Constructible Invariants

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Abstract

A local numerical invariant is a map ω which assigns to a local ring R a natural number $\omega(R)$. It induces on any scheme X a partition given by the sets consisting of all points x of X for which $\omega(\mathcal{O}_{X,x})$ takes a fixed value. Criteria are given for this partition to be constructible, in case X is a scheme of finite type over a field. It follows that if the partition is constructible, then it is finite, so that the invariant takes only finitely many different values on X. Examples of local numerical invariants to which these results apply, are the regularity defect, the Cohen-Macaulay defect, the Gorenstein defect, the complete intersection defect, the Betti numbers and the (twisted) Bass numbers.

As an application, we obtain that an affine scheme of finite type over a field is 'asymptotically a complete intersection'.

Key words: Constructible property, invariant, Betti number, Bass number, regularity defect, complete intersection defect, Gorenstein defect, Cohen-Macaulay defect.

1 Introduction

In [2, Chap. IV, §9], Grothendieck studies in detail the nature of the subset on a scheme X consisting of all points which have a certain property, or the fiber of which with respect to a map of finite type $Y \to X$ has a certain property. To name a few of these properties, points (or rather, their local rings) could be regular, complete intersections, Gorenstein or Cohen-Macaulay, and fibers could be non-empty, reduced or regular. Subsets defined by these conditions often turn out to be open (or closed). This is particularly useful in arguments using induction on the dimension, especially for the study of fibers of a map. In fact, all one needs to know is that the set (or its complement) is dense for the induction to go through.

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The aim of this paper is to extend this *qualitative* analysis of [2] to a *quantitative* one in the following sense. Rather than study properties, we will study numerical (and other) invariants which, in some sense, describe the defect that a particular property holds. For instance, let $(R, \mathfrak{m})$ be a Noetherian local ring. Let d be its dimension and e its embedding dimension, that is to say, the minimal number of generators of $\mathfrak{m}$. Then always $d \leq e$, with equality if and only if R is regular. Hence the numerical invariant $e - d$ measures the defect of R being regular and accordingly is called the *regularity defect* of R. The goal is now to study the collection X_s of all points on a scheme X with a prescribed regularity defect $s \in \mathbb{N}$ (the regularity defect of a point is the regularity defect of its local ring). Our techniques will show that at least for schemes X of finite type over a field, such a set X_s is constructible. In particular, as we let the regularity defect run over all possible values, we get a constructible partition of X (after dismissing those X_s which are empty, of course). Such a partition is necessarily finite: indeed, except for the finitely many X_s containing a generic point, their dimension is strictly smaller than the dimension of X and hence by induction on the dimension, only finitely many can be non-empty. In particular, there are only finitely many regularity defects which can occur on a fixed scheme. This extends to include various other singularity defects, where we mean with a *singularity defect* any of the following defects (see Section 7 for their definition): regularity defect, Cohen-Macaulay defect, Gorenstein defect or complete intersection defect.

Theorem 1.1 *For each scheme X of finite type over a field K, the collection of points for which a singularity defect has a fixed value s, is a constructible subset of X. In particular, a singularity defect takes only finitely many different values on a scheme X.*

More generally, if $f\colon Y \to X$ is a map of finite type of schemes of finite type over K, then the collection of points y in Y for which the fiber $f^{-1}(f(y))$ has a prescribed singularity defect at y, is constructible, and only finitely many possibilities for these values occur.

The first part, under the additional assumption that K is algebraically closed, is Theorem 7.1 below; the full version then follows from this by the results in the last two sections. Applying the theorem for the complete intersection defect gives the following corollary (see the end of Section 8 for a proof).

Theorem 1.2 *For each map $f\colon Y \to X$ of finite type of schemes of finite type over a field K, there exists a number $D_f \in \mathbb{N}$, such that for each n and each point $x \in X$, if the fiber $Y_x := f^{-1}(x)$ is embedded as a closed subscheme of $\mathbb{A}^n_{k(x)}$, where $k(x)$ is the residue field of the point x, then Y_x is the (scheme-theoretic) intersection of at most $D_f + n$ hypersurfaces.*

Corollary 1.3 *Let X be an affine scheme of finite type over a field K and for each closed immersion $i\colon X \hookrightarrow \mathbb{A}^{n(i)}_K$, let $\rho(i)$ be the minimal number of hypersurfaces* *needed to define X scheme-theoretically. Then X is* asymptotically a complete intersection *in the sense that the limit of $\rho(i)/n(i)$ for $n(i)$ going to infinity is equal to one.*

Proof. By Theorem 1.2, there exists a bound $D := D(X)$ such that $\rho(i) \leq D+n(i)$. On the other hand, since $d := \dim(X)$ is at least $n(i) - \rho(i)$ by Krull's Principal Ideal Theorem, we have $n(i) - d \leq \rho(i) \leq n(i) + D$, proving that in the limit $\rho(i)/n(i)$ is one. □

Let me briefly describe the strategy for obtaining the constructibility results stated in Theorem 1.1. To simplify the exposition, assume that ω denotes a numerical invariant, that is to say, ω assigns to a Noetherian local ring R a natural number $\omega(R)$. (In the text we will deal with more complicated invariants, involving finitely generated modules and maps). Let X be a scheme of finite type over an algebraically closed field K (the generalization to arbitrary fields is postponed until the last section). We want to determine the nature of the *level* set

$$X_s := \{\, x \in X \mid \omega(\mathcal{O}_{X,x}) = s \,\}\,, \tag{1}$$

where s is a natural number. In order to prove that X_s is constructible, we first show that $X_{\max} \cap X_s$ is constructible, where $X_{\max}$ denotes the (topological) space of all closed (that is to say, K-rational) points on X. Next we show that X_s is *saturated*, in the following sense: for any $x \in X_s$, we can find a *specialization* $y \in X_{\max}$ of x which also lies in X_s. It then follows, by a general argument discussed in Section 4, that X_s is constructible. In fact, to prove that X_s is saturated, it suffices to show that it is *devissable* (from the French *dévissage*), meaning that for each non-closed point x in X_s, we can find an open $U \subseteq X$ containing x with the following property. If y is an immediate specialization of x lying in U, then y lies also in X_s. Under some additional assumptions, namely, if X is Cohen-Macaulay and ω *deforms well* (that is to say, is stable under reduction modulo a regular element; see Definition 5.3), it suffices to check this for x a generic point of X. It follows by an easy induction on dimension that a devissable set is saturated.

In summary, our task is twofold. Given an invariant ω and a scheme X over an algebraically closed field K, in order to prove that the level sets X_s are constructible we have to establish the following two facts:

(1.3.1) Each level set X_s when restricted to the space of closed points $X_{\max}$ is constructible (in the induced Zariski topology).
(1.3.2) Each level set X_s is devissable.

It turns out that the first condition is model-theoretic in nature and the second is algebraic. To solve problem (1.3.1) for the invariants mentioned in the abstract, the necessary research has already been carried out in [6] and I only need to discuss how to translate the results from that paper into the geometric language of this paper. This is carried out in the second section. The model-theoretic approach guarantees that these constructibility results will be base field free whence also characteristic free. The advantage of this is the applicability of the Lefschetz Principle and is demonstrated in length in the papers [6,7]. It also provides us with a more uniform and functorial result, which is needed for the second part of Theorem 1.1.

The third and fourth section develop the general theory. The two subsequent sections put this general theory to use by showing the constructibility of the Betti and Bass numbers and the singularity defects. Most of the work here goes to proving devissability, that is to say, to solving problem (1.3.2). In fact, in view of the algebraic nature, this part can be carried out in a more general setup: often it suffices that the scheme is excellent. The penultimate section deals with a relative version needed for the second part of Theorem 1.1 and the final section explains how these results can be extended to base fields which are no longer algebraically closed, using some form of faithfully flat descent.

Notation. In this paper, except in the last section, K will always denote some algebraically closed field. Schemes will always be understood to be Noetherian, and often, they will be of finite type over K. If a scheme X is defined over $\mathbb{Z}$, then $X(K)$ will denote the set of K-rational points of X and X_K the base change to K, that is to say, $X_K = X \times \operatorname{Spec} K$. In other words, $X(K) = (X_K)_{\text{max}}$. An *affine* algebra is an algebra essentially of finite type over a field.

The difference of two sets F and G is denoted by $F - G$. Whenever it is clear in which ambient set X we work, we will denote the complement $X - F$ of a subset F of X simply by $-F$.

2 Local invariants

All rings and schemes in this paper will be understood to be Noetherian. Let $\mathbb{S}$ be an arbitrary set; often $\mathbb{S}$ will just be the set of natural numbers $\mathbb{N}$.

Definition 2.1 *With a* (local, $\mathbb{S}$-valued, ring) invariant, *we mean a function ω which assigns to a Noetherian local ring R an element $\omega(R)$ in $\mathbb{S}$. With a* (local, $\mathbb{S}$-valued) module invariant ω, *we mean a function which assigns to a pair (R, M), an element $\omega(R, M) \in \mathbb{S}$, where R is a Noetherian local ring and M a finitely generated R-module.*

If the ring R is understood, we might just write $\omega(M)$ for $\omega(R, M)$. Sometimes we simply talk about a local invariant and leave it to the context whether a module invariant or a ring invariant is meant, or which values this invariant takes. In case $\mathbb{S} \subseteq \mathbb{Z}$ (possibly including also $\pm\infty$), we call ω a *numerical invariant*. For naturally occurring invariants, we often have to restrict the scope of ω to a subclass of pairs

(R, M), although we could formally circumvent this by adding a symbol to $\mathbb{S}$ which we then assign to a pair with undefined ω-value. Anyway, at times, we will be only interested in an invariant restricted to a certain subclass, and we will make this then explicit.

Let X be a scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Given a point $x \in X$, we say that ω is *defined* (for $\mathcal{F}$) at x if it is defined on $\mathcal{O}_{X,x}$ (respectively, on the pair $(\mathcal{O}_{X,x}, \mathcal{F}_x)$). When this is the case, we put

$$\omega_X(x) := \omega(\mathcal{O}_{X,x}), \quad \text{respectively} \quad \omega_X(x, \mathcal{F}) := \omega(\mathcal{O}_{X,x}, \mathcal{F}_x),$$

where we may leave out the subscript X if the underlying scheme X is understood. We say that ω is *defined* (for $\mathcal{F}$) on X, if its defined (for $\mathcal{F}$) at each point of X. Since we are especially interested in schemes of finite type over an algebraically closed field, we reserve a special name for any invariant that is defined on them: we will say that such an invariant is *of finite type*. Assume ω is defined (for $\mathcal{F}$) on X. For $s \in \mathbb{S}$, we define the *level set* to be the set

$$\omega_X^{-1}(s) := \{ x \in X \mid \omega(x) = s \}$$

or, in case of a module invariant, the set

$$\omega_{X,\mathcal{F}}^{-1}(s) := \{ x \in X \mid \omega(x, \mathcal{F}) = s \} .$$

3 Geometrically constructible sets

Let X be a Noetherian scheme. With a *subset* T of X we really mean a subset of the underlying set of points of X. The *Zariski closure* of T will be denoted by $\operatorname{cl}(T)$. Recall that T is called *constructible* if it is a finite Boolean combination of Zariski closed subsets. The *constructible topology* on X has as opens precisely the constructible subsets of X. We denote the collection of all closed points of X by X_{max}, and view it with its induced Zariski topology. More generally, for an arbitrary subset $T \subseteq X$, we put $T_{\text{max}} := X_{\text{max}} \cap T$.

Geometrically constructible sets. A subset T of X is called *geometrically constructible*, if there exists a constructible subset F of X, such that $F_{\text{max}} = T_{\text{max}}$. In other words, T is geometrically constructible if T_{max} is constructible in X_{max}. Recall that a scheme is called *Jacobson* if it admits a finite open covering by affine schemes $\operatorname{Spec} A_i$ with each A_i a Noetherian *Jacobson* ring, that is to say, a Noetherian ring in which each radical ideal is equal to the intersection of all maximal ideals containing it. Any scheme of finite type over a field is Jacobson; more generally, so is any scheme of finite type over a Noetherian Jacobson ring ([1, Theorem A.17]). We proved in [10, Theorem 1.13] that X is Jacobson if and only if every closed subset of dimension $d > 0$ contains infinitely many irreducible closed subsets of dimension $d-1$, if and only if any constructible subset has the same dimension as its closure. Here are some further characterizations.

Lemma 3.1 *For a Noetherian scheme X, the following are equivalent:*

- *X is Jacobson;*
- *if $F, G \subseteq X$ are constructible and $F_{max} = G_{max}$, then $F = G$;*
- *X_{max} is dense in the constructible topology.*

Proof. Note that X_{max} being dense in the constructible topology means that F_{max} is non-empty, whenever F is a non-empty constructible subset. Applying this criterion to the symmetric difference $(F - G) \cup (G - F)$, we see that the last two conditions are equivalent. Remains to prove the equivalence with the first condition. Since the problem is local, we may assume that $X = \operatorname{Spec} A$ is affine. Assume first that A is Jacobson and suppose F is a non-empty constructible subset. Since we want to show that $F_{\text{max}} \neq \emptyset$, we may reduce to the case that $F = V \cap U$ is locally closed, with V a closed subset and U an open subset. Since V is also Jacobson, we may replace X by V and hence assume that F is a non-empty open subset, say of the form $X - \mathrm{V}(\mathfrak{a})$, with $\mathfrak{a}$ a radical ideal. Since $\mathfrak{a}$ is the intersection of all maximal ideals containing it and since it is not nilpotent lest F be empty, there must be at least one maximal ideal $\mathfrak{m}$ of A not containing $\mathfrak{a}$. This maximal ideal then determines a closed point inside F, as we wanted to show.

Conversely, let $\mathfrak{a}$ be a radical ideal and let $\mathfrak{b}$ be the intersection of all maximal ideals containing $\mathfrak{a}$. Let F and G be the closed subsets defined by $\mathfrak{a}$ and $\mathfrak{b}$ respectively. By construction, $F_{\text{max}} = G_{\text{max}}$, and hence by assumption, $F = G$. By the Nullstellensatz, this in turn implies $\mathfrak{a} = \mathfrak{b}$. □

In order to solve problem (1.3.1) from the Introduction, that is, to show that the level sets are geometrically constructible, we restrict to the case of a scheme of finite type over an algebraically closed field K. As we need to study the behavior of a local invariant in families, we need the notion of a *family of affine local rings*. Moreover, we also want to include finitely generated modules in our treatment. Algebraic geometry does not provide us with such families in a straightforward way, so that we need the following device.

Let $g\colon Y \to U$ be a map of finite type between schemes of finite type over $\mathbb{Z}$. By the definability results in [5,6], there exists a constructible subset $\mathbf{Irr}_g$ of U, such that for each algebraically closed field K, a K-rational point u of $U(K)$ lies in $\mathbf{Irr}_g(K)$ if and only if $g^{-1}(u)$ is irreducible (as a scheme over K). If Y itself is irreducible, then $\mathbf{Irr}_g$ is dense.

Definition 3.2 *Let X be a scheme of finite type over $\mathbb{Z}$. With an* abstract family $\mathfrak{R}$

of local rings on X *we mean a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\gamma} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[\pi]{} & T \end{array} \tag{2}$$

of maps of finite type over $\mathbb{Z}$.

One verifies that for each $u \in U$, the image of $g^{-1}(u)$ under γ is mapped inside $f^{-1}(\pi(u))$. By the same argument as above, there is a constructible subset $\mathbf{Irr}_{\mathfrak{R}}$ of U, such that for each algebraically closed field K and each $u \in U(K)$, the Zariski closure of $\gamma(g^{-1}(u))$ is irreducible if and only if u lies in $\mathbf{Irr}_{\mathfrak{R}}(K)$. Clearly $\mathbf{Irr}_g$ is contained in $\mathbf{Irr}_{\mathfrak{R}}$, but the latter set might be bigger. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then we call the pair $\mathfrak{M} = (\mathfrak{R}, \mathcal{F})$ an *abstract family of local modules on X*. For an algebraically closed field K, these yield families of affine local K-algebras and finitely generated modules as follows. For each K-rational point u in $\mathbf{Irr}_{\mathfrak{R}}(K)$, let $\mathfrak{R}_u$ be the localization of the coordinate ring of $f^{-1}(\pi(u))$ at the prime ideal defining the closure of $\gamma(g^{-1}(u))$ in the former fiber. In other words, $\mathfrak{R}_u$ is the stalk of $f^{-1}(\pi(u))$ at the point η, where η is the generic point of $\gamma(g^{-1}(u))$. For instance, if all schemes in $\mathfrak{R}$ are affine with a corresponding commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array} \tag{3}$$

of finitely generated $\mathbb{Z}$-algebra homomorphisms (so that $X = \operatorname{Spec} A$, etc.), then $\mathfrak{R}_u$ is isomorphic to

$$(A_K/(\mathfrak{n} \cap C_K)A_K)_{\mathfrak{n}B_K \cap A_K}$$

where $\mathfrak{n}$ is the maximal ideal of D_K associated to the K-rational point $u \in U(K)$ and where a subscript $_K$ denotes the base change to K. To obtain a family of finitely generated $\mathfrak{R}_u$-modules, let $\mathfrak{M}_u$ be the base change $\mathcal{F} \otimes \mathfrak{R}_u$. An affine local K-algebra $\mathfrak{R}_u$ or a finitely generated module $\mathfrak{M}_u$ will be referred to as an *actualization over K* of the abstract family.

3.2.1 Family of closed stalks. An example of an abstract family is the *family of closed stalks* of a scheme X over $\mathbb{Z}$ defined as follows. Let U be equal to X, Y equal to $X \times X$ and T equal to $\operatorname{Spec} \mathbb{Z}$, with g and γ the projections onto the second component and π and f the canonical maps to $\operatorname{Spec} \mathbb{Z}$. For $x \in X(K)$, the fiber $g^{-1}(x)$ is mapped under γ to the singleton $\{x\}$, whereas $f^{-1}(\pi(x))$ is X_K, so that $\mathfrak{R}_x \cong \mathcal{O}_{X_K,x}$. If, moreover, we have an abstract family of local modules $\mathfrak{M}$ over this family of closed stalks given by a coherent $\mathcal{O}_X$-module $\mathcal{F}$, then its actualizations are exactly the stalks $(\mathcal{F}_K)_x$ (where $\mathcal{F}_K$ is the base change of $\mathcal{F}$ to K).

3.2.2 Definability in families. We say that an $\mathbb{S}$-valued invariant ω of finite type is *definable in families*, if for each scheme X of finite type over $\mathbb{Z}$, for each abstract family of local rings $\mathfrak{R}$ on X as in (2) and for each $s \in \mathbb{S}$, there exists a constructible subset $L_{\mathfrak{R},s} \subseteq \mathbf{Irr}_{\mathfrak{R}}$ (defined over $\mathbb{Z}$), such that for each algebraically closed field K, a K-rational point u of $\mathbf{Irr}_{\mathfrak{R}}(K)$ lies in $L_{\mathfrak{R},s}(K)$ if and only if $\omega(\mathfrak{R}_u) = s$. Similarly, an $\mathbb{S}$-valued module invariant ω of finite type is *definable in families*, if for each abstract family of local modules $\mathfrak{M} = (\mathfrak{R}, \mathcal{F})$ on X and for each $s \in \mathbb{S}$, there is a constructible subset $L_{\mathfrak{M},s}$ of $\mathbf{Irr}_{\mathfrak{R}}$, such that $u \in L_{\mathfrak{M},s}(K)$ if and only if $\omega(\mathfrak{R}_u, \mathfrak{M}_u) = s$, for every algebraically closed field K and every $u \in \mathbf{Irr}_{\mathfrak{R}}(K)$.

Theorem 3.3 *Let ω be an $\mathbb{S}$-valued invariant which is definable in families. For each scheme X of finite type over an algebraically closed field K, for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ and for each $s \in \mathbb{S}$, the level set $\omega^{-1}_{X,\mathcal{F}}(s)$ (respectively, the level set $\omega^{-1}_X(s)$ in the ring invariant case) is geometrically constructible.*

Proof. Let X' be a scheme of finite type over $\mathbb{Z}$ such that $X'_K = X$. Now apply the definition to the family of closed stalks of X' defined in §3.2.1. □

In [6], I laid out the basis to prove that many of the invariants encountered in commutative algebra and algebraic geometry are definable in families. The key observation is that many invariants are defined using (co)homology, and in particular, using *Tor* and *Ext* groups. Therefore, the main results in that paper, are derived from the fact that these cohomology groups are definable in families. This in turn follows from the fact that they are bounded in the sense that $\operatorname{Tor}_i^R(M,N)$ has degree complexity (see below) uniformly bounded by the degree complexities of R, M and N. In the remainder of this section, I will briefly explain the notion of *degree complexity* and show how the present definition of being definable in families is identical with the model-theoretic one in [6,7].

Let us fix some notation. Let A be a finitely generated K-algebra, say of the form $K[\xi]/I$ for some ideal I of $K[\xi]$ and for some fixed set of variables $\xi = (\xi_1, \dots, \xi_n)$. Let $\mathfrak{p}$ be a prime ideal of A and let $R := A_{\mathfrak{p}}$, so that R is an example of an *affine local* K-algebra. Finally, let M be finitely generated R-module and choose an exact sequence $R^b \to R^a \to M \to 0$. Since the first map is given by a matrix $\mathbb{A}_M$ over R, we simply say that M is given as the *cokernel of* $\mathbb{A}_M$.

Definition 3.4 *We say that A (respectively, R) has* degree complexity *at most d, if $n \leq d$ and if I (respectively, I and $\mathfrak{p}$) is generated by polynomials of degree at* *most d. If, moreover, $a, b \leq d$ and each entry of the matrix $\mathbb{A}_M$ can be written as a fraction p/q with p and q of degree at most d and $q \notin \mathfrak{p}$, then we say that M has* degree complexity *at most d.*

If $I = (f_1, \dots, f_s)K[\xi]$, with f_i of degree at most d, then the tuple $\mathbf{a}_A$ of all coefficients of the f_i, listed in a once and for all fixed order, completely determines A. Similarly, if $\mathfrak{p} = (g_1, \dots, g_t)K[\xi]$, with g_j of degree at most d, then the tuple $\mathbf{a}_R$ of all coefficients of the f_i and the g_j completely determines R. We call the tuples $\mathbf{a}_A$ and $\mathbf{a}_R$ *codes* for A and R. Moreover, one checks that the length of these tuples is completely determined by d. The tuple of all coefficients of all entries of $\mathbb{A}_M$ together with a code for R, is a *code* $\mathbf{a}_M$ for M. Clearly, the length of this code depends only on the degree complexity. (For these definitions, we do not need to assume that K is algebraically closed).

In [6], a property $\mathbf{P}$ of affine local algebras (respectively, of finitely generated modules over affine local algebras) is called *asymptotically definable*, if, for each d, there is a first order formula $\psi_{d,\mathbf{P}}$, without parameters, such that a code $\mathbf{a}_R$ of an affine local K-algebra R of degree complexity at most d (respectively, a code $\mathbf{a}_M$ of a finitely generated R-module M of degree complexity at most d), satisfies the formula $\psi_{d,\mathbf{P}}$ if and only if R (respectively, M) has property $\mathbf{P}$. It is important to note that these formulae are independent from the field K. Let ω be an $\mathbb{S}$-valued invariant and let $s \in \mathbb{S}$. Let us write $\mathbf{P}_{\omega,s}$ for the property that a local ring (or a module) has ω-value s. In Theorem 3.5 below, I will show that ω is definable in families if and only if the property $\mathbf{P}_{\omega,s}$ is asymptotically definable, for each $s \in \mathbb{S}$. For the proof, we need to describe the family of all affine local rings of degree complexity at most d.

3.4.1 Universal families. Let $\xi = (\xi_1, \dots, \xi_n)$ be a fixed set of variables and d a positive integer. Let F be the *general polynomial* of degree d in the variables ξ given by

$$F(t, \xi) := \sum_{\alpha} t_\alpha \xi^\alpha$$

where α runs over all indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_1 + \dots + \alpha_n \leq d$ and where t is a tuple of variables, say of length $N = N(d)$. Let τ_i be N-tuples of variables and let τ be the tuple of all these τ_i, for $i = 1, \dots, N$. Let A be the quotient of $\mathbb{Z}[\tau, \xi]$ modulo the ideal generated by all $F(\tau_i, \xi)$, for $i = 1, \dots, N$. In other words, one could think of $X := \operatorname{Spec} A$ as the intersection of N general hypersurfaces of τ-degree one and ξ-degree at most d. Let T be the affine N^2-space $\operatorname{Spec} \mathbb{Z}[\tau]$. The closed fibers of $f\colon X \to T$ are precisely the finitely generated K-algebras of degree complexity at most d (just observe that any ideal generated by polynomials of degree at most d requires at most N generators).

To obtain local affine algebras, we essentially duplicate this construction: let τ'_i be new N-tuples of variables and let σ be the tuple of all the τ_i and τ'_i, for $i = 1, \dots, N$.

Let Y be the closed subscheme of affine $2N^2+n$ space defined by all $F(\tau_i,\xi)$ and all $F(\tau_i',\xi)$ and let U be affine σ-space. This yields an abstract family $\mathfrak{R}^{(d)}$ given by a commutative diagram (2), called the *universal family of affine local algebras of degree complexity at most* d. The actualizations of this family are then precisely the affine local K-algebras of degree complexity at most d. Indeed, if $u=(t,t')\in \mathbf{Irr}_{\mathfrak{R}^{(d)}}(K)$, then $g^{-1}(u)$ and $f^{-1}(t)$ have coordinate rings $K[\xi]/\mathfrak{p}$ and $K[\xi]/I$ respectively, where I is the ideal generated by all $F(t_i,\xi)$ and where $\mathfrak{p}$ is the ideal generated by I and all $F(t_i',\xi)$. By construction, $\mathfrak{p}$ is prime. Therefore, $\mathfrak{R}^{(d)}_u$ is the localization of $K[\xi]/I$ at $\mathfrak{p}$ and so is an affine local K-algebra of degree complexity at most d. Conversely, any affine local K-algebra of degree complexity at most d is realized in this way.

The reason for calling the families $\mathfrak{R}^{(d)}$ 'universal' is because any abstract family is a subfamily of some $\mathfrak{R}^{(d)}$, in the sense that every actualization of the former is isomorphic to some actualization of the latter, over any algebraically closed field. Indeed, choose d bigger than the degree of any polynomial defining the schemes and the maps occurring in the commutative diagram of an abstract family (since everything is of finite type and locally affine, there is such a maximal value). In fact, this 'embedding' of an abstract family in a universal family can be carried out in a constructible way, which is what we need to prove the equivalence of the two definitions.

Theorem 3.5 *An $\mathbb{S}$-valued invariant ω of finite type is definable in families if and only if for each $s\in\mathbb{S}$, the property $\mathbf{P}_{\omega,s}$ is asymptotically definable.*

Proof. By construction of the families $\mathfrak{R}^{(d)}$, it is clear that $\mathbf{P}_{\omega,s}$ is asymptotically definable if ω is definable in families. Conversely, assume $\mathbf{P}_{\omega,s}$ is asymptotically definable, for a fixed $s\in\mathbb{S}$. I will only treat the ring invariant case; the module invariant case is completely analogous. Let X be a scheme and let $\mathfrak{R}$ be an abstract family of local rings on X. We need to show that there exists a constructible subset $L_{\mathfrak{R},s}$ of $\mathbf{Irr}_{\mathfrak{R}}$, such that for each algebraically closed field K, a K-rational point u in $\mathbf{Irr}_{\mathfrak{R}}(K)$ lies in $L_{\mathfrak{R},s}(K)$ if and only if $\omega(\mathfrak{R}_u)=s$. Since the property we seek to prove is local in the constructible topology, we may assume without loss of generality that all schemes in $\mathfrak{R}$ are affine. Let

$$\begin{array}{ccc} C & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & B \end{array} \qquad (4)$$

be the corresponding commutative diagram of finitely generated $\mathbb{Z}$-algebras. As before, we will write a subscript $_K$ to denote the base change to an algebraically closed field K. Let $\mathfrak{n}$ be the maximal ideal of D_K corresponding to a K-rational point u in $\mathbf{Irr}_{\mathfrak{R}}$. By definition

$$\mathfrak{R}_u = (A_K/(\mathfrak{n} \cap C_K)A_K)_{\mathfrak{n}B_K \cap A_K}$$

where by assumption $\mathfrak{n}B_K \cap A_K$ is a prime ideal. Since U and T are closed subschemes of affine spaces, we may assume without loss of generality that $C = \mathbb{Z}[\tau]$ and $D = \mathbb{Z}[\sigma]$, for some tuples of variables τ and σ. Suppose $C \to D$ is given by $\tau = P(\sigma)$, for some tuple P of polynomials with integer coefficients. Since A and B are finitely generated over C and D respectively, we may write $A \cong \mathbb{Z}[\tau, \xi]/I$ and $B \cong \mathbb{Z}[\sigma, \zeta]/J$ for some tuples of variables ξ and ζ and some ideals I and J. In view of the commutativity of diagram (4), the homomorphism $A \to B$ is given by $\tau = P(\sigma)$ and $\xi = Q(\sigma, \zeta)$, for some tuple Q of polynomials with integer coefficients. Take $d \in \mathbb{N}$ larger than the degree of any polynomial involved, that is to say, each entry of P and Q has degree at most d, and the ideals I and J can be generated by polynomials of degree at most d.

With this notation, $\mathfrak{n}$ is the ideal in $D_K = K[\sigma]$ generated by the linear forms $\sigma_i - a_i$, where the a_i are the coordinates of the point u. Therefore, $\mathfrak{n} \cap C_K$ is generated by the linear forms $\tau_i - P_i(\mathbf{a}_u)$, where $\mathbf{a}_u$ is the tuple of coordinates a_i of u. If we put A_u equal to $A_K/(\mathfrak{n} \cap C_K)A_K$, then A_u is isomorphic to $K[\xi]/I(P(\mathbf{a}_u))$, where $I(P(\mathbf{a}_u))$ denotes the ideal in $K[\xi]$ obtained from I by substituting $P(\mathbf{a}_u)$ for the variables τ. In particular, A_u has degree complexity at most d^2. Moreover, there exists a map $h\colon U \to \mathbb{A}^N_{\mathbb{Z}}$, such that its base change h_K sends u to a code of the K-algebra A_u of degree complexity at most d.

Next, we want to describe a code for the prime ideal $\mathfrak{n}B_K \cap A_K$. Note that if we localize A_u with respect to this prime ideal, we get $\mathfrak{R}_u$. It follows from [6, Theorem 2.7] that $\mathfrak{n}B_K \cap A_K$ is generated by (images of) polynomials of degree at most d', where d' only depends on d (and not on u nor on K). A polynomial in $K[\tau, \xi]$ of degree at most d' can be written in the form $F(\mathbf{w}, \tau, \xi)$, for some tuple $\mathbf{w}$ over K and some polynomial F with integer coefficients of degree at most $d' + 1$. One checks that such a polynomial $F(\mathbf{w}, \tau, \xi)$ lies in $\mathfrak{n}B_K \cap A_K$ if and only if $F(\mathbf{w}, P(\mathbf{a}_u), Q(\mathbf{a}_u, \zeta))$ lies in $J(\mathbf{a}_u)$, where $J(\mathbf{a}_u)$ denotes the ideal in $K[\zeta]$ obtained from J by substituting $\mathbf{a}_u$ for the variables σ. It follows from the arguments in [6] that there exists a first order formula ψ_d without parameters (not depending on K nor on u but solely on d), such that $(\mathbf{a}_u, \mathbf{w})$ satisfies ψ_d if and only if $F(\mathbf{w}, \tau, \xi)$ lies in $\mathfrak{n}B_K \cap A_K$. To obtain a code for $\mathfrak{R}_u$, we now do the following. Consider the condition Ψ_d on a tuple $(\mathbf{a}_u, \mathbf{w}_1, \dots, \mathbf{w}_{N'})$ expressing that each $(\mathbf{a}_u, \mathbf{w}_i)$ satisfies ψ_d and, for any other tuple $\mathbf{w}$, if $(\mathbf{a}_u, \mathbf{w})$ satisfies ψ_d, then $F(\mathbf{w}, \tau, \xi)$ is a linear combination of the $F(\mathbf{w}_i, \tau, \xi)$ modulo $J(\mathbf{a}_u)$. Here we take N' equal to the number of monomials in $N + d$ variables of degree at most $d' + 1$ (it follows that $\mathfrak{n}B_K \cap A_K$ is generated by at most N' elements). Another application of [6] shows that Ψ_d is a first order statement. Moreover, a tuple $(\mathbf{a}_u, \mathbf{w}_1, \dots, \mathbf{w}_{N'})$ satisfies Ψ_d if and only if $(\mathbf{w}_1, \dots, \mathbf{w}_{N'})$ is a code for the prime ideal $\mathfrak{n}B_K \cap A_K$.

In summary, any tuple $(h_K(u), \mathbf{w}_1, \dots, \mathbf{w}_{N'})$ for which $(\mathbf{a}_u, \mathbf{w}_1, \dots, \mathbf{w}_{N'})$ satisfies Ψ_d is a code for $\mathfrak{R}_u$. By the asymptotical definability of $\mathbf{P}_{\omega,s}$, there exists a first order formula $\varphi_{d,s}$, such that if a tuple $(h_K(u), \mathbf{w}_1, \dots, \mathbf{w}_{N'})$ satisfies $\varphi_{d,s}$ and $(\mathbf{a}_u, \mathbf{w}_1, \dots, \mathbf{w}_{N'})$ satisfies Ψ_d, then $\omega(\mathfrak{R}_u) = s$. Therefore, let $\Phi_{d,s}$ be the formula stating that $u \in \mathbf{Irr}_{\mathfrak{R}}$ and that there exist tuples $\mathbf{w}_i$ such that $(h_k(u), \mathbf{w}_1, \dots, \mathbf{w}_{N'})$ satisfies $\varphi_{d,s}$ and $(\mathbf{a}_u, \mathbf{w}_1, \dots, \mathbf{w}_{N'})$ satisfies Ψ_d. It follows that $\mathbf{a}_u$ satisfies $\Phi_{d,s}$ if and only if $\omega(\mathfrak{R}_u) = s$. Since the theory of algebraically closed fields has Quantifier Elimination, the set defined by the formula $\Phi_{d,s}$ is a constructible subset $L_{\mathfrak{R},s}$ of $\mathbf{Irr}_{\mathfrak{R}}$, which therefore has the required properties. $\square$

4 Constructible sets

In this section, X denotes an arbitrary Noetherian scheme and $T \subseteq X$ an arbitrary subset (of points of X). A point $y \in X$ is called a *specialization* of a point $x \in X$ or an *x-specialization*, if y lies in $\operatorname{cl}(\{x\})$. We say that y is an *immediate x-specialization* if y is minimal in $\operatorname{cl}(\{x\}) - \{x\}$. If y is an (immediate) x-specialization, then we will also say that that x is an (*immediate*) *y-generalization*. If $X = \operatorname{Spec} A$ is affine and $\mathfrak{p}$ and $\mathfrak{q}$ are the prime ideals of A corresponding respectively to x and y, then y is a x-specialization if and only if $\mathfrak{q}$ is an *overprime* of $\mathfrak{p}$, that is to say, $\mathfrak{p} \subseteq \mathfrak{q}$; and y is an immediate x-specialization, if there is no prime ideal strictly in between $\mathfrak{p}$ and $\mathfrak{q}$, in which case we say that $\mathfrak{q}$ is an *immediate overprime* of $\mathfrak{p}$. Note that $\mathfrak{q}$ is an immediate overprime of $\mathfrak{p}$ if and only if its image in $A/\mathfrak{p}$ has height one.

Definition 4.1 (Saturated Sets) *We say that T is* saturated, *if for each $x \in T$, we can find a closed point $y \in T$ which is a specialization of x.*

Lemma 4.2 *Let X be a scheme and T a subset of X. If*

(4.2.1) T is geometrically constructible, and,
(4.2.2) for each open subset U of X, both $T \cap U$ and $(-T) \cap U$ are saturated in U,

then T is constructible.

Proof. As the problem is local, we may assume without loss of generality that $X = \operatorname{Spec} A$ is affine. Assume that

$$F_{\max} = T_{\max} \tag{5}$$

with F a constructible set of the form

$$(\mathrm{V}(\mathfrak{a}_1) \cap U_1) \cup \cdots \cup (\mathrm{V}(\mathfrak{a}_s) \cap U_s)$$

with each $\mathfrak{a}_i$ an ideal of A and U_i a Zariski open of $\operatorname{Spec} A$. Let $x \in F$ and let $\mathfrak{p}$ be the prime ideal in A corresponding to x. Hence for some i, say for $i = 1$, we have

$\mathfrak{a}_1 \subseteq \mathfrak{p}$ and $\mathfrak{p} \in U_1$. Suppose that $x \notin T$. As $(-T) \cap U_1$ is saturated, we can find a maximal ideal $\mathfrak{m}$ of A, containing $\mathfrak{p}$ and belonging to $(-T) \cap U_1$. It follows that $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}_1) \cap U_1 \subseteq F$ and hence by (5), that $\mathfrak{m} \in T$, contradiction.

In other words, we showed that $F \subseteq T$. By the same argument, this time applied to $-T$ and $-F$, and using that $T \cap U$ is saturated, it follows that also $-F \subseteq -T$. Putting these two inclusions together, we obtain that $F = T$. □

Note that conversely, if X is Jacobson, then each non-empty constructible subset contains a closed point by Lemma 3.1, and therefore each constructible set satisfies Conditions (4.2.1) and (4.2.2) of Lemma 4.2. On the other hand, if A is a local ring, then the only subsets of $X = \operatorname{Spec} A$ satisfying (4.2.2) are $\emptyset$ and X. Let us call a subset T *universally saturated*, if $T \cap U$ is saturated, for every open U in X.

Definition 4.3 *We call T* devissable *if, for each non-closed point $x \in T$, we can find an open U of X containing x, such that any immediate x-specialization y in U belongs to T.*

Of interest is also the following stronger variant: we call T *strongly devissable* if for each point $x \in T$, we can find an open U containing x, such that $U \cap \mathrm{cl}(\{x\}) \subseteq T$, that is to say, if any x-specialization inside U belongs to T. Any subset of $X_{\max}$ is trivially devissable, showing that in general, devissable subsets need not be constructible (but the converse does hold by Theorem 4.4 below). It is not hard to see that an arbitrary union or a finite intersection of (strongly) devissable subsets is again (strongly) devissable. Recall that T is said to be *ind-constructible*, if it is an arbitrary union of constructible subsets. The complement of an ind-constructible subset, that is to say, an arbitrary intersection of constructible subsets, is called a *pro-constructible* subset.

Theorem 4.4 *Let X be a Noetherian scheme and T a subset of X. Consider the following properties the subset T can have*

*(*4.4.1*) constructible;*
*(*4.4.2*) ind-constructible;*
*(*4.4.3*) strongly devissable;*
*(*4.4.4*) devissable;*
*(*4.4.5*) universally saturated;*
*(*4.4.6*) saturated.*

Then we have implications

$$(4.4.1) \Rightarrow (4.4.2) \Leftrightarrow (4.4.3) \Rightarrow (4.4.4) \quad \textit{and} \quad (4.4.5) \Rightarrow (4.4.6).$$

Moreover, if X is Jacobson, then (4.4.4) $\Rightarrow$ (4.4.5)*.*

If T is geometrically constructible, and both T and $-T$ are universally saturated, then T is constructible.

Proof. The last statement is just Lemma 4.2, so that we only need to prove the stated implications. The implications (4.4.1) $\Rightarrow$ (4.4.2), (4.4.3) $\Rightarrow$ (4.4.4) and (4.4.5) $\Rightarrow$ (4.4.6) are immediate. Hence assume T ind-constructible, say the union of locally closed sets $V_i \cap U_i$, where V_i is closed and U_i is open. For $x \in T$, say $x \in V_{i_0} \cap U_{i_0}$, it suffices to take $U = U_{i_0}$ in the definition of strong devissability. This proves (4.4.2) $\Rightarrow$ (4.4.3). Conversely, if T is strongly devissable, then we can find for each $x \in T$ an open U_x such that the locally closed set $U_x \cap \mathrm{cl}(\{x\})$ is contained in T. Therefore, T is the union of all the $U_x \cap \mathrm{cl}(\{x\})$ whence is ind-constructible.

Remains to show (4.4.4) $\Rightarrow$ (4.4.5) under the additional assumption that X is Jacobson. If T is devissable, then so is $T \cap U$ for all open U. Hence it suffices to show that if T is devissable, then it is saturated. Let us prove by downward induction on the dimension of $\mathcal{O}_{X,x}$ that any non-closed point $x \in T$ admits a closed x-specialization in T. By assumption, there exists an open U containing x, such that any immediate x-specialization $y \in U$ belongs to T. Let $F := U \cap \mathrm{cl}(\{x\})$. Since $F_{\max}$ is non-empty by Lemma 3.1, there exists at least one immediate x-specialization $y \in F$. By the choice of U, the point y belongs to T. By induction, there exists $z \in T_{\max}$ generalizing to y, whence to x, as we wanted to show. □

In fact, we can add the following characterization to the ones in Lemma 3.1: every constructible subset of X is saturated if and only if X is Jacobson. Indeed, we just proved one direction. For the other, it suffices by Lemma 3.1 to show that $F_{\max}$ is non-empty whenever F is non-empty, and this is clear since by assumption, if $x \in F$, then there exists a specialization of x which lies in $F_{\max}$.

For the reader's convenience, I have included the following well-known results on the constructible topology.

Proposition 4.5 *Let X be a Noetherian scheme. If F_i, for $i \in I$, are constructible subsets of X whose union is equal to X, then already finitely many cover X. In other words, X is quasi-compact in the constructible topology.*

Proof. We will prove this by Noetherian induction, which means that we may assume that it holds for any proper closed subset of X and we now have to show it for X itself. In particular, we may assume X is irreducible. Without loss of generality, since a constructible set is a finite union of locally closed sets, we may also assume that each F_i is locally closed, that is to say, of the form $U_i \cap Z_i$ with U_i Zariski open and Z_i Zariski closed. Let η be the generic point of X and assume F_{i_0} contains η. Therefore, Z_{i_0}, being a closed set containing the generic point, must be equal to X. In other words, F_{i_0} is Zariski open. Let X_0 be the complement of F_{i_0}. Clearly, the collection of all $F_i - F_{i_0}$ cover X_0, so that by Noetherian induction, already finitely many cover X_0, say for $i \in I_0$ with I_0 a finite subset of I. It is now clear that X is the union of F_{i_0} and all F_i with $i \in I_0$. □

Corollary 4.6 *Let X be a Noetherian scheme and F a subset of X. Then F is* *constructible if and only if it is pro-constructible and ind-constructible.*

Proof. Let F be pro-constructible and ind-constructible. In particular, we can write

$$F = \bigcup_{i \in I} F_i \qquad \text{and} \qquad -F = \bigcup_{j \in J} G_j,$$

with F_i and G_j constructible subsets. The F_i together with the G_j form a covering of X. By Proposition 4.5, we can find subsets $I_0 \subseteq I$ and $J_0 \subseteq J$, such that the F_i and G_j cover X, for $i \in I_0$ and $j \in J_0$. One checks that F is the union of all F_i with $i \in I_0$ whence is constructible. □

Corollary 4.7 *Let X be a Noetherian scheme and $\mathcal{F}$ a finite partition of X. If each member of $\mathcal{F}$ is strongly devissable, then $\mathcal{F}$ is constructible.*

Proof. By Theorem 4.4, each $F \in \mathcal{F}$ is ind-constructible. Moreover, F is the union of the complements of the other members, and therefore is pro-constructible since the partition is finite. Hence, F is constructible by Corollary 4.6. □

In particular, a subset T is constructible if and only if T and its complement are strongly devissable. Let us consider the following weaker variant: call $T \subseteq X$ *bi-devissable* if T and $-T$ are both devissable. If X is a one-dimensional scheme or a semi-local two-dimensional scheme, then a subset T is constructible if and only if it is bi-devissable. Indeed, we only need to prove sufficiency, and for that we may assume X is irreducible and affine, since the problem is local. Replacing T by its complement if necessary, we may assume that T contains the generic point. Since T is devissable, there is some non-empty open U such that any height one prime in U belongs to T. Since we may choose U disjoint from $X_{\max}$ in the semi-local case, we get $U \subseteq T$. Since $-U$ is finite, T is easily seen to be constructible.

This last result is no longer true in higher dimensions. For instance, let X be the affine plane over $\mathbb{C}$ and let $T \subseteq X$ consist of all closed points with coordinates (e^n, n), for $n \in \mathbb{N}$. Clearly, T is ind-constructible whence devissable, but not constructible. Since T lies on the transcendental curve $\xi_1 = e^{\xi_2}$, any (algebraic) curve in X meets T only in finitely many points. In particular, $\mathrm{cl}(\{x\}) - T$ is a constructible subset for each non-closed point x other than the generic point η. Since $-T - \{\eta\}$ is the union of all these constructible subsets, it is also ind-constructible. Furthermore, any immediate η-specialization belongs to $-T$. These two results together prove that $-T$ is devissable and hence that T is bi-devissable but not constructible. We can build a similar example of a non-constructible bi-devissable subset in a local scheme of dimension three or higher. In particular, we cannot leave out the Jacobson condition in the next result.

Theorem 4.8 *Let X be a Jacobson scheme and $\mathcal{F}$ a partition of X. If each member of $\mathcal{F}$ is geometrically constructible and devissable, then $\mathcal{F}$ is constructible and finite.*

Proof. Note that if $F \in \mathcal{F}$ is devissable, then so is its complement, since devissability is preserved under arbitrary unions and since $-F$ is the union of the other members in $\mathcal{F}$. Hence $\mathcal{F}$ is constructible by Theorem 4.4, whence finite by Proposition 4.5. □

We may replace devissability in the statement by the weaker condition that the invariant is universally saturated. We conclude this section with a generalization, which might be useful when dealing with arbitrary schemes.

4.8.2 Γ-constructible subsets. Let X be a scheme and Γ a subset of X. We say that a subset $T \subseteq X$ is Γ-*constructible* if there exists a constructible subset $F \subseteq X$ such that $T \cap \Gamma = F \cap \Gamma$. In other words, T is Γ-constructible, if $T \cap \Gamma$ is constructible in the induced topology on Γ. Moreover, we will say that T is Γ-*saturated*, if each $x \in T$ admits a specialization belonging to $T \cap \Gamma$. As before, we then say that T is *universally* Γ-*saturated*, if $T \cap U$ is Γ-saturated in U, for any open $U \subseteq X$. Note that if $\Gamma = X_{\text{max}}$, then we recover the homonymous concepts defined previously. Inspecting the proof of Lemma 4.2, we immediately get the following generalization.

Lemma 4.9 *Let X be a scheme and T a subset of X. If T is Γ-constructible, and, both T and $-T$ are universally Γ-saturated, then T is constructible. The converse holds if Γ is dense in the constructible topology.*

5 Constructible invariants

Let ω be an $\mathbb{S}$-valued invariant. Let $\mathbf{P}$ be one of the properties (4.4.1)–(4.4.6) in Theorem 4.4, or for that matter any property of subsets of a scheme.

Definition 5.1 *We say that ω has property $\mathbf{P}$, if for each scheme X, for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ and for each $s \in \mathbb{S}$, the level set $\omega^{-1}_{X,\mathcal{F}}(s)$ (or, in the ring case, the level set $\omega^{-1}_X(s)$) has property $\mathbf{P}$.*

Of course, our convention for partially defined invariants is still in effect, meaning that we only quantify over those schemes or sheaves for which ω is defined. For instance, if ω is a of finite type, then in the above definition X is assumed to be of finite type over an algebraically closed field. Since ω is only a property about local rings (and their modules), any saturated invariant is universally saturated. In this new terminology, Theorem 3.3 states that any invariant which is definable in families is geometrically constructible. On occasion, we will use the following algebraic translation of what it means for an invariant ω to be devissable: for every Noetherian ring A, every finitely generated A-module M and every non-maximal prime ideal $\mathfrak{g}$ in A, there exists $c \notin \mathfrak{g}$ such that $\omega(A_\mathfrak{g}, M_\mathfrak{g}) = \omega(A_\mathfrak{p}, M_\mathfrak{p})$ for all height one prime ideals $\mathfrak{p}$ in $A_c/\mathfrak{g}A_c$. Here we have identified the height one prime ideals of $A_c/\mathfrak{g}A_c$ with the immediate overprimes of $\mathfrak{g}$ not containing c, via the natural locally closed immersion $\operatorname{Spec}(A_c/\mathfrak{g}A_c) \hookrightarrow \operatorname{Spec} A$. A similar criterion exists for strong devissability, where we now impose no restriction on the height of $\mathfrak{p}$ in the above.

Theorem 5.2 *Let ω be an $\mathbb{S}$-valued invariant of finite type. If ω is definable in families and devissable (or, saturated), then it is constructible. In particular, if X is a scheme of finite type over an algebraically closed field K and if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $\omega(x,\mathcal{F})$ takes only finitely many different values as x runs over all points of X.*

Moreover, if X is irreducible, then there is some non-empty open U of X such that $\omega(\cdot,\mathcal{F})$ is constant on U.

Proof. Immediate from Theorem 4.8 and Theorem 3.3. For the last statement, let η be the generic point of X and let $s := \omega(\eta,\mathcal{F})$. The level set $\omega^{-1}_{X,\mathcal{F}}(s)$ is a finite union of locally closed subsets, one of which contains η and therefore is open. □

More generally, if X is a Jacobson scheme and ω is an arbitrary invariant which is geometrically constructible and devissable on X, then it is constructible on X by Theorem 4.8. The value at the generic point of an irreducible scheme is sometimes referred to as the *generic value*. The last statement in Theorem 5.2 justifies this terminology. If ω is a ring invariant and X is moreover integral, then the generic value is equal to $\omega(K(X))$, where $K(X)$ is the function field of X. Often, an invariant is preserved under scalar extensions (see definition 9.1 below), so that in that case, the generic value is equal to $\omega(K)$ and even to $\omega(\mathbb{F})$, where $\mathbb{F}$ is the prime field of the same characteristic as K. In other words, the generic value only depends on the characteristic of the base field. For instance the singularity defects (see Section 7 below) all have generic value zero. In the next two sections, we will treat in detail some numerical invariants: Betti numbers, Bass numbers and defects. Here are some more examples. In the next two examples, let C be a finitely generated $\mathbb{Z}$-algebra.

5.2.1 Height. Let I be an ideal in C and define a ring invariant on C-algebras by putting $\omega^{\mathrm{ht}}_I(R) := \operatorname{ht}(IR)$, for any local C-algebra R. Here we take the convention that the unit ideal has height ∞, so that ω^{ht}_I is an invariant with $\mathbb{S} = \mathbb{N} \cup \{\infty\}$. It follows from [6, Proposition 5.1] in conjunction with Theorem 3.5 that ω^{ht}_I, or rather, the invariant of finite type determined by it, is definable in families in the sense that for any abstract family $\mathfrak{R}$ of C-algebras, the set of closed points u in $\mathbf{Irr}_{\mathfrak{R}}$ for which $I\mathfrak{R}_u$ has a fixed height, is (geometrically) constructible. We next argue that ω^{ht}_I is also devissable. Namely, let A be a Noetherian C-algebra and let $\mathfrak{g}$ be a non-maximal prime ideal. Suppose $\operatorname{ht}(IA_{\mathfrak{g}}) = s$. If $s = \infty$, meaning that $I \not\subseteq \mathfrak{g}$, then we can take for U the open of all prime ideals not containing I. So we may assume $s < \infty$. Let $\mathfrak{q}_i$, for $i = 1, \dots, m$, be the minimal prime ideals of IA and renumber in such way that the n first ones lie in $\mathfrak{g}$ and the remaining ones do not. It follows that s is the minimum of the heights of the $\mathfrak{q}_i$ for $i = 1, \dots, n$. Therefore, if we let U be the complement of $\mathrm{V}(\mathfrak{q}_{n+1}) \cup \dots \cup \mathrm{V}(\mathfrak{q}_m)$, then $\mathrm{ht}(IA_\mathfrak{p}) = s$, for any overprime $\mathfrak{p}$ of $\mathfrak{g}$ in U, showing that ω_I^{ht} is strongly devissable. In conclusion, by Theorem 5.2, the invariant ω_I^{ht} is constructible on schemes of finite type over an algebraically closed field.

5.2.2 Regular sequence.

As above, C is a finitely generated $\mathbb{Z}$-algebra. Let $\mathbf{a}$ be a (finite) tuple in C. We define a module invariant $\omega_\mathbf{a}^{\mathrm{reg}}$ as follows: for R a local C-algebra and M a finitely generated R-module, let $\omega_\mathbf{a}^{\mathrm{reg}}(M)$ be either one or zero, according to whether $\mathbf{a}$ is an M-regular sequence or not. Here we can prove directly that this is a constructible invariant. By induction on the length of the tuple, we may reduce to the case that we have a single element $a \in C$. Given a Noetherian C-algebra A and a finitely generated A-module M, one easily checks that a is $M_\mathfrak{p}$-regular if and only if $\mathfrak{p}$ belongs to the support of M/aM and $\mathrm{Ann}_A(\mathrm{Ann}_M(a))$ is not contained in $\mathfrak{p}$. The former is a closed condition and the latter an open, showing that ω_a^{reg} is a constructible invariant.

5.2.3 Hilbert series.

The following example will be studied in more detail in a future paper. Let $\mathbb{S}$ be the polynomial ring $\mathbb{Z}[T]$ in a single variable T over the integers. Let R be a Noetherian local ring and let M be a finitely generated R-module. The *Hilbert series* of M is defined as the formal power series

$$H_M(T) := \sum_n \ell_R(\mathfrak{m}^n M/\mathfrak{m}^{n+1}M)T^n,$$

where $\ell_R(H)$ denotes the length of an arbitrary R-module H. It is shown (see for instance [1, Chapter 4]) that H_M is of the form

$$H_M(T) = \frac{Q_M(T)}{(1-T)^h} \tag{6}$$

where Q_M is a polynomial over $\mathbb{Z}$ with $Q_M(1) \neq 0$ and h is the dimension of M (that is to say, the dimension of $R/\mathrm{Ann}_R(M)$). The assignment of Q_M to the pair (R, M) is an example of a module invariant. It turns out that this is a constructible invariant on schemes of finite type over an algebraically closed field.

5.2.4 Singularities.

Properties of local rings or of their finitely generated modules provide also examples of invariants. This time we let $\mathbb{S} := \{0, 1\}$. Let $\mathbf{P}$ be a property of local rings (for instance, to be regular, complete intersection, Gorenstein or Cohen-Macaulay), then we set $\omega^\mathbf{P}(R)$ equal to 1 or 0, according to whether the local ring R does or does not have the property $\mathbf{P}$. For each of the above mentioned properties, this is indeed a constructible invariant on excellent schemes, as shown in [2, Chap. IV,§9].

Definition 5.3 (Deformations) *We say that an $\mathbb{S}$-valued module invariant ω* deforms well, *if for each $a \in \mathfrak{m} - \mathfrak{m}^2$ which is simultaneously R-regular and M-regular, we have*

$$\omega(R, M) = \omega(R/aR, M/aM),$$

where R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated R-module. In case ω is a ring invariant, we require that $\omega(R) = \omega(R/aR)$ for every R-regular element $a \in \mathfrak{m} - \mathfrak{m}^2$.

The following well-known result (see for instance [4, §18 Lemma 2]) is very useful in combination with deformation.

Lemma 5.4 *Let R be a ring and let M and H be R-modules. If $a \in R$ annihilates H and is both R-regular and M-regular, then we have isomorphisms*

$$\begin{aligned}\mathrm{Tor}_i^R(H, M) &\cong \mathrm{Tor}_i^{R/aR}(H, M/aM)\\ \mathrm{Ext}_R^i(M, H) &\cong \mathrm{Ext}_{R/aR}^i(M/aM, H)\\ \mathrm{Ext}_R^{i+1}(H, M) &\cong \mathrm{Ext}_{R/aR}^i(H, M/aM),\end{aligned}$$

for each $i \geq 0$.

6 Betti and Bass numbers

Let R be a Noetherian local ring with residue field k and let M be a finitely generated R-module. We define the Betti and Bass invariants as follows. Let $\omega_i^{\mathrm{Betti}}$ be the numerical invariant given as the i-th Betti number

$$\omega_i^{\mathrm{Betti}}(M) := \dim_k \mathrm{Tor}_i^R(k, M).$$

Suppose M has depth q. Let ω_i^{Bass} be the numerical invariant given as

$$\omega_i^{\mathrm{Bass}}(M) := \dim_k \mathrm{Ext}_R^{q+i}(k, M).$$

In other words, $\omega_i^{\mathrm{Bass}}(M)$ is the $(q+i)$-th Bass number of M. Note that by [4, Theorem 16.7], we have $\mathrm{Ext}_R^j(k, M) = 0$, for $j < q$. By [6, Theorem 4.5] in conjunction with Theorem 3.5, the Betti numbers and the Bass numbers are definable in families. Therefore, so are the invariants $\omega_i^{\mathrm{Betti}}$ and ω_i^{Bass}. Note that the Bass numbers themselves cannot be constructible invariants: if A is a Gorenstein ring and $\mathfrak{p}$ a prime ideal of A, then the i-th Bass number of $\mathfrak{p}$ equals one, if i is the height of $\mathfrak{p}$, and is zero otherwise. This example motivates the dimension shift in the definition of ω_i^{Bass}. We will show that the $\omega_i^{\mathrm{Betti}}$ and the ω_i^{Bass} are devissable as well and therefore constructible.

Theorem 6.1 *For each i, the numerical invariants $\omega_i^{\mathrm{Betti}}$ and ω_i^{Bass} are devissable on excellent schemes.*

Proof. For the duration of this proof, let A be an excellent ring, M a finitely generated A-module and $\mathfrak{g}$ a non-maximal prime ideal.

6.1.1 The following subset will be useful later on as well. Let $\mathbf{Reg}_{\mathfrak{g}}$ be the collection of all prime ideals $\mathfrak{p}$ for which $A_{\mathfrak{p}}/\mathfrak{g}A_{\mathfrak{p}}$ is either regular or zero. If we identify $\operatorname{Spec}(A/\mathfrak{g})$ with the closed subset of $\operatorname{Spec} A$ consisting of all prime ideals of A containing $\mathfrak{g}$, then $\mathbf{Reg}_{\mathfrak{g}} \cap \operatorname{Spec}(A/\mathfrak{g})$ is exactly the regular locus of $A/\mathfrak{g}$. Since $A/\mathfrak{g}$ is an excellent domain, this regular locus is a non-empty open subset. On the other hand, the complement of $\operatorname{Spec}(A/\mathfrak{g})$ is contained in $\mathbf{Reg}_{\mathfrak{g}}$. Therefore, if W is any open subset containing $\mathfrak{g}$, then $W \cap \mathbf{Reg}_{\mathfrak{g}}$ is also open. In particular, whenever we want to do so, we may shrink some open W containing $\mathfrak{g}$ so that it is entirely contained in $\mathbf{Reg}_{\mathfrak{g}}$.

This has the following advantage. Suppose W is an open inside $\mathbf{Reg}_{\mathfrak{g}}$ containing $\mathfrak{g}$ and suppose $\mathfrak{p} \in W$ is an immediate overprime of $\mathfrak{g}$. The latter means that $A_{\mathfrak{p}}/\mathfrak{g}A_{\mathfrak{p}}$ has dimension one, and hence is a discrete valuation ring, since $W \subseteq \mathbf{Reg}_{\mathfrak{g}}$. Therefore, the image of any element $a \in \mathfrak{p} - (\mathfrak{p}^2 + \mathfrak{g})$ is a uniformizing parameter in $A_{\mathfrak{p}}/\mathfrak{g}A_{\mathfrak{p}}$. In other words, we have an equality

$$\mathfrak{p}A_{\mathfrak{p}} = \mathfrak{g}A_{\mathfrak{p}} + aA_{\mathfrak{p}}. \tag{7}$$

Moreover, suppose Q is an arbitrary finitely generated A-module such that $Q_{\mathfrak{g}} \neq 0$. If $\mathfrak{p}$ is not an associated prime of Q–a condition that can be enforced by shrinking W since Q has only finitely many associated primes–, then by prime avoidance, we may assume that a is $Q_{\mathfrak{p}}$-regular.

6.1.2 We first treat the invariant $\omega_0^{\mathrm{Betti}}$. By Nakayama's Lemma, $\omega_0^{\mathrm{Betti}}(M_{\mathfrak{g}})$ is equal to $\mu(M_{\mathfrak{g}})$, the minimal number of generators of $M_{\mathfrak{g}}$. It is well known (see for instance [4, Theorem 4.10]) that the minimal number of generators is a constructible invariant, whence in particular strongly devissable. Let us choose for an arbitrary finitely generated A-module Q an open $\mathbf{Gen}_{Q,\mathfrak{g}}$ of $\operatorname{Spec} A$ containing $\mathfrak{g}$, such that $\omega_0^{\mathrm{Betti}}(Q_{\mathfrak{p}})$ is constant for all overprimes $\mathfrak{p}$ of $\mathfrak{g}$ inside $\mathbf{Gen}_{Q,\mathfrak{g}}$. This constant value is of course equal to $\omega_0^{\mathrm{Betti}}(Q_{\mathfrak{g}})$. This settles the case of $\omega_0^{\mathrm{Betti}}$ by taking for open set $\mathbf{Gen}_{M,\mathfrak{g}}$.

6.1.3 Before treating the remaining invariants, we need a devissage result on depth. I claim that for each finitely generated A-module Q, there exists an open set

$\mathbf{Dep}_{Q,\mathfrak{g}}$ of $\operatorname{Spec} A$ containing $\mathfrak{g}$ with the property that for any immediate overprime $\mathfrak{p}$ of $\mathfrak{g}$ in $\mathbf{Dep}_{Q,\mathfrak{g}}$, we have

$$\operatorname{depth} Q_{\mathfrak{p}} = \operatorname{depth} Q_{\mathfrak{g}} + 1.$$

Let us first prove the claim in case $Q_{\mathfrak{g}}$ has depth zero. This means that $\mathfrak{g}$ is an associated prime of Q. Therefore, there is some $m \in Q$ for which $\operatorname{Ann}_A(m) = \mathfrak{g}$. Choose $\mathbf{Dep}_{Q,\mathfrak{g}}$ so that it does not contain any associated prime of A, Q or $N := Q/Am$ other than $\mathfrak{g}$. Moreover, by §6.1.1, we may choose $\mathbf{Dep}_{Q,\mathfrak{g}}$ inside $\mathbf{Reg}_{\mathfrak{g}}$. Let $\mathfrak{p} \in \mathbf{Dep}_{Q,\mathfrak{g}}$ be an immediate overprime of $\mathfrak{g}$. It follows that we may choose an $a \in \mathfrak{p}$ satisfying (7) which is simultaneously $A_{\mathfrak{p}}$-regular, $Q_{\mathfrak{p}}$-regular and $N_{\mathfrak{p}}$-regular. From the exact sequence

$$0 \to Am \to Q \to N \to 0$$

and $Am \cong A/\mathfrak{g}$, we get after localizing at $\mathfrak{p}$ and then applying $\operatorname{Hom}_{A_{\mathfrak{p}}}(k(\mathfrak{p}), \cdot)$, an exact sequence

$$\operatorname{Hom}_{A_{\mathfrak{p}}}(k(\mathfrak{p}), N_{\mathfrak{p}}) \to \operatorname{Ext}^1_{A_{\mathfrak{p}}}(k(\mathfrak{p}), A_{\mathfrak{p}}/\mathfrak{g}A_{\mathfrak{p}}) \to \operatorname{Ext}^1_{A_{\mathfrak{p}}}(k(\mathfrak{p}), Q_{\mathfrak{p}}) \tag{8}$$

Since $\mathfrak{p}$ is not an associated prime of N, the depth of $N_{\mathfrak{p}}$ is positive. Consequently, the left most module in (8) is zero. Using Lemma 5.4 and the fact that a is $A_{\mathfrak{p}}$-regular, we get

$$\operatorname{Ext}^1_{A_{\mathfrak{p}}}(k(\mathfrak{p}), A_{\mathfrak{p}}/\mathfrak{g}A_{\mathfrak{p}}) \cong \operatorname{Hom}_{A_{\mathfrak{p}}/aA_{\mathfrak{p}}}(k(\mathfrak{p}), A_{\mathfrak{p}}/(\mathfrak{g}A_{\mathfrak{p}} + aA_{\mathfrak{p}})).$$

By (7), this latter module is simply $k(\mathfrak{p})$. Therefore, $\operatorname{Ext}^1_{A_{\mathfrak{p}}}(k(\mathfrak{p}), Q_{\mathfrak{p}})$ is non-zero, by (8), showing that $Q_{\mathfrak{p}}$ has depth one, by [4, Theorem 16.7], as required.

Assume next that $Q_{\mathfrak{g}}$ has depth $q > 0$. Let $(a_1, \dots, a_q)$ be a maximal $Q_{\mathfrak{g}}$-regular sequence, with $a_i \in \mathfrak{g}$. Let $B := A/(a_1, \dots, a_q)A$ and $H := Q/(a_1, \dots, a_q)Q$. It follows that $H_{\mathfrak{g}B}$ has depth zero. Let $\mathbf{Dep}_{H,\mathfrak{g}B}$ be the open subset of $\operatorname{Spec} B$ for the depth zero B-module H defined above. In other words, for any immediate overprime $\mathfrak{P}$ of $\mathfrak{g}B$ inside $\mathbf{Dep}_{H,\mathfrak{g}B}$, the depth of $H_{\mathfrak{P}}$ is one. The canonical closed immersion $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$ given by $\mathfrak{P} \mapsto \mathfrak{p} := \mathfrak{P} \cap A$ induces a bijection between the immediate overprimes of $\mathfrak{g}B$ and $\mathfrak{g}$ respectively. Since $(a_1, \dots, a_q)$ is $Q_{\mathfrak{g}}$-regular, we can find an open U containing $\mathfrak{g}$, such that $(a_1, \dots, a_q)$ is $Q_{\mathfrak{p}}$-regular, for any $\mathfrak{p} \in U$ containing $\mathfrak{g}$ by §5.2.2. Therefore, if we let $\mathbf{Dep}_{Q,\mathfrak{g}}$ be the intersection of $\mathbf{Dep}_{H,\mathfrak{g}B}$ and U, then we get from [1, Proposition 1.2.10] that $Q_{\mathfrak{p}}$ has depth $q+1$, for every immediate overprime $\mathfrak{p}$ of $\mathfrak{g}$ inside $\mathbf{Dep}_{Q,\mathfrak{g}}$, proving the claim.

6.1.4 We now treat the remaining invariants simultaneously. Suppose $M_{\mathfrak{g}}$ has depth q. For any A-algebra B, let $\mathrm{C}_i(B)$ be either the module $\operatorname{Tor}_i^B(B/\mathfrak{g}B, M \otimes_A B)$ or the module $\operatorname{Ext}_B^{q+i}(B/\mathfrak{g}B, M \otimes_A B)$ according to whether we are in the

Betti case or in the Bass case. Note that if $A \to B$ is flat, then $\mathrm{C}_i(A) \otimes B \cong \mathrm{C}_i(B)$. Fix $i \in \mathbb{N}$ and let b be respectively $\omega_i^{\mathrm{Betti}}(M_\mathfrak{g})$ or $\omega_i^{\mathrm{Bass}}(M_\mathfrak{g})$. By definition, b is the dimension of $\mathrm{C}_i(A_\mathfrak{g}) = (\mathrm{C}_i(A))_\mathfrak{g}$ over $k(\mathfrak{g})$. Therefore, b is also the minimal number of generators of $\mathrm{C}_i(A_\mathfrak{g})$. Let U be an open inside

$$\mathbf{Gen}_{\mathrm{C}_i(A),\mathfrak{g}} \cap \mathbf{Dep}_{M,\mathfrak{g}}$$

as defined in §6.1.2 and §6.1.3 respectively. Moreover, we can choose U so that it does not contain any associated prime of A, of M or of $\mathrm{C}_{i+1}(A)$ other than $\mathfrak{g}$. Fix an immediate overprime $\mathfrak{p}$ of $\mathfrak{g}$ in U. By the choice of U, we have that $\mathrm{C}_i(A_\mathfrak{p}) = (\mathrm{C}_i(A))_\mathfrak{p}$ is minimally generated by b elements and $M_\mathfrak{p}$ has depth $q+1$. Since $\mathfrak{p}$ lies in $\mathbf{Reg}_\mathfrak{g}$, we may choose an $a \in \mathfrak{p}$ satisfying (7) which is simultaneously $A_\mathfrak{p}$-regular, $M_\mathfrak{p}$-regular and $\mathrm{C}_{i+1}(A_\mathfrak{p})$-regular. Let us write a bar to indicate that we take reduction modulo a, so that for instance $\overline{A} = A/aA$ and $\overline{M} = M/aM$. By Nakayama's Lemma, $\overline{\mathrm{C}_i(A_\mathfrak{p})}$ is also minimally generated by b elements. I claim that

$$\overline{\mathrm{C}_i(A_\mathfrak{p})} \cong \mathrm{C}_i(\overline{A}_\mathfrak{p}). \tag{9}$$

Assuming the claim, it follows that $\mathrm{C}_i(\overline{A}_\mathfrak{p})$ is minimally generated by b elements. By (7), we have an isomorphism $\overline{A}_\mathfrak{p}/\mathfrak{g}\overline{A}_\mathfrak{p} \cong k(\mathfrak{p})$, so that $\mathrm{C}_i(\overline{A}_\mathfrak{p})$ is in fact a b-dimensional $k(\mathfrak{p})$-vector space. More precisely, in the Betti case, $\mathrm{C}_i(\overline{A}_\mathfrak{p})$ is the module $\mathrm{Tor}_i^{\overline{A}_\mathfrak{p}}(k(\mathfrak{p}), \overline{M}_\mathfrak{p})$. Since a is $A_\mathfrak{p}$-regular and $M_\mathfrak{p}$-regular, Lemma 5.4 implies that this latter module is isomorphic to $\mathrm{Tor}_i^{A_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$. Combining these isomorphisms, we get that $\omega_i^{\mathrm{Betti}}(M_\mathfrak{p}) = b$. In the Bass case,

$$\mathrm{C}_i(\overline{A}_\mathfrak{p}) = \mathrm{Ext}^{q+i}_{\overline{A}_\mathfrak{p}}(k(\mathfrak{p}), \overline{M}_\mathfrak{p}).$$

By Lemma 5.4 the right hand side is isomorphic to $\mathrm{Ext}^{q+i+1}_{A_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$. Since $M_\mathfrak{p}$ has depth $q+1$, it follows that $b = \omega_i^{\mathrm{Bass}}(M_\mathfrak{p})$, as required.

6.1.5 So remains to prove isomorphism (9). Consider the exact sequence

$$0 \to M_\mathfrak{p} \xrightarrow{a} M_\mathfrak{p} \to \overline{M}_\mathfrak{p} \to 0.$$

Applying respectively the functor $A_\mathfrak{p}/\mathfrak{g}A_\mathfrak{p} \otimes_{A_\mathfrak{p}} \cdot$ or $\mathrm{Hom}_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{g}A_\mathfrak{p}, \cdot)$ to this sequence yields part of a long exact sequence

$$\mathrm{C}_i(A_\mathfrak{p}) \xrightarrow{a} \mathrm{C}_i(A_\mathfrak{p}) \to T \xrightarrow{\delta} \mathrm{C}_{i+1}(A_\mathfrak{p}) \xrightarrow{a} \mathrm{C}_{i+1}(A_\mathfrak{p}).$$

where T is respectively $\mathrm{Tor}_i^{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{g}A_\mathfrak{p}, \overline{M}_\mathfrak{p})$ or $\mathrm{Ext}^{q+i}_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{g}A_\mathfrak{p}, \overline{M}_\mathfrak{p})$. Since multiplication by a is injective on $C_{i+1}(A_\mathfrak{p})$, we get that δ is the zero homomorphism. It follows that $\overline{\mathrm{C}_i(A_\mathfrak{p})} \cong T$. On the other hand, since a is $A_\mathfrak{p}$-regular and is not contained in $\mathfrak{g}$, we get by Lemma 5.4 an isomorphism $T \cong \mathrm{C}_i(\overline{A}_\mathfrak{p})$, proving (9). □

As an immediate corollary, we get from Theorem 6.1 and Theorem 5.2 the following result.

Theorem 6.2 *For each $i \geq 0$, the numerical invariants $\omega_i^{\mathrm{Betti}}$ and ω_i^{Bass} are constructible on schemes of finite type over an algebraically closed field. In particular, if X is such a scheme and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then the points of X for which the Betti or the twisted Bass number of $\mathcal{F}$ are equal to some fixed number form a constructible set and only finitely many possibilities for these numbers occur.*

Corollary 6.3 *The invariant $\omega^{\mathrm{ProjDim}}$ assigning to a finitely generated R-module M its projective dimension is constructible on schemes of finite type over an algebraically closed field.*

Proof. Note that $\omega^{\mathrm{ProjDim}}$ takes values in $\mathbb{N} \cup \{\infty\}$. However, for each fixed base ring R, there are only finitely many possibilities, to wit, all values up to the dimension of R together with ∞. Using this observation in conjunction with [6, Proposition 6.3], we see that $\omega^{\mathrm{ProjDim}}$ is definable in families.

We will show devissability for all excellent schemes; constructibility then follows by Theorem 5.2. Let A be an excellent ring, let M be a finitely generated A-module and let $\mathfrak{g}$ be a non-maximal prime ideal of A. Suppose $\omega^{\mathrm{ProjDim}}(M_{\mathfrak{g}}) = q$. If $q = \infty$, then $\omega^{\mathrm{ProjDim}}(M_{\mathfrak{p}}) = \infty$, for all prime ideals $\mathfrak{p}$ containing $\mathfrak{g}$, since $M_{\mathfrak{g}}$ is a localization of $M_{\mathfrak{p}}$. Therefore, assume q finite. By the Auslander-Buchsbaum Formula (see [4, Theorem 19.1]),

$$q = \mathrm{depth}(A_{\mathfrak{g}}) - \mathrm{depth}(M_{\mathfrak{g}}).$$

By §6.1.3, if we take for U the intersection $\mathbf{Dep}_{A,\mathfrak{g}} \cap \mathbf{Dep}_{M,\mathfrak{g}}$ and if $\mathfrak{p} \in U$ is an immediate overprime of $\mathfrak{g}$, then $\mathrm{depth}(A_{\mathfrak{p}}) = \mathrm{depth}(A_{\mathfrak{g}}) + 1$ and $\mathrm{depth}(M_{\mathfrak{p}}) = \mathrm{depth}(M_{\mathfrak{g}}) + 1$. By another application of the Auslander-Buchsbaum Formula, we get $\omega^{\mathrm{ProjDim}}(M_{\mathfrak{p}}) = q$, as required. □

The invariant which assigns to an R-module M its injective dimension $\mathrm{injdim}(M)$ is not constructible, as the injective dimension is either infinite or equal to the depth of R. However, the difference $\mathrm{injdim}(M) - \mathrm{depth}(R)$ is definable in families by [6, Corollary 5.5] and devissable (it is either 0 or ∞ according to whether M has finite injective dimension or not), and therefore, it is constructible on schemes of finite type over an algebraically closed field. Consequently, the locus of points on such a scheme X for which the stalk of a coherent $\mathcal{O}_X$-module $\mathcal{F}$ has finite injective dimension, is constructible. In Section 8, we will use the following result to obtain a uniform version of Theorem 6.2.

Proposition 6.4 *The numerical invariants $\omega_i^{\mathrm{Betti}}$ and ω_i^{Bass} deform well.*

Proof. Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and M a finitely generated R-module. Let $a \in \mathfrak{m} - \mathfrak{m}^2$ be R-regular and M-regular. By Lemma 5.4, we have isomorphisms

$$\begin{aligned}\operatorname{Tor}_i^R(M,k) &\cong \operatorname{Tor}_i^{R/aR}(M/aM,k)\\ \operatorname{Ext}_R^{i+1}(k,M) &\cong \operatorname{Ext}_{R/aR}^{i}(k,M/aM)\end{aligned}$$

for all $i \geq 0$. Since the depth of M/aM as an R/aR-module is one less than the depth of M as an R-module, the statement follows. □

7 Singularity defects

In this section, we study several numerical ring invariants which measure the failure that some property holds. Using the general theory developed in the first part, we will show that they are constructible. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$.

Regularity defect. We define the *regularity defect* of R to be the difference between its embedding dimension and its (Krull) dimension and we denote it by $\omega^{\text{RegDef}}(R)$. Recall that the *embedding dimension* $\operatorname{embdim} R$ of R is by definition the minimal number of generators of its maximal ideal, which by Nakayama's Lemma is equal to the dimension of $\mathfrak{m}/\mathfrak{m}^2$ viewed as a vector space over the residue field k of R. Therefore, the embedding dimension is also equal to $\omega_0^{\text{Betti}}(\mathfrak{m}) = \omega_1^{\text{Betti}}(k)$, where k is the residue field of R. By definition, a Noetherian local ring is regular if and only if $\omega^{\text{RegDef}}(R) = 0$.

Complete intersection defect. We define the *complete intersection defect* of R to be the number

$$\omega^{\text{CIDef}}(R) := \omega_2^{\text{Betti}}(k) - \frac{\omega_1^{\text{Betti}}(k)^2 + \omega_1^{\text{Betti}}(k)}{2} + \dim R. \tag{10}$$

It follows from [1, Theorem 2.3.3] that $\omega^{\text{CIDef}}(R)$ is always non-negative and that R is a complete intersection if and only if $\omega^{\text{CIDef}}(R) = 0$. See (17) below for an alternative formula for ω^{CIDef} which better explains its name.

Cohen-Macaulay defect. We define the *Cohen-Macaulay defect* of R to be the number

$$\omega^{\text{CMDef}}(R) := \dim R - \operatorname{depth} R.$$

Note that $\omega^{\text{CMDef}}(R)$ is always non-negative, and equal to zero precisely when R is Cohen-Macaulay.

Gorenstein defect. We define the *Gorenstein defect* of R to be the number

$$\omega^{\mathrm{GorDef}}(R) := \omega^{\mathrm{CMDef}}(R) + \mathrm{type}(R) - 1 \tag{11}$$

where $\mathrm{type}(R)$ denotes the type of R. Recall that the *type* of R is by definition the zero-th twisted Bass number $\omega_0^{\mathrm{Bass}}(R)$, that is to say, the q-th ordinary Bass number of R, where q is the depth of R. Since $\omega_0^{\mathrm{Bass}}(R)$ is positive, $\omega^{\mathrm{GorDef}}(R)$ is always non-negative and is equal to zero if and only if $\omega^{\mathrm{CMDef}}(R) = 0$ and $\mathrm{type}(R) = 1$, and this in turn is equivalent with R being Gorenstein by [1, Theorem 3.2.10].

We will refer to the above four invariants as *singularity defects*. The following result in combination with Theorem 4.8 proves already the first assertion of Theorem 1.1 over an algebraically closed field.

Theorem 7.1 *Each singularity defect is definable in families and deforms well. Moreover, each singularity defect is devissable on any scheme admitting a closed immersion into an excellent regular scheme.*

Proof. Definability in families of each singularity defect follows from the results of [6] together with Theorem 3.5. More precisely, apart from the Betti and (twisted) Bass numbers, which were discussed in the previous section, we only need to consider depth and dimension of a local algebra R. This, however, is covered by [6, Proposition 5.1].

If $a \in \mathfrak{m} - \mathfrak{m}^2$ is an R-regular element, then both embedding dimension, depth and dimension have dropped by one for R/aR. In other words, $\omega^{\mathrm{RegDef}}(R) = \omega^{\mathrm{RegDef}}(R/aR)$ and $\omega^{\mathrm{CMDef}}(R) = \omega^{\mathrm{CMDef}}(R/aR)$, so that ω^{RegDef} and ω^{CMDef} deform well. Complete intersection defect deforms well by [1, Theorem 2.3.4]. Finally, since

$$\mathrm{Ext}_R^q(k, R) \cong \mathrm{Ext}_{R/aR}^{q-1}(k, R/aR)$$

by Lemma 5.4, we get $\mathrm{type}(R) = \mathrm{type}(R/aR)$, from which it follows that also ω^{GorDef} deforms well.

Hence remains to prove that these singularity defects are devissable on any closed subscheme of an excellent regular scheme. Let A be a homomorphic image of an excellent regular ring and let $\mathfrak{g}$ be a non-maximal prime ideal of A. We need to find an open U containing $\mathfrak{g}$, such that for any immediate overprime $\mathfrak{p}$ of $\mathfrak{g}$ in U, the localizations $A_{\mathfrak{g}}$ and $A_{\mathfrak{p}}$ have the same defect. Moreover, we will always choose U inside $\mathbf{Reg}_{\mathfrak{g}}$ so that the results of §6.1.1 apply. In particular, we will take an $a \in \mathfrak{p} - (\mathfrak{p}^2 + \mathfrak{g})$ (possibly subject to some other constraints), so that equality (7) holds. We fix the above notation and treat each singularity defect separately.

7.1.1 Regularity defect.

Suppose $A_{\mathfrak{g}}$ has embedding dimension e. Apply the fact that $\omega_0^{\mathrm{Betti}}$ is devissable to the A-module $M = \mathfrak{g}$ at the prime ideal $\mathfrak{g}$. In other words, if we take U inside $\mathbf{Gen}_{\mathfrak{g},\mathfrak{g}}$ as defined in §6.1.2, then

$$\mu(\mathfrak{g}A_{\mathfrak{p}}) = \mu(\mathfrak{g}A_{\mathfrak{g}}) = e. \tag{12}$$

Our aim is to show that

$$\mu(\mathfrak{p}A_{\mathfrak{p}}) = e + 1. \tag{13}$$

To this end, consider the exact sequence

$$0 \to aA_{\mathfrak{p}} \to \mathfrak{p}A_{\mathfrak{p}} \to \mathfrak{p}A_{\mathfrak{p}}/aA_{\mathfrak{p}} \to 0$$

and tensor it with the residue field $k(\mathfrak{p})$ of $A_{\mathfrak{p}}$ to get an exact sequence

$$k(\mathfrak{p}) \to \mathfrak{p}A_{\mathfrak{p}}/\mathfrak{p}^2A_{\mathfrak{p}} \to (\mathfrak{p}A_{\mathfrak{p}}/aA_{\mathfrak{p}}) \otimes k(\mathfrak{p}) \to 0. \tag{14}$$

The first homomorphism in this sequence is not zero since $a \notin \mathfrak{p}^2$. Therefore, it must be injective. I claim that the last module in (14) has length e, from which (13) then follows. Now, in view of (7), this claim is equivalent with showing that $\mathfrak{p}A_{\mathfrak{p}}/aA_{\mathfrak{p}} = \mathfrak{g}A_{\mathfrak{p}}/aA_{\mathfrak{p}}$ is minimally generated by e elements. By (12), we can find elements $a_1, \dots, a_e \in \mathfrak{g}$ which minimally generate $\mathfrak{g}A_{\mathfrak{p}}$. So we only need to verify that they also form a minimal set of generators for $\mathfrak{g}A_{\mathfrak{p}}/aA_{\mathfrak{p}}$. If not, then after renumbering, we would have an equation

$$a_1 = c_0a + \sum_{i=2}^{e} c_ia_i \tag{15}$$

in $A_{\mathfrak{p}}$, for some $c_i \in A_{\mathfrak{p}}$. However, from $a \notin \mathfrak{g}$ we get $c_0 \in \mathfrak{g}A_{\mathfrak{p}}$, so that we can write $c_0 = \sum d_ia_i$ for some $d_i \in A_{\mathfrak{p}}$. Substituting this in (15) yields

$$0 = \sum_{i=1}^{e}(c_i + d_ia)a_i$$

in $A_{\mathfrak{p}}$, where we let $c_1 := -1$. By Nakayama's Lemma, this violates the fact that the a_i minimally generate $\mathfrak{g}A_{\mathfrak{p}}$. Hence we showed the validity of (13). Next, we may assume, by shrinking U if necessary, that any overprime $\mathfrak{q}$ of $\mathfrak{g}$ in U contains exactly the same minimal prime ideals as $\mathfrak{g}$. In particular, since A is catenary, the height of $\mathfrak{q}$ is equal to the height of $\mathfrak{g}$ plus the height of $\mathfrak{q}(A/\mathfrak{g})$. Applied to the immediate overprime $\mathfrak{p}$, we get that the dimension of $A_{\mathfrak{g}}$ is one less than the dimension of $A_{\mathfrak{p}}$. Together with (13), this shows that $A_{\mathfrak{g}}$ and $A_{\mathfrak{p}}$ have the same regularity defect.

7.1.2 Cohen-Macaulay defect. Suppose $A_{\mathfrak{g}}$ has depth q. Take U inside the open $\mathbf{Dep}_{A,\mathfrak{g}}$ defined in §6.1.3 applied with $Q = A$. It follows that $A_{\mathfrak{p}}$ has depth $q+1$, so that $\omega^{\mathrm{CMDef}}(A_{\mathfrak{g}}) = h - q = \omega^{\mathrm{CMDef}}(A_{\mathfrak{p}})$.

7.1.3 Gorenstein defect. Using the previous case, we only need to show that we can maintain the type of $A_{\mathfrak{g}}$, since the sum of devissable invariants is again devissable. Since the type is equal to the zero-th Bass number of the module $A_{\mathfrak{g}}$, devissability follows from Theorem 6.1 applied with $M = A$.

7.1.4 Complete intersection defect. One might be tempted to infer directly from the devissability of the Betti numbers proven in Theorem 6.1 that ω^{CIDef} is devissable. However, the Betti numbers as they appear in (10) vary with the point: at each point, we take a different module, to wit, the residue field of that point. In other words, the ring invariant which assigns to a local ring R the i-th Betti number $\omega_i^{\mathrm{Betti}}(k)$ of its residue field k is not devissable. For instance, if $i = 1$ then $\omega_1^{\mathrm{Betti}}(k) = \operatorname{embdim} R$, which is clearly not devissable.

Therefore, we need an alternative description of $\omega^{\mathrm{CIDef}}(R)$. It follows from [1, Theorem 2.3.2] that

$$\omega^{\mathrm{CIDef}}(R) = \epsilon_1(R) - \omega^{\mathrm{RegDef}}(R) \tag{16}$$

where $\epsilon_1(R)$ is the length of the first Koszul homology $H_1(R)$ of a system of parameters of R (this is independent from the choice of system of parameters; see [1, §2.3]). Moreover, if R a homomorphic image $S/\mathfrak{a}$ of a regular local ring S, then we have

$$\epsilon_1(R) = \operatorname{embdim} R - \dim S + \mu(\mathfrak{a})$$

by [4, Theorem 21.1]. Putting these two equations together, we get

$$\omega^{\mathrm{CIDef}}(R) = \dim R - \dim S + \mu(\mathfrak{a}) = \mu(\mathfrak{a}) - \operatorname{ht}(\mathfrak{a}), \tag{17}$$

where the last equality holds since S is a regular local ring.

Let B be an excellent regular ring such that $A = B/\mathfrak{a}$ for some ideal $\mathfrak{a}$ in B and let $f\colon \operatorname{Spec} A \hookrightarrow \operatorname{Spec} B$ be the corresponding closed immersion. Let $\mathfrak{G} := \mathfrak{g} \cap B$ and let W be an open in $\operatorname{Spec} B$ containing $\mathfrak{G}$ witnessing the strong devissability of $\omega_{\mathfrak{a}}^{\mathrm{ht}}$ proven in §5.2.1. Choose W moreover in $\mathbf{Gen}_{\mathfrak{a},\mathfrak{G}}$ as given by §6.1.2 applied to the B-module $\mathfrak{a}$. Let $U := f^{-1}(W)$ and let $\mathfrak{P} := \mathfrak{p} \cap B$, where as before $\mathfrak{p}$ is an immediate overprime of $\mathfrak{g}$ inside U. It follows that $\mathfrak{P}$ is an immediate overprime of $\mathfrak{G}$ inside W. Strong devissability of $\omega_{\mathfrak{a}}^{\mathrm{ht}}$ gives that $\mathfrak{a}B_{\mathfrak{P}}$ and $\mathfrak{a}B_{\mathfrak{G}}$ have the same height. On the other hand, devissability of the minimal number of generators yields $\mu(\mathfrak{a}B_{\mathfrak{P}}) = \mu(\mathfrak{a}B_{\mathfrak{G}})$, showing by (17) applied with R equal to respectively $A_{\mathfrak{g}} = B_{\mathfrak{G}}/\mathfrak{a}B_{\mathfrak{G}}$ and $A_{\mathfrak{p}} = B_{\mathfrak{P}}/\mathfrak{a}B_{\mathfrak{P}}$, that $\omega^{\mathrm{CIDef}}(A_{\mathfrak{g}}) = \omega^{\mathrm{CIDef}}(A_{\mathfrak{p}})$. □

From the proof it is clear that all singularity defects other than the complete intersection defect are devissable on any excellent scheme. However, the latter defect seems to require some type of Noether normalization.

Corollary 7.2 *The invariant assigning to an affine local algebra R its* first deviation $\epsilon_1(R)$ *is constructible. The same is true for the invariant which assigns to R its type.*

Proof. Immediate from equalities (16) and (11), together with the following fact: if ω_i are constructible numerical invariants, then so is any polynomial expression

$$\omega := P(\omega_1, \dots, \omega_n)$$

in the ω_i with $P \in \mathbb{Z}[\xi_1, \dots, \xi_n]$. To prove the latter fact, observe that the ω_i take only finitely many values on each scheme X, say given by the finite subset S of $\mathbb{Z}$. Therefore, $\omega_X^{-1}(s)$ consists of all points $x \in X$, for which there exist $s_i \in S$ with $s = P(s_1, \dots, s_n)$ and $\omega_i(x) = s_i$, and hence is constructible. □

This raises the question whether the higher deviations ϵ_p (that is to say, the length of the Koszul homologies $H_p(R)$) are also constructible on schemes of finite type over an algebraically closed field. Definability in families follows from [6, Theorem 4.7] and the fact that we can choose a system of parameters of bounded degree complexity. In case $p = 2$, we can use alternatively [1, Theorem 2.3.12] to show definability in families. Moreover, assuming the devissability of the Poincare series, it follows from the expression in [1, Theorem 2.3.12] for ϵ_2, that it is devissable whence constructible. For the higher deviations, additional work seems to be required.

Definition 7.3 *We call a subset T of a scheme X* generically devissable *if, for each generic point η of X which belongs to T, we can find an open U of X containing η, such that any immediate η-specialization $y \in U$ belongs to T.*

In particular, any subset omitting all the generic points is automatically generically devissable. We call a ring invariant ω *generically devissable*, if for each scheme X and for each generic point $\eta \in X$, the level set $\omega_X^{-1}(\omega(\eta))$ is generically devissable in X.

Proposition 7.4 *Let ω be a ring invariant defined on the class of all excellent Cohen-Macaulay schemes. If ω deforms well and is generically devissable, then it is devissable.*

Proof. In view of the local nature of the assertion, we may reduce the proof to the following special case. Let A be an excellent Cohen-Macaulay ring and $\mathfrak{g}$ a non-maximal prime ideal in A. We need to show that there exists an $c \notin \mathfrak{g}$, such that $\omega(A_\mathfrak{g}) = \omega(A_\mathfrak{p})$ for every height one prime ideal $\mathfrak{p}$ in $A_c/\mathfrak{g}A_c$.

We will prove this statement for all pairs $(A, \mathfrak{g})$ by induction on the height h of $\mathfrak{g}$, where the case $h = 0$ holds by assumption. So assume $h > 0$ and let $s := \omega(A_\mathfrak{g})$. Since $A_\mathfrak{g}$ is Cohen-Macaulay, there exists an $A_\mathfrak{g}$-regular element x, which we may choose moreover outside $\mathfrak{g}^2$. Let $B := A/xA$. Since ω deforms well, $\omega(B_{\mathfrak{g}B}) = s$.

Since B is Cohen-Macaulay and since $\mathfrak{g}B$ has height $h-1$, our induction hypothesis implies the existence of an element $c \notin \mathfrak{g}$ such that $\omega(B_{\mathfrak{p}B}) = s$ for any height one prime ideal $\mathfrak{p}$ in $B_c/\mathfrak{g}B_c = A_c/\mathfrak{g}A_c$. Replacing c by some multiple of it (which corresponds to shrinking the open defined by $c \neq 0$), we may moreover assume by §7.1.1, applied respectively in B and A, that $B_{\mathfrak{g}B}$ and $B_{\mathfrak{p}B}$ have the same regularity defect, and so do $A_{\mathfrak{g}}$ and $A_{\mathfrak{p}}$. Moreover, by §5.2.2, we may assume that x is $A_{\mathfrak{p}}$-regular whenever $\mathfrak{p}$ belongs to $A_c/\mathfrak{g}A_c$.

Let us verify that this c satisfies the desired properties. Take a height one prime ideal in $A_c/\mathfrak{g}A_c$, and let us denote the corresponding immediate overprime of $\mathfrak{g}$ in A by $\mathfrak{p}$. Since $x \notin \mathfrak{g}^2$, the embedding dimension of $B_{\mathfrak{g}B}$ is one less than the embedding dimension of $A_{\mathfrak{g}}$ by Nakayama's Lemma. Hence both rings have the same regularity defect, which is then by choice of c also the same regularity defect of $A_{\mathfrak{p}}$ and $B_{\mathfrak{p}B}$. This in turn implies that the embedding dimension of $B_{\mathfrak{p}B}$ is one less than the embedding dimension of $A_{\mathfrak{p}}$. By another application of Nakayama's Lemma, $x \notin \mathfrak{p}^2$. Since x is $A_{\mathfrak{p}}$-regular and ω deforms well, $\omega(A_{\mathfrak{p}}) = \omega(B_{\mathfrak{p}B})$. Since $\mathfrak{p}(B_c/\mathfrak{g}B_c) = \mathfrak{p}(A_c/\mathfrak{g}A_c)$ has height one, we get from our choice of c that $\omega(B_{\mathfrak{p}B}) = s$. In conclusion, we showed that $\omega(A_{\mathfrak{p}}) = s$ for every height one prime in $A_c/\mathfrak{g}A_c$. □

8 Constructible families

So far we have been dealing with ring and module invariants, but it should be obvious that the present techniques allow us to treat more general situations. Given a local ring R, we call an R-algebra S a *local R-algebra* if S is a local ring and $R \to S$ is a local homomorphism.

Definition 8.1 (Relative Invariants) *A map ν which assigns to a pair (R, S) a value in a set $\mathbb{S}$, where R is a Noetherian local ring and S a Noetherian local R-algebra, will be called a* relative ($\mathbb{S}$-valued ring) invariant.

One can similarly define a relative module invariant; details are left to the reader. We say that ω is *of finite type*, if we moreover impose that R is essentially of finite type over an algebraically closed field and S is essentially of finite type over R. If $f\colon Y \to X$ is a map of schemes and y a point of Y, then we write

$$\nu(y, f) := \nu(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y})$$

where $x = f(y)$.

As before, the *level sets* of ν are defined for a map $Y \to X$, as the collection of all points $y \in Y$ for which $\nu(y, f) = s$, for some $s \in \mathbb{S}$. Note that they form a partition of Y. We call ν *saturated* (respectively, *devissable*, *geometrically constructible*, *constructible*), if each of its level sets is. It is immediate from Theorem 4.8 that a relative invariant of finite type which is geometrically constructible and devissable, is in fact constructible. As before, most invariants only behave properly on some subcategory $\mathcal{C}$ of schemes, and to emphasize this we may say that ν is defined for schemes (or maps) in $\mathcal{C}$.

For our purposes, the following construction of a relative invariant will be the only example used in this paper. Namely, we start from an $\mathbb{S}$-valued ring invariant ω. To ω, we associate a relative invariant, denoted $\tilde{\omega}$, as follows. Given a local map $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ of Noetherian local rings, we set

$$\tilde{\omega}(R, S) := \omega(S/\mathfrak{m}S). \tag{18}$$

Let us study a little closer this definition in case we have a map $f\colon Y \to X$ of schemes and a point $y \in Y$. Let $R := \mathcal{O}_{X,x}$ and $S := \mathcal{O}_{Y,y}$, where $x = f(y)$. If $\mathfrak{m}$ denotes the maximal ideal in R, then $S/\mathfrak{m}S$ is the local ring of the fiber $f^{-1}(x)$ at the point y, and therefore

$$\tilde{\omega}(y, f) = \omega(\mathcal{O}_{f^{-1}(x),y}). \tag{19}$$

Proposition 8.2 *Let ω be an $\mathbb{S}$-valued invariant and let $\tilde{\omega}$ denote the associated relative invariant. If ω is strongly devissable and deforms well, then $\tilde{\omega}$ is strongly devissable for flat maps.*

Proof. Let $f\colon Y \to X$ be a flat map and fix some $s \in \mathbb{S}$. Since the property we seek to prove is local, we may assume without loss of generality that $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ are affine. Let $\mathfrak{G}$ be a prime ideal in B corresponding to a point $y \in Y$ and let $\mathfrak{g} := \mathfrak{G} \cap A$ be the prime ideal corresponding to $x = f(y)$. Since the base change $A/\mathfrak{g} \to B/\mathfrak{g}B$ has the same fibers as $A \to B$, we may reduce to the case that $\mathfrak{g} = 0$. Let $s := \tilde{\omega}(y, f)$. Hence, by definition, $s = \omega(B_{\mathfrak{G}})$. Applying our strong devissability hypothesis in Y at the prime ideal $\mathfrak{G}$, we can find an open set $V \subseteq Y$, such that for all overprimes $\mathfrak{P}$ of $\mathfrak{G}$ in V, we have

$$s = \omega(B_{\mathfrak{P}}). \tag{20}$$

Let U be a non-empty open set of X contained in $\mathbf{Reg}_{\mathfrak{g}}$ as defined in §6.1.1. Let $z \in V \cap f^{-1}(U)$ be an x-specialization and let $\mathfrak{P}$ be the overprime of $\mathfrak{G}$ corresponding to z. Hence $\mathfrak{p} := \mathfrak{P} \cap A$ corresponds to the point $f(z) \in U$. Let h be the height of $\mathfrak{p}$. Since $A_{\mathfrak{p}}$ is regular of dimension h, we can find a regular sequence $(x_1, \dots, x_h)$ in $\mathfrak{p}$ such that

$$(x_1, \dots, x_h)A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}. \tag{21}$$

Since $A \to B$ is flat, $(x_1, \dots, x_h)$ is also $B_{\mathfrak{P}}$-regular, and hence

$$s = \omega(B_{\mathfrak{P}}) = \omega(B_{\mathfrak{P}}/(x_1, \dots, x_h)B_{\mathfrak{P}}) = \omega(B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}}) = \tilde{\omega}(z, f)$$

where the first equality follows from (20), the second by deformation, the third from (21) and the final by definition of $\tilde{\omega}$. In conclusion, we showed that $\tilde{\omega}$ is strongly devissable. □

Presumably, we can replace strong devissability by devissability and drop the flatness assumption, but for our purposes, the above result suffices.

Theorem 8.3 *Let ω be an $\mathbb{S}$-valued invariant of finite type, which is definable in families, devissable and deforms well. If $f\colon Y \to X$ is a map of finite type of schemes of finite type over an algebraically closed field, then there exists a finite constructible partition $Y = Y_{s_1} \cup \dots \cup Y_{s_m}$, indexed by elements $s_k \in \mathbb{S}$, such that for each $y \in Y_{s_k}$, we have*

$$\omega(\mathcal{O}_{f^{-1}(f(y)),y}) = s_k.$$

Proof. In view of (19), all we need to do is show that the associated invariant $\tilde{\omega}$ is constructible. By [2, Corollary **IV**.6.9.3], we can find a constructible partition $X = X_1 \cup \dots \cup X_s$, such that each base change $f^{-1}(X_i) \to X_i$ is flat. Since the local rings of the fibers of $f^{-1}(X_i) \to X_i$ and of f are the same, we may therefore pass to one of these base changes and assume from the start that f is moreover flat. By Theorem 5.2, the invariant ω is constructible, whence in particular strongly devissable by Theorem 4.4. Hence $\tilde{\omega}$ is strongly devissable by Proposition 8.2.

So remains to show that $\tilde{\omega}$ is geometrically constructible in view of Theorem 5.2. Fix some $s \in \mathbb{S}$. We need to show that the subset of $Y(K)$ consisting of all K-rational points y for which $\tilde{\omega}(y, f) = s$, is constructible in $Y(K)$. Consider the abstract family $\mathfrak{R}$ given by the commutative diagram

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{\ \gamma\ } & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\ f\ ]{} & X \end{array} \tag{22}$$

where $\gamma = g$ is the projection onto the second coordinate. For $y \in Y(K)$, we have $\gamma(g^{-1}(y)) = y$. Therefore, $\mathfrak{R}_y$ is the local ring of the fiber $f^{-1}(f(y))$ at the point y. By (19), we get $\tilde{\omega}(y, f) = \omega(\mathfrak{R}_y)$. Since ω is definable in families, the collection of all $y \in Y(K)$ for which $\tilde{\omega}(y, f) = s$ is therefore constructible, as required. □

By Proposition 6.4, the invariants $\omega_i^{\mathrm{Betti}}$ and ω_i^{Bass} deform well and so we can apply Theorem 8.3 to them. The same is true for the singularity defects from Section 7 in view of Theorem 7.1. In particular, this proves Theorem 1.1 over algebraically closed fields; the case of an arbitrary base field is then covered by the arguments in the next section. The next theorem gives a similar application of good deformation; this time we get a constructible partition in the target space.

Theorem 8.4 *Let ω be an $\mathbb{S}$-valued invariant of finite type. Assume ω is definable in families, devissable and deforms well. Let $f\colon Y \to X$ be a map of finite type of* *schemes of finite type over an algebraically closed field. For each $x \in X$, let*

$$\mathrm{Val}_\omega(x) := \left\{ \omega(\mathcal{O}_{f^{-1}(x),y}) \mid y \in f^{-1}(x) \right\}.$$

Then $\mathrm{Val}_\omega(x)$ is finite.

Moreover, for an arbitrary subset $\mathbb{T}$ of $\mathbb{S}$, let

$$F_\mathbb{T} := \{ x \in X \mid \mathrm{Val}_\omega(x) = \mathbb{T} \}.$$

Then the partition of X consisting of the non-empty sets $F_\mathbb{T}$, where $\mathbb{T}$ runs through all subsets of $\mathbb{S}$, is constructible. In particular, only finitely many finite subsets $\mathbb{T}$ of $\mathbb{S}$ occur as a set of the form $\mathrm{Val}_\omega(x)$.

Proof. By (19), we have

$$\mathrm{Val}_\omega(x) = \left\{ \tilde{\omega}(y, f) \mid y \in f^{-1}(x) \right\},$$

where $\tilde{\omega}$ is the relative invariant associated to ω. By Theorem 8.3, the collection of (non-empty) level sets

$$G_s := \{ y \in Y \mid \tilde{\omega}(y, f) = s \}$$

of $\tilde{\omega}$, is a constructible partition, where s runs over all possible values of $\mathbb{S}$. In particular, this partition is finite so that only finitely many values in $\mathbb{S}$ can occur. Therefore, also each $\mathrm{Val}_\omega(x)$ is finite.

Let $\mathbb{T}$ be a finite subset of $\mathbb{S}$. Let us write

$$\tilde{Y}_\mathbb{T} := \bigcup_{s \notin \mathbb{T}} G_s.$$

Since the partition $\{G_s\}$ is finite and constructible, each $\tilde{Y}_\mathbb{T}$ is constructible. I claim that

$$F_\mathbb{T} = \left(\bigcap_{t \in \mathbb{T}} f(G_t) \right) - f(\tilde{Y}_\mathbb{T}). \tag{23}$$

Assuming the claim, the result then follows by Chevalley's Theorem. To prove the claim, assume $x \in F_\mathbb{T}$. Since then $\mathrm{Val}_\omega(x) = \mathbb{T}$, we get $\tilde{\omega}(y, f) \in \mathbb{T}$ for each $y \in f^{-1}(x)$. In other words

$$f^{-1}(x) \cap \bigcup_{s \notin \mathbb{T}} G_s = \emptyset$$

which shows that x does not lie in $f(\tilde{Y}_\mathbb{T})$. On the other hand, for each $t \in \mathbb{T} = \mathrm{Val}_\omega(x)$, we can find an $y \in G_t$ with $f(y) = x$, so that x lies indeed in the right hand side of (23).

Conversely, if x lies in the right hand side of (23), then we can find for each $t \in \mathbb{T}$, a $y \in G_t$, such that $x = f(y)$, showing that $\mathbb{T} \subseteq \mathrm{Val}_\omega(x)$. However, since x does not lie in $f(\tilde{Y}_{\mathbb{T}})$, one checks that no other value in $\mathbb{S}$ can occur, so that $\mathbb{T} = \mathrm{Val}_\omega(x)$, as required. □

Suppose $\mathbf{P}$ is a property of local rings, such as being regular or Cohen-Macaulay. We say that a scheme X has property $\mathbf{P}$ if each of its local rings has. Let $\omega^{\mathbf{P}}$ be the associated invariant which takes the values 1 or 0 according to whether the property holds or not. Applying Theorem 8.4 to the singleton $\mathbb{T} = \{1\}$, we see that the collection of all points x in X for which the fiber $f^{-1}(x)$ has property $\mathbf{P}$, is a constructible set when X is of finite type over an algebraically closed field. This yields an alternative approach to the results from [2, Chap. IV, §9].

Proof of Theorem 1.2. Let $f\colon Y \to X$ be of finite type over an algebraically closed field K (but using the results from the next section, K can in fact be any field) and let x be a point of X. Suppose $f^{-1}(x)$ is embedded as a closed subscheme of $\mathbb{A}^n_{k(x)}$. Let I be the ideal defining this embedding. We need to show that $\mu(I) - n$ is bounded independently from x, I or n.

Since everything is of finite type, we may assume that both schemes are affine, so that f corresponds to a K-algebra homomorphism $A \to B$ of finite type. By Theorem 8.3 applied to the complete intersection defect ω^{CIDef}, there is a bound D such that, if y is a point of some fiber $f^{-1}(x)$, then $\mathcal{O}_{f^{-1}(x),y}$ has complete intersection defect at most D. In other words, $R := B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$ has complete intersection defect at most D, where $\mathfrak{q}$ is the prime ideal of B corresponding to y and $\mathfrak{p} = \mathfrak{q} \cap A$ the prime ideal corresponding to x.

On the other hand, by assumption, the coordinate ring $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ of $f^{-1}(x)$ is isomorphic to C/I, for $C = k(\mathfrak{p})[\xi]$ with $\xi = (\xi_1, \dots, \xi_n)$ some variables and for I some ideal in C. Therefore, $R = C_{\mathfrak{Q}}/IC_{\mathfrak{Q}}$, where $\mathfrak{Q} = \mathfrak{q}B_{\mathfrak{p}} \cap C$. By (17),

$$\omega^{\mathrm{CIDef}}(R) = \mu(IC_{\mathfrak{Q}}) - \mathrm{ht}(IC_{\mathfrak{Q}}).$$

In particular, $\mu(IC_{\mathfrak{Q}})$ is at most $D+n$. Since this estimate holds for any prime ideal $\mathfrak{Q}$ of C/I, we obtain from the Forster-Swan Theorem that $\mu(I) \leq D + n + \dim B$ (use for instance [8, Corollary 3.2]). □

Applying Theorem 1.2 to the universal family of finitely generated algebras of degree complexity at most d defined in §3.4.1, we get:

Corollary 8.5 *For each $d \in \mathbb{N}$, there exists a bound $d' \in \mathbb{N}$, such that for any affine K-algebra A of degree complexity at most d over a field K and for any presentation $A = K[\xi_1, \dots, \xi_n]/I$, we have $\mu(I) \leq d' + n$.*

9 Constructible invariants over arbitrary base fields

In this section, we will drop the restriction that the base field K is algebraically closed. Let us call a local homomorphism $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ of Noetherian local rings a *scalar extension*, if it is faithfully flat and $\mathfrak{m}S = \mathfrak{n}$. For some properties of this notion, including the reason for its terminology, see [9]. For our purposes, the following example of a scalar extension is the only one used in this paper: let A be an algebra over a field K and let $B := A \otimes_K L$ be its base change over some algebraic field extension L of K. Then for any prime ideal $\mathfrak{q}$ of B, the localization $A_\mathfrak{p} \to B_\mathfrak{q}$ is a scalar extension, where $\mathfrak{p} = \mathfrak{q} \cap A$. Indeed, the fibers of $A \to B$ are all finite since $A \to B$ is integral. Hence $\mathfrak{p}B_\mathfrak{p}$ is the Jacobson radical of $B_\mathfrak{p}$ and therefore, after localizing, we get $\mathfrak{p}B_\mathfrak{q} = \mathfrak{q}B_\mathfrak{q}$.

Definition 9.1 *Let ω be an $\mathbb{S}$-valued ring invariant. We say that ω* is preserved under scalar extensions, *if for each scalar extension $R \to S$, we have $\omega(R) = \omega(S)$.*

In case ω is a module invariant, then we require for each finitely generated R-module M that $\omega(R, M) = \omega(S, M \otimes_R S)$.

Theorem 9.2 *Let ω be an $\mathbb{S}$-valued invariant of finite type. Suppose ω is definable in families and devissable (or, saturated). If ω is preserved under scalar extensions, then for every scheme X of finite type over a field K and every coherent $\mathcal{O}_X$-module $\mathcal{F}$, the (non-empty) level sets*

$$\omega^{-1}_{X,\mathcal{F}}(s) := \{\, x \in X \mid \omega(\mathcal{O}_{X,x}, \mathcal{F}_x) = s \,\}$$

form a constructible partition of X.

Proof. In case K is algebraically closed, this is just Theorem 5.2. For K an arbitrary field, let $\overline{K}$ denote its algebraic closure. Let $\overline{X} := X \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}$ and $\overline{\mathcal{F}} := \mathcal{F} \otimes \mathcal{O}_{\overline{X}}$ be the base changes of X and $\mathcal{F}$ over $\overline{K}$. Let $\overline{x}$ be a point in $\overline{X}$ and let $x := \pi(\overline{x})$, where $\pi \colon \overline{X} \to X$ denotes the canonical map. Since $K \subseteq \overline{K}$ is algebraic, the natural homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{\overline{X},\overline{x}}$ is a scalar extension by our previous discussion. Preservation under scalar extensions then yields

$$\omega_{\overline{X}}(\overline{x}, \overline{\mathcal{F}}) = \omega_X(x, \mathcal{F}).$$

It follows that

$$\pi(\omega^{-1}_{\overline{X},\overline{\mathcal{F}}}(s)) = \omega^{-1}_{X,\mathcal{F}}(s), \tag{24}$$

for all $s \in \mathbb{S}$. By Theorem 5.2, the level sets on $\overline{X}$ are constructible. In particular, only finitely many are non-empty. Since π is surjective, it follows from (24) that all but finitely many level sets on X are empty. Moreover, by [3, Proposition 6.F], each level set in X is pro-constructible, since it is the image of a constructible set by (24). In particular, since each level set is the intersection of the complements of the other level sets and since a finite intersection of ind-constructible sets is again ind-constructible, it follows that each level set is also ind-constructible. Corollary 4.6 then yields that each level set is constructible. □

Proposition 9.3 *The invariants $\omega_i^{\mathrm{Betti}}$, ω_i^{Bass} and all the singularity defects are preserved under scalar extensions.*

Proof. Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local homomorphism with $\mathfrak{m}S = \mathfrak{n}$ and let M be a finitely generated R-module. Let k be the residue field of R. Hence $S \otimes k = S/\mathfrak{m}S$ is the residue field l of S. By [4, Theorem 15.1], the invariant given by Krull dimension, and by [4, Theorem 23.3], the invariant given by depth are both preserved under scalar extensions. In particular, $\operatorname{depth} M = \operatorname{depth} M \otimes_R S$. Since we have isomorphisms

$$\operatorname{Tor}_i^R(k,M) \otimes_R S \cong \operatorname{Tor}_i^S(l, M \otimes_R S) \tag{25}$$

$$\operatorname{Ext}_R^i(k,M) \otimes_R S \cong \operatorname{Ext}_S^i(l, M \otimes_R S), \tag{26}$$

it follows that also $\omega_i^{\mathrm{Betti}}$ and ω_i^{Bass} are preserved under scalar extensions. As the singularity defects are made up of dimension, depth, Betti and/or Bass numbers, they are all preserved under scalar extensions as well. □

Proposition 9.3 together with Theorem 9.2 proves Theorem 1.1 in full.

References

[1] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, 1993.

[2] A. Grothendieck and J. Dieudonné, *Elements de géométrie algébrique I-IV*, Inst. Hautes Études Sci. Publ. Math., vol. 4, 8, 11, 17, 20, 24, 28, Presses universitaires de France, Paris, 1960-1965.

[3] H. Matsumura, *Commutative algebra*, W.A. Benjamin, 1970.

[4] ______, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[5] K. Schmidt and L. van den Dries, *Bounds in the theory of polynomial rings over fields. A non-standard approach*, Invent. Math. **76** (1984), 77–91.

[6] H. Schoutens, *Bounds in cohomology*, Israel J. Math. **116** (2000), 125–169.

[7] ______, *Uniform bounds in algebraic geometry and commutative algebra*, Connections between model theory and algebraic and analytic geometry, Quad. Mat., vol. 6, Dept. Math., Seconda Univ. Napoli, Caserta, 2000, pp. 43–93.

[8] ______, *Number of generators of a Cohen-Macaulay ideal*, J. Algebra **259** (2003), 235–242.

[9] ———, *Approximation metric*, (2004) preprint, in preparation.

[10] ———, *The first-order theory of Noetherian spaces*, (2004) preprint, in preparation.