CLASSIFYING SINGULARITIES UP TO ANALYTIC EXTENSIONS OF SCALARS

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ABSTRACT. The singularity space consists of all germs (X, x), with X a Noetherian scheme and x a point, where we identify two such germs if they become the same after an analytic extension of scalars. This is a Polish space for the metric given by the order to which infinitesimal neighborhoods agree after base change. In other words, the classification of singularities up to analytic extensions of scalars is a smooth problem in the sense of descriptive set-theory.

1. INTRODUCTION

Roughly speaking, a classification problem consists of a class of objects together with an equivalence relation telling us which objects to identify; a solution to this problem is then an 'effective' or 'concrete' description of the quotient, preferably by a 'system of complete invariants'. What constitutes a reasonably concrete or effective solution to a classification problem, however, might depend on one's purposes or even one's taste. Descriptive set-theory proposes smoothness to be the decisive indication that a classification is explicit and/or concrete (see for instance [6] for a discussion). More precisely, recall that a *Polish* space is a complete metric space containing a countable dense subset. Considering a Polish space to be concrete is justified by the fact that its underlying Borel structure is in essence equal to the standard Borel space $\mathbb{R}$. With this in mind, an equivalence relation on a Polish space is called *smooth* if its quotient space is (Borel) isomorphic to a Polish space.

In this paper, we concern ourselves with a *local* classification problem from algebraic geometry: to describe all germs of points on arbitrary Noetherian schemes. Associating to a point its local ring, the problem reduces to the study of the category $\mathbb{Noe}$ of all Noetherian local rings. However, as part of this problem, we would have to classify already all fields, and even for countable fields [4] or fields of finite transcendence degree [15] these are non-smooth problems. Hence to circumvent this arithmetical obstruction, we will allow for 'extensions of scalars'–to be made more precise below–, resulting in the identification of any two fields of the same characteristic. Even after this modification, the local classification problem is probably still not smooth. We introduce one further identification, inspired in part by Grothendieck's suggestion to substitute the etale topos for the (classical) Zariski topos. A down-to-earth interpretation of this point of view is that two local rings can be considered identical if they have a common etale extension, or more generally, if they have the same completion. In summary, we say that two Noetherian local rings are *similar* if they can be made isomorphic by an *analytic extension of scalars*, that is to say, by the process of extending scalars and taking completion. To also make sense of this in mixed

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characteristic, we subsume these types of extensions under the larger class consisting of all unramified, faithfully flat extensions. We will show that *similar* points (meaning that their corresponding local rings are similar) have the same type of singularity (see Theorem 5.1). As a spinoff of this investigation, we obtain a flatness criterion generalizing a result of Kollár (Theorem 4.14).

Our assertion that classifying points up to similarity is smooth is established by effectively putting a metric on the space of similarity classes $\mathbb{N}\widetilde{\mathfrak{oe}}$, called the *deformation metric*. We will prove that the induced topology is complete, and that the collection of similarity classes of Artinian local rings with a finitely generated residue field is a countable dense subset. This shows that $\mathbb{N}\widetilde{\mathfrak{oe}}$ is a Polish space and hence classification up to similarity is a smooth problem. The deformation metric on $\mathbb{N}\widetilde{\mathfrak{oe}}$ is induced by a metric on $\mathbb{N}\mathfrak{oe}$ and in terms of (germs of) points, it measures to which order the infinitesimal neighborhoods of two points agree. So far, all concepts are algebraic-geometric in nature, but the existence of limits relies on a tool from model-theory, to wit, the ultraproduct construction. Of course, the ultraproduct of Noetherian local rings is in general no longer Noetherian. However, if we have a Cauchy sequence of Noetherian local rings, then their *separated ultraproduct* obtained by killing all infinitesimals in the ultraproduct, yields a complete Noetherian local ring, which, up to similarity, is the limit of the sequence.

In a future paper, we will investigate more properties of the deformation metric and we content ourselves here with quoting, without proof, two examples which should convince the reader of the naturality of the metric.

- The map $\mathrm{Hilb}\colon \mathbb{N}\mathfrak{oe} \to \mathbb{Z}[[t]]$ associating to a Noetherian local ring its Hilbert series, is continuous when we view $\mathbb{Z}[[t]]$ in its t-adic topology.
- Among the d-dimensional Noetherian local rings, the collection of all Cohen-Macaulay local rings is closed, or put differently, any sufficiently small deformation of a non-Cohen-Macaulay local ring preserving the dimension is again non-Cohen-Macaulay. Note that we have to fix the dimension as this is only an upper-semicontinuous invariant.

2. Deformation metric

In this paper, a *local* ring $(R, \mathfrak{m})$ means a (commutative) ring R with a unique maximal ideal $\mathfrak{m}$. As a rule, we will identify two local rings when they are isomorphic. Except for a sporadic occurrence of an ultraproduct, all local rings will moreover be Noetherian. Let $\mathbb{N}\mathfrak{oe}$ denote the category whose objects are (isomorphism classes of) Noetherian local rings, and whose morphisms are local ring homomorphisms. Note that $\mathbb{N}\mathfrak{oe}$ is a proper class, that is to say, *not* a set. However, in §4, we will define a quotient of $\mathbb{N}\mathfrak{oe}$ which is a set.

2.1. **Deformation metric.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. The *n-th infinitesimal deformation* of R is by definition the (Artinian) residue ring $R/\mathfrak{m}^n$ and will be denoted $R/^n$. Recall that the ($\mathfrak{m}$-adic) completion $\widehat{R}$ of R is the inverse limit of all n-th infinitesimal deformations of R, and that $R/^n \cong \widehat{R}/^n$. We define a semi-metric on $\mathbb{N}\mathfrak{oe}$, called the *deformation metric*, as follows. Given two Noetherian local rings R and S, let $\mathrm{d}(R, S)$ be the infimum of the numbers 2^{-n} for which $R/^n \cong S/^n$. In words, the distance between two local rings is at most 2^{-n} if their n-th infinitesimal deformations agree. One easily verifies that this distance function satisfies all the axioms of a metric, except that two distinct elements can be at distance zero, so that $\mathrm{d}(\cdot, \cdot)$ is only a semi-metric. By definition of completion, $\mathrm{d}(R, S) = 0$ if and only if $\widehat{R} \cong \widehat{S}$, so that $\mathrm{d}(\cdot, \cdot)$ induces a metric on the

full subcategory of all complete Noetherian local rings $\mathbb{N}\mathfrak{oecomp}$. Since a semi-metric is a metric if and only if the induced topology is Hausdorff, we call $\mathbb{N}\mathfrak{oecomp}$ the *Hausdorffication* of $\mathbb{N}\mathfrak{oe}$. The distance function satisfies the *ultrametric* (or, *non-archimedean*) triangle inequality $\mathrm{d}(R,T) \leq \max\{\mathrm{d}(R,S), \mathrm{d}(S,T)\}$, for all R, S and T in $\mathbb{N}\mathfrak{oe}$. As a result the topology induced by the semi-metric, called the *deformation topology*, is totally disconnected.

By convention, the zero-th infinitesimal deformation of a ring is zero (since we think of $\mathfrak{m}^0$ as the unit ideal). It follows that the distance between any two local rings is at most one. Immediately from the definitions we also get:

2.2. **Lemma.** *If $\mathrm{d}(R,S) < 1$, then R and S have the same residue field; if $\mathrm{d}(R,S) < 1/2$, then R and S have the same embedding dimension.*

In particular, embedding dimension is a continuous map from $\mathbb{N}\mathfrak{oe}$ to the discrete space $\mathbb{Z}$. This is no longer true for dimension: for instance $R := k[[X]]$ and $R_n := R/X^nR$ lie at distance 2^{-n}, yet their dimensions are not the same. One can show, however, that dimension is upper-semicontinuous.

By an (open) *ball* $\mathbf{B}$ in $\mathbb{N}\mathfrak{oe}$ with *center* R and *radius* $0 < \delta \leq 1$, we mean the collection of all S in $\mathbb{N}\mathfrak{oe}$ such that $\mathrm{d}(R,S) < \delta$. Since the metric is non-archimedean, any member of a ball is its center and every ball is both open and closed in the deformation topology, that is to say, is a *clopen*. Because the distance function only takes discrete values (the powers of $1/2$), any two radii which lie between two consecutive powers of $1/2$ yield the same ball. Therefore, with the *radius* of a ball $\mathbf{B}$, we mean twice the largest distance between two members of $\mathbf{B}$; this is always a power of $1/2$. (We need to take twice the distance since we used a strict inequality in the definition of a ball.)

A *unit ball* is a ball $\mathbf{B}$ with radius 1 and hence consists of all local rings with the same residue field. We call this common residue field the *residue field* of $\mathbf{B}$. This gives a one-one correspondence between unit balls and fields. More generally, to every ball $\mathbf{B}$, we associate an Artinian local ring $R_{\mathbf{B}}$, called the *residue ring* of $\mathbf{B}$, given as the unique local ring such that $R/^n \cong R_{\mathbf{B}}$, for all $R \in \mathbf{B}$, where 2^{-n+1} is the radius of $\mathbf{B}$. Note that $R_{\mathbf{B}}$ is a center of $\mathbf{B}$ and, moreover, the radius of $\mathbf{B}$ is determined by $R_{\mathbf{B}}$: it is equal to 2^{-n+1} where n is the nilpotency index of $R_{\mathbf{B}}$. In conclusion, there is a one-one correspondence between balls $\mathbf{B}$ and Artinian local rings.

2.3. **Cohen's structure theorems.** Cohen's structure theorems for Noetherian local rings will play an essential role in this paper, so we quickly review the relevant properties; a good reference for all this is [10, §29]. For each field k of prime characteristic p, there exists a uniquely defined complete discrete valuation ring V of characteristic zero whose residue field is k and whose maximal ideal is pV; we call V the *complete p-ring* over k. Let R be a Noetherian local ring with residue field k. We say that R has *equal characteristic* if R and k have the same characteristic; in the remaining case, we say that R has *mixed characteristic*. Assume R is moreover complete and let X be a finite tuple of indeterminates. Cohen's structure theorems now claim, among other things, the following:

- if R has equal characteristic, then it is a homomorphic image of $k[[X]]$;
- if R has mixed characteristic, then it is a homomorphic image of $V[[X]]$, where V is the complete p-ring over k.

2.4. **Proposition.** *Every ball is a set.*

Proof. It suffices to prove this for a unit ball $\mathbf{B}$. The result will follow if we can show that there is a cardinal number λ so that every member of $\mathbf{B}$ has cardinality at most λ. Let

k be the residue field of $\mathbf{B}$ and let $R \in \mathbf{B}$. Since the cardinality of a Noetherian local ring is at most the cardinality of its completion, we may assume that R is complete. By Cohen's structure theorems, R is a homomorphic image of $V[[X]]$, where X is a finite tuple of indeterminates and V is equal to k in the equal characteristic case, and equal to the complete p-ring over k in the mixed characteristic case. It is clear that in either case, the cardinality of $V[[X]]$ is bounded in terms of the cardinality of k, whence so is its homomorphic image R. □

Note that each ball $\mathbf{B}$ is infinite: if $R_{\mathbf{B}}$ is its residue ring, then the latter is of the form S/I, where $(S, \mathfrak{n})$ is a power series ring $V[[X]]$ as in §2.3. If n is the nilpotency index of $R_{\mathbf{B}}$, then $S/J \in \mathbf{B}$ for any ideal $J \subseteq S$ such that $J + \mathfrak{n}^n = I$.

3. Limits and ultraproducts

Let (M, d) be a semi-metric space and assume d is non-archimedean. To include the deformation metric in our treatment, we allow for M to be merely a class. A *sequence* $\mathbf{r}$ in M is simply a map $\mathbb{N} \to M$; often we view $\mathbf{r}$ as the collection of all $\mathbf{r}(w)$ for $w \in \mathbb{N}$. We call $\mathbf{r}$ a *Cauchy sequence* in (M, d), if for each $\epsilon > 0$, there exists N, such that $d(\mathbf{r}(w), \mathbf{r}(w+1)) < \epsilon$, for all $w > N$. We say that two Cauchy sequences $\mathbf{r}$ and $\mathbf{s}$ are *equivalent*, denoted $\mathbf{r} \sim \mathbf{s}$, if for each $\epsilon > 0$, there exists N, such that $d(\mathbf{r}(w), \mathbf{s}(w)) < \epsilon$, for all $w > N$. Let $\mathrm{Cau}(M, d)$, or simply, $\mathrm{Cau}(M)$, denote the set of all Cauchy sequences in M. We make $\mathrm{Cau}(M)$ into a semi-metric space by defining the *distance* between two Cauchy sequences $\mathbf{r}$ and $\mathbf{s}$, denoted $d(\mathbf{r}, \mathbf{s})$, as the lim-sup of the distances $d(\mathbf{r}(w), \mathbf{s}(w))$. Whence two Cauchy sequences are equivalent if and only if their distance is zero. There is a natural isometry $M \to \mathrm{Cau}(M)$ sending x to the constant sequence $\mathbf{x}$ given as $\mathbf{x}(w) := x$; we will identify the element x with its constant sequence in $\mathrm{Cau}(M)$.

A *limit* of a sequence $\mathbf{r}$ is an element $x \in M$ such that $\mathbf{r} \sim x$. It is easy to see that if $\mathbf{r}$ has a limit, then it must be Cauchy. We call (M, d) *complete* if every Cauchy sequence has a unique limit. This implies in particular that d is a metric. The *completion* of (M, d) is the metric space $\widehat{M} := \mathrm{Cau}(M)/\sim$ with its induced metric described above; it is a complete metric space containing M as a dense subspace.

For the remainder of this section, we work in $\mathbb{N}\mathfrak{oe}$ with its deformation metric (in fact, we may even restrict our study to the Hausdorffication $\mathbb{N}\mathfrak{oecomp}$). We will see that in order to understand $\widehat{\mathbb{N}\mathfrak{oe}}$, we need a notion from model-theory: the ultraproduct construction (some references for ultraproducts are [3], [8, §9.5] or the brief review in [12, §2]) .

Ultraproducts. Let $\mathbf{R}$ be a sequence in $\mathbb{N}\mathfrak{oe}$, that is to say, a collection of Noetherian local rings $(R_w, \mathfrak{m}_w)$, for $w \in \mathbb{N}$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. The *ultraproduct* of the R_w with respect to $\mathcal{U}$, denoted $\mathbf{R}_{\mathcal{U}}$, is defined as a certain homomorphic image of the product of the R_w. It is again a local ring, with maximal ideal $\mathfrak{m}_{\mathcal{U}}$ given as the ultraproduct of the $\mathfrak{m}_w$. In general, however, $\mathbf{R}_{\mathcal{U}}$ will no longer be Noetherian. If almost all R_w have embedding dimension at most n, then so does $\mathbf{R}_{\mathcal{U}}$. Call an element $r \in \mathbf{R}_{\mathcal{U}}$ an *infinitesimal* if it is contained in each power of $\mathfrak{m}_{\mathcal{U}}$. The set of all infinitesimals is an ideal in $\mathbf{R}_{\mathcal{U}}$ and the residue ring obtained by modding out this ideal is called the *separated ultraproduct* of the R_w and is denoted $\mathbf{R}_{(\mathcal{U})}$. If almost all R_w have embedding dimension at most n, then so does $\mathbf{R}_{(\mathcal{U})}$. Moreover, by the saturatedness property of ultraproducts, the separated ultraproduct is a complete local ring, whence Noetherian by [10, Theorem 29.4] (for more details see [14, Lemma 10.1]). In particular, if $\mathbf{R}$ is a Cauchy sequence, then by Lemma 2.2, almost of all its members have the same residue field, called the *residue field*

of $\mathbf{R}$, and the same embedding dimension, so that $\mathbf{R}_{(\mathcal{U})}$ is a complete Noetherian local ring.

Henceforth, we will only consider separated ultraproducts of Noetherian local rings of bounded embedding dimension, so that we tacitly assume that they are complete and Noetherian. Moreover, the particular choice of ultrafilter $\mathcal{U}$ will no longer matter, and we will denote the respective ultraproduct and separated ultraproduct of $\mathbf{R}$ simply by $\mathbf{R}_\infty$ and $\mathbf{R}_{(\infty)}$.[1] In case $\mathbf{R}$ is a constant sequence, given by a Noetherian local ring R, then we also write R_∞ and $R_{(\infty)}$ for the (separated) ultraproduct and call it the (separated) *ultrapower* of R. By Łos' Theorem, ultrapowers commute with base change, that is to say $(R/I)_\infty \cong R_\infty/IR_\infty$; the same is true for separated ultrapowers by [14, Corollary 5.3]:

3.1. **Lemma.** *If R is a Noetherian local ring and I an ideal in R, then $(R/I)_{(\infty)} = R_{(\infty)}/IR_{(\infty)}$.*

3.2. **Proposition.** *If $\mathbf{R}$ and $\mathbf{S}$ are Cauchy sequences in $\mathbb{N}\mathfrak{oe}$, then $\mathrm{d}(\mathbf{R}_{(\infty)}, \mathbf{S}_{(\infty)}) = \mathrm{d}(\mathbf{R}, \mathbf{S})$. In particular, $\mathbf{R} \sim \mathbf{S}$ if and only if $\mathbf{R}_{(\infty)} \cong \mathbf{S}_{(\infty)}$.*

Proof. The last assertion is immediate by the first. Suppose $\mathrm{d}(\mathbf{R}, \mathbf{S}) \leq 2^{-n}$. This means that for some N and all $w > N$, we have ${R_w}/^n \cong {S_w}/^n$, where R_w and S_w are the components of $\mathbf{R}$ and $\mathbf{S}$ respectively. By Lemma 3.1, the n-th infinitesimal deformations ${\mathbf{R}_{(\infty)}}/^n$ and ${\mathbf{S}_{(\infty)}}/^n$ are isomorphic, showing that $\mathrm{d}(\mathbf{R}_{(\infty)}, \mathbf{S}_{(\infty)}) \leq 2^{-n}$. In conclusion, we showed $\mathrm{d}(\mathbf{R}_{(\infty)}, \mathbf{S}_{(\infty)}) \leq \mathrm{d}(\mathbf{R}, \mathbf{S})$. To prove the converse, we may assume that $\mathrm{d}(\mathbf{R}, \mathbf{S}) = 2^{-n}$ for some $n > 0$. Since 2^{-n} is an isolated value, there is some N (namely the one corresponding to $\epsilon := 2^{-n-1}$), such that

$$\mathrm{d}(R_w, S_w) = 2^{-n}, \tag{1}$$

for all $w > N$. Towards a contradiction, suppose $\mathrm{d}(\mathbf{R}_{(\infty)}, \mathbf{S}_{(\infty)}) < 2^{-n}$. Hence

$${R_w}/^{n+1} \cong {S_w}/^{n+1}$$

for almost all w, contradicting (1). □

In particular, the separated ultraproduct of all n-th approximations of a Noetherian local ring R is equal to the separated ultrapower of R, and in fact, to the separated ultrapower of any Noetherian local ring having the same completion as R.

4. Scalar extensions

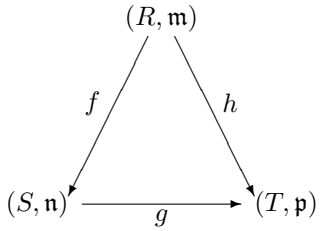
Proposition 3.2 is a step in the direction of finding limits in $\mathbb{N}\mathfrak{oe}$. However, the residue field of a separated ultraproduct $\mathbf{R}_{(\infty)}$ of a Cauchy sequence $\mathbf{R}$ is the ultrapower of the residue field of $\mathbf{R}$, by Lemma 3.1, and hence there is no chance that $\mathbf{R}_{(\infty)}$ is a limit of $\mathbf{R}$ in $\mathbb{N}\mathfrak{oe}$. In this section, we will find a way to control this change of base field phenomenon. This will allow us in the next section to define the similarity relation, which will then solve our limit problem.

Let $(R, \mathfrak{m})$ be a Noetherian local with residue field k and let l be a field extension of k. With a *scalar extension of R over l* we mean a local R-algebra $(S, \mathfrak{n})$ with residue field l such that $R \to S$ is faithfully flat, $\mathfrak{n} = \mathfrak{m}S$ and $R \to S$ induces the embedding $k \subseteq l$ on the residue fields. A *scalar extension* of a local ring R is then a scalar extension of R over some field extension of its residue field. The condition that $\mathfrak{n} = \mathfrak{m}S$ is also expressed

[1] There is really no reason to restrict only to ultraproducts on a countable index set, although it is the only type we will use in this paper. However, for the separated ultraproduct to be Noetherian and complete, we do have to impose that the ultrafilter is countably incomplete, which automatically holds on countable index sets and can always be arranged on arbitrary index sets.

by saying that $R \to S$ has *trivial closed fiber* or that it is *unramified*. By [5, 0_{III} 10.3.1], for any Noetherian local ring R and any extension l of its residue field, at least one scalar extension of R over l exists; we will reprove this in Corollary 4.4 below.

4.1. **Proposition.** *Given a commutative triangle of local homomorphisms between Noetherian local rings*

(2)


If any two are scalar extensions, then so is the third.

Proof. It is clear that the composition of two scalar extensions is again scalar. Assume g and h are scalar extensions. Then f is faithfully flat and $\mathfrak{m}T = \mathfrak{p} = \mathfrak{n}T$. Since g is faithfully flat, we get $\mathfrak{m}S = \mathfrak{m}T \cap S = \mathfrak{n}T \cap S = \mathfrak{n}$, showing that f is also a scalar extension. Finally, assume f and h are scalar extensions. Let

$$\dots R^{b_2} \to R^{b_1} \to R \to R/\mathfrak{m} \to 0 \tag{3}$$

be a free resolution of $R/\mathfrak{m}$. Since S is flat over R, tensoring yields a free resolution

$$\dots S^{b_2} \to S^{b_1} \to S \to S/\mathfrak{m}S \to 0. \tag{4}$$

By assumption $S/\mathfrak{m}S$ is the residue field l of S. Therefore, we can calculate $\operatorname{Tor}_1^S(T, l)$ as the homology of the complex

$$\dots T^{b_2} \to T^{b_1} \to T \to T/\mathfrak{m}T \to 0 \tag{5}$$

obtained from (4) by tensoring over S with T. However, (5) can also be obtained by tensoring (3) over R with T. Since T is flat over R, the sequence (5) is exact, whence $\operatorname{Tor}_1^S(T, l) = 0$. By the local flatness criterion, T is flat over S. Since $\mathfrak{n} = \mathfrak{m}S$ and $\mathfrak{p} = \mathfrak{m}T$, we get $\mathfrak{p} = \mathfrak{n}T$, showing that also g is a scalar extension. □

Three important examples of scalar extensions are given by the following proposition.

4.2. **Proposition.** *Let R be a Noetherian local ring.*

(4.2.1) *The natural map $R \to \widehat{R}$ is a scalar extension.*
(4.2.2) *Any etale map is a scalar extension.*
(4.2.3) *The natural map $R \to R_{(\infty)}$ is a scalar extension, where $R_{(\infty)}$ is a separated ultrapower of R.*

Proof. The first two assertions are well-known, so remains to show the last. Let $\mathfrak{m}$ be the maximal ideal of R. It is easy to show that $\mathfrak{m}R_{(\infty)}$ is the maximal ideal of $R_{(\infty)}$. So remains to prove that $R \to R_{(\infty)}$ is flat. Since $R_{(\infty)}$ is complete, we get a natural homomorphism $\widehat{R} \to R_{(\infty)}$ and it suffices to show that this is flat. Hence, without loss of generality, we may assume that R is complete. In particular, R is a homomorphic image of a regular local ring and if we prove the corresponding result for this regular local ring, then we get the desired result by Lemma 3.1. Therefore, we may moreover assume that R is regular. Since $\mathfrak{m}R_{(\infty)}$ is the maximal ideal of $R_{(\infty)}$ and since $R_{(\infty)}$ is also regular by [14, Theorems 4.2 and 10.6], of the same dimension as R, the flatness of $R \to R_{(\infty)}$ then follows from [10, Theorem 23.1]. □

In fact, (4.2.2) has the following converse: if $R \to S$ is essentially of finite type inducing a finite separable extension on the residue fields, then $R \to S$ is a scalar extension if and only if it is etale. In this sense, scalar extensions are generalizations of etale maps. This shows already that classification up to scalar extension is a reasonable and interesting problem. To gather further support for this claim, we will now explore how closely related scalar extensions are to isomorphisms. An important observation in that direction, one we will use several times below, is that a scalar extension of complete Noetherian local rings inducing an isomorphism on their residue fields is itself an isomorphism; see [10, Theorem 8.4]. Hence it is of interest to generate scalar extensions $R \to S$ with S complete. We will see that there exists a canonical choice over any field.

4.3. **Completions along a residual extension.** Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and let l be a field extension of k. The *completion of R along l* is the (unique) local R-algebra $R_l^\frown$ solving the following universal problem: given an arbitrary Noetherian local R-algebra S with residue field l, if S is complete, then there exists a unique local R-algebra homomorphism $R_l^\frown \to S$. When $k = l$, we recover the usual completion of R, that is to say, $R_k^\frown := \widehat{R}$. Here and elsewhere, we say that there is a *unique* homomorphism with certain properties, when we actually mean that there exists a unique homomorphism *up to isomorphism*; this is consistent with our practice of identifying two local rings when they are isomorphic.

To prove the existence of a completion along l, we have to treat the equal and mixed characteristic cases separately. Firstly, assume R has equal characteristic (this case is also discussed in [7, (6.3)]). By Cohen's structure theorems, there exists an embedding $k \to \widehat{R}$. Let $R_l^\frown$ be the $\mathfrak{m}(\widehat{R} \otimes_k l)$-adic completion of $\widehat{R} \otimes_k l$. To see that this is a completion along l, let S be a Noetherian local R-algebra with residue field l and assume S is complete. By the universal property of ordinary completions, we get a unique homomorphism $\widehat{R} \to S$. Since S is complete, we can find an embedding $l \to S$ which agrees on the subfield k of l with the composition $k \to \widehat{R} \to S$. By the universal property of tensor products, the two maps $\widehat{R} \to S$ and $l \to S$ combine to a unique local R-algebra homomorphism $\widehat{R} \otimes_k l \to S$, and using once more the universal property of completion, this then yields a unique R-algebra homomorphism $R_l^\frown \to S$.

In the mixed characteristic case, coefficient fields no longer exist and we now proceed as follows. Let V be the (unique) complete p-ring with residue field k, where p is the characteristic of k (see §2.3). We first define the completion of V along l, that is to say, $V_l^\frown$, as the unique complete p-ring with residue field l. That the latter satisfies the universal property of a completion along l is proven in [10, Theorem 29.2]. To define $R_l^\frown$, let S be any Noetherian local R-algebra with residue field l extending k and assume S is complete. As before, we have a unique local R-algebra homomorphism $\widehat{R} \to S$. By Cohen's structure theorems, there exists a commutative diagram of local homomorphisms

(6)
$$\begin{array}{ccc} V & \longrightarrow & V_l^\frown \\ \downarrow & & \downarrow \\ \widehat{R} & \longrightarrow & S. \end{array}$$

By the universal property of tensor products, we get a unique R-algebra homomorphism $\widehat{R} \otimes_V V_l\widehat{} \to S$. Define $R_l\widehat{}$ now as the $\mathfrak{m}(\widehat{R} \otimes_V V_l\widehat{})$-adic completion of $\widehat{R} \otimes_V V_l\widehat{}$, so that we get a unique local R-algebra homomorphism $R_l\widehat{} \to S$, as required. □

4.4. **Corollary.** *For every Noetherian local ring R and every extension field l of its residue field, $R_l\widehat{}$, the completion of R along l, exists and is unique. For every ideal I in R, the completion of R/I along l is equal to $R_l\widehat{}/IR_l\widehat{}$.*

Moreover, the natural map $R \to R_l\widehat{}$ is a scalar extension over l.

Proof. Existence was proven above; uniqueness then follows formally from being a solution to a universal problem. To prove the second assertion, assume $R/I \to S$ is a local homomorphism with S a complete Noetherian local ring with residue field l. The composition $R \to R/I \to S$ yields by definition a unique local R-algebra homomorphism $R_l\widehat{} \to S$. Since $IS = 0$, the latter homomorphism factors through $R_l\widehat{}/IR_l\widehat{}$. As for the last assertion, in the equal characteristic case, the base change $\widehat{R} \to \widehat{R} \otimes_k l$ of $k \subseteq l$ is faithfully flat. Since completion is exact, each map in

$$R \to \widehat{R} \to \widehat{R} \otimes_k l \to R_l\widehat{}$$

is faithfully flat, whence so is their composition. In the mixed characteristic case, $V_l\widehat{}$ is torsion-free whence flat over V. Hence by the same argument as in the equal characteristic case, the composite map

$$R \to \widehat{R} \to \widehat{R} \otimes_V V_l\widehat{} \to R_l\widehat{}$$

is faithfully flat. By our second assertion, $R_l\widehat{}/\mathfrak{m}R_l\widehat{}$ is the completion of $R/\mathfrak{m} \cong k$ along l in either characteristic. In other words, $R_l\widehat{}/\mathfrak{m}R_l\widehat{} \cong l$ and hence in particular, $\mathfrak{m}R_l\widehat{}$ is the maximal ideal of $R_l\widehat{}$. This proves that $R \to R_l\widehat{}$ is a scalar extension. □

4.5. **Proposition.** *Let $R \to S$ be a scalar extension over l. If S is complete, then $S \cong R_l\widehat{}$.*

Proof. By the universal property, we have a local R-algebra homomorphism $R_l\widehat{} \to S$. It follows from [10, Theorem 8.4] that $R_l\widehat{} \to S$ is surjective. Since $R \to R_l\widehat{}$ and $R \to S$ are scalar extensions by Corollary 4.4 and by assumption respectively, $R_l\widehat{} \to S$ is faithfully flat by Proposition 4.1, whence injective. □

4.6. **Corollary** (Lifting of scalar extensions)**.** *Let $R \to S$ be a scalar extension with S complete. If R is the homomorphic image of a Noetherian local ring A, then there exists a scalar extension $A \to B$ whose base change is $R \to S$, that is to say, $S = B \otimes_A R$.*

Proof. We leave it to the reader to verify that, after taking completions, we may assume that also A and R are complete. By Cohen's structure theorems, A and R are the homomorphic images of $V[[X]]$ modulo some ideals $J \subseteq I$ respectively, where V is either their common residue field or otherwise a complete p-ring with that residue field and where X is a finite tuple of indeterminates. Moreover, $S \cong R_l\widehat{}$ by Proposition 4.5, where l is the residue field of S. In particular, $S \cong V_l\widehat{}[[X]]/IV_l\widehat{}[[X]]$. Hence putting $B := V_l\widehat{}[[X]]/JV_l\widehat{}[[X]]$ yields a scalar extension $A \to B$ with $A/IA = R \to B/IB = S$, as required. □

The following result is a sharpening of [11, Theorem 2.4].

4.7. **Corollary.** *Let R be a Noetherian local ring with residue field k. If k_∞ is the ultrapower of k, then $R_{k_\infty}\widehat{}$ is equal to the ultrapower $R_{(\infty)}$.*

Proof. By Lemma 3.1, the residue field of $R_{(\infty)}$ is k_∞. Since $R \to R_{(\infty)}$ is a scalar extension by (4.2.3), and since $R_{(\infty)}$ is complete, $R_{(\infty)} \cong R_{k_\infty}\widehat{}$ by Proposition 4.5. □

4.8. **Corollary.** *Let $R \to S$ be a finite local homomorphism inducing a trivial extension on the residue fields. For every extension l of this common residue field, $S_l^\frown \cong R_l^\frown \otimes_R S$.*

Proof. The base change $S \to R_l^\frown \otimes_R S$ is faithfully flat. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals of R and S respectively. Since

$$(R_l^\frown \otimes_R S)/\mathfrak{n}(R_l^\frown \otimes_R S) \cong (R_l^\frown/\mathfrak{m}R_l^\frown) \otimes_{R/\mathfrak{m}} (S/\mathfrak{n}) \cong l \otimes_k k = l$$

the ideal $\mathfrak{n}(R_l^\frown \otimes_R S)$ is a maximal ideal. Since the base change $R_l^\frown \to R_l^\frown \otimes_R S$ is finite with trivial residue field extension and since $R_l^\frown$ is complete whence Henselian, $R_l^\frown \otimes_R S$ is a complete local ring. Hence we showed that $S \to R_l^\frown \otimes_R S$ is a scalar extension and since the latter ring is complete with residue field equal to l, it is isomorphic to $S_l^\frown$ by Proposition 4.5. □

4.9. **Corollary.** *Let $k \subseteq l$ be an extension of fields and let $\mathbf{B}_k$ and $\mathbf{B}_l$ be the unique unit balls in $\mathbb{Noe}$ with residue field k and l respectively. The map sending a ring in $\mathbf{B}_k$ to its completion along l is an isometry $\mathbf{B}_k \to \mathbf{B}_l$.*

Proof. Take $R, S \in \mathbf{B}_k$. Clearly, the completions $R_l^\frown$ and $S_l^\frown$ along l belong both to $\mathbf{B}_l$. Suppose $\mathrm{d}(R,S) \leq 2^{-n}$, that is to say, their n-th infinitesimal deformations $R/^n$ and $S/^n$ are isomorphic. By Corollary 4.4, the completions of $R/^n$ and $S/^n$ along l are respectively $R_l^\frown/^n$ and $S_l^\frown/^n$, and therefore are isomorphic, showing that $\mathrm{d}(R_l^\frown, S_l^\frown) \leq 2^{-n}$. □

4.10. **Corollary.** *Suppose R is an excellent local ring. If $R \to S$ is a scalar extension inducing a separable extension on the residue fields, then $R \to S$ is a regular homomorphism.*

Proof. By [10, Theorem 28.10], the scalar extension $R \to S$ is formally smooth, since it is unramified and the residue field extension is separable. The assertion now follows from a result by André in [1] (see also [10, p. 260]). □

In fact, with aid of Proposition 4.5, Corollary 4.6 and Cohen's structure theorems, one reduces to proving that $V[[X]] \to V_l^\frown[[X]]$ is regular, where V is either a field or a complete p-ring and where l is a separable extension of the residue field of V. This approach circumvents the use of André's deep result.

4.11. **Definition.** A Noetherian local ring R is called *analytically irreducible* if $\widehat{R}$ is a domain; it is called *absolutely analytically irreducible*, if $R_{k^{\mathrm{alg}}}^\frown$ is a domain, where k^{alg} is the algebraic closure of the residue field of R; and it is called a *universally irreducible* if any scalar extension of R is a domain.

4.12. **Corollary.** *If R is an excellent normal local domain with perfect residue field, then R is universally irreducible.*

Proof. Let S be a scalar extension of R. By Corollary 4.10, the map $R \to S$ is regular and hence S is again normal by [10, Theorem 32.2], whence a domain. □

4.13. **Proposition.** *If R is absolutely analytically irreducible, then it is universally irreducible.*

Proof. Since we will make no essential use of this result, we only give a sketch of a proof. We may reduce to the case that R is a complete Noetherian local domain with algebraically closed residue field k. We need to show that $R_l^\frown$ is a domain, where l is an arbitrary extension field of k. By Cohen's structure theorems, there exists a finite extension $S := V[[X]] \subseteq R$, where V is either k or the complete p-ring over k and X is a tuple of

indeterminates. Write $R = S[Y]/\mathfrak{p}$ for some finite tuple of indeterminates Y, so that $\mathfrak{p}$ is in particular a prime ideal. Since the fraction field of $S_l^\frown = V_l^\frown[[X]]$ is a regular extension of the fraction field of $S = V[[X]]$, the same argument as in the proof of [2, Lemma 5.21] then shows that $\mathfrak{p}S_l^\frown[Y]$ is a prime ideal. Hence we are done, since $R_l^\frown = S_l^\frown[Y]/\mathfrak{p}S_l^\frown[Y]$ by Corollary 4.8. □

We are ready to formulate a flatness criterion generalizing [9, Theorem 8].

4.14. **Theorem.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume R is universally irreducible, e.g., an excellent normal local domain with perfect residue field, or a complete local domain with algebraically closed residue field. If $R \to S$ is unramified and $\dim(R) = \dim(S)$, then $R \to S$ is faithfully flat, whence a scalar extension.*

Proof. Recall that $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ being unramified means that $\mathfrak{n} = \mathfrak{m}S$. It suffices to prove the assertion under the additional assumption that both R and S are complete. Indeed, if $R \to S$ is arbitrary, then $\widehat{R} \to \widehat{S}$ satisfies again the hypotheses of the theorem and therefore by assumption is faithfully flat. By an easy descent argument, $R \to S$ is then also faithfully flat.

So assume R and S are complete and let l be the residue field of S. By assumption, $R_l^\frown$ is a domain, of the same dimension as R. By the universal property of the completion along l, we get a local R-algebra homomorphism $R_l^\frown \to S$. By [10, Theorem 8.4], this homomorphism is surjective. It is also injective, since $R_l^\frown$ and S have the same dimension and $R_l^\frown$ is a domain. Hence $R_l^\frown \cong S$, so that $R \to S$ is a scalar extension. □

We end this section with a convergence criterion in terms of scalar extensions.

4.15. **Theorem.** *Let $\mathbf{R}$ be a Cauchy sequence in $\mathbb{N}\mathfrak{oe}$ and let S be a Noetherian local ring with the same residue field k as $\mathbf{R}$. The Cauchy sequence $\mathbf{R}$ converges to S if and only if the separated ultraproduct $\mathbf{R}_{(\infty)}$ is a scalar extension of S.*

In fact, let l be any extension field of the ultrapower k_∞ of k. If $\mathbf{R}_l^\frown$ denotes the sequence of rings obtained by taking the completions along l of all members of $\mathbf{R}$, then $\mathbf{R}_l^\frown$ is a Cauchy sequence converging to $(\mathbf{R}_{(\infty)})_l^\frown$.

Proof. By Proposition 3.2, if S is the limit of $\mathbf{R}$, then $S_{(\infty)} \cong \mathbf{R}_{(\infty)}$. Since $S \to S_{(\infty)}$ is a scalar extension by (4.2.3), we proved the direction implication. For the converse, assume $S \to \mathbf{R}_{(\infty)}$ is a scalar extension. By Corollary 4.7, we have an isomorphism $S_{(\infty)} \cong S_{k_\infty}^\frown$. Since $\mathbf{R}_{(\infty)}$ is complete with residue field k_∞, it is also isomorphic to $S_{k_\infty}^\frown$ by Proposition 4.5. It follows then from Proposition 3.2 that $\mathrm{d}(\mathbf{R}, S) = 0$, that is to say, that S is a limit of $\mathbf{R}$.

To prove the last assertion, let R_w be the rings in $\mathbf{R}$ and fix some n. Since $\mathbf{R}$ is Cauchy, there exists w_n so that for $w > w_n$, all $R_w/^n$ are isomorphic, say to T. By Lemma 3.1, the n-th infinitesimal deformation $\mathbf{R}_{(\infty)}/^n$ is isomorphic to the separated ultrapower $T_{(\infty)}$; the latter is isomorphic to $T_{k_\infty}^\frown$ by Corollary 4.7; and this in turn is isomorphic to $\big((R_w)_{k_\infty}^\frown\big)/^n$, for all $w > w_n$ by Corollary 4.4. In summary, we showed that $\mathrm{d}((R_w)_{k_\infty}^\frown, \mathbf{R}_{(\infty)}) \leq 2^{-n}$, for all $w > w_n$. In view of Corollary 4.4, taking completions along l yields $\mathrm{d}((R_w)_l^\frown, (\mathbf{R}_{(\infty)})_l^\frown) \leq 2^{-n}$, for all $w > w_n$. Since this holds for all n, the assertion follows. □

5. SIMILARITY SPACE

We now introduce an equivalence relation on $\mathbb{N}\mathfrak{oe}$ which, although coarser than the isomorphism relation, preserves most local singularity properties (see for instance Theorem 5.1 below). Namely, we say that two Noetherian local rings R and S are *similar*, denoted $R \approx S$, if they admit a common scalar extension. Let T be this common scalar extension. Its completion is again a scalar extension and by Proposition 4.5, it is therefore isomorphic to both $R_l^\wedge$ and $S_l^\wedge$, where l is the residue field of T. In other words, we showed that $R \approx S$ if and only if $R_l^\wedge \cong S_l^\wedge$ for some sufficiently large common extension l of their respective residue fields. It follows easily from this that $\approx$ is an equivalence relation. The collection of all local rings similar to a given Noetherian local ring R is called the *similarity class* of R and is denoted $[R]$. Immediately from the results in [10, §23] and [13, Proposition 9.3] (where the notion of a *singularity defect* is introduced), we get:

5.1. **Theorem.** *If two Noetherian local rings are similar, then they have the same dimension, depth and Hilbert series, and one is regular (respectively, Cohen-Macaulay, Gorenstein, complete intersection) if and only if the other is. More generally, any two similar local rings have the same singularity defects.*

Using Corollary 4.10, other properties, such as being reduced or normal, are also invariant under the similarity relation, provided the rings are excellent with perfect residue field. Note that being a domain is not preserved under the similarity relation, necessitating definitions 4.11.

5.2. **Proposition.** *Any two separated ultrapowers of a Noetherian local ring, or more generally, any two Noetherian local rings which are elementary equivalent, are similar.*

More generally, let $\mathbf{R} := (R_w)_w$ *and* $\mathbf{S} := (S_w)_w$ *be sequences of Noetherian local rings of embedding dimension at most* d*. If almost each* R_w *is similar to* S_w*, then the respective separated ultraproducts* $\mathbf{R}_{(\infty)}$ *and* $\mathbf{S}_{(\infty)}$ *are also similar.*

Proof. Suppose R and S are elementary equivalent Noetherian local rings. By the Keisler-Shelah theorem (see [8, Theorem 9.5.7]), some ultrapower of R and S are isomorphic, whence so are their corresponding separated ultrapowers (strictly speaking, the underlying index set will in general no longer be countable, so that we have to make some minor modifications alluded to in footnote (1); details are left to the reader). By Proposition 4.2, these are scalar extensions of R and S respectively, proving the first assertion.

To prove the second assertion, we may without loss of generality assume that all rings are complete. By our discussion above, we may further reduce to the case that $S_w \cong (R_w)_{l_w}^\wedge$, for some field extension l_w of the residue field of R_w. Since R_w is a homomorphic image of a d-dimensional regular local ring by Cohen's structure theorems, and since the property we seek to prove is preserved under homomorphic images by Lemma 3.1 and Corollary 4.4, we may moreover assume by Corollary 4.6 that each R_w is regular, of dimension d. By Theorem 5.1, almost each S_w is regular, also of dimension d. By [14, Theorems 4.2 and 10.6], the separated ultraproducts $\mathbf{R}_{(\infty)}$ and $\mathbf{S}_{(\infty)}$ are therefore also d-dimensional regular local rings. Since $\mathbf{R}_{(\infty)} \to \mathbf{S}_{(\infty)}$ is unramified by Lemma 3.1, it is faithfully flat by [10, Theorem 23.1], whence a scalar extension, as we wanted to show. □

We denote the collection of all similarity classes of Noetherian local rings by $\mathbb{N}\widetilde{\mathfrak{oe}}$. By (4.2.1), the similarity relation restricted to $\mathbb{N}\mathfrak{oecomp}$ has the same quotient $\mathbb{N}\widetilde{\mathfrak{oe}}$. Although $\mathbb{N}\mathfrak{oe}$ was only a class, we do no longer have this complication for its quotient:

5.3. **Proposition.** *The quotient* $\mathbb{N}\widetilde{\mathfrak{oe}}$ *is a set.*

Proof. Let $[R]$ be a similarity class of Noetherian local rings and let k be the residue field of R. Since $R \approx \widehat{R}$, we may assume that R is complete, whence by Cohen's structure theorems, the homomorphic image of $S := V[[X]]$ with V either equal to k or to the complete p-ring over k, and with X a finite tuple of indeterminates. Suppose $I = (f_1, \ldots, f_s)S$. We may choose a subring V_0 of V of size at most the continuum so that it contains all coefficients of the f_i and so that V_0 is again a field or a complete p-ring. Let $S_0 := V_0[[X]]$ and $I_0 = (f_1, \ldots, f_s)S_0$, so that $S \cong (S_0)^\wedge_k$. Hence by base change $R \cong S/I$ is a scalar extension of S_0/I_0, so that $S_0/I_0 \approx R$. In conclusion, we showed that every similarity class contains a ring of size at most the continuum, and therefore $\mathbb{N}\widetilde{\mathfrak{oe}}$ is a set. □

5.4. **Similarity metric.** We want to extend the metric on $\mathbb{N}\mathfrak{oe}$ to a metric on the similarity space $\mathbb{N}\widetilde{\mathfrak{oe}}$, in such a way that the natural map $\mathbb{N}\mathfrak{oe} \to \mathbb{N}\widetilde{\mathfrak{oe}}$ preserves the distance. For two similarity classes $[R]$ and $[S]$, let $\mathrm{d}([R],[S])$ be equal to the infimum of all $\mathrm{d}(R',S')$ with $R' \approx R$ and $S' \approx S$. Alternatively, recall that for a (semi-)metric space (M,d), the distance between two subclasses U and V is defined to be the infimum of all $\mathrm{d}(x,y)$ with $x \in U$ and $y \in V$; hence $\mathrm{d}([R],[S])$ is just the distance between $[R]$ and $[S]$ viewed as subclasses of $\mathbb{N}\mathfrak{oe}$.

5.5. **Lemma.** *For all Noetherian local rings R and S and all $n \in \mathbb{N}$, we have $\mathrm{d}([R],[S]) \leq 2^{-n}$ if and only if $R/^n \approx S/^n$.*

Proof. Choose $R' \approx R$ and $S' \approx S$ so that $\mathrm{d}([R],[S]) = \mathrm{d}(R',S')$. Without loss of generality, we may assume R' and S' to be complete. Let $R/^n$, $R'/^n$, $S/^n$ and $S'/^n$ be the n-th infinitesimal deformations of R, R', S and S' respectively. If $\mathrm{d}(R',S') \leq 2^{-n}$, then $R'/^n \cong S'/^n$ and hence $R/^n \approx S/^n$. Conversely, assume $R/^n \approx S/^n$ and let T be a common scalar extension of $R/^n$ and $S/^n$. Let l be the residue field of T. By Corollary 4.4, the n-th infinitesimal deformations of $R^\wedge_l$ and $S^\wedge_l$ are equal to T. In other words, $\mathrm{d}(R^\wedge_l, S^\wedge_l) \leq 2^{-n}$. Since $\mathrm{d}([R],[S])$ is defined as an infimum, $\mathrm{d}([R],[S]) \leq 2^{-n}$. □

5.6. **Corollary.** *The quotient $\mathbb{N}\widetilde{\mathfrak{oe}}$ endowed with the distance function d is an ultrametric space.*

Proof. The ultrametric triangle inequality follows immediately from Lemma 5.5 and the fact that $\approx$ is an equivalence relation. To show that it is a metric suppose $\mathrm{d}([R],[S]) = 0$. By Lemma 5.5, the n-th infinitesimal deformations $R/^n$ and $S/^n$ of R and S are similar, for all n. Hence there exists a common scalar extension T_n of $R/^n$ and $S/^n$. We may inductively choose T_{n+1} to have a residue field containing the residue field of T_n by Corollary 4.9, since scalar extensions can only make the distance smaller. Let l be the union of all these residue fields. By Corollary 4.4, the n-th infinitesimal deformations of $R^\wedge_l$ and $S^\wedge_l$ are equal to $(T_n)^\wedge_l$. Since this holds for all n, we showed that $\mathrm{d}(R^\wedge_l, S^\wedge_l) = 0$ whence $R^\wedge_l \cong S^\wedge_l$ and hence $[R] = [S]$. □

It follows from Theorem 4.15 that given a Cauchy sequence $\mathbf{R}$ in $\mathbb{N}\mathfrak{oe}$, the sequence $\mathbf{R}^\wedge_{k_\infty}$ has a limit, where k_∞ is the ultrapower of the residue field of $\mathbf{R}$. Since the corresponding members of $\mathbf{R}$ and $\mathbf{R}^\wedge_{k_\infty}$ are similar, we showed that every Cauchy sequence becomes convergent after replacing each of its components by an appropriately chosen similar ring. Therefore, the next result should not come as a surprise:

5.7. **Theorem.** *The ultrametric space $\mathbb{N}\widetilde{\mathfrak{oe}}$ is complete.*

Proof. We will define an isometry $\widehat{c}\colon \widehat{\mathbb{N}\widetilde{\mathbb{o}\mathbb{e}}} \to \mathbb{N}\widetilde{\mathbb{o}\mathbb{e}}$ as follows. We start with defining a map $c\colon \mathrm{Cau}(\mathbb{N}\widetilde{\mathbb{o}\mathbb{e}}) \to \mathbb{N}\widetilde{\mathbb{o}\mathbb{e}}$. Let $\tilde{\mathbf{R}} := ([R_w])_w$ be a Cauchy sequence in $\mathbb{N}\widetilde{\mathbb{o}\mathbb{e}}$ and let $\mathbf{R}_{(\infty)}$ be the separated ultraproduct of the R_w (this is a complete Noetherian local ring since almost all R_w have the same embedding dimension). Define $c(\tilde{\mathbf{R}}) := [\mathbf{R}_{(\infty)}]$. By Proposition 5.2, the map c is well-defined, that is to say, does not depend on the choice of representatives R_w. Suppose $\tilde{\mathbf{S}} := ([S_w])_w$ is a second Cauchy sequence which is equivalent to $\tilde{\mathbf{R}}$ and let $\mathbf{S}_{(\infty)}$ be the separated ultraproduct of the S_w. For a fixed n, we have $\mathrm{d}([R_w],[S_w]) \leq 2^{-n}$ for all sufficiently large w. By Lemma 5.5, the n-th infinitesimal deformations of R_w and S_w are therefore similar, for all sufficiently large w. By Proposition 5.2, then so are the n-th infinitesimal deformations of $\mathbf{R}_{(\infty)}$ and $\mathbf{S}_{(\infty)}$, so that $\mathrm{d}([\mathbf{R}_{(\infty)}],[\mathbf{S}_{(\infty)}]) \leq 2^{-n}$ by another application of Lemma 5.5. Since this holds for all n, we showed that $[\mathbf{R}_{(\infty)}] = [\mathbf{S}_{(\infty)}]$. By definition of completion, c therefore factors through a map

$$\widehat{c}\colon \widehat{\mathbb{N}\widetilde{\mathbb{o}\mathbb{e}}} \to \mathbb{N}\widetilde{\mathbb{o}\mathbb{e}}.$$

We leave it to the reader to check that $\widehat{c}$ preserves the metric. Note that $\widehat{c}$ restricted to $\mathbb{N}\widetilde{\mathbb{o}\mathbb{e}}$ is the identity, since a separated ultrapower is a scalar extension by Proposition 5.2. Hence $\widehat{c}$ must be surjective. To prove injectivity, assume $\mathbf{R}$ and $\mathbf{S}$ are Cauchy sequences in $\mathbb{N}\mathbb{o}\mathbb{e}$ whose respective separated ultraproducts $\mathbf{R}_{(\infty)}$ and $\mathbf{S}_{(\infty)}$ are similar. Let l be a large enough field extension so that

$$(\mathbf{R}_{(\infty)})_l^{\widehat{}} \cong (\mathbf{S}_{(\infty)})_l^{\widehat{}}.$$

By Theorem 4.15, the (component-wise) completion $\mathbf{R}_l^{\widehat{}}$ along l converges to $(\mathbf{R}_{(\infty)})_l^{\widehat{}}$, and likewise $\mathbf{S}_l^{\widehat{}}$ converges to $(\mathbf{S}_{(\infty)})_l^{\widehat{}}$. Therefore, $\mathbf{R}_l^{\widehat{}}$ and $\mathbf{S}_l^{\widehat{}}$, as they converge to the same limit, are equivalent, which proves that $\widehat{c}$ is injective. □

5.8. **Theorem.** *The ultrametric space $\mathbb{N}\widetilde{\mathbb{o}\mathbb{e}}$ is a Polish space. In particular, the similarity relation is smooth.*

Proof. In view of Theorem 5.7, it remains to show that $\mathbb{N}\widetilde{\mathbb{o}\mathbb{e}}$ contains a countable dense subset. We already observed that there is a one-one correspondence between balls and Artinian local rings in $\mathbb{N}\mathbb{o}\mathbb{e}$, so that the Artinian local rings form a dense subset of $\mathbb{N}\mathbb{o}\mathbb{e}$ whence of $\mathbb{N}\widetilde{\mathbb{o}\mathbb{e}}$. Let R be an Artinian local ring with residue field k. By Cohen's structure theorems, R is of the form $V[[X]]/I$, where V is either k or the complete p-ring over k, and where X is a tuple of indeterminates. Since R is Artinian, it is in fact finitely generated over V. Hence, by an argument similar to the one in the proof of Proposition 5.3, there exists a finitely generated subfield $k_0 \subseteq k$ and an Artinian local ring R_0 with residue field k_0, such that $R_0 \approx R$. Since there are only countably many finitely generated fields, the collection of all these R_0 is again countable. □

5.9. *Remark.* Instead of working with the category $\mathbb{N}\mathbb{o}\mathbb{e}$ in the above, we can do exactly the same thing with the category $\mathbb{N}\mathbb{o}\mathbb{e}_Z$ of all Noetherian local Z-algebras, for Z a Noetherian ring, so that the morphisms are now given by local Z-algebra homomorphisms.

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