

A GEOMETRIC APPROACH TO DEFINABLE SUBSETS

Goal. The goal of this problem is to explore an alternative definition of definable subset. There are several questions, some of which I only want you to **verify** and others to **prove**. Whenever I say **verify**, you should check for yourself that the claim is correct. For a good understanding of a mathematical subject, it is crucial that at least once in your lifetime, you check all its tedious but easy facts in gruesome detail. Once you have convinced yourself (or your peers—I greatly encourage working together in group), there is no need to ever bother with these again. The problems where I ask you to **prove** something, I want you to hand in a solution in sufficient detail. You are allowed to refer to any property proven in the book, and even to assertions stated as exercises (but then you should check for yourself whether you can actually do the exercise).

By the way, if you want to produce smooth looking math papers, I strongly recommend you to learn TeX. You will definitely want to learn this once you start writing your thesis. There are several TeX variants, but the most commonly used one nowadays is LaTeX. This assignment was written in an enriched form of LaTeX, developed by the American Mathematical Society, called AMS-LaTeX.

Definitions. Let M be a set, and let $\mathfrak{P}(M)$ be its power set, that is to say, the set of all subsets of M . We want to define a structure on M , but instead of using signatures and languages, we take a more geometrical approach. We need a couple of definitions. By a *Boolean subalgebra* $\mathfrak{D} \subseteq \mathfrak{P}(M)$ on M , we mean a collection of subsets of M closed under (finite) intersections, unions and complements, and containing the empty set \emptyset as well as the whole set M . Fix a positive integer n . The *diagonal* D_{ij} in M^n , where $1 \leq i < j \leq n$, is the collection of all n -tuples which have the same i -th entry as j -th entry, that is to say, all (x_1, \dots, x_n) with $x_i = x_j$. Given a second positive integer k , let $\pi_{n+k,n}: M^{n+k} \rightarrow M^n$ be the *projection* onto the first n coordinates, given by sending an $(n+k)$ -tuple (x_1, \dots, x_{n+k}) to the tuple (x_1, \dots, x_n) . Given a k -tuple $\bar{a} \in M^k$ and a subset $C \subseteq M^{n+k}$, we define the *fiber* of C at \bar{a} as the subset $C_{\bar{a}} \subseteq M^n$ consisting of all n -tuples \bar{x} such that the $(n+k)$ -tuple (\bar{x}, \bar{a}) belongs to C . Given a function $f: M^n \rightarrow M$ we define its *graph* as the subset in M^{n+1} given by all $(n+1)$ -tuples (a_1, \dots, a_n, b) such that $f(a_1, \dots, a_n) = b$.

We are now ready to define the notion of a *weak geometry* \mathfrak{D} on a set M . It consists of a collection $\mathfrak{D} = (\mathfrak{D}_n)_n$, where each $\mathfrak{D}_n \subseteq \mathfrak{P}(M^n)$ is a Boolean subalgebra on M^n with the following three properties:

- (i) for all $1 \leq i < j \leq n$, the diagonal $D_{ij} \subseteq M^n$ belongs to \mathfrak{D}_n ;
- (ii) for every $C \in \mathfrak{D}_n$ and $B \in \mathfrak{D}_k$, the product $C \times B \subseteq M^{n+k}$ belongs to \mathfrak{D}_{n+k} ;
- (iii) for every $C \in \mathfrak{D}_{n+k}$, its projection under $\pi_{n+k,n}: M^{n+k} \rightarrow M^n$ is in \mathfrak{D}_n .

If, moreover, also the following property holds

- (iv) for every $C \in \mathfrak{D}_{n+k}$ and every $\bar{a} \in M^k$, the fiber $C_{\bar{a}}$ belongs to \mathfrak{D}_n ,

then we will say that \mathfrak{D} is a *geometry*. Given a subset C in some Cartesian power M^k , we say that C *belongs* to the geometry $\mathfrak{D} = (\mathfrak{D}_n)_n$, if $C \in \mathfrak{D}_k$.

Properties. Verify that if $\mathfrak{D} = (\mathfrak{D}_n)_n$ and $\mathfrak{D}' = (\mathfrak{D}'_n)_n$ are (weak) geometries, then so is their intersection $\mathfrak{D} \cap \mathfrak{D}' := (\mathfrak{D}_n \cap \mathfrak{D}'_n)_n$. In fact, the same is also true for infinite intersections. Let $G = (G_n)_n$ be any collection of subsets $G_n \subseteq \mathfrak{P}(M^n)$. We say that G is contained in a (weak) geometry $\mathfrak{D} = (\mathfrak{D}_n)_n$, if $G_n \subseteq \mathfrak{D}_n$, for all n . To define the (weak) geometry *generated* by G , consider the class of all (weak) geometries \mathfrak{D} containing G and let $\mathfrak{D}(G)$ be their intersection. By what we just argued, $\mathfrak{D}(G)$ is a (weak) geometry, and is called therefore the (weak) geometry *generated by* G . Verify that $\mathfrak{D}(G)$ is the smallest, with respect to inclusion, (weak) geometry containing G , and that any subset belonging to $\mathfrak{D}(G)$ can be obtained by a finite number of applications of the rules (i)–(iv) (or, in the weak case, only using the first three rules) to the subsets in G .

Our next goal is to associate to an L -structure $\mathcal{M} = (M; \mathbf{C}^{\mathcal{M}}; \mathbf{F}^{\mathcal{M}}; \mathbf{R}^{\mathcal{M}})$ (in the language L with signature $(\mathbf{C}, \mathbf{F}, \mathbf{R})$) a (weak) geometry as follows. We start with defining for each n , a collection G_n of subsets of M^n . If $n = 1$, then G_1 consists precisely of all singletons $\{c\}$ where c is some constant in $\mathbf{C}^{\mathcal{M}}$ together with all subsets defined by a unary predicate in \mathbf{R} . For $n > 1$, let G_n consist of all n -ary subsets $R \in \mathbf{R}^{\mathcal{M}}$ as well as of all graphs of $(n - 1)$ -ary functions $f \in \mathbf{F}^{\mathcal{M}}$. Let $\mathfrak{D}_{\mathcal{M}}^w$ be the weak geometry generated by $G := (G_n)_n$ and let $\mathfrak{D}_{\mathcal{M}}$ be the geometry generated by $G := (G_n)_n$, called respectively the weak geometry and geometry *associated* to \mathcal{M} . The main result I want you to prove is then:

Theorem. *Let \mathcal{M} be an L -structure and let $\mathfrak{D}_{\mathcal{M}}^w$ and $\mathfrak{D}_{\mathcal{M}}$ be the respective weak geometry and geometry associated to \mathcal{M} . A subset $C \subseteq M^n$ is definable without parameters if and only if it belongs to $\mathfrak{D}_{\mathcal{M}}^w$; and it is definable with parameters if and only if it belongs to $\mathfrak{D}_{\mathcal{M}}$.*

Here are some hints that might help you in the proof: Exercise (3.3.6) can make the presentation more streamlined. You also might want to prove the following fact separately: *if $C \subseteq M^{n+k}$ is definable, then so is its projection under $\pi_{n+k,n}: M^{n+k} \rightarrow M^n$.* The key to understand this last assertion is to see that projection corresponds to taking an existential quantifier.

There is also a converse to the Theorem, which I only want you to verify. Namely, given a weak geometry $\mathfrak{D} = (\mathfrak{D}_n)_n$ on a set M , construct a signature which has a n -ary relation symbol \underline{C} for each subset $C \in \mathfrak{D}_n$ (where we identify 1-ary relations with constants) and let $L_{\mathfrak{D}}$ be the corresponding language. Make M into an $L_{\mathfrak{D}}$ -structure $\mathcal{M}_{\mathfrak{D}}$ by interpreting each relation symbol \underline{C} by the subset $C \subseteq M^n$. Verify that the associated weak geometry of this $L_{\mathfrak{D}}$ -structure is the same as \mathfrak{D} . Conversely, if we start from an L -structure \mathcal{M} , associate to it its weak geometry $\mathfrak{D}_{\mathcal{M}}^w$ and then take the corresponding structure $\mathcal{M}_{\mathfrak{D}_{\mathcal{M}}^w}$, then we get an $L_{\mathfrak{D}_{\mathcal{M}}^w}$ -structure which is isomorphic with \mathcal{M} when we view both structures in the expanded language $L \cup L_{\mathfrak{D}_{\mathcal{M}}^w}$ (or more precisely, both structures admit an isomorphic expansion to the larger language).

I leave you with the following puzzling fact (intentionally designed to confuse you, I admit): we know that if $\mathcal{M} = (M, \dots)$ is an L -structure and $N \subseteq M$ just a subset, then we cannot always define an L -structure \mathcal{N} on N such that the inclusion map $N \hookrightarrow M$ induces a homomorphism of L -structures (give a concrete example). However, verify that if $\mathfrak{D} = (\mathfrak{D}_n)_n$ is a (weak) geometry on M , then the collection $\mathfrak{D}|_N$ consisting of all $C \cap N^n$ with $C \in \mathfrak{D}_n$, for all n , is a (weak) geometry on N , called the *restriction* of \mathfrak{D} to N . So we could start with \mathcal{M} , associate to it the weak geometry $\mathfrak{D}_{\mathcal{M}}^w$, then take its restriction $\mathfrak{D}_{\mathcal{M}}^w|_N$ to N and, finally, take the associated first-order structure $\mathcal{M}_{\mathfrak{D}_{\mathcal{M}}^w|_N}$, to get a structure on N . How can you reconcile both facts?