessary conditions for this partition to be *constructible*, meaning that each level set is a constructible set. We say that ω is *asymptotically definable*, if for each d and each $s \in \mathbb{S}$, we can find a formula $(\texttt{Val} = s)_d$, such that a triple $(X, x, \mathcal{F})$ satisfies the formula $(\texttt{Val} = s)_d$ over an algebraically closed field K, if and only if, $\omega(x, \mathcal{F}) = s$, where $\mathcal{F}$ is a coherent module of degree complexity at most d on the scheme X (over K) and where x is a point of degree complexity at most d on X. One verifies without too much effort that asymptotic definability implies that the partition induced by the level sets on $X(K)$ is constructible. Our goal is to extend this to the whole space X. Whence the following definition and proposition.

We say that a subset F of a scheme X is *saturated*, if for each $x \in F$, we can find a K-rational point $y \in F$ which lies in the Zariski closure of x (i.e., y is a *specialization* of x). Note that a constructible set is saturated. Indeed, without loss of generality, we may assume that $X = \operatorname{Spec} A$ is affine and F is locally closed, given by

$$f_1 = \cdots = f_s = 0 \wedge f_0 \neq 0 \tag{63}$$

with $f_i \in A$. Let $x \in F$ be an arbitrary point corresponding to the prime ideal $\mathfrak{p}$ of A. Hence $f_i \in \mathfrak{p}$, for $i = 1, \dots, s$ and $f_0 \notin \mathfrak{p}$. In particular,

$$B := \frac{A_{f_0}}{\mathfrak{p}A_{f_0}} \tag{64}$$

is not the zero ring and hence contains at least one maximal ideal $\mathfrak{m}$. If we still write $\mathfrak{m}$ for its preimage in A, then this is again a maximal ideal, containing $\mathfrak{p}$ but not containing f_0. Since $f_i \in \mathfrak{p}$, for $i = 1, \dots, s$, we get that the K-rational point y corresponding to $\mathfrak{m}$ lies in F and is a specialization of x.

We call ω *saturated*, if each of its level sets is saturated, for all pairs $(X, \mathcal{F})$.

11.2. Proposition. *Let $\mathbb{S}$ be a set and ω an $\mathbb{S}$-invariant. If ω is asymptotically definable and saturated, then it is constructible.*

Moreover, in view of Proposition 1.5, ω takes only finitely many different values on a scheme X for a fixed coherent $\mathcal{O}_X$-module $\mathcal{F}$.

So, our next goal is to find necessary conditions for an invariant to be saturated. The tools we'll use for this are geometric in nature, so we will only briefly describe them. The idea is the following. Suppose that $X = \mathrm{SpecA}$ is just an affine curve, i.e., is one dimensional, reduced and irreducible. Let F be a subset of X. There is only one non-trivial instance for F to be saturated. Namely, if the generic point η of X belongs to F, then we have to show that F contains also a K-rational point. Now, if $F = X_{\mathcal{F}}^{-1}(s)$, then we need to show that for some K-rational point x of X, we have that

$$\omega(x, \mathcal{F}) = \omega(\eta, \mathcal{F}). \tag{65}$$

In other words, if M is the finitely generated A-module corresponding to $\mathcal{F}$, then we need to find a maximal ideal $\mathfrak{m}$, such that

$$\omega(L, M \otimes L) = \omega(A_{\mathfrak{m}}, M_{\mathfrak{m}}) \tag{66}$$

where L is the fraction field (or function field) of A (note that $L = \mathcal{O}_{X,\eta}$). If X has higher dimension, then we could imitate the previous argument, by going up *one dimension at the time*. We call this *devissage*.

11.3. Definition. We say that an $\mathbb{S}$-invariant ω is *devissable*, if for all affine rings A, all finitely generated A-modules M and each prime ideal $\mathfrak{g}$ of A, we can find a Zariski open U of $\operatorname{Spec} A$ containing $\mathfrak{g}$, such that for all prime ideals $\mathfrak{p}$ of height one higher than $\mathfrak{g}$, containing $\mathfrak{g}$ and lying in U, we have an equality

$$\omega(A_{\mathfrak{g}}, M_{\mathfrak{g}}) = \omega(A_{\mathfrak{p}}, M_{\mathfrak{p}}). \tag{67}$$

Note that a constructible invariant is devissable. Indeed, as in (63), we may assume that $X = \operatorname{Spec} A$ is affine, M is the finitely generated A-module associated to $\mathcal{F}$ and $s \in \mathbb{S}$, so that $F = \omega_{\mathcal{F}}^{-1}(s)$ is given by (a disjunction) of equations (63). Let $\mathfrak{g}$ be a prime ideal belonging to $\omega_{\mathcal{F}}^{-1}(s)$. Let U be the Zariski open set of all prime ideals not containing f_0, then U satisfies the requirement in Definition 11.3, since any prime ideal in U and containing $\mathfrak{g}$ then belongs to $\omega_{\mathcal{F}}^{-1}(s)$, so that both sides of Equation (67) equal s.

11.4. Proposition. *Any devissable invariant is saturated.*

Strictly speaking, we do not need to find an open set U of 'good' liftings, but a single lifting suffices, to go up all the way to a maximal ideal and whence prove saturatedness. However, it will always come for free that we can do this *generically* and we will be able to derive some stronger results from this assumption. As a rule, we will always take the open U in Definition 11.3 inside the regular locus of $A/\mathfrak{g}$ (which itself is open, since A is excellent). Under this extra assumption on U and with $\mathfrak{p}$ as before in U, containing $\mathfrak{g}$ and of height one higher than $\mathfrak{g}$, we get that $A_{\mathfrak{p}}/\mathfrak{g}A_{\mathfrak{p}}$ is a *discrete valuation ring*, that is to say, its maximal ideal is generated by a single element f (called a *uniformizing parameter*). With this extra information, Equality (67) is often easier to verify.

Sometimes we can reduce the situation to the case that $\mathfrak{g}$ is even a minimal prime of A (so that $A_{\mathfrak{g}}$ has finite length). This can be achieved by using regular sequences provided we assume that A is moreover Cohen-Macaulay, see [13] for details. Let us give some examples.

11.5. Definition. We already encountered some Betti numbers as the dimension of certain *Tor* modules; Bass numbers are similarly obtained, using *Ext* modules in stead. More precisely, let R be an affine local ring with residue field k and let M be a finitely generated R-module. We define the i-th *Betti* number of M to be

$$\omega_i^{\text{Betti}}(R,M) := \dim_k \operatorname{Tor}_i^R(k,M). \tag{68}$$

This makes ω_i^{Betti} into a numerical invariant. It follows from Theorem 7.7 together with Proposition 7.2, that this is an asymptotically definable invariant. Similarly, we define the i-th *twisted Bass* number of M to be

$$\omega_i^{\text{Bass}}(R,M) := \dim_k \operatorname{Ext}_R^{q+i}(k,M), \tag{69}$$

where q is the *depth* of M (i.e., the length of a maximal M-regular sequence inside the maximal ideal of R). The reason why we add the twist by the depth, is that otherwise, the invariant would not be devissable. In fact, $\operatorname{Ext}_R^i(M,k) = 0$, for all $i < q$. The same argument as for the Betti numbers, shows that the twisted Bass numbers ω_i^{Bass} are asymptotically definable. The zero-th twisted Bass number is also known as the *type* of a module (not to be confused with the model-theoretic notion of type; there is no connection whatsoever). The crucial result is now the following.

11.6. Theorem. *The numerical invariants ω_i^{Betti} and ω_i^{Bass} are devissable whence constructible.*

Let us just give some indications why this is so. For M a finitely generated R-module, we have that $\omega_0^{\text{Betti}}(R,M)$ equals the minimal number of generators of M by Nakayama's Lemma. This invariant is well-known to be constructible whence in particular devissable. The result for the higher Betti numbers and the Bass numbers then follows from this case by taking a finite free resolution.

The two final examples are cases where $\mathbb{S}$ is not the natural numbers, but the ring $\mathbb{Z}[[T]]$ of all formal power series over $\mathbb{Z}$ in one variable T.

11.7. Definition. We define the *Hilbert series* of a finitely generated R-module over an affine local ring R with maximal ideal $\mathfrak{m}$ as the formal power series

$$H_M(T) := \sum_{n=0}^{\infty} \ell\left(\frac{\mathfrak{m}^n M}{\mathfrak{m}^{n+1}M}\right)T^n \tag{70}$$

in the variable T. It is a classical result from Commutative Algebra that H_M is rational. More precisely, it is of the form

$$H_M(T) = \frac{Q_M(T)}{(1-T)^h} \tag{71}$$

where h is the dimension of M and where Q_M is a polynomial over $\mathbb{Z}$ with $Q_M(1) \neq 0$. The degree of Q_M is bounded by the so-called Castelnuovo-Mumford regularity of M. One shows that the Castelnuovo-Mumford regularity of M, and hence the degree of Q_M, is uniformly bounded in terms of the degree complexity of M.

Our invariant under investigation here will be the *Hilbert numerator* ω^{Hilbert}, which assigns to the pair (R, M) the polynomial $Q_M(T)$.

Now, let γ be an arbitrary polynomial in $\mathbb{Z}[T]$. We can write down a formula expressing that $Q_M = \gamma$ as follows. Let d be the degree complexity of M and D the bound on the degree of Q_M. We may obviously assume that γ has degree at most D as well. Now, by the results in Section 7, there exists a bound D', such that each of the modules $\mathfrak{m}^n M/\mathfrak{m}^{n+1}M$ has degree complexity at most D', for all $n \leq D$. In fact, there is a formula which expresses its specific value. Therefore, by working out the quotient (71), we can express that γ and Q_M have the same coefficients up to order D and whence must be equal. This shows that the invariant ω^{Hilbert} is asymptotically definable. More is true:

11.8. Theorem. *The invariant ω^{Hilbert} is saturated whence constructible.*

11.9. Definition. We define the *Poincare series* of an affine local ring R with residue field k as the formal power series

$$\zeta(R) := (1+T)^{-h} \sum_{n=0}^{\infty} \dim_k \operatorname{Tor}_n^R(k,k) T^n, \tag{72}$$

where h is the (Krull) dimension of R. Again, the initial factor is put there to guarantee that the invariant will be devissable. Note that $(1+T)$ is invertible in $\mathbb{Z}[[T]]$. Also note that this is an invariant only defined on the collection of local rings (without a module). All what has been said so far about invariants for modules adopts to this situation without any effort.

It is still not well understood when this power series is rational, but is so for complete intersections. For regular local rings it is a polynomial by Serre's Theorem, and one verifies that in fact $\zeta(R) = 1$. However, there exist counterexamples to its rationality in general. Therefore, let us call a scheme X *Poincare rational*, if for each point $x \in X$, we have that $\zeta(\mathcal{O}_{X,x})$ is rational.

Due to the existence of schemes which are not Poincare rational, we seem to stand little chance in proving that ζ is an asymptotically definable invariant; the argument used in the Hilbert case completely falls apart here. To get some uniformity, let us define the m-th truncation of ζ to be the power series (72) truncated at degree m, i.e., restricting n in the summation to run from 0 to m only.

11.10. Theorem. *The truncations of the $\mathbb{Z}[T]$-invariant ζ are asymptotically definable and devissable whence constructible. In particular, the $\mathbb{Z}[T]$-invariant ζ is pro-constructible.*

Recall that a subset is called *pro-constructible*, if it is a (possible infinite) intersection of constructible subsets. Let us show that even this partial result has some *pure* consequences.

11.11. Corollary. *If the Poincare series of each K-rational point on a scheme X is rational, then X is Poincare rational.*

Proof. Let x be an arbitrary point on X and let $\gamma(T)$ be its Poincare series, i.e., $\zeta(\mathcal{O}_{X,x}) = \gamma$. Let F be the subset of all points on X for which the Poincare series is γ, i.e., $F = \zeta^{-1}(\gamma)$. By Theorem 11.10, F is pro-constructible. However, as a constructible set is saturated and as the intersection of any number of saturated sets is saturated again, we conclude that F is saturated. Since by construction $x \in F$, there exists a K-rational point $y \in F$ which is a specialization of x. By assumption the Poincare series of y is rational and since $y \in F$, this series is precisely γ. ■

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