Dimension and singularity theory for local rings of finite embedding dimension

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Abstract

In this paper, we develop an algebraic theory for local rings of finite embedding dimension. Several extensions of (Krull) dimension are proposed, which are then used to generalize singularity notions from commutative algebra. Finally, variants of the homological theorems are shown to hold in equal characteristic.

This theory is then applied to Noetherian local rings in order to get: (i) over a Cohen-Macaulay local ring, uniform bounds on the Betti numbers of a Cohen-Macaulay module in terms of dimension and multiplicity, and similar bounds for the Bass numbers of a finitely generated module; (ii) a characterization for being respectively analytically unramified, analytically irreducible, unmixed, quasi-unmixed, normal, Cohen-Macaulay, pseudo-rational, or weakly F-regular in terms of certain uniform arithmetic behavior; (iii) in mixed characteristic, the Improved New Intersection Theorem when the residual characteristic or ramification index is large with respect to dimension (and some other numerical invariants).

Key words: commutative algebra; local rings; Betti numbers; homological conjectures; ultraproducts

1. Introduction

This paper is devoted to the study of local rings of finite embedding dimension, where by a *local ring*,[2] we mean a not necessarily Noetherian, commutative ring R with a unique maximal ideal $\mathfrak{m}$, and where the *embedding dimension* of R, denoted $\operatorname{embdim}(R)$, is the minimal number of elements generating $\mathfrak{m}$. We will see that there are various ways of extending the dimension and singularity theory of Noetherian local rings to this larger class. The motivation for this study comes from the subclass of *ultra-Noetherian* local rings: these are the ultraproducts of Noetherian local rings of fixed embedding dimension. I had used these ultra-Noetherian rings in my previous

[1]Partially supported by the National Science Foundation and a PSC-CUNY grant
[2]Other authors may call these *quasi-local*.

work on transfer from positive to zero characteristic ([38, 41]) and on non-standard tight closure ([4, 45, 40, 43, 48, 49]), but the actual study of their properties was only prompted by the papers [39, 47], where it was essential to have a generalized dimension and singularity theory to get asymptotic versions of various homological theorems in mixed characteristic. It was this realization that led me to develop a systematic 'local algebra' for these rings. Consequently, we will be able to derive from this study some improved asymptotic versions in the final section. For some other recent papers studying ultraproducts of Noetherian rings, see [31, 32, 33].

Closely related to a local ring of finite embedding dimension are two local rings which are always Noetherian: its graded ring and its completion. Especially through the latter the study of local rings of finite embedding dimension is greatly facilitated. Accordingly, I will use the modifier *cata-* to indicate that a property is inherited by completion. In contrast, for ultra-Noetherian local rings, the prefix *ultra-* is used to refer to properties that are inherited by the ultraproduct. The main goal is now to find conditions under which both versions agree, which often requires the introduction of a third, intrinsic (*pseudo-*) variant. To study these variants, we introduce the notion of a *cataproduct*, defined as the completion of the ultraproduct. In fact, the cataproduct is obtained from the ultraproduct by factoring out the *ideal of infinitesimals*, that is to say, the ideal of elements lying in each power of the maximal ideal. In [50], both the ultraproduct and the cataproduct are called *chromatic products*, inspired by our musical notation $R_\natural$ and $R_\sharp$ respectively (a third chromatic product, not discussed in this paper, is called the *proto-product* and denoted $R_\flat$).

What follows is a brief outline of the present paper. To illustrate the methods and concepts, I will here only treat the special case that $(R_\natural, \mathfrak{m}_\natural)$ is an ultra-Noetherian local ring, realized as the ultraproduct of Noetherian local rings $(R_n, \mathfrak{m}_n)$ of the same embedding dimension. Section 2 contains general facts of local rings of finite embedding dimension, by far the most important of which is the already mentioned result that its completion is Noetherian (Theorem 2.2). In particular, the cataproduct $R_\sharp$ is Noetherian.[3] Our first task is now to develop a good dimension theory, which is done in §§3–5. Krull dimension in this context is of minor use, as it is always infinite for example for ultra-Noetherian local rings, except when almost all R_n are Artinian of a fixed length l, in which case $R_\natural$ is also zero-dimensional and has length l. A first variant, called *geometric dimension*, is inspired by the geometric intuition that dimension is the least number of hypersurfaces cutting out a finitely supported subscheme. Specifically, the *geometric dimension*, $\operatorname{gdim}(R_\natural)$, of $R_\natural$ is the least number d of non-units $x_1, \dots, x_d$ such that $R_\natural/(x_1, \dots, x_d)R_\natural$ is Artinian. Other variants are obtained by the general principle discussed above: the *ultra-dimension*, $\operatorname{udim}(R_\natural)$, of $R_\natural$ is the common dimension of almost all R_n; and its *cata-dimension* is the dimension of its completion, that is to say, of $R_\sharp$. It turns out that the cata-dimension is equal to the geometric dimension (Theorem 3.4). These dimensions also have a combinatorial nature: whereas Krull dimension is the combinatorial dimension of the full spectrum $\operatorname{Spec}(R_\natural)$, the ultra-dimension of $R_\natural$ is equal to the combinatorial dimension of the subset of all associated prime ideals of finitely generated ideals; the cata-dimension

[3] Special cases of this result were already observed and used by various authors [4, 6, 32].

is equal to the combinatorial dimension of the subset of all $\mathfrak{m}_\natural$-adically closed prime ideals (Theorem 5.19; see also [33] for some related results). The ultra-dimension of $R_\natural$ is at most its cata-dimension, with equality precisely when almost all R_n have the same parameter degree (Theorem 5.23).

Our next step is to develop a singularity theory for local rings of finite embedding dimension. Three options present themselves to us: cata-singularities via completions (§6); ultra-singularities via ultraproducts (§8); and pseudo-singularities via homological algebra (§7). For instance, $R_\natural$ is called *cata-regular* if $R_\sharp$ is regular; *ultra-regular* if almost all R_n are regular; and *pseudo-regular* if its depth equals its embedding dimension. Requiring each one of the first three quantities

$$\operatorname{depth}(R) \leq \operatorname{udim}(R_\natural) \leq \operatorname{gdim}(R_\natural) \leq \operatorname{embdim}(R_\natural) \tag{1}$$

in this chain of inequalities to be equal to the last turns out to determine these regularity conditions, in decreasing order of strength: pseudo-regularity, ultra-regularity, and cata-regularity respectively (note that we do not observe such a distinction in the Noetherian case). In fact, the two first conditions are equal (Theorem 8.1). Moreover, Serre's criterion for regularity extends to this larger class (Theorem 7.11). In particular, for coherent local rings of finite embedding dimension, regular in the sense of Bertin ([7, 18]) implies pseudo-regular, and the converse holds for uniformly coherent local rings of finite embedding dimension (Theorem 7.18). Next, variants of the Cohen-Macaulay property are analyzed—for instance, by equating the first quantity in (1) with respectively the second and third, we get the notions of ultra-Cohen-Macaulay and pseudo-Cohen-Macaulay local rings. Unfortunately, these variants behave less well. For instance, although the class of local Cohen-Macaulay rings of fixed dimension and multiplicity is closed under cataproducts (Corollary 8.8), the converse need not be true, that is to say, $R_\sharp$ can be Cohen-Macaulay without the R_n being Cohen-Macaulay. At the source of these discrepancies lies the fact that a sequence can be quasi-regular without being regular in non-Noetherian rings. In 5.20, we present an example showing that all of the four quantities in (1) can be different. Although $R_\natural$ is rarely coherent, under an additional pseudo-Cohen-Macaulay assumption, it behaves much like one: any $\mathfrak{m}_\natural$-primary ideal, and more generally, any finitely generated ultra-Cohen-Macaulay module is finitely presented. Another generalization of the Cohen-Macaulay condition for local rings of finite embedding dimension, motivated by model-theoretic considerations, was introduced in [37]; we show that up to a Nagata extension of the ring (which can be taken to be trivial in the ultra-Noetherian case), this condition is equivalent with being pseudo-Cohen-Macaulay (Theorem 7.26). Some further characterizations of the various types of Cohen-Macaulay singularities are given in §9 by means of an analogue of Noether Normalization for the class of local rings of finite embedding dimension.

Once we have developed a sufficiently well-behaved singularity theory, we analyze the homological theory of the class of local rings of finite embedding dimension; this is the contents of §10. We show that most homological theorems, properly restated, hold in an arbitrary equicharacteristic local ring of finite embedding dimension. The main tool is the existence of an analogue of big Cohen-Macaulay algebras for this class of rings. In fact, it suffices to assume that only the completion is equicharacteristic, which is a strictly weaker condition, as I will explain below. As an application, we provide the following partial answer to a question raised by Glaz ([19]) about the extent to which split subrings of coherent regular local rings are Cohen-Macaulay (compare with [20, Corollary 4.5]).

1.1 Corollary. *If $(R, \mathfrak{m})$ is a local ring of finite embedding dimension containing a field, and if S a coherent regular local ring locally of finite type over R, such that $R \to S$ is cyclically pure (e.g., split), then there exists a (Noetherian) regular local subring $(A, \mathfrak{p})$ of R such that each A-regular sequence is a quasi-regular sequence in R, and each $R/\mathfrak{p}^n R$ is a finite, free $A/\mathfrak{p}^n$-module.*

In the final three sections, we apply the theory to ultra-Noetherian rings to obtain new results about Noetherian local rings. In §11, we derive uniform bounds on Betti and Bass numbers. In the literature, one often studies the asymptotic growth of the *Betti numbers* $\beta_n(M) = \dim_k(\operatorname{Tor}_n^R(M, k))$, as n goes to infinity, for M a finitely generated module over a Noetherian local ring R with residue field k. In contrast, varying the module and fixing n, we show in Theorem 11.1 that over a local Cohen-Macaulay ring R, the n-th Betti number of a module M of finite length is bounded by a function which only depends on the dimension and multiplicity of R and the length of M. In particular, if $P_R(t) := \sum_n \beta_n(k)t^n$ denotes the *Poincare series* of R, then we show:

1.2 Corollary. *For each $d, e \geq 0$, there exists a power series $P_{d,e}(t) \in \mathbb{Z}[[t]]$ such that the Poincare series $P_R(t)$ of any d-dimensional local Cohen-Macaulay ring R of multiplicity e is dominated by $P_{d,e}(t)$, meaning that $P_{d,e}(t) - P_R(t)$ has non-negative coefficients.*

Recall that a Cohen-Macaulay local ring R is called of *bounded multiplicity type* if there is a bound $\epsilon(R)$ on the multiplicity of all of its indecomposable maximal Cohen-Macaulay modules. According to the Brauer-Thrall conjectures such a ring is expected to be of *finite representation type*, meaning that there exist only finitely many indecomposable maximal Cohen-Macaulay modules. The conjecture is known to hold for certain reduced, excellent Henselian isolated singularities by the work of [11, 34, 55]. In support of this, we prove the following two results:

1.3 Theorem (Brauer-Thrall for Isolated Singularities)**.** *Let R be an equicharacteristic, unramified local isolated singularity with an uncountable algebraically closed residue field. If R has bounded multiplicity type, then it has finite representation type.*

Proof. Immediate from Theorem 11.6 below and [34, Theorem 1.2]. □

1.4 Corollary (Effective Brauer-Thrall)**.** *Let d, e, and ϵ be positive integers for which the Brauer-Thrall conjecture holds, in the sense that every d-dimensional Cohen-Macaulay local ring of multiplicity e and multiplicity type at most ϵ, has finite presentation type. Then there is a bound on the number of indecomposable maximal Cohen-Macaulay modules which only depends on d, e and ϵ.*

1.5 Corollary. *Suppose R is a local Cohen-Macaulay ring of bounded multiplicity type. There exists an R-algebra Z, and a complex of finite free Z-modules $\mathcal{F}_\bullet$, such that for every indecomposable maximal Cohen-Macaulay module M, there exists a section $Z \to R$, such that $\mathcal{F}_\bullet \otimes_Z R$ is a free resolution of M.*

The theory also gives applications to preservation of properties under infinitesimal deformations, of which the next result is but an example (recall that an *invertible ideal* is a principal ideal generated by a non zero-divisor):

1.6 Corollary. *Let R be a local Cohen-Macaulay ring and let $I \subseteq R$ be an invertible ideal. There exists a positive integer $a := a(I)$ with the property that if $J \subseteq R$ such that R/J is Cohen-Macaulay of multiplicity at most the multiplicity of R/I, and such that $I + \mathfrak{m}^a = J + \mathfrak{m}^a$, then J is invertible.*

It is not clear yet whether similar bounds exist if we drop the Cohen-Macaulay assumption in these results. In §12, we characterize ring-theoretic properties in terms of uniform arithmetic in the ring. For instance, in Theorem 12.1, we reprove, as an illustration of our methods, that multiplication is bounded in R if and only if R is analytically irreducible. Whereas the ultraproduct method only gives the existence of a uniform bound, we know in this particular case, by the work of Hübl-Swanson [27, 54], that these bounds can be taken to be linear. Nonetheless, our method is far more versatile, allowing us to derive in §12.8 many more characterizations of ring-theoretic properties in terms of certain uniform asymptotic behavior of ($\mathfrak{m}$-adic) *order* and (parameter) *degree*. For instance, one can characterize the Cohen-Macaulay property as follows:

Theorem (12.14)**.** *For each quadruple (d, e, a, b) there exists a bound $\delta(d, e, a, b)$ with the following property. A d-dimensional Noetherian local ring $(R, \mathfrak{m})$ of multiplicity e is Cohen-Macaulay if and only if for each ideal I generated by $d-1$ elements, and for any two elements $x, y \in R$, if $R/(I + xR)$ has length at most a and y does not belong to $I + \mathfrak{m}^b$, then xy does not belong to $I + \mathfrak{m}^{\delta(d,e,a,b)}$.*

As already mentioned, our methods only prove the existence of uniform bounds (and possibly their dependence on other invariants), but say nothing about the nature of these bounds. It would be interesting to see whether for instance these new bounds also have a linear character.

However, the main application of this paper is discussed in the final section. Here we derive some asymptotic versions of the homological theorems in mixed characteristic. Whereas the papers [39, 47] relied on a deep result from model theory, the so-called Ax-Kochen-Ershov theorem, to carry out transfer from mixed to equal characteristic,[4] the present paper departs from the following simple observation: if the $(R_n, \mathfrak{m}_n)$ have mixed characteristic p_n, then their cataproduct $R_\sharp$ is equicharacteristic in the following two cases: (i) the p_n grow unboundedly (in which case the ultraproduct $R_\natural$ is already equicharacteristic), or (ii), almost all p_n are equal to a fixed prime number p, but the ramification index, that is to say, the $\mathfrak{m}_n$-adic order of p, grows unboundedly (in which case, however, $R_\natural$ still has mixed characteristic p). Thus we prove:

Theorem (13.6, Asymptotic Improved New Intersection Theorem)**.** *For each triple of positive integers (m, r, l) there exists a bound $\kappa(m, r, l)$ with the following property.*

[4] In fact, although not mentioned explicitly in these papers (but see [50, §14] or [49, §6]), these methods make heavily use of proto-products, one of the chromatic products not studied in this paper.

Let $(R, \mathfrak{m})$ be a mixed characteristic Noetherian local ring of embedding dimension m and let $F_\bullet$ be a finite complex of finitely generated free R-modules of rank at most r. If each $H_i(F_\bullet)$, for $i > 0$, has length at most l and $H_0(F_\bullet)$ has a non-zero minimal generator generating a submodule of length at most l, then the length of $F_\bullet$ is at least the dimension of R, provided either the residual characteristic or the ramification index of R is at least $\kappa(m, r, l)$.

It should be noted that some Homological Conjectures, such as the Direct Summand Conjecture and the Hochster-Roberts theorem on the Cohen-Macaulayness of pure subrings of regular local rings, at present elude our methods, and so no asymptotic versions in the style of this paper are known (but see [47, §9 and §10] for different asymptotic versions).

I conclude the paper with a sketch of an argument that derives the full version from its asymptotic counterpart, provided the bounding function does not grow too fast. For example, if for some prime p, the bound $\kappa(m, r, l)$ on the ramification in the above theorem can be taken to be of the form $c(m, r)l^{\alpha(m,r)}$, for some real valued functions $c(m, r)$ and $\alpha(m, r)$ with $\alpha(m, r) < 1$, for all m and r, then the Improved New Intersection Theorem holds in mixed characteristic p.

2. Finite embedding dimension

Although we will mainly be interested in the maximal adic topology of a local ring, we start our exposition in a more general setup.

2.1. Filtrations

Recall that a *filtration* $\mathfrak{I} = (I_n)_n$ on a ring A is a descending chain of ideals $A = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \ldots$. An important instance of a filtration is obtained by taking the powers of a fixed ideal $I \subseteq A$, that is to say, $I_n := I^n$; we call this the *I-adic filtration* on A. A filtration $\mathfrak{I}$ defines a topology on A, called the *$\mathfrak{I}$-adic topology* of A, by taking for basic open subsets all cosets of all I_n. If B is an A-algebra, then $\mathfrak{I}B$ is the *extended filtration* on B given by the ideals I_nB, and hence the natural homomorphism $A \to B$ is continuous with respect to the respective adic topologies. The intersection of all I_n will be denoted by $\mathfrak{I}_\infty$. Hence the $\mathfrak{I}$-adic topology is Haussdorf (separated) if and only if $\mathfrak{I}_\infty = (0)$. Accordingly, the quotient $A/\mathfrak{I}_\infty$ is called the *$\mathfrak{I}$-adic separated quotient* of A. The *$\mathfrak{I}$-adic completion* of A is defined as the inverse limit of the A/I_n and is denoted $\widehat{A}_\mathfrak{I}$. There is a natural map $A \to \widehat{A}_\mathfrak{I}$ whose kernel is equal to $\mathfrak{I}_\infty$. In fact, A and its $\mathfrak{I}$-adic separated quotient have the same $\mathfrak{I}$-adic completion. In general, $\widehat{A}_\mathfrak{I}$, although complete in the inverse limit topology, need not be complete in the $\mathfrak{I}\widehat{A}_\mathfrak{I}$-adic topology.

Given a filtration $\mathfrak{I} = (I_n)_n$ we define its *associated graded module*, where we view A with its trivial grading, as the direct sum

$$\operatorname{gr}_\mathfrak{I}(A) := \bigoplus_{n=0}^{\infty} I_n/I_{n+1}.$$

The *initial form* $\mathrm{in}_{\mathfrak{I}}(a) \in \mathrm{gr}_{\mathfrak{I}}(A)$ and the $\mathfrak{I}$*-adic order* $\mathrm{ord}_{\mathfrak{I}}(a)$ of an element $a \in A$ are defined as follows. If $a \in I_n \setminus I_{n+1}$ for some n, then we set $\mathrm{ord}_{\mathfrak{I}}(a) := n$ and we let $\mathrm{in}_{\mathfrak{I}}(a)$ be the image of a in I_n/I_{n+1}; otherwise $a \in \mathfrak{I}_\infty$, in which case we set $\mathrm{ord}_{\mathfrak{I}}(a) := \infty$ and $\mathrm{in}_{\mathfrak{I}}(a) := 0$. For J an ideal in A, we let $\mathrm{in}_{\mathfrak{I}}(J)$ be the ideal in $\mathrm{gr}_{\mathfrak{I}}(A)$ generated by all $\mathrm{in}_{\mathfrak{I}}(a)$ with $a \in J$. If $J = (a_1, \dots, a_n)A$, then $\mathrm{in}_{\mathfrak{I}}(J)$ is in general larger than the ideal generated by the $\mathrm{in}_{\mathfrak{I}}(a_i)$ (even if A is Noetherian!).

Alternatively, we may think of a filtration as given by a function $f \colon A \to \bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ such that $f(a+b)$ and $f(ab)$ are greater than or equal to respectively the minimum and the maximum of $f(a)$ and $f(b)$; we express this by calling f *filtering*. Given a filtering function f, the ideals I_n of all elements $a \in A$ for which $f(a) \geq n$ form a filtration. Conversely, given a filtration $\mathfrak{I}$, the function $\mathrm{ord}_{\mathfrak{I}}(\cdot)$ is filtering. Suppose f is filtering. If $f(ab) \geq f(a) + f(b)$, then we call f *multiplicative* (this then corresponds to the property that $I_n I_m \subseteq I_{n+m}$); and if 0 is the only element of infinite f-value (so that the corresponding filtration is separated) and $f(ab) = f(a) + f(b)$, then f is called a *valuation*. If $\mathfrak{I}$ is multiplicative, then $\mathrm{gr}_{\mathfrak{I}}(A)$ admits the structure of a ring and as such is graded. This applies in particular to any ideal adic filtration.

We now specify these notions to the case of interest, where $\mathfrak{I}$ is the $\mathfrak{m}$-adic filtration of a local ring $(R, \mathfrak{m})$. The topology on R is always assumed to be the $\mathfrak{m}$-adic topology, so that when we say that R is separated or complete, we are always referring to this topology. With this in mind, the *ideal of infinitesimals* of R is the intersection of all $\mathfrak{m}^n$, and will be denoted $\mathfrak{I}_R$. The $\mathfrak{m}$-adic order of an element $x \in R$ is denoted $\mathrm{ord}_R(x)$ or just $\mathrm{ord}(x)$. The ($\mathfrak{m}$-adic) separated quotient $R/\mathfrak{I}_R$ is denoted R; the graded ring associated to $\mathfrak{m}$ is denoted $\mathrm{gr}(R)$; and the completion of R is denoted $\widehat{R}$. By construction, $\widehat{R}$ is a complete local ring whose maximal ideal is equal to the inverse limit of the $\mathfrak{m}/\mathfrak{m}^n$. However, this maximal ideal may be strictly larger than $\mathfrak{m}\widehat{R}$, so that $\widehat{R}$ need not be complete in the $\mathfrak{m}\widehat{R}$-adic topology.

Let $(S, \mathfrak{n})$ be a second local ring and let $R \to S$ be a ring homomorphism. We call this homomorphism *local*, or we say that S is a *local* R-algebra, if $\mathfrak{m}S \subseteq \mathfrak{n}$; if we have equality, then we call the homomorphism *unramified*. A local homomorphism induces local homomorphisms $R \to S$ and $\widehat{R} \to \widehat{S}$. The natural map $R \to \widehat{R}$ is local. It is flat if R is Noetherian, but no so in general.

Finite embedding dimension

Suppose from now on that R has moreover finite embedding dimension, that is to say, that $\mathfrak{m}$ is finitely generated. Since $\mathrm{gr}(R)$ is generated by $\mathfrak{m}/\mathfrak{m}^2$ as an algebra over the field $R/\mathfrak{m}$, it is itself a Noetherian local ring. For each n, let $\widehat{\mathfrak{m}}_n$ be the kernel of the natural map $\widehat{R} \to R/\mathfrak{m}^n$. It follows that $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \widehat{\mathfrak{m}}_n/\widehat{\mathfrak{m}}_{n+1}$, so that $\mathrm{gr}(R)$ is equal to the graded ring $\mathrm{gr}_{\mathfrak{M}}(\widehat{R})$ associated to the filtration $\mathfrak{M} := (\widehat{\mathfrak{m}}_n)_n$ on $\widehat{R}$. By [12, Proposition 7.12], an ideal $I \subseteq \widehat{R}$ is generated by elements $a_1, \dots, a_n$ if its initial from $\mathrm{in}_{\mathfrak{M}}(I)$ in $\mathrm{gr}_{\mathfrak{M}}(\widehat{R})$ is generated by the initial forms $\mathrm{in}_{\mathfrak{M}}(a_1), \dots, \mathrm{in}_{\mathfrak{M}}(a_n)$. Therefore, since $\mathrm{gr}_{\mathfrak{M}}(\widehat{R}) \cong \mathrm{gr}(R)$ is Noetherian, so is $\widehat{R}$. Moreover, since $\mathfrak{m}^n\widehat{R}$ has the same initial form as $\widehat{\mathfrak{m}}_n$, both ideals are equal. In particular, for each n, we have an isomorphism $R/\mathfrak{m}^n \cong \widehat{R}/\mathfrak{m}^n\widehat{R}$. In conclusion, we have proven:

2.2 Theorem. *If $(R, \mathfrak{m})$ is a local ring of finite embedding dimension, then its completion $\widehat{R}$ is a complete Noetherian local ring with maximal ideal $\mathfrak{m}\widehat{R}$.* □

2.3 Corollary. *If a local ring $(R, \mathfrak{m})$ has finite embedding dimension, then each $\mathfrak{m}$-primary ideal is finitely generated.*

Proof. Immediate from the fact that $R/\mathfrak{m}^n$ is Artinian and $\mathfrak{m}^n$ is finitely generated, for every n. □

An ideal I in a local ring $(R, \mathfrak{m})$ is called *closed* if it is closed in the $\mathfrak{m}$-adic topology, that is to say, if I is equal to the intersection of all $I + \mathfrak{m}^n$ with $n \in \mathbb{N}$.

2.4 Lemma. *Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension and let I be an arbitrary ideal in R. The completion of R/I is $\widehat{R}/I\widehat{R}$. In particular, $I\widehat{R} \cap R = I$ if and only if I is closed.*

Proof. Let $\bar{R} := R/I$ and let $S := \widehat{R}/I\widehat{R} = \widehat{R} \otimes_R \bar{R}$. The isomorphism $R/\mathfrak{m}^n \cong \widehat{R}/\mathfrak{m}^n\widehat{R}$ induces by base change an isomorphism $\bar{R}/\mathfrak{m}^n\bar{R} \cong S/\mathfrak{m}^n S$. Hence $\bar{R}$ and S have the same completion. However, since $\widehat{R}$ is complete, so is S, showing that it is the completion of $\bar{R}$.

Applied with I an $\mathfrak{m}$-primary ideal, we get an isomorphism $R/I \cong \widehat{R}/I\widehat{R}$ showing that $I\widehat{R} \cap R = I$, that is to say, that I is *contracted from* $\widehat{R}$. Since this property is preserved under arbitrary intersections, every closed ideal I is contracted from $\widehat{R}$, as it is the intersection of the $\mathfrak{m}$-primary ideals $I + \mathfrak{m}^n$. Conversely, if $I\widehat{R} \cap R = I$, then R/I embeds in $\widehat{R}/I\widehat{R}$, and by the first assertion, this is its completion. In particular, R/I is separated, that is to say, I is closed. □

The above proof shows that the closure of an ideal I is equal to $I\widehat{R} \cap R$. In particular, any closed ideal is the closure of a finitely generated ideal, since $\widehat{R}$ is Noetherian by Theorem 2.2. Moreover, the ascending chain condition holds for closed ideals in R: if $I_1 \subseteq I_2 \subseteq \dots$ is an increasing chain of closed ideals in R, then, since $\widehat{R}$ is Noetherian, their extension to $\widehat{R}$ must become stationary, say $I_n\widehat{R} = I_{n+k}\widehat{R}$ for all k, and hence contracting back to R gives $I_n = I_{n+k}$ for all k. This immediately yields:

2.5 Corollary. *A local ring is Noetherian if and only if it has finite embedding dimension and every ideal is closed.* □

2.6 Corollary. *A closed ideal in a local ring R of finite embedding dimension has finitely many minimal primes and each of them is closed.*

Proof. Let I be a closed ideal and let $\mathfrak{Q}_1, \dots, \mathfrak{Q}_s$ be the minimal prime ideals of $I\widehat{R}$. Let $\mathfrak{q}_i := \mathfrak{Q}_i \cap R$ and let J be their product. Hence $J^n \subseteq I\widehat{R}$ for some n. By Lemma 2.4, we have $J^n \subseteq I\widehat{R} \cap R = I$. Hence any prime ideal $\mathfrak{p}$ of R containing I contains one of the $\mathfrak{q}_i$. This shows that all minimal prime ideals of I must be among the $\mathfrak{q}_i$. □

2.7 Corollary. *If $(R, \mathfrak{m})$ is a local ring of finite embedding dimension, then the image of the map $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$ consists precisely of the closed prime ideals of R.*

Proof. By Lemma 2.4, the image of the map consists of closed prime ideals. To prove the converse, let $\mathfrak{p}$ be an arbitrary closed prime ideal of R. By Lemma 2.4, we have $\mathfrak{p} = \mathfrak{p}\widehat{R} \cap R$. Let $\mathfrak{N}$ be maximal in $\widehat{R}$ with the property that $\mathfrak{p} = \mathfrak{N} \cap R$. I claim that

$\mathfrak{N}$ is a prime ideal, showing that $\mathfrak{p}$ lies in the image of $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$. To prove the claim, suppose $fg \in \mathfrak{N}$, but $f, g \notin \mathfrak{N}$. By maximality, there exist $a, b \in R \setminus \mathfrak{p}$ such that $a \in \mathfrak{N} + f\widehat{R}$ and $b \in \mathfrak{N} + g\widehat{R}$. Hence $ab \in \mathfrak{N} + fg\widehat{R} = \mathfrak{N}$ and since $ab \in R$, we get $ab \in \mathfrak{N} \cap R = \mathfrak{p}$, contradicting that $\mathfrak{p}$ is prime. □

2.8 Lemma. *If the completion of a local ring $(R, \mathfrak{m})$ of finite embedding dimension is Artinian, then so is R.*

Proof. By assumption, $\mathfrak{m}^n\widehat{R} = 0$, for some n. Since $R/\mathfrak{m}^{n+1} \cong \widehat{R}/\mathfrak{m}^{n+1}\widehat{R} = \widehat{R}$, we get $\mathfrak{m}^n/\mathfrak{m}^{n+1} = 0$. Since $\mathfrak{m}$ is finitely generated, we may apply Nakayama's Lemma and conclude that $\mathfrak{m}^n = 0$, which implies that R is Artinian. □

2.9. Infinite ramification

We conclude this section with a note on ramification in mixed characteristic, which we will use occasionally. Let $(R, \mathfrak{m})$ be a local ring with residue field k. We say that R is *equicharacteristic* (or has *equal characteristic*) if R and k have the same characteristic; in the remaining case, that is to say, if R has characteristic 0 and k characteristic p, we say that R has *mixed characteristic* p. A local ring is equicharacteristic if and only if it contains a field.

For the next definition, assume that the residue field of R has characteristic p. We call $\operatorname{ord}(p)$ the *ramification index* of R. We say R is *unramified* if its ramification index is one; and *infinitely ramified*, if its ramification index is infinite, that is to say, if $p \in \mathfrak{I}_R$. If R is infinitely ramified and Noetherian (or just separated), then in fact it has equal characteristic p (in the literature this is also deemed as an instance of an 'unramified' local ring, but for us, it will be more useful to make the distinction). However, in the general case, a local ring can have characteristic zero and be infinitely ramified (see Lemma 13.5 below). It follows that the separated quotient and the completion of an infinitely ramified local ring are both equicharacteristic.

3. Geometric dimension

The *dimension* $\dim(A)$ of a ring A will always mean its Krull dimension, that is to say, the maximal length (possible infinite) of a chain of prime ideals in A. The *dimension* of an ideal $I \subseteq A$ is the dimension of its residue ring A/I. If R is local and Noetherian, then its dimension is always finite, but without the Noetherian assumption, it is generally infinite. In this section, we propose a first substitute for Krull dimension for an arbitrary local ring $(R, \mathfrak{m})$; other alternatives will be discussed in §4.

3.1 Definition. We define the *geometric dimension* of R recursively as follows. We say that R has geometric dimension zero, and we write $\operatorname{gdim}(R) = 0$, if and only if R is Artinian. For arbitrary d, we say that $\operatorname{gdim}(R) \leq d$, if there exists $x \in \mathfrak{m}$ such that $\operatorname{gdim}(R/xR) \leq d - 1$. Finally, we say that R has geometric dimension equal to d if $\operatorname{gdim}(R) \leq d$, but not $\operatorname{gdim}(R) \leq d - 1$, and we simply write $\operatorname{gdim}(R) := d$. If there is no d such that $\operatorname{gdim}(R) \leq d$, then we set $\operatorname{gdim}(R) := \infty$.

It follows that $\mathrm{gdim}(R) \leq \mathrm{embdim}(R)$. In fact, R has finite geometric dimension if and only if it has finite embedding dimension. If R has finite embedding dimension then $\mathrm{gdim}(R) = 0$ if and only if $\mathfrak{m}$ is nilpotent. The following fact is immediate from the definition.

3.2 Lemma. *If $(R, \mathfrak{m})$ is a local ring and $a \in \mathfrak{m}$, then*

$$\mathrm{gdim}(R) - 1 \leq \mathrm{gdim}(R/aR) \leq \mathrm{gdim}(R).$$

□

The geometric dimension can be formulated, as in the Noetherian case, in terms of the minimal number of generators of an $\mathfrak{m}$-primary ideal (showing that geometric dimension and Krull dimension agree for Noetherian local rings):

3.3 Lemma. *The geometric dimension of a local ring $(R, \mathfrak{m})$ of finite embedding dimension is the least possible number of elements generating an $\mathfrak{m}$-primary ideal.*

Proof. Let $d := \mathrm{gdim}(R)$. By Lemma 3.2, there exists no sequence $\mathbf{y}$ of length less than d such that $R/\mathbf{y}R$ has geometric dimension zero. It follows that any $\mathfrak{m}$-primary ideal is generated by at least d elements. So remains to show that there exists a tuple of length d generating an $\mathfrak{m}$-primary ideal. We induct on d, where the case $d = 0$ is clear, since then (0) is $\mathfrak{m}$-primary. By definition, we can choose $x_1 \in \mathfrak{m}$ such that $\mathrm{gdim}(R/x_1R) = d - 1$. By induction, there exist elements $x_2, \dots, x_d$ whose image in R/x_1R generate an $\mathfrak{m}(R/x_1R)$-primary ideal. Hence $(x_1, \dots, x_d)R$ is $\mathfrak{m}$-primary. □

3.4 Theorem. *Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension. The following numbers are all equal.*

- *the geometric dimension d of R;*
- *the least possible number of elements d' generating an $\mathfrak{m}$-primary ideal;*
- *the dimension $\widehat{d}$ of the completion $\widehat{R}$ of R;*
- *the dimension $\overline{d}$ of the graded ring $\mathrm{gr}(R)$ associated to R;*
- *the degree $\underline{d}$ of the* Hilbert-Samuel polynomial HS_R, *where HS_R is the unique polynomial with rational coefficients for which $\mathrm{HS}_R(n)$ equals the length of $R/\mathfrak{m}^n$ for all large n;*
- *the geometric dimension $\dot{d}$ of the separated quotient $\dot{R}$;*

Proof. The equality of d and d' is given by Lemma 3.3. We already observed that $\mathrm{gr}(R)$ and $\widehat{R}$ are Noetherian and that we have isomorphisms

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \mathfrak{m}^n\widehat{R}/\mathfrak{m}^{n+1}\widehat{R}$$

for all n. Hence $\mathrm{HS}_R = \mathrm{HS}_{\widehat{R}}$ and $\mathrm{gr}(R) \cong \mathrm{gr}(\widehat{R})$. It follows that $\underline{d} = \widehat{d}$, by the Hilbert-Samuel theory and that $\overline{d} = \widehat{d}$ by [29, Theorem 13.9]. This shows already that $\overline{d} = \widehat{d} = \underline{d}$.

Let $(y_1, \dots, y_d)$ be a tuple generating an $\mathfrak{m}$-primary ideal. Since $(y_1, \dots, y_d)\widehat{R}$ is then $\mathfrak{m}\widehat{R}$-primary, $\widehat{d} \leq d$. Finally, let $(\xi_1, \dots, \xi_{\overline{d}})$ be a homogeneous system of parameters of $\operatorname{gr}(R)$ and choose $x_i \in R$ such that $\xi_i = \operatorname{in}(x_i)$. Let $I := (x_1, \dots, x_{\overline{d}})R$. By [12, Exercise 5.3], we have an isomorphism

$$\operatorname{gr}(R)/\operatorname{in}(I) \cong \operatorname{gr}(R/I).$$

Since $(\xi_1, \dots, \xi_{\overline{d}}) \operatorname{gr}(R) \subseteq \operatorname{in}(I)$, we see that $\operatorname{gr}(R)/\operatorname{in}(I)$ is Artinian, whence so is $\operatorname{gr}(R/I)$. This in turn means that R/I has a nilpotent maximal ideal, so that $d \leq \overline{d}$ by definition of geometric dimension. This proves that the first five numbers in the statement are equal. That they are also equal to the last, d, follows by applying the result to R together with the fact that R and R have the same completion. □

3.5 Remark. If the leading coefficient of the Hilbert-Samuel polynomial is written as $e/d!$, with $d := \operatorname{gdim}(R)$, then we call e the *multiplicity* of R and we denote it $\operatorname{mult}(R)$. It follows that R has the same multiplicity as its completion and as its separated quotient.

3.6 Corollary. *If R is a local ring of geometric dimension one, then there exists $N \in \mathbb{N}$ such that every closed ideal is the closure of an N-generated ideal.*

Proof. By Theorem 3.4, the completion $\widehat{R}$ is a one-dimensional Noetherian local ring, and hence by the Akizuki-Cohen theorem ([1, 10]), there is some N such that every ideal in $\widehat{R}$ is generated by at most N elements. Let $I \subseteq R$ be an arbitrary ideal. Since $I\widehat{R}$ is generated by at most N elements, we may choose by Nakayama's Lemma $a_1, \dots, a_N \in I$ such that $I\widehat{R} = (a_1, \dots, a_N)\widehat{R}$. Contracting this equality back to R shows, by Lemma 2.4, that I is the closure of $(a_1, \dots, a_N)R$. □

It is well-known that one may take N to be equal to the multiplicity of R, in case the latter is Cohen-Macaulay. In view of Remark 3.5 and our definition in §6 below, the same holds true under the assumption that R is cata-Cohen-Macaulay.

3.7. Generic sequences

A tuple $\mathbf{x}$ is called *generic*, if it generates an $\mathfrak{m}$-primary ideal and its length is equal to the geometric dimension of R; it is called *part of a generic sequence*, if it can be extended to a generic sequence. If x is a single element which is part of a generic sequence, then we simply call x a *generic element*.

3.8 Lemma. *Let $(R, \mathfrak{m})$ be a local ring of geometric dimension d. A tuple $(x_1, \dots, x_e)$ is part of a generic sequence if and only if $R/(x_1, \dots, x_e)R$ has geometric dimension $d - e$.*

In particular, x is generic if and only if $\operatorname{gdim}(R/xR) = \operatorname{gdim}(R) - 1$.

Proof. Suppose $(x_1, \dots, x_e)$ is part of a generic sequence and enlarge it to a generic sequence $(x_1, \dots, x_d)$. One checks that (the image of) $(x_{e+1}, \dots, x_d)$ is a generic sequence in $R/(x_1, \dots, x_e)R$. This shows that $\operatorname{gdim}(R/(x_1, \dots, x_e)R) = d - e$. Conversely, assume $\operatorname{gdim}(R/(x_1, \dots, x_e)R) = d - e$. Choose a tuple $(x_{e+1}, \dots, x_d)$ in R so that its image in $R/(x_1, \dots, x_e)R$ is a generic sequence. Since $(x_1, \dots, x_d)$ generates an $\mathfrak{m}$-primary ideal and has length d, it is generic. □

3.9 Proposition. *Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension. A sequence in R is generic if and only if its image in $\widehat{R}$ is a system of parameters.*

Proof. One direction has already been noted, so let $\mathbf{x}$ be a tuple in R whose image in $\widehat{R}$ is a system of parameters. By Theorem 3.4, the geometric dimension of R is equal to the length of this tuple. Let $J := \mathbf{x}R$. By Lemma 2.4, the completion of R/J is $\widehat{R}/J\widehat{R}$. As the latter is Artinian, so must the former be by Lemma 2.8, showing that $\mathbf{x}$ is generic. $\square$

It follows that $(x_1, \dots, x_d)$ is generic if and only if so is $(x_1^{n_1}, \dots, x_d^{n_d})$. However, this does in general not imply that $(\mathrm{in}(x_1), \dots, \mathrm{in}(x_d))$ is a system of parameters in $\mathrm{gr}(R)$ (this even fails in the Noetherian case as the example $\{\xi^2, \xi\zeta + \zeta^3\}$ in $k[[\xi, \zeta]]$ shows). Immediately from Proposition 3.9 and [29, Theorem 14.5] we get:

3.10 Corollary. *Any generic sequence $\mathbf{x}$ in R is* analytically independent *in the sense that if $F(\xi)$ is a homogeneous form over R such that $F(\mathbf{x}) = 0$, then all coefficients of $F(\xi)$ lie in the maximal ideal of R.* $\square$

3.11. Threshold primes

By Proposition 3.9, an element x is generic if and only if the image of x in $\widehat{R}$ is part of a system of parameters. More concretely, let d be the geometric dimension of R and let $\mathfrak{p}_1, \dots \mathfrak{p}_s$ be the d-dimensional prime ideals of $\widehat{R}$. Note that $\widehat{R}$ itself has dimension d by Theorem 3.4, so that all its d-dimensional primes are minimal (but there may be other minimal prime ideals, of lower dimension). We call the $\mathfrak{q}_i := \mathfrak{p}_i \cap R$ the *threshold* primes of R. By Corollary 2.7, every threshold prime $\mathfrak{q}$ is closed and contains no proper closed prime ideals. Moreover, $R/\mathfrak{q}$ has the same geometric dimension as R by Theorem 3.4, since $\widehat{R}/\mathfrak{q}\widehat{R}$ has the same dimension as $\widehat{R}$. By a *threshold prime* of an ideal I, we mean a threshold prime of its residue ring R/I. Proposition 3.9 yields the following criterion for genericity.

3.12 Corollary. *An element $x \in R$ is generic if and only if it is not contained in any threshold prime of R. In particular, the product of any two generic elements is again generic.* $\square$

3.13 Corollary. *Any $\mathfrak{m}$-primary ideal contains a generic sequence. More precisely, if R is a Z-algebra and $I \subseteq Z$ an ideal such that IR is $\mathfrak{m}$-primary, then there exists a tuple $\mathbf{x}$ over Z with entries in I such that its image in R is a generic sequence.*

Proof. We prove the last assertion by induction on the geometric dimension d of R. Since there is nothing to show if $d = 0$, we may assume $d > 0$. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the threshold primes of R. Towards a contradiction, suppose I is contained in the union of the $\mathfrak{q}_i \cap Z$. By prime avoidance, there is some i such that $I \subseteq \mathfrak{q}_i \cap Z$. But then $IR \subseteq \mathfrak{q}_i$, forcing $\mathfrak{q}_i = \mathfrak{m}$, thus contradicting by Corollary 3.12 that $d > 0$. Hence there exists $x \in I$ so that its image in R lies outside every threshold prime of R, and therefore is generic by Corollary 3.12. By Lemma 3.8, the geometric dimension of R/xR is $d - 1$. Therefore, by induction, we can find a tuple $\mathbf{y}$ of length $d - 1$ with entries in I so that its image in R/xR is generic. The desired sequence is now given by adding x to this tuple $\mathbf{y}$. $\square$

In [20], the authors introduce the notion of a *strong parameter sequence*. It should be noted that this is different from our present notion of generic sequence. For example, if V is an ultra-discrete valuation ring (see Example 6.3 for more details), and x a non-zero infinitesimal in V, then x is V-regular, whence a strong parameter by [20, Proposition 3.3(f)], but x is clearly not generic (in fact, the unique threshold prime of V is the ideal of infinitesimals $\mathfrak{I}_V$).

3.14. Geometric codimension

Given an ideal I in a local ring $(R, \mathfrak{m})$ of finite embedding dimension, we call its *geometric codimension* the maximal length of a tuple in I that is part of a generic sequence and we denote it $\operatorname{gcodim}(I)$. In particular, an ideal is $\mathfrak{m}$-primary if and only if its geometric codimension equals the geometric dimension of R. Our terminology is justified by the next result.

3.15 Proposition. *Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension. For every ideal $I \subseteq R$, we have an equality $\operatorname{gcodim}(I) = \operatorname{gdim}(R) - \operatorname{gdim}(R/I)$.*

Proof. Let d be the geometric dimension of R and let h be the geometric codimension of I. Choose a tuple $\mathbf{y}$ in I of length h which is part of a generic sequence of R. Put $S := R/\mathbf{y}R$, so that $\operatorname{gdim}(S) = d - h$ by Lemma 3.8. Since IS contains no generic element, it must be contained in some threshold prime $\mathfrak{q}$ of S by Corollary 3.12. From the inclusions $IS \subseteq \mathfrak{q}$ we get $\operatorname{gdim}(S) \geq \operatorname{gdim}(S/IS) \geq \operatorname{gdim}(S/\mathfrak{q}) = \operatorname{gdim}(S)$, and hence all these geometric dimensions are equal to $d - h$. Since $S/IS = R/I$, we are done. □

3.16. Parameter degree and degree

We conclude this section with another genericity criterion, in terms of an invariant which was introduced for Noetherian rings in [46, 47] and which will play a crucial role in what follows. The *parameter degree* of a local ring R of finite embedding dimension is by definition the minimal length of a residue ring $R/\mathbf{x}R$, where $\mathbf{x}$ runs over all possible generic sequences of R. We denote the parameter degree of R by $\operatorname{pardeg}(R)$. We will show in Lemma 6.11 below that the multiplicity of R is bounded by its parameter degree and indicate when they are equal.

Closely related to this is an invariant, which for want of a better name, we call *degree* and which is defined as follows. Let R be a local ring of geometric dimension $d \geq 1$. We define the *degree* $\deg_R(x)$ of an element x to be the least possible length of a residue ring $R/(xR + \mathbf{y}R)$, where $\mathbf{y}$ runs over all tuples of length $d - 1$ inside the maximal ideal. Hence, if x is a unit, its degree is zero; if x is generic, its degree is the parameter degree of R/xR; and in the remaining case, its degree is infinite. In particular, we showed:

3.17 Corollary. *An non-unit in a non-Artinian local ring R of finite embedding dimension is generic if and only if its degree is finite. Moreover, the parameter degree of R is the minimum of the degrees of all non-units in R.* □

4. Extended dimensions

In this section, we introduce several other dimension notions for a local ring $(R, \mathfrak{m})$. With an *extended dimension*, we mean an invariant on the class of local rings taking values in $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ which agrees with Krull dimension on the subclass of all Noetherian local rings. Clearly, Krull dimension itself is an extended dimension, and so is geometric dimension by the results from the previous section. Note, however, that embedding dimension is *not* an extended dimension.

Recall that a partially ordered set Γ has *combinatorial dimension* (or, *height*) d if any proper (ascending) chain in Γ has length at most d (meaning that it contains at most $d + 1$ elements). Hence, the dimension of a ring A is the combinatorial dimension of $\operatorname{Spec}(A)$ (the set of all prime ideals ordered by inclusion). Given ideals $J \subseteq \mathfrak{p}$ in A with $\mathfrak{p}$ prime, we say that $\mathfrak{p}$ is an *associated* prime of J if $\mathfrak{p}$ is of the form $(J : a)$; a *minimal* prime of J if no prime ideal is properly contained between J and $\mathfrak{p}$; and a *minimal associated* prime of J if it is associated and no associated prime of J is properly contained between J and $\mathfrak{p}$.

4.1. Cl-dimension

Let $\operatorname{CL-Spec}(R)$ be the subset of $\operatorname{Spec}(R)$ consisting of all closed prime ideals of R. Note that the maximal ideal as well as the threshold primes (see §3.11) belong to $\operatorname{CL-Spec}(R)$. In fact, we showed in Corollary 2.7 that $\operatorname{CL-Spec}(R)$ is the image of the canonical map $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$. We call the combinatorial dimension of $\operatorname{CL-Spec}(R)$ the *cl-dimension* of R and denote it $\operatorname{cldim}(R)$. It is clear that $\operatorname{cldim}(R) = \dim(R)$ when R is Noetherian, showing that cl-dimension is an extended dimension.

4.2. Fr-dimension

We say that an ideal $I \subseteq R$ is n*-generated*, if there exists a tuple $\mathbf{x}$ of length n such that $\mathbf{x}R = I$. We say that an ideal $\mathfrak{a} \subseteq R$ is n*-related* if it is of the form $\mathfrak{a} = (I : a)$ with I an n-generated ideal. An ideal $\mathfrak{a}$ is called *finitely related* if it is n-related for some $n < \infty$. Let $\operatorname{FR-Spec}(R)$ be the subset of $\operatorname{Spec}(R)$ consisting of all finitely related prime ideals, that is to say, all associated prime ideals of finitely generated ideals of R. We call the combinatorial dimension of $\operatorname{FR-Spec}(R)$ the *fr-dimension* of R and denote it $\operatorname{frdim}(R)$. When R is Noetherian, every ideal is finitely related whence $\operatorname{frdim}(R) = \dim(R)$, showing that fr-dimension is an extended dimension. Let us call a prime ideal $\mathfrak{p}$ *strongly finitely related* if it is of the form $(I : a)$ with I finitely generated and $a \notin \mathfrak{p}$. A priori, not every finitely related prime ideal is strong, but see Corollaries 5.3 and 5.27.

4.3. Pi-dimension

We say that R has *pi-dimension* at most d, if $\mathfrak{m}$ is a minimal associated prime of a d-generated ideal. The pi-dimension, $\operatorname{pidim}(R)$, of R is then the least d such that R has pi-dimension at most d. That pi-dimension is an extended dimension follows from Krull's Principal Ideal theorem (from which it borrows its name; see for instance [29, Theorem 8.10]).

4.4 Theorem. *For an arbitrary local ring* $(R, \mathfrak{m})$*, we have the following inequalities between extended dimensions:*

4.4.1. $\operatorname{frdim}(R), \operatorname{cldim}(R) \leq \dim(R)$*;*
4.4.2. $\operatorname{pidim}(R) \leq \operatorname{gdim}(R)$*;*
4.4.3. $\operatorname{cldim}(R) \leq \operatorname{gdim}(R)$*, with equality if* $\operatorname{gdim}(R)$ *is finite.*

Moreover, each of these inequalities can be strict.

Proof. Inequalities (4.4.1) are immediate from the definition. In order to show inequality (4.4.2), we may assume that $\operatorname{gdim}(R) = d < \infty$. By definition, R/I is an Artinian local ring for some d-generated ideal I. It follows that $\mathfrak{m}$ is a minimal associated prime of I, whence the pi-dimension of R is at most d.

So remains to prove (4.4.3). There is nothing to show if R has infinite geometric dimension, so assume R has finite geometric dimension, say, d (whence also finite embedding dimension). By Corollary 2.7, there is a surjective map $\operatorname{Spec}(\widehat{R}) \to \text{CL-Spec}(R)$. In particular, the combinatorial dimension of $\text{CL-Spec}(R)$ is at most the dimension of $\widehat{R}$, that is to say, in view of Theorem 3.4, at most d. So remains to prove the other inequality by induction on d. There is nothing to show if $d = 0$, so we may assume $d > 0$. By Corollary 2.7, the minimal elements in $\text{CL-Spec}(R)$ are the contractions of the minimal primes of $\widehat{R}$. Hence there are only finitely many of them, all different from the maximal ideal $\mathfrak{m}$. By prime avoidance, we may choose $x \in \mathfrak{m}$ outside all these finitely many prime ideals. In particular, since the threshold primes are among these, x is generic and hence R/xR has geometric dimension $d-1$. By induction, the combinatorial dimension of $\text{CL-Spec}(R/xR)$ is $d-1$. By Lemma 2.4, the completion of R/xR is $\widehat{R}/x\widehat{R}$. The homomorphism $\widehat{R} \to \widehat{R}/x\widehat{R}$ induces an injection $\operatorname{Spec}(\widehat{R}/x\widehat{R}) \hookrightarrow \operatorname{Spec}(\widehat{R})$, whose image is the subset of all prime ideals of $\widehat{R}$ containing x. It follows that the canonical injection $\operatorname{Spec}(R/xR) \hookrightarrow \operatorname{Spec} R$ maps $\text{CL-Spec}(R/xR)$ into the subset of $\text{CL-Spec}(R)$ consisting of all closed prime ideals containing x. Using this and the fact that the combinatorial dimension of $\text{CL-Spec}(R/xR)$ is $d-1$, we can find a proper chain of closed primes ideals $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2 \subsetneq \cdots \subsetneq \mathfrak{q}_d = \mathfrak{m}$ in R with $x \in \mathfrak{q}_1$. Let $\mathfrak{q}_0$ be a minimal element of $\text{CL-Spec}(R)$ lying inside $\mathfrak{q}_1$. Since by construction $x \notin \mathfrak{q}_0$, the $\mathfrak{q}_i$ form a proper chain of length d, showing that the combinatorial dimension of $\text{CL-Spec}(R)$ is at least d. This proves (4.4.3).

Finally, the local ring in Example 4.5 (respectively, in Example 4.6) shows that in general, the inequalities (4.4.1) and (4.4.2) (respectively, inequality (4.4.3)) are strict. □

4.5 Example. Let $R_\natural$ be the ultraproduct (see §5 for more details) of the $A/\mathfrak{p}^n$ for $n = 1, 2 \dots$, where $(A, \mathfrak{p})$ is a d-dimensional Noetherian local ring, for $d > 0$. Its pi-dimension and fr-dimension are equal to zero, its geometric dimension and cl-dimension are equal to d, and its Krull dimension is infinite.

4.6 Example. Let $(R_\natural, \mathfrak{m}_\natural)$ be the ultraproduct of the $A_n/\mathfrak{m}_n^2$ for $n = 1, 2 \dots$, where $(A_n, \mathfrak{m}_n)$ is the power series ring over a field k in n indeterminates. Since $\mathfrak{m}_\natural^2 = 0$ in $R_\natural$, the local ring $R_\natural$ has cl-dimension and Krull dimension equal to zero, but its embedding dimension, whence its geometric dimension, is infinite.

There is a more instructive way to see (4.4.2): the geometric dimension of a local ring $(R, \mathfrak{m})$ of finite embedding dimension is at most d if and only if $\mathfrak{m}$ is a minimal prime of a d-generated ideal (that is to say, the same definition as for pi-dimension, but omitting the term 'associated').

Let 'e-dim' be some extended dimension. We call e-dim *first-order* if the property $\text{e-dim}(\cdot) = d$ is first-order in the sense of §5.5 below, for every $d \in \mathbb{N}$. Moreover, to prove this, it suffices to show that the property $\text{e-dim}(\cdot) \le d$ is first-order.

4.7 Lemma. *Fr-dimension and pi-dimension are first-order; geometric dimension, cl-dimension and Krull dimension are not.*

Proof. The assertion is obvious for pi-dimension, since we can express in a first-order way that the maximal ideal $\mathfrak{m}$ of a local ring is of the form $(I : a)$ for some d-generated ideal I such that no prime ideal of the form $(I : b)$ is properly contained in $\mathfrak{m}$ (note that $\mathfrak{m}$ admits a first-order definition as the collection of all non-units). As for fr-dimension, for each n, let $\tau_{n,d}$ be the statement expressing that there does not exist a proper chain of length $d+1$ consisting of n-related prime ideals. Hence a local ring has fr-dimension at most d if and only if $\tau_{n,d}$ holds in it, for all n.

The local ring in Example 4.5 shows that Krull dimension, cl-dimension and geometric dimension are not first-order. □

5. Ultra-Noetherian rings

Before we further develop the 'local algebra' of local rings of finite embedding dimension, we introduce an important subclass, arising as ultraproducts of Noetherian local rings. Fix an infinite index set W and a non-principal ultrafilter on W. We will moreover assume that the ultrafilter is countably incomplete. This is equivalent with the existence of a function $f : W \to \mathbb{N}$ such that for each k, the set of all $w \in W$ for which $f(w) \ge k$ belongs to the ultrafilter. If W is countable, then any non-principal ultrafilter is countably incomplete, and this is the situation we will find ourselves in all applications.[5] For each $w \in W$, let R_w be a local ring and let $R_\natural$ be the *ultraproduct* of the R_w (for a quick review on ultraproducts, see [40, §1]; for more details see for instance, [14, 26, 35, 50]). It is important to note that R_w are not uniquely defined by $R_\natural$ (not even almost all; see the example in §5.5). By Łos' Theorem, $R_\natural$ is a local ring with maximal ideal $\mathfrak{m}_\natural$ equal to the ultraproduct of the maximal ideals $\mathfrak{m}_w$. If for some m, almost all R_w have embedding dimension at most m, then we say that the R_w have *bounded embedding dimension*; a similar usage will be applied to other numerical invariants. Hence if the R_w have bounded embedding dimension, then $R_\natural$ has finite embedding dimension, whence finite geometric dimension. In case all R_w are equal to a single local ring R, we refer to $R_\natural$ as the *ultrapower* of R.

When dealing with ultraproducts, Łos' Theorem is an extremely useful tool for transferring properties between almost all R_w and $R_\natural$. However, this only applies to

[5] In fact, it is consistent with ZF to assume that every non-principal ultrafilter on any infinite set is countably incomplete. Moreover, for most of what we say, we will not need to assume that the ultrafilter is countably incomplete; it is only used explicitly in Lemma 5.6 below.

first-order properties (see §5.5 below for more details). In view of this, we introduce the following more general set-up for discussing transfer through ultraproducts. Let $\mathbf{P}$ be a property of local rings of finite embedding dimension and let R be a local ring. We call R *cata*-$\mathbf{P}$ if it has finite embedding dimension and its completion has property $\mathbf{P}$. In particular, by Theorem 2.2, any such ring is, in our newly devised terminology, cata-Noetherian. We call a local ring *ultra*-$\mathbf{P}$ if it is equal to an ultraproduct $R_\natural$ of local rings R_w of bounded embedding dimension almost all of which satisfy property $\mathbf{P}$. In particular, $R_\natural$ has finite embedding dimension too. In fact, according to this terminology, an *ultra-ring* is any ultraproduct of local rings of bounded embedding dimension; and an *ultra-Noetherian* ring is any ring isomorphic to an ultraproduct of Noetherian local rings of bounded embedding dimension. It is important to notice that the well-known duality between rings and affine schemes breaks down under ultraproducts:

5.1 Proposition. *Let R_w be Noetherian local rings of bounded embedding dimension and let $R_\natural$ be their ultraproduct. Then the ultraproduct of the $\operatorname{Spec}(R_w)$ is equal to $\operatorname{FR-Spec}(R_\natural)$.*

Proof. Recall that $\operatorname{FR-Spec}(R_\natural)$ consists of all finitely related prime ideals of $R_\natural$ (see §4.2). If $I_\natural$ is a finitely generated ideal in $R_\natural$, say of the form $(x_{1\natural}, \dots, x_{n\natural})R_\natural$, and if $x_{iw} \in R_w$ are such that their ultraproduct is equal to $x_{i\natural}$, then the ultraproduct of the ideals $I_w := (x_{1w}, \dots, x_{nw})R_w$ is equal to $I_\natural$. Moreover, if $y_\natural \in R_\natural$ is the ultraproduct of elements $y_w \in R_w$, then $(I_\natural : y_\natural)$ is equal to the ultraproduct of the $(I_w : y_w)$. If $(I_\natural : y_\natural)$ is prime, then so are almost all $(I_w : y_w)$ by Łos' Theorem. Hence any finitely related prime ideal in $R_\natural$ belongs to the ultraproduct of the $\operatorname{Spec}(R_w)$.

Conversely, for each w, let $\mathfrak{p}_w$ be a prime ideal in R_w, and let $\mathfrak{p}_\natural$ be their ultraproduct. By Łos' Theorem, $\mathfrak{p}_\natural$ is prime. Since the R_w have bounded embedding dimension, they also have bounded dimension. Therefore, there is a d such that almost each R_w has dimension d (in the terminology of §5.18 below, d is the ultra-dimension of $R_\natural$). By Krull's Principal Ideal theorem, almost each $\mathfrak{p}_w$ is d-related, whence so is $\mathfrak{p}_\natural$ by Łos' Theorem. □

In particular, the ultraproduct of the $\operatorname{Spec}(R_w)$ does not depend on the choice of the R_w having as ultraproduct $R_\natural$. The local algebra of rings of finite embedding dimension is hampered by the fact that very few localizations have finite embedding dimension. We will discuss one case here (see Corollary 8.3 for another one). We first prove a bound for Noetherian rings.[6] For a Noetherian ring A, let $\gamma(A) \in \mathbb{N} \cup \{\infty\}$ be the supremum of all $\operatorname{embdim}(A_\mathfrak{p})$, where $\mathfrak{p}$ runs through all prime ideals of A.

5.2 Proposition. *If A is a d-dimensional, excellent ring, then $\gamma(A) < \infty$. In fact, if A is equicharacteristic and local, then $\gamma(A) \leq d + \rho$, where ρ is the parameter degree of A.*

Proof. We prove the first statement by induction on d. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal prime ideals of A, and let N be a bound on their number of generators. Since

[6] In §§11 and 12, we adopt the reverse strategy, by developing bounds from our local algebra results.

any prime ideal $\mathfrak{p}$ contains one of the $\mathfrak{p}_i$, we see that $\gamma(A)$ is bounded by the maximum of all $\gamma(A/\mathfrak{p}_i) + N$. Hence we may assume without loss of generality that A is an excellent domain. Therefore, its regular locus is non-empty and open. Let $U = \operatorname{Spec} A_f$ be a non-empty affine open contained in the regular locus of A. By regularity, $\operatorname{embdim}(A_\mathfrak{p}) \leq d$, for any $\mathfrak{p} \in U$, and so we only need to show a bound for those prime ideals containing f. Put $\bar{A} := A/fA$. Note that $\bar{A}$ has Krull dimension $d-1$ and is again excellent, so that by induction $\gamma(\bar{A}) < \infty$. Therefore, for any prime ideal $\mathfrak{p}$ of A containing f, we have an estimate $\operatorname{embdim}(A_\mathfrak{p}) \leq \gamma(\bar{A}) + 1$, finishing the proof of the first assertion.

Assume next that A is moreover equicharacteristic and local, with parameter degree ρ. I claim that $\gamma(A) \leq \gamma(\widehat{A})$, where $\widehat{A}$ is the completion of A. Assuming the claim, we may take A to be complete, since parameter degree does not change under completion. By the Cohen structure theorem, A contains a d-dimensional regular local subring R over which it is finite. Moreover, by [46, Proposition 3.5], we may choose R so that A is generated by ρ elements as an R-module. Let $\mathfrak{p}$ be a prime ideal in A and put $\mathfrak{g} := \mathfrak{p} \cap R$. By base change, the fiber ring $A_\mathfrak{g}/\mathfrak{g}A_\mathfrak{g}$ has dimension ρ over the residue field of $\mathfrak{g}$. Moreover, $A_\mathfrak{p}/\mathfrak{g}A_\mathfrak{p}$ is a direct summand of $A_\mathfrak{g}/\mathfrak{g}A_\mathfrak{g}$ by the structure theorem of Artinian local rings ([12, Corollary 2.16]), whence has length at most ρ. In particular, $\operatorname{embdim}(A_\mathfrak{p}/\mathfrak{g}A_\mathfrak{p}) \leq \rho$. Since R is regular, $\mathfrak{g}R_\mathfrak{g}$ is generated by at most d elements, whence so is $\mathfrak{g}A_\mathfrak{p}$. It follows that $\operatorname{embdim}(A_\mathfrak{p}) \leq \rho + d$, as we wanted to show.

To prove the claim, let $\mathfrak{q}$ be a minimal prime ideal of $\mathfrak{p}\widehat{A}$. Since $A/\mathfrak{p}$ is excellent, its completion $\widehat{A}/\mathfrak{p}\widehat{A}$ is reduced. Therefore, the localization of $\widehat{A}/\mathfrak{p}\widehat{A}$ at $\mathfrak{q}$ is a field, showing that $\mathfrak{p}\widehat{A}_\mathfrak{q} = \mathfrak{q}\widehat{A}_\mathfrak{q}$, an ideal generated by at most $\gamma(\widehat{A})$ elements. Since $A_\mathfrak{p} \to \widehat{A}_\mathfrak{q}$ is faithfully flat, $\mathfrak{p}A_\mathfrak{p}$ is therefore also generated by at most $\gamma(\widehat{A})$ elements, showing that $\gamma(A) \leq \gamma(\widehat{A})$. □

5.3 Corollary. *If R is an excellent local ring, then any localization of its ultrapower $R_\natural$ at a finitely related prime ideal has finite embedding dimension. Moreover, every finitely related prime ideal of $R_\natural$ is strong.*

Proof. Let $\mathfrak{p}$ be a finitely related prime ideal of $R_\natural$. By Proposition 5.1, we can find prime ideals $\mathfrak{p}_w$ in R with ultraproduct equal to $\mathfrak{p}$. Let $\gamma(R)$ be the bound given by Proposition 5.2 on the embedding dimension of all $R_{\mathfrak{p}_w}$. Since $(R_\natural)_\mathfrak{p}$ is the ultraproduct of the $R_{\mathfrak{p}_w}$, its embedding dimension is at most $\gamma(R)$ as well. In fact, we can find ideals $I_w \subseteq \mathfrak{p}_w$ generated by at most $\gamma(R)$ elements, so that $I_w R_{\mathfrak{p}_w} = \mathfrak{p}_w R_{\mathfrak{p}_w}$. Hence, there exists $a_w \notin \mathfrak{p}_w$, such that $(I_w : a_w) = \mathfrak{p}_w$. Taking ultraproducts, we see that $\mathfrak{p}$ is strongly finitely related (see §4.2 for the definition). □

In fact, we have the following more general version of the second assertion.

5.4 Proposition. *A finitely related prime ideal $\mathfrak{p}$ in an ultra-Noetherian local ring $R_\natural$ is strongly finitely related if and only if $(R_\natural)_\mathfrak{p}$ has finite geometric dimension.*

Proof. Note that a local ring has finite geometric dimension if and only if it has finite embedding dimension. One direction is true in any ring A: if $\mathfrak{p}$ is strongly finitely related, say, of the form $(I : s)$ with $I \subseteq A$ finitely generated and $s \notin \mathfrak{p}$, then $\mathfrak{p}A_\mathfrak{p} = IA_\mathfrak{p}$, showing that $A_\mathfrak{p}$ has finite embedding dimension.

Conversely, suppose $(R_\natural)_\mathfrak{p}$ has finite geometric dimension, whence finite embedding dimension. In particular, there exists a finitely generated ideal $I \subseteq \mathfrak{p}$ such that $I(R_\natural)_\mathfrak{p} = \mathfrak{p}(R_\natural)_\mathfrak{p}$. By Łos' Theorem and Proposition 5.1, we can find ideals $I_w \subseteq \mathfrak{p}_w$ so that their respective ultraproducts are I and $\mathfrak{p}$. In particular, almost all $\mathfrak{p}_w$ are prime and $I_w(R_w)_{\mathfrak{p}_w} = \mathfrak{p}_w(R_w)_{\mathfrak{p}_w}$, for almost all w. Hence, we can find $s_w \notin \mathfrak{p}_w$ such that $\mathfrak{p}_w = (I_w : s_w)$. Letting $s_\natural$ be the ultraproduct of the s_w, we get $\mathfrak{p} = (I : s_\natural)$ and $s_\natural \notin \mathfrak{p}$, showing that $\mathfrak{p}$ is strong. □

5.5. *First-order properties*

A property $\mathbf{P}$ of rings is called *first-order* if there exists a first-order theory $\mathcal{T}$, in the language of rings, such that R is a model of $\mathcal{T}$ if and only if R satisfies $\mathbf{P}$. Łos' Theorem states that if $\mathbf{P}$ is first-order, then ultra-$\mathbf{P}$ implies $\mathbf{P}$. Although we will not use this here, the converse is also true, due to a theorem of Keisler-Shelah (see for instance [26, Theorem 9.5.7]). It follows that if $\mathbf{P}$ is not first-order, then there exists an ultra-ring $S_\natural$ which is at the same time ultra-$\mathbf{P}$ and ultra-non-$\mathbf{P}$. Indeed, by what we just said, there exist R_w of bounded embedding dimension satisfying $\mathbf{P}$ so that there ultraproduct $R_\natural$ does not satisfy $\mathbf{P}$. Let $S_\natural$ be any ultrapower of $R_\natural$. Since $S_\natural$ is then also an ultraproduct of the R_w, but for a larger underlying index set, $S_\natural$ is both ultra-$\mathbf{P}$ and ultra-non-$\mathbf{P}$.

For an ultra-Noetherian example, consider the property $\mathbf{C}_0$: 'being a Noetherian local ring of characteristic zero'. The ultraproduct $V_\natural$ of all the rings of p-adic integers $\mathbb{Z}_p$ (with respect to some non-principal ultrafilter on the set of prime numbers) is ultra-$\mathbf{C}_0$, but by the Ax-Kochen-Ershov theorem, this ring can also be realized as the ultraproduct of non-$\mathbf{C}_0$ local rings, to wit, the $\mathbb{F}_p[[t]]$, where t is a single indeterminate and $\mathbb{F}_p$ is the p element field (see also Example 9.7 below).

Cataproducts

Let R_w be Noetherian local rings of bounded embedding dimension and let $R_\natural$ be their ultraproduct. The separated quotient of $R_\natural$, that is to say, the factor ring $R_\sharp := R_\natural/\mathfrak{I}_{R_\natural}$, is called the *cataproduct* of the R_w. If all R_w are equal to a single ring R, then we call $R_\sharp$ the *catapower* of R. This terminology is justified by:

5.6 Lemma. *The cataproduct of local rings of bounded embedding dimension is equal to the completion of their ultraproduct, whence in particular is Noetherian.*

Proof. Let $(R_\natural, \mathfrak{m}_\natural)$ be the ultraproduct of Noetherian local rings $(R_w, \mathfrak{m}_w)$ of embedding dimension at most e, and let $R_\sharp$ be their cataproduct, that is to say, $R_\natural/\mathfrak{I}_{R_\natural}$. We start with showing that any Cauchy sequence $\mathbf{a}_\natural : \mathbb{N} \to R_\natural$ has a limit. After taking a subsequence if necessary, we may assume that $\mathbf{a}_\natural(n) \equiv \mathbf{a}_\natural(n+1) \mod \mathfrak{m}_\natural^n$, for all n. For each n, choose $\mathbf{a}_w(n) \in R_w$ such that their ultraproduct is equal to $\mathbf{a}_\natural(n)$. By Łos' Theorem, we have for a fixed n that

$$\mathbf{a}_w(n) \equiv \mathbf{a}_w(n+1) \mod \mathfrak{m}_w^n \tag{2}$$

for almost all w, say, for all w in D_n. I claim that we can modify the $\mathbf{a}_w(n)$ in such way that (2) holds for all n and all w. More precisely, for each n there exist $\tilde{\mathbf{a}}_w(n)$

with ultraproduct equal to $\mathbf{a}_\natural(n)$, such that

$$\tilde{\mathbf{a}}_w(n) \equiv \tilde{\mathbf{a}}_w(n+1) \mod \mathfrak{m}_w^n \tag{3}$$

for all n and w. We will construct the $\tilde{\mathbf{a}}_w(n)$ recursively from the $\mathbf{a}_w(n)$. When $n = 0$, no modification is required (since by assumption $\mathfrak{m}_w^0 = R_w$), and hence we set $\tilde{\mathbf{a}}_w(0) := \mathbf{a}_w(0)$ and $\tilde{\mathbf{a}}_w(1) := \mathbf{a}_w(1)$. So assume we have defined already the $\tilde{\mathbf{a}}_w(j)$ for $j \leq n$ such that (3) holds for all w. Now, for those w for which (2) fails for some $j \leq n$, that is to say, for $w \notin (D_0 \cup \dots \cup D_n)$, let $\tilde{\mathbf{a}}_w(n+1)$ be equal to $\tilde{\mathbf{a}}_w(n)$; for the remaining w, that is to say, for almost all w, we make no changes: $\tilde{\mathbf{a}}_w(n+1) := \mathbf{a}_w(n+1)$. It is now easily seen that (3) holds for all w. Since, for every n, almost each $\tilde{\mathbf{a}}_w(n)$ is equal to $\mathbf{a}_w(n)$, their ultraproduct is $\mathbf{a}_\natural(n)$, thus establishing our claim.

So we may assume (2) holds for all n and w. Let $f\colon W \to \mathbb{N}$ be a function on the index set W such that for each n, almost all $f(w) \geq n$ (this is where we use that the ultrafilter is countably incomplete; if $W = \mathbb{N}$, we can of course simply take the identity map). Let $b_\natural$ be the ultraproduct of the $\mathbf{a}_w(f(w))$. Since $\mathbf{a}_w(f(w)) \equiv \mathbf{a}_w(n) \mod \mathfrak{m}_w^n$ for almost all w by (3), Łos' Theorem yields $b_\natural \equiv \mathbf{a}_\natural(n) \mod \mathfrak{m}_\natural^n$, for each n, showing that $b_\natural$ is a limit of $\mathbf{a}_\natural$. Although this limit might not be unique, it will be in the separated quotient $R_\sharp$, showing that the latter is a complete local ring, equal therefore to $\widehat{R_\natural}$. Noetherianity now follows from Theorem 2.2. □

5.7 Corollary. *The closure of an ideal I in an ultra-Noetherian ring $R_\natural$ is equal to $I + \mathfrak{I}_{R_\natural}$. In particular, if $R_\sharp$ is the cataproduct of the R_w, and $I_\natural$ the ultraproduct of ideals $I_w \subseteq R_w$, then $R_\sharp/I_\natural R_\sharp$ is the cataproduct of the R_w/I_w.*

Proof. Since $R_\sharp := R_\natural/\mathfrak{I}_{R_\natural}$ is Noetherian by Lemma 5.6, the ideal $IR_\sharp$ is closed by Krull's intersection theorem. All assertions now follow from Lemma 2.4. □

5.8 Corollary. *The cataproduct $R_\sharp$ of Noetherian local rings R_w of bounded embedding dimension is equal to the cataproduct $S_\sharp$ of their completions.*

Proof. Let $(R_\natural, \mathfrak{m}_\natural)$ and $(S_\natural, \mathfrak{n}_\natural)$ be the ultraproduct of respectively the R_w and the $\widehat{R}_w$. By Łos' Theorem, $\mathfrak{m}_\natural S_\natural = \mathfrak{n}_\natural$ and $R_\natural$ is dense in $S_\natural$. Hence both rings have the same completion, which by Lemma 5.6 is respectively the cataproduct of the R_w and of the $\widehat{R}_w$. □

However, this is not the only case in which different rings can have the same cataproduct. Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension. A filtration $\mathfrak{I} = (I_n)_n$ on R is called *analytic* if its extension $\mathfrak{I}\widehat{R}$ induces a Haussdorf topology on $\widehat{R}$, or, equivalently, if the intersection of all $I_n\widehat{R}$ is zero. In particular, the $\mathfrak{m}$-adic filtration is analytic by Theorem 2.2. Given two filtrations $\mathfrak{I} = (I_n)_n$ and $\mathfrak{J} = (J_n)_n$, we say that $\mathfrak{I}$ is *bounded* by $\mathfrak{J}$, if the $\mathfrak{I}$-adic topology is stronger than or equal to the $\mathfrak{J}$-adic topology, that is to say, for each fixed N, we have $I_n \subseteq J_N$ for all sufficiently big n.

5.9 Lemma (Chevalley). *A filtration on a Noetherian local ring $(R, \mathfrak{m})$ is analytic if and only if it is bounded by the $\mathfrak{m}$-adic filtration.*

Proof. If $\mathfrak{I} = (I_n)_n$ is analytic, then the intersection of all $I_n\widehat{R}$ is zero. By Chevalley's theorem (see for instance [29, Exercise 8.7]) we have for fixed N an inclusion $I_n\widehat{R} \subseteq \mathfrak{m}^N\widehat{R}$ for n sufficiently big. By faithful flatness, $I_n \subseteq \mathfrak{m}^N$ for $n \gg 0$. The converse is immediate from Krull's intersection theorem (see for instance [29, Theorem 8.10]). $\square$

5.10 Corollary. *If $(I_n)_n$ is an analytic filtration on a Noetherian local ring R, then the catapower $R_\sharp$ of R is isomorphic to the cataproduct $S_\sharp$ of the R/I_n.*

Proof. Without loss of generality, we may assume R is complete. The natural surjections $R \to R/I_n$ induce a map $R_\sharp \to S_\sharp$, which is again surjective by Łos' Theorem. Let $x_\natural$ be an element in the ultrapower $R_\natural$ of R so that its image in $R_\sharp$ is in the kernel of $R_\sharp \to S_\sharp$. Choose $x_n \in R$ with ultraproduct equal to $x_\natural$ and fix N. Since $x_\natural \in \mathfrak{I}_{S_\natural}$, almost each $x_n \in \mathfrak{m}^N(R/I_n)$. By Lemma 5.9, almost each $I_n \subseteq \mathfrak{m}^N$ and hence almost each $x_n \in \mathfrak{m}^N$. By Łos' Theorem, $x_\natural \in \mathfrak{m}^N R_\natural$. Since N was arbitrary, $x_\natural$ lies in $\mathfrak{I}_{R_\natural}$ and hence its image is zero in $R_\sharp$, showing that $R_\sharp \to S_\sharp$ is also injective. $\square$

It should be noted that the corresponding ultraproducts $R_\natural$ and $S_\natural$, however, are far from equal, as, for instance, $\text{FR-Spec}(S_\natural)$ is always a singleton by Proposition 5.1. Contrary to the Noetherian case, the natural map $R \to \widehat{R}$ does not need to be flat if R has finite embedding dimension. We nevertheless expect some vestige of (faithful) flatness to hold. One example of this is given by Lemma 2.4, namely $I = I\widehat{R} \cap R$ for any closed ideal I. It is well-known (see for instance [44, Theorem 2.2]) that the latter property already follows from the vanishing of $\operatorname{Tor}_1^R(\widehat{R}, k)$, where k is the residue field of R. For ultra-Noetherian local rings, where completion and separated quotient coincide by Lemma 5.6, this latter property does indeed hold:

5.11 Proposition. *For every ultra-Noetherian local ring $R_\natural$ with residue field $k_\natural$, we have $\operatorname{Tor}_1^{R_\natural}(R_\sharp, k_\natural) = 0$.*

Proof. From the exact sequence

$$0 \to \mathfrak{I}_{R_\natural} \to R_\natural \to R_\sharp \to 0$$

we get after tensoring over $k_\natural$ an exact sequence

$$0 \to \operatorname{Tor}_1^{R_\natural}(R_\sharp, k_\natural) \to \mathfrak{I}_{R_\natural}/\mathfrak{m}_\natural\mathfrak{I}_{R_\natural} \to k_\natural \to k_\natural \to 0,$$

where $\mathfrak{m}_\natural$ is the maximal ideal of $R_\natural$. In particular, the first Betti number of $R_\sharp$ vanishes if and only if $\mathfrak{m}_\natural\mathfrak{I}_{R_\natural} = \mathfrak{I}_{R_\natural}$. To prove the latter equality, let $(R_w, \mathfrak{m}_w)$ be Noetherian local rings with ultraproduct $R_\natural$. Let $a_\natural$ be a non-zero element in $\mathfrak{I}_{R_\natural}$ and choose non-zero $a_w \in R_w$ so that their ultraproduct is equal to $a_\natural$. Let $\mathfrak{m}_\natural$ be generated by $x_{1\natural}, \dots, x_{e\natural}$ and, for each i, choose $x_{iw} \in R_w$ whose ultraproduct equals $x_{i\natural}$. By Łos' Theorem, $\mathfrak{m}_w = (x_{1w}, \dots, x_{ew})R_w$. If a_w has order n_w, then we can find $b_{iw} \in R_w$ of order $n_w - 1$ such that $a_w = x_{1w}b_{1w} + \dots + x_{ew}b_{ew}$. Let $b_{i\natural}$ be the ultraproduct of the b_{iw}. Fix some N. Since $a_\natural \in \mathfrak{I}_{R_\natural}$, its order is strictly bigger than N and hence so is almost each n_w. Therefore, almost each b_{iw} has order at least N and hence $b_{i\natural} \in \mathfrak{m}_\natural^N$. Since this holds for all N, we get $b_{i\natural} \in \mathfrak{I}_{R_\natural}$. Since $a_\natural = x_{1\natural}b_{1\natural} + \dots + x_{e\natural}b_{e\natural}$ by Łos' Theorem, we are done. $\square$

5.12 Corollary. *Let $(R_\natural, \mathfrak{m}_\natural)$ be an ultra-Noetherian local ring and I an ideal in $R_\natural$. If I is closed, then so is $I\mathfrak{m}_\natural^n$ for every n.*

Proof. By Corollary 5.7, we have $\mathfrak{I}_{R_\natural} \subseteq I$. Since $\mathfrak{I}_{R_\natural} = \mathfrak{m}_\natural^n \mathfrak{I}_{R_\natural}$ by the proof of Proposition 5.11, we get $\mathfrak{I}_{R_\natural} \subseteq I\mathfrak{m}_\natural^n$, showing that $I\mathfrak{m}_\natural^n$ is closed by another application of Corollary 5.7. □

We may extend the notion of cataproduct to modules as well: for each w, let M_w be an R_w-module, and let $M_\natural$ be their ultraproduct. It follows that $M_\natural$ is an $R_\natural$-module. We define the *cataproduct* of the M_w as the $R_\sharp$-module $M_\sharp := M_\natural \otimes_{R_\natural} R_\sharp = M_\natural/\mathfrak{I}_{R_\natural} M_\natural$ given by base change. If $N_w \subseteq M_w$ are submodules, then $N_\natural \subseteq M_\natural$. However, the induced homomorphism $N_\sharp \to M_\sharp$ may fail to be injective. The following result is an exercise on Łos' Theorem (see for instance [36]), and the proof is left to the reader.

5.13 Proposition. *Let $M_\natural$ and $M_\sharp$ be the respective ultraproduct and cataproduct of the M_w. Almost each M_w is minimally generated by s elements (respectively, has length s), if and only if $M_\natural$ is minimally generated by s elements (respectively, has length s), if and only if so does $M_\natural$.* □

Flatness of catapowers

A key result about catapowers, one which will be used frequently in our characterizations through uniform behavior in §12, is the following theorem and its corollary:

5.14 Theorem. *Let R be a Noetherian local ring and $R_\sharp$ its catapower. There is a canonical homomorphism $R \to R_\sharp$ which is faithfully flat and unramified. In particular, R and $R_\sharp$ have the same dimension.*

Proof. Let $R_\natural$ be the ultrapower of R and $R \to R_\natural$ the diagonal embedding. Composed with the canonical surjection $R_\natural \to R_\sharp = R_\natural/\mathfrak{I}_{R_\natural}$, we get the map $R \to R_\sharp$. By Corollary 5.8 and the fact that completion is faithfully flat, we may already assume that R is complete. Since $\mathfrak{m}R_\sharp$ is the maximal ideal of $R_\sharp$, the map $R \to R_\sharp$ is unramified. So remains to show that this map is flat. Let us first prove this under the additional assumption that R is regular. We induct on its dimension. Let x be a regular parameter of R, that is to say, an element of order one. I claim that x is $R_\sharp$-regular. This follows for instance from the results in §8 (proving among other things that $R_\sharp$ is then regular), but we can give a direct argument here. Indeed, suppose $s_\natural \in R_\natural$ is such that $xs_\natural \in \mathfrak{I}_{R_\natural}$. If $s_w \in R$ have ultraproduct equal to $s_\natural$, then for a fixed N, almost each $xs_w \in \mathfrak{m}^N$. Since R is regular and x has order one, $s_w \in \mathfrak{m}^{N-1}$ and hence by Łos' Theorem, $s_\natural \in \mathfrak{m}^{N-1}R_\natural$. Since this holds for all N, we get $s_\natural \in \mathfrak{I}_{R_\natural}$, showing that x is $R_\sharp$-regular. It is not hard to see that $R_\sharp/xR_\sharp$ is the catapower of the regular local ring R/xR, so that by induction, $R/xR \to R_\sharp/xR_\sharp$ is faithfully flat. Since any R/xR-regular sequence is then $R_\sharp/xR_\sharp$-regular, $R_\sharp$ is a balanced big Cohen-Macaulay algebra over R. Since R is regular, $R \to R_\sharp$ is therefore faithfully flat (see for instance [42, Theorem IV.1] or [25, Lemma 2.1(d)]).

For the general case, we may write R as a homomorphic image S/I of a complete regular local ring S by Cohen's theorem. By what we just proved, $S \to S_\sharp$ is faithfully flat, where $S_\sharp$ is the catapower of S. Hence the base change $R = S/I \to R_\sharp = S_\sharp/IS_\sharp$ is also flat. The equality of dimension is now a well-known consequence of the first assertion. ☐

5.15 Corollary. *Let R be an excellent local ring (e.g., a complete Noetherian local ring) with catapower $R_\sharp$. The natural map $R \to R_\sharp$ is regular. In particular, R is regular (respectively, normal, reduced, Cohen-Macaulay or Gorenstein), if and only if, so is $R_\sharp$.*

Proof. The second assertion is a well-known consequence of the first (see for instance [29, Theorem 32.2]). As for the first, let us first show this in the special case that $R = k$ is a field. Note that in this case, the catapower is equal to the ultrapower $k_\natural$ of k. Hence, we need to show that $k \to k_\natural$ is separable, and so we may assume that k has positive characteristic p. We will establish separatedness by verifying MacLane's criterion (see for instance [29, Theorem 26.4]). Let $b_1, \dots, b_n$ be elements in $k^{1/p}$ which are linearly independent over k. Suppose $x_{1\natural}b_1 + \cdots + x_{n\natural}b_n = 0$ for some $x_{i\natural} \in k_\natural$. Choose $x_{iw} \in k$ with ultraproduct equal to $x_{i\natural} \in k_\natural$. Taking p-th powers, using Łos' Theorem and then taking p-th roots, we get $x_{1w}b_1 + \cdots + x_{nw}b_n = 0$ for almost all w. Since the b_i are linearly independent over k, almost all x_{iw} are zero. By Łos' Theorem, each $x_{i\natural}$ is zero, showing that the b_i, viewed as elements in $k_\natural^{1/p}$, remain linearly independent over $k_\natural$, as we wanted to show.

For R arbitrary, Theorem 5.14 yields that $R \to R_\sharp$ is faithfully flat and unramified. By what we just proved, the induced residue field extension is separable. Therefore $R \to R_\sharp$ is formally smooth by [29, Theorem 28.10]. Regularity then follows from a result by André in [2] (see also [29, p. 260]). ☐

5.16 Corollary. *If $(R, \mathfrak{m})$ is an equicharacteristic complete Noetherian local ring with residue field k, then its catapower $R_\sharp$ is isomorphic to the $\mathfrak{m}$-adic completion of $R \otimes_k k_\natural$, where $k_\natural$ is the ultrapower of k.*

Proof. By Cohen's structure theorem, R is a homomorphic image of a power series ring $k[[x]]$, with x an n-tuple of indeterminates. Since catapowers commute with homomorphic images by Corollary 5.7, we may assume $R = k[[x]]$. So remains to show that $R_\sharp \cong k_\sharp[[x]]$. However, this is clear by Cohen's structure theorem, since $R_\sharp$ is regular by Corollary 5.15, with residue field $k_\sharp$, and dimension n, by Theorem 5.14. ☐

5.17 Proposition. *Let $R \subseteq S$ be an injective, local homomorphism between Noetherian local rings and let $R_\sharp \to S_\sharp$ be the induced map of catapowers.*

5.17.1. *If $R \subseteq S$ is finite, then $R_\sharp \to S_\sharp$ is finite and injective.*

5.17.2. *If $R \subseteq S$ is cata-injective, that is to say, if $\widehat{R} \to \widehat{S}$ is injective, then $R_\sharp \to S_\sharp$ is injective too.*

Proof. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals of respectively R and S. Assume $R \subseteq S$ is finite, so that $\mathfrak{n}^a \subseteq \mathfrak{m}S$ for some a. By the Artin-Rees Lemma, $\mathfrak{m}^n S \cap R \subseteq \mathfrak{m}^{n-c}$ for some c and all $n \geq c$. Hence $\mathfrak{n}^{na} \cap R \subseteq \mathfrak{m}^{n-c}$ for all $n \geq c$ and hence by Łos' Theorem, the same inclusions hold in the extension $R_\natural \subseteq S_\natural$ of ultrapowers. Using this, it is not hard to show that $\mathfrak{I}_{S_\natural} \cap R_\natural = \mathfrak{I}_{R_\natural}$, showing that $R_\sharp \subseteq S_\sharp$ is injective (and clearly also finite).

If $R \subseteq S$ is cata-injective, then the filtration $\mathfrak{n}^k \cap R$, for $k = 0, 1, \dots$, is easily seen to be analytic, whence bounded by the $\mathfrak{m}$-adic filtration by Lemma 5.9. Again one derives from this that $\mathfrak{I}_{S_\natural} \cap R_\natural = \mathfrak{I}_{R_\natural}$, whence that $R_\sharp \subseteq S_\sharp$ is injective. □

5.18. Extended dimensions in ultra-Noetherian local rings

We extend the nomenclature introduced in the beginning of this section to include invariants. In particular, we define the *cata-dimension* of R, denoted $\operatorname{cdim}(R)$, as the (Krull) dimension of its completion $\widehat{R}$. For an ultra-Noetherian local ring $R_\natural$ given as the ultraproduct of Noetherian local rings R_w of embedding dimension at most m, we define its *ultra-dimension*, denoted $\operatorname{udim}(R_\natural)$, as the dimension of almost all R_w. Since almost all R_w have dimension at most m, the ultra-dimension of $R_\natural$ is finite.

5.19 Theorem. *For an ultra-Noetherian local ring $R_\natural$, we have inequalities*

$$\begin{aligned}\operatorname{depth}(R_\natural) \le \operatorname{pidim}(R_\natural) = \operatorname{frdim}(R_\natural) = \operatorname{udim}(R_\natural)\\ \le \operatorname{cldim}(R_\natural) = \operatorname{gdim}(R_\natural) = \operatorname{cdim}(R_\natural) \le \operatorname{embdim}(R_\natural).\end{aligned} \tag{4}$$

Proof. By Theorems 3.4 and 4.4, the cata-dimension of $R_\natural$ is equal to its geometric dimension and to its cl-dimension. Łos' Theorem and Lemma 4.7 yield that the ultra-dimension of $R_\natural$ coincides with its pi-dimension and its fr-dimension. Depth is also first-order, as it is cast in terms of the vanishing of the Koszul homology of a generating set of $\mathfrak{m}$ (see §7.1 below for more details). Since in a Noetherian local ring depth never exceeds dimension, the first inequality is then also clear. □

There are no further constraints on the above invariants of an ultra-Noetherian ring, as the following examples show (in the discussion of these examples, we will also use some terminology from later sections).[7]

5.20 Example. Let $e \le h \le d \le m$. We will construct an ultra-Noetherian local ring $R_\natural$ with depth e, ultra-dimension h, cata-dimension d, and embedding dimension m. First we introduce some notation. Let $R_\natural$ be the ultraproduct of the R_w and let $n_\natural$ be a non-standard positive integer, that is to say, an ultraproduct of an unbounded sequence of positive integers n_w. For an element $a_\natural \in R_\natural$, realized as an ultraproduct of elements $a_w \in R_w$, we write $a_\natural^{n_\natural}$ to denote the ultraproduct of the elements $a_w^{n_w}$; one verifies that this is independent of the choice of a_w or n_w. Let $S_\natural$ be the ultrapower of $S := k[[\xi]]$, for some indeterminates $\xi := (\xi_1, \dots, \xi_m)$ and some field k, let

$$I := (\xi_{e+1}^{n_\natural}\xi_m, \dots, \xi_h^{n_\natural}\xi_m, \xi_{h+1}^{n_\natural}, \dots, \xi_d^{n_\natural}, \xi_{d+1}^2, \dots, \xi_m^2)S_\natural$$

and put $R_\natural := S_\natural/I$. By Łos' Theorem, $(\xi_1, \dots, \xi_e)$ is $R_\natural$-regular and since the maximal ideal of $R_\natural/(\xi_1, \dots, \xi_e)R_\natural$ is annihilated by the element $\xi_{e+1}^{n_\natural-1} \cdots \xi_d^{n_\natural-1}$.

[7] One should note that for Noetherian rings, other than the obvious restriction that pi-dimension and dimension agree, we also have the remarkable fact that when dimension and embedding dimension agree, that is to say, when the ring is regular, then this common value must also be equal to its depth.

$\xi_{d+1}\cdots\xi_m$, we see that $R_\natural$ has depth e. Since $\xi_{d+1},\dots,\xi_m$ are nilpotent, we get from Proposition 5.22 below that the ultra-dimension of $R_\natural$ is the same as the ultra-dimension of

$$R_\natural/(\xi_{d+1},\dots,\xi_m)R_\natural = S_\natural/(\xi_{h+1}^{n_\natural},\dots,\xi_d^{n_\natural},\xi_{d+1},\dots,\xi_m)S_\natural,$$

that is to say, equal to h. On the other hand, $IS_\sharp = (\xi_{d+1}^2,\dots,\xi_m^2)S_\sharp$, where $S_\sharp$ is the catapower of S (note that $S_\sharp \cong k_\natural[[\xi]]$, where $k_\natural$ is the ultrapower of k; see for instance [4, Proposition 3.1]). Hence the catapower $R_\sharp$ of R has dimension d. By Lemma 5.6, the cata-dimension of $R_\natural$ is therefore d. Finally, it follows from Łos' Theorem that $R_\natural$ has embedding dimension m. Note that since $R_\sharp$ is Cohen-Macaulay, $R_\natural$ is cata-Cohen-Macaulay (see §6.8).

More generally, let q be any number between e and d and let $R'_\natural := S_\natural/I'$, where I' is the sum of the ideal I above and the ideal $(\xi_{q+1}\xi_m,\dots,\xi_d\xi_m)S_\natural$. Then $R'_\natural$ has still the same depth, ultra-dimension, cata-dimension and embedding dimension as $R_\natural$, but now the depth of $R_\sharp$, that is to say, the *cata-depth* of $R_\natural$, is q, since $(\xi_1,\dots,\xi_q)$ is a regular sequence.

5.21 Example. The previous example might one lead to think that the depth of R is always at most its cata-depth. However, this is not the case as the following example shows. Let $S_\natural$ be as in the previous example with $m = 3$, and let $R_\natural := S_\natural/(\xi_1^2,\xi_1\xi_2,\xi_1\xi_3-\xi_2^{n_\natural})S_\natural$, with $n_\natural$ a non-standard positive integer. Since ξ_3 is $R_\natural$-regular and since $R_\natural$ has ultra-dimension one, the depth of $R_\natural$ is one by Theorem 5.19. On the other hand, $R_\sharp$ is equal to $S_\sharp/(\xi_1^2,\xi_1\xi_2,\xi_1\xi_3)S_\sharp$, whence has depth zero. Note that $R_\sharp$ has dimension two, so that $R_\natural$ itself has cata-dimension two. Hence $R_\natural$ is ultra-Cohen-Macaulay(see §8.5), but not cata-Cohen-Macaulay.

Isodimensionality

We call a local ring R of finite embedding dimension *isodimensional* if (4.4.2) is an equality, that is to say, if the geometric dimension of R is equal to its pi-dimension. In view of Theorem 5.19, an ultra-Noetherian local ring is isodimensional if and only if its ultra-dimension is equal to its cata-dimension.

5.22 Proposition. *Let $R_\natural$ be an ultra-Noetherian local ring. If $\mathfrak{a}$ is a finitely related ideal contained in $\mathrm{nil}(R_\natural)$, then $R_\natural$ and $R_\natural/\mathfrak{a}$ have the same ultra-dimension. In particular, $R_\natural$ is isodimensional if and only if $R_\natural/\mathfrak{a}$ is.*

Proof. Let h be the ultra-dimension of $R_\natural$, and write it as the ultraproduct of h-dimensional Noetherian local rings R_w of bounded embedding dimension. Since $\mathfrak{a}$ is finitely related, it can be realized as the ultraproduct of finitely related ideals $\mathfrak{a}_w$ by the argument in the proof of Proposition 5.1. By Łos' Theorem, almost each $\mathfrak{a}_w$ is nilpotent, and therefore $R_w/\mathfrak{a}_w$ has again dimension h. Hence $R_\natural/\mathfrak{a}$ has ultra-dimension h as well.

The final assertion follows from the fact that $R_\natural$ and $R_\natural/\mathfrak{a}$ have the same geometric dimension (this is true in general, since $\mathfrak{a}$ is contained in every threshold prime of $R_\natural$). □

For ultra-Noetherian local rings, we have the following important criterion for isodimensionality:

5.23 Theorem. *Let $R_\natural$ and $R_\sharp$ be the respective ultraproduct and cataproduct of Noetherian local rings R_w of bounded embedding dimension. The following are equivalent:*

5.23.1. *$R_\natural$ is isodimensional;*

5.23.2. *almost all R_w have dimension equal to $\operatorname{gdim}(R_\natural)$;*

5.23.3. *almost all R_w have the same dimension as $R_\sharp$;*

5.23.4. *almost all R_w have the same parameter degree (which is then also the parameter degree of $R_\natural$ and of $R_\sharp$).*

Proof. The equivalence of (5.23.2) and (5.23.3) follows from Lemma 5.6 and Theorem 3.4. Let $d \leq m$ be the respective geometric dimension and embedding dimension of $R_\natural$. By Theorem 5.19, the cata-dimension of $R_\natural$ is d. Since $\dim(R_w) \leq m$, almost all R_w have a common dimension $h \leq m$, which is then the ultra-dimension of $R_\natural$ by definition, from which we get the equivalence of (5.23.1) and (5.23.2).

Remains to show the equivalence of (5.23.2) and (5.23.4). Suppose $\operatorname{pardeg}(R_w) = e$ for almost all w. In each R_w, choose an h-tuple $\mathbf{x}_w$ so that almost all $R_w/\mathbf{x}_w R_w$ have length e. Let $\mathbf{x}_\natural$ be the ultraproduct of the $\mathbf{x}_w$. By Proposition 5.13, the length of $R_\natural/\mathbf{x}_\natural R_\natural$, being the ultraproduct of the $R_w/\mathbf{x}_w R_w$, is also e. It follows that $R_\natural$ has geometric dimension at most h. We already argued that its geometric dimension is at least h, so that we get $h = d$. In particular, the parameter degree of R is at most e, and by reversing this argument, one can also show that it cannot be less than e, whence must be equal to e.

Conversely, assume $h = d$. Let $\mathbf{x}_\natural$ be a generic sequence in $R_\natural$ and choose d-tuples $\mathbf{x}_w$ whose ultraproduct is $\mathbf{x}_\natural$. By Łos' Theorem, almost each $\mathbf{x}_w$ generates an $\mathfrak{m}_w$-primary ideal, and therefore must be a system of parameters in R_w, since almost each R_w has dimension $h = d$. Let l be the length of $R_\natural/\mathbf{x}_\natural R_\natural$. By Proposition 5.13, almost each $R_w/\mathbf{x}_w R_w$ has length l, showing that $\operatorname{pardeg}(R_w) \leq l$, for almost w. □

5.24 Example. We cannot replace parameter degree with multiplicity in the previous result as the following example shows. Fix some $e > 0$ and put

$$R_w := S/(\xi^w, \xi^e \zeta^{w-e})S$$

for each $w \geq e$, where $S := k[[\xi, \zeta]]$ and k is a field. Let $R_\natural$ be the ultraproduct of the R_w, let $k_\natural$ be the ultrapower of k and let $S_\sharp \cong k_\natural[[\xi, \zeta]]$ be the catapower of S. Since the ultraproduct of the ξ^w and the $\xi^e \zeta^{w-e}$ are infinitesimals, the cataproduct of the R_w is $R_\sharp = S_\sharp$, showing that $R_\natural$ is not isodimensional (since the R_w are one-dimensional and $R_\sharp$ is two-dimensional). Therefore, by the theorem, the parameter degree of the R_w is unbounded (in fact, equal to w). On the other hand, ζ is a parameter in each R_w so that we can calculate the multiplicity of R_w by Lech's lemma ([29, Theorem 14.12]) as the limit of e_{wn}/n as n tends to infinity, where e_{wn} is the length of $R_w/\zeta^n R_w$. One calculates that $e_{wn} = w(w-1) + e(n-w+2)$ and hence $\operatorname{mult}(R_w) = e$. This shows, in view of Remark 3.5, that multiplicity is in general not first-order.

5.25 Remark. In view of Theorem 5.23, we will often require that a collection of Noetherian local rings R_w have (almost all) the same embedding dimension and the same parameter degree, to ensure that their cataproduct is again Noetherian of the same dimension. In fact, we can replace this requirement with the more natural requirement that (almost all) R_w have the same dimension and parameter degree. Indeed, if a Noetherian local ring R has dimension d and parameter degree e, then its embedding dimension is at most $d + e - 1$.

Note that by Lemma 6.11 below, if almost all R_w are Cohen-Macaulay we may further simplify this to the requirement that almost all R_w have the same dimension and multiplicity. The previous example shows that this is no longer true without the Cohen-Macaulay assumption.

5.26 Corollary. *If $R_\natural$ is an isodimensional ultra-Noetherian local ring and $x_\natural$ the ultraproduct of elements x_w, then $x_\natural$ is generic if and only if $\deg(x_w)$ is bounded.*

Proof. Let R_w be Noetherian local rings with ultraproduct $R_\natural$. By Theorem 5.23, almost each R_w has dimension $d := \operatorname{gdim}(R_\natural)$. Suppose $x_\natural$ is generic. Hence, $R_\natural/x_\natural R_\natural$ has geometric dimension $d - 1$, whence ultra-dimension at most $d - 1$. In particular, almost each $R_w/x_w R_w$ must have dimension $d - 1$. Hence x_w is generic in R_w and $R_\natural/x_\natural R_\natural$ is again isodimensional. By Theorem 5.23, this means that the $R_w/x_w R_w$ must have bounded parameter degree, proving the direct implication.

Conversely, suppose the $\deg(x_w)$ are bounded, that is to say, almost all x_w are generic and the parameter degrees of the $R_w/x_w R_w$ are bounded. By Theorem 5.23 once more, $R_\natural/x_\natural R_\natural$ has geometric dimension $d - 1$, showing that $x_\natural$ is generic. □

Without the isodimensional assumption, the result is false: for instance if $R_\natural$ has ultra-dimension zero (e.g., the ultraproduct of the $R/\mathfrak{m}^n$), then no element in $R_\natural$ is realized as an ultraproduct of elements of finite degree. We can now give the following improvement of Corollary 5.3.

5.27 Corollary. *Let $R_\natural$ be an ultra-Noetherian local ring, realized as the ultraproduct of equicharacteristic excellent local rings R_w. If $R_\natural$ is isodimensional, then any localization at a finitely related prime ideal has finite embedding dimension, and any finitely related prime ideal is strong.*

Proof. Let $\mathfrak{p} \in \text{FR-Spec}(R_\natural)$. By Proposition 5.1, there exist prime ideals $\mathfrak{p}_w \subseteq R_w$ with ultraproduct equal to $\mathfrak{p}$. By Theorem 5.23, there is some ρ, such that almost each R_w has parameter degree ρ. Hence, by Proposition 5.2, almost each $(R_w)_{\mathfrak{p}_w}$ has embedding dimension at most $d + \rho$, where d is the common dimension of almost all R_w (that is to say, the ultra-dimension, whence geometric dimension, of $R_\natural$). Since $(R_\natural)_\mathfrak{p}$ is the ultraproduct of the $(R_w)_{\mathfrak{p}_w}$, its embedding dimension is at most $d + \rho$. Proposition 5.4 then implies that $\mathfrak{p}$ is strong. □

We actually showed that the stalk of $\operatorname{Spec}(R_\natural)$ at a point in $\text{FR-Spec}(R_\natural)$ has embedding dimension at most $d + \rho$, where d is the geometric dimension of $R_\natural$ and ρ its parameter degree. Inspecting the proof of Proposition 5.2, we see that almost each $(R_w)_{\mathfrak{p}_w}$ has parameter degree at most ρ, showing that each stalk is also isodimensional, of ultra-dimension, whence geometric dimension, at most d.

6. Cata-singularities

According to the definitions in §5, a local ring of finite embedding dimension is *cata-regular* if its completion is a regular (Noetherian) local ring.

6.1 Theorem. *Let $(R,\mathfrak{m})$ be a local ring of geometric dimension d and let k be its residue field. The following are equivalent:*

6.1.1. *R is cata-regular;*
6.1.2. *R is cata-regular;*
6.1.3. $\operatorname{gdim}(R) = \operatorname{embdim}(R)$*;*
6.1.4. *$\mathfrak{m}$ is generated by a generic sequence;*
6.1.5. *$\mathfrak{m}$ is generated by a quasi-regular sequence;*
6.1.6. *$\operatorname{gr}(R)$ is isomorphic to $k[\xi]$, with ξ a d-tuple of indeterminates.*

Proof. The equivalence of (6.1.1) and (6.1.2) is clear since R has the same completion as R, and their equivalence with (6.1.6) follows from [29, Theorem 14.4], since we have an isomorphism of graded rings $\operatorname{gr}(R) \cong \operatorname{gr}(\widehat{R})$. The equivalence of (6.1.3) and (6.1.4) is clear from the definition of geometric dimension. Suppose (6.1.4) holds, so that $\mathfrak{m}$ is generated by a generic sequence $(x_1,\dots,x_d)$. There is a natural surjective homomorphism $k[\xi] \to \operatorname{gr}(R)$ which maps ξ_i to $\operatorname{in}(x_i)$, where $\xi = (\xi_1,\dots,\xi_d)$. Since both rings have the same dimension by Theorem 3.4, the kernel must be zero, proving (6.1.6). Conversely, assume $\operatorname{gr}(R) \cong k[\xi]$. Hence $\mathfrak{m}/\mathfrak{m}^2$ is generated by d elements, and therefore, by Nakayama's Lemma $\mathfrak{m}$ is generated by d elements, showing that (6.1.4) holds.

Remains to show the equivalence of the other conditions with (6.1.5). Recall that $\mathbf{x}$ is quasi-regular if $F(\mathbf{x}) = 0$, for a homogeneous polynomial $F \in R[\xi]$, implies that F has all its coefficients in $I := \mathbf{x}R$. This is equivalent with the natural epimorphism $(R/I)[\xi_1,\dots,\xi_d] \to \operatorname{gr}_I(R)$ being injective, whence an isomorphism (see for instance [29, §16]). Hence taking $I = \mathfrak{m}$, we see that (6.1.5) is equivalent with (6.1.6). $\square$

6.2 Remark. In the above proof, we actually showed that if R is cata-regular of geometric dimension d, then any d-tuple generating $\mathfrak{m}$ is quasi-regular. We will shortly show (Theorem 6.9 below) that, in fact, every generic sequence is quasi-regular. The ring R in the next example shows that a generic sequence generating the maximal ideal in a cata-regular local ring is not necessarily a regular sequence.

6.3 Example. A local ring of geometric dimension zero is cata-regular if and only if it is a field. A local ring of geometric dimension one is cata-regular if and only if its maximal ideal is generated by a non-nilpotent element. For instance, let $V_\natural$ be an ultraproduct of discrete valuation rings (an *ultra-DVR* for short), or more generally, a valuation ring of finite embedding dimension (which is then automatically one). If x is an element in the ideal of infinitesimals $\mathfrak{I}_{V_\natural}$ of $V_\natural$, then $R := V_\natural/xV_\natural$ is cata-regular of geometric dimension one. If $x \neq 0$, then R is not a domain. In fact, R has then depth zero (and so is not pseudo-regular in the sense of §7.6 below).

The following fact, however, is noteworthy: if R is moreover separated, then any quasi-regular element is regular; see for instance [29, Theorem 16.3]. In fact, we have the following result:

6.4 Corollary. *If a cata-regular local ring is separated, then it is a domain. More generally, the separated quotient of a cata-regular local ring is a domain.*

Proof. Immediate from the fact that R embeds in $\widehat{R}$ and the fact that Noetherian regular local rings are always domains. □

6.5 Corollary. *If R is cata-regular, then so is any homomorphic image R/I, for $I \subseteq \mathfrak{I}_R$.*

Proof. Since R and R/I have the same separated quotient, the result follows from Theorem 6.1. □

6.6 Corollary. *For each d, the class of cata-regular local rings of geometric dimension d is first-order definable.*

Proof. Observe that a ring is local if and only if any sum of two non-units is again a non-unit. In fact, an element lies in the maximal ideal of a local ring if and only if it is not a unit. Therefore, the maximal ideal of a local ring is definable, as is expressing that some element lies in the maximal ideal. In particular, the formula $\lambda_{d,n}(\mathbf{x}, \mathbf{a})$ is first order, where $\lambda_{d,n}(\mathbf{x}, \mathbf{a})$ is the formula in the variables $\mathbf{x} := (x_1, \dots, x_d)$ and $\mathbf{a} := (a_\nu)_\nu$, for ν running over all d-tuples in $\mathbb{N}^d$ whose sum $|\nu|$ is equal to n, expressing that

$$\begin{aligned} &\textit{if } \mathbf{x} \textit{ generates the maximal ideal and if } \sum_{|\nu|=n} a_\nu \mathbf{x}^\nu = 0, \\ &\textit{then some } a_\nu \textit{ lies in the maximal ideal.} \end{aligned} \tag{5}$$

Let T_d be the theory consisting of all sentences $(\forall \mathbf{x}, \forall \mathbf{a})\lambda_{d,n}(\mathbf{x}, \mathbf{a})$, for $n = 1, 2, \dots,$ together with the sentence σ_d expressing that the maximal ideal is generated by some d-tuple. I claim that T_d axiomatizes the class of cata-regular local rings of geometric dimension d. Indeed, suppose that $(R, \mathfrak{m})$ satisfies T_d. By σ_d, there is a d-tuple $\mathbf{x}$ such that $\mathfrak{m} = \mathbf{x}R$. Since $\lambda_{d,n}(\mathbf{x}, \mathbf{a})$ holds for all tuples $\mathbf{a}$ in R, we see that $\mathbf{x}$ is quasi-regular. Hence R is cata-regular by Theorem 6.1. Conversely, if R is cata-regular of geometric dimension d, then it satisfies T_d by Remark 6.2. □

This immediately gives a large class of cata-regular local rings. Namely, any ultra-product of regular local rings of dimension d is cata-regular, of geometric dimension d. We will address this situation further in §8 below.

6.7 Corollary. *A local ring R of geometric dimension one is cata-regular if and only if R is a discrete valuation ring.*

Proof. Assume R is cata-regular so that $\widehat{R}$ is a discrete valuation ring with valuation $\operatorname{ord}_{\widehat{R}}(\cdot)$. Since $\operatorname{ord}_R(a) = \operatorname{ord}_{\widehat{R}}(a)$ for all $a \in R$, also $\operatorname{ord}_R(\cdot)$ is a valuation, showing that R is a discrete valuation ring. Conversely, if R is a discrete valuation ring, then R is cata-regular by Theorem 6.1. □

6.8. Cata-Cohen-Macaulay local rings

We now turn to the study of cata-Cohen-Macaulay local rings of finite embedding dimension, that is to say, local rings whose completion is Cohen-Macaulay. Clearly, any cata-regular local ring is cata-Cohen-Macaulay.

6.9 Theorem. *A local ring of finite embedding dimension is cata-Cohen-Macaulay if and only if its separated quotient is cata-Cohen-Macaulay if and only if some (equivalently, every) generic sequence is quasi-regular.*

Proof. Let $(R, \mathfrak{m})$ be a local ring of geometric dimension d and let $\mathbf{x}$ be a generic sequence. Since $\mathrm{gr}_{\mathbf{x}R}(R) \cong \mathrm{gr}_{\mathbf{x}\widehat{R}}(\widehat{R})$, the sequence $\mathbf{x}$ is quasi-regular in R if and only if it is so in $\widehat{R}$. Since R and R have the same completion, we only need to show the equivalence of the first and last condition. Assume the last condition, and choose a generic tuple $\mathbf{x}$. By our previous observation, $\mathbf{x}$ is $\widehat{R}$-quasi-regular, and therefore $\widehat{R}$-regular by [29, Theorem 16.3] and the fact that $\widehat{R}$ is Noetherian. Since $\widehat{R}$ has dimension d by Theorem 3.4, it is Cohen-Macaulay, showing that R is cata-Cohen-Macaulay.

Conversely, suppose $\widehat{R}$ is Cohen-Macaulay, and let $\mathbf{x}$ be any generic tuple. Since $\mathbf{x}$ is a system of parameters in $\widehat{R}$ by Proposition 3.9, it is $\widehat{R}$-regular, whence $\widehat{R}$-quasi-regular. By the above, $\mathbf{x}$ is then quasi-regular in R. □

6.10 Corollary. *A local ring of finite embedding dimension is cata-regular if and only if it is cata-Cohen-Macaulay and has multiplicity one.*

Proof. If a local ring R is cata-regular, its completion $\widehat{R}$ is regular, whence has multiplicity one. Since R and its completion $\widehat{R}$ have the same multiplicity by Remark 3.5, the direct implication is clear. Conversely, if R is cata-Cohen-Macaulay of multiplicity one, then $\widehat{R}$ is Cohen-Macaulay with $\mathrm{mult}(\widehat{R}) = 1$ by Remark 3.5. Since $\widehat{R}$ is unmixed, it is regular by [30, Theorem 40.6], showing that R is cata-regular. □

6.11 Lemma. *The multiplicity of R is at most its parameter degree. If R has infinite residue field then we have equality if and only if R is cata-Cohen-Macaulay.*

Proof. Let $\mathbf{x}$ be a generic sequence of R. By Proposition 3.9, it is a system of parameters in $\widehat{R}$ and $R/\mathbf{x}R \cong \widehat{R}/\mathbf{x}\widehat{R}$ by Lemma 2.4. The common length of the latter two quotients is at least the multiplicity of the ideal $\mathbf{x}\widehat{R}$ by [29, Theorem 14.10] which in turn is at most $\mathrm{mult}(\widehat{R})$ by [29, Formula 14.4]. The desired inequality now follows from this, since R and $\widehat{R}$ have the same multiplicity by Remark 3.5.

The last assertion holds if R is Noetherian by [46, Lemma 3.3]. The general case follows from this since R and $\widehat{R}$ have the same multiplicity and the same parameter degree. □

6.12 Theorem. *A local ring of finite embedding dimension is cata-Gorenstein (respectively, a cata-'complete intersection') if and only if so is its separated quotient, if and only if it admits a quasi-regular, generic sequence $\mathbf{x}$ such that $R/\mathbf{x}R$ is Gorenstein (respectively, a complete intersection).*

Proof. Let $(R, \mathfrak{m})$ be a local ring of geometric dimension d. Since R and R have the same completion, we only need to show the equivalence of the first and last condition. Suppose $\mathbf{x}$ is a quasi-regular, generic sequence. In particular, R is cata-Cohen-Macaulay by Theorem 6.9, whence $\widehat{R}$ is Cohen-Macaulay and $\mathbf{x}$ is $\widehat{R}$-regular. Moreover, $R/\mathbf{x}R \cong \widehat{R}/\mathbf{x}\widehat{R}$ by Lemma 2.4. Therefore the former is Gorenstein (respectively, a complete intersection) if and only if the latter is, if and only if $\widehat{R}$ is (see [9, Theorem 2.3.4 and Proposition 3.1.19]). ☐

6.13 Proposition. *A local ring of finite embedding dimension is cata-Gorenstein if and only if there exists a quasi-regular, generic sequence generating an irreducible ideal. When this is the case, every generic sequence is quasi-regular and generates an irreducible ideal.*

Proof. Let $\mathbf{x}$ be a quasi-regular, generic sequence. The result is now immediate from the fact that $\mathbf{x}R$ is irreducible if and only if $R/\mathbf{x}R$ is Gorenstein. ☐

7. Pseudo-singularities

The cata-singularities from the previous section do not always correspond to their 'ultra' versions (which will be treated in the next section). To this end we will define some stronger versions of these cata-singularities, defined intrinsically, that is to say, without reference to the completion. Throughout this section, $(R, \mathfrak{m})$ is a local ring of finite embedding dimension.

7.1. Grade and depth.

Let A be an arbitrary ring and I a finitely generated ideal in A. Choose a tuple of generators $\mathbf{x} = (x_1, \dots, x_n)$ of I. The *grade* of I, denoted $\operatorname{grade}(I)$, is by definition equal to $n - h$, where h is the largest value i for which the i-th Koszul homology $H_i(\mathbf{x}; A)$ is non-zero. One shows that the grade of I does not depend on the choice of generators $\mathbf{x}$. For a local ring R of finite embedding dimension, we define its *depth* as the grade of its maximal ideal; it is non-zero if and only if its maximal ideal is not an associated prime.

Grade, and hence depth, *deforms well*, in the sense that the

$$\operatorname{grade}(I(A/\mathbf{x}A)) = \operatorname{grade}(I) - |\mathbf{x}| \tag{6}$$

for every A-regular sequence $\mathbf{x}$ contained in I. If R has geometric dimension d, then its depth is at most d. Indeed, by definition, the grade of a finitely generated ideal never exceeds its minimal number of generators, and by [9, Proposition 9.1.3], the depth of R is equal to the grade of any of its $\mathfrak{m}$-primary ideals, and the assertion now follows from Theorem 3.4.

The relationship between depth and the length of a regular sequence (sometimes called the *naive depth* of R) is less straightforward in the non-Noetherian case and requires an additional definition. For a local ring $(R, \mathfrak{m})$ and a finite tuple of indeterminates $\xi := (\xi_1, \dots, \xi_n)$, we will denote the localization of $R[\xi]$ at the ideal $\mathfrak{m}R[\xi]$ by $R(\xi)$ (this is sometimes called the n-fold *Nagata extension* of R). It follows that

$R \to R(\xi)$ is faithfully flat and unramified, with closed fiber equal to the residue field extension $k \subseteq k(\xi)$, where k is the residue field of R and $k(\xi)$ the field of fractions of $k[\xi]$.

7.2 Lemma. *Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension and let ξ be a tuple of indeterminates. Then R and $R(\xi)$ have the same geometric dimension and the same depth.*

Proof. Let d be the geometric dimension of R and e its depth. We will induct on d to show that $\operatorname{gdim}(R(\xi)) = d$. It is easy to see that R is Artinian if and only if $R(\xi)$ is, thus proving the case $d = 0$. In the general case, we may choose $x \in \mathfrak{m}$ so that $\operatorname{gdim}(R/xR) = d - 1$. By induction, $(R/xR)(\xi) \cong R(\xi)/xR(\xi)$ has geometric dimension $d - 1$, showing that $\operatorname{gdim}(R(\xi)) \leq d$. On the other hand, induction also shows that $\operatorname{gdim}(R(\xi)) > d - 1$, so that we get $\operatorname{gdim}(R(\xi)) = d$, as required.

As for depth, this follows from [9, Proposition 9.1.2] since $R \to R(\xi)$ is faithfully flat. □

We can now characterize depth in terms of regular sequences:

7.3 Lemma. *For a local ring R of finite embedding dimension, its depth is equal to the maximal length of an $R(\xi)$-regular sequence, where ξ runs over all finite tuples of indeterminates. More precisely, if R has depth e, then we can find a regular sequence $(y_1, \dots, y_e)$ in $R(\xi_1, \dots, \xi_e)$ which is part of a generic sequence.*

Proof. In view of Lemma 7.2, it suffices to prove the second assertion. To this end, we need to construct, by Lemma 3.8, an $R(\xi)$-regular sequence $(y_1, \dots, y_e)$ such that the geometric dimension of $R(\xi)/(y_1, \dots, y_e)R(\xi)$ is $d - e$, where $\xi := (\xi_1, \dots, \xi_e)$. We induct on the depth e of R, where there is nothing to show if $e = 0$. Let $(x_1, \dots, x_d)$ be a generic sequence and let $\mathfrak{n}$ be the ideal generated by this sequence. Since $\mathfrak{n}$ is then $\mathfrak{m}$-primary, its grade is e. By [9, Proposition 9.1.3], the element

$$y_1 := x_1 + x_2\xi_1 + \cdots + x_d\xi_1^{d-1}$$

is an $R[\xi_1]$-regular element. Since $R[\xi_1] \to R(\xi_1)$ is flat, y_1 is $R(\xi_1)$-regular. Let $S := R(\xi_1)/y_1R(\xi_1)$. Since $S/(x_2, \dots, x_d)S \cong (R/\mathfrak{n})(\xi_1)$, it is Artinian. Therefore, the geometric dimension of S is at most $d-1$. By Lemma 7.2, the geometric dimension of S cannot be less, and hence it is equal to $d - 1$. In particular, we are done in case $e = 1$.

Assume therefore $e > 1$. It follows from Lemma 7.2 and (6) that S has depth $e - 1$. By induction, there exists an $S(\xi_2, \dots, \xi_e)$-regular sequence $(y_2, \dots, y_e)$ such that $S(\xi_2, \dots, \xi_e)/(y_2, \dots, y_e)S(\xi_2, \dots, \xi_e)$ has geometric dimension $d - e$. Hence with $\xi := (\xi_1, \dots, \xi_e)$, the sequence $(y_1, \dots, y_e)$ is $R(\xi)$-regular and part of a generic sequence. □

7.4 Remark. The argument even shows that, for a given generic sequence $(x_1, \dots, x_d)$, we may choose an $R(\xi)$-regular sequence $(y_1, \dots, y_e)$ so that

$$(y_1, \dots, y_e, x_{e+1}, \dots, x_d)R(\xi) = (x_1, \dots, x_d)R(\xi).$$

In particular, if R is moreover cata-regular, then we may take $(y_1, \dots, y_e)$ equal to a generating set of the maximal ideal of $R(\xi)$.

For ultra-Noetherian rings, no such extension is necessary, since depth is first-order definable:

7.5 Proposition. *The depth of an ultra-Noetherian local ring R is equal to the maximal length of an R-regular sequence.* □

7.6. Pseudo-singularities

We now introduce some singularity variants that are based on depth. Let R be a local ring of finite embedding dimension. If the depth of R is equal to its embedding dimension, then we call R *pseudo-regular*, and if it is equal to its geometric dimension, we call R *pseudo-Cohen-Macaulay*. Immediate from the definitions we get:

7.7 Proposition. *A local ring of finite embedding dimension is pseudo-regular if and only if it is cata-regular and pseudo-Cohen-Macaulay.* □

In order to derive a homological characterization of pseudo-regularity analogous to Serre's characterization for regularity, we need some additional definitions.

7.8. Finite presentation type

We say that an R-module M admits a *finite free resolution* (of length n), if there exists an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0 \tag{7}$$

with each F_i a finitely generated free R-module. The alternating sum of the ranks of the F_i is called the *Euler number* $\mathrm{Eul}(M)$ of M. It follows from Schanuel's Lemma that $\mathrm{Eul}(M)$ does not depend on the choice of finite free resolution, and by [29, Theorem 19.7], it is always non-negative. Also, if

$$0 \to H \to G_m \to G_{m-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

is an arbitrary exact sequence with all G_i finitely generated free R-modules, then H is also finitely generated, and $\mathrm{Eul}(M)$ is the alternating sum of the ranks of the G_i and of $\mathrm{Eul}(H)$ (see [29, §19] for more details).

In general, very few modules admit a finite free resolution, and hence we introduce the following weaker version: we say that an R-module is *finitely n-presented*, if it admits finitely generated i-th syzygies for $i = 0, \dots, n$, or equivalently, if there exists an exact sequence as in (7), but without the initial zero, with all F_i finitely generated free R-modules. Hence M is finitely 0-presented if and only if it is finitely generated, and M is finitely 1-presented if and only if it is finitely presented. We will say that an R-module has *finite presentation type*, if it is finitely n-presented, for all n. Although these definitions do not require R to be local, the next one does: we call a R-module complex $(G_\bullet, d_\bullet)$ *minimal* if the kernel of each morphism d_i lies inside $\mathfrak{m}G_i$.

7.9 Lemma. *Let $(R, \mathfrak{m})$ be a local ring with residue field k. An R-module M is finitely n-presented if and only if there exist a minimal exact sequence*

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0 \tag{$F_\bullet$}$$

with each F_i a finitely generated free R-module. Moreover, if this is the case then the i-th Betti number $\beta_i^R(M)$ of M, that is to say, the vector space dimension of $\operatorname{Tor}_i^R(M,k)$, is equal to the rank of F_i, for all $i \leq n$, showing that $F_\bullet$ is unique up to isomorphism.

Proof. One direction is immediate and the other can by induction be reduced to the case that M is finitely 0-presented, that is to say, finitely generated. This case is then simply a reformulation of Nakayama's Lemma. To prove the last assertion, augment $F_\bullet$ by adding on the left a free module F_{n+1}, possibly of infinite rank, which maps onto the kernel of $F_n \to F_{n-1}$. Tensoring this exact sequence with k gives a complex in which all morphisms are zero and hence its i-th homology is $F_i \otimes k$, for $i = 0, \dots, n$. Since this homology is also equal to $\operatorname{Tor}_i^R(M,k)$, we proved the second assertion. □

Since a projective module over a local ring is always free ([29, Theorem 2.5]), a necessary and sufficient condition for an R-module M to have a finite free resolution is that M has projective dimension $n < \infty$ and is finitely n-presented. By the previous result, such a module then admits a unique minimal finite free resolution.

7.10 Lemma. *Any direct summand of an R-module with a finite free resolution has itself a finite free resolution. Similarly, any direct summand of a finitely n-presented module is again finitely n-presented.*

Proof. We prove both results simultaneously. Suppose $M \oplus N$ has a finite free resolution of length n as in (7) (respectively, of the form $F_\bullet$). We will show by induction on n that M has a finite free resolution (respectively, is finitely n-presented). If $n = 0$, that is to say, if $M \oplus N$ is free, then M is projective whence free (respectively, if $n = 0$, that is to say, $M \oplus N$ is finitely generated, then so is M). Hence assume $n > 0$ and choose an exact sequence

$$0 \to K \to R^m \to M \oplus N \to 0 \tag{8}$$

such that K admits a finite free resolution of length $n-1$ (respectively is finitely $n-1$-presented). Clearly, M and N must also be finitely generated, so that we can choose exact sequences

$$0 \to G \to R^a \to M \to 0$$
$$0 \to H \to R^b \to N \to 0.$$

Taking the direct sum of these last two exact sequences and comparing it with (8), we get from Schanuel's Lemma an isomorphism $K \oplus R^a \oplus R^b \cong G \oplus H \oplus R^n$. Since the module at the left hand side has a finite free resolution of length $n-1$ (respectively, is finitely $n-1$-presented), our induction hypothesis yields that G has a finite free resolution (respectively, is finitely $n-1$-presented), whence so does M (respectively, whence M is finitely n-presented). □

7.11 Theorem. *A local ring of finite embedding dimension is pseudo-regular if and only if its residue field admits a finite free resolution.*

Proof. Suppose first that $(R, \mathfrak{m})$ is pseudo-regular of geometric dimension d. Let $\mathbf{x}$ be a generic sequence generating $\mathfrak{m}$. Since R has depth d, all $H_i(\mathbf{x}; R)$ vanish, showing that the Koszul complex $K_\bullet(\mathbf{x})$ of $\mathbf{x}$ is exact, yielding the desired finite free resolution of the residue field k.

Conversely, assume that k has a finite free resolution

$$0 \to R^{a_n} \to R^{a_{n-1}} \to \cdots \to R^{a_1} \to R \to k \to 0.$$

Let m be the embedding dimension of R (so that we may choose $a_1 = m$). Observe that both hypothesis and conclusion are invariant under taking a Nagata extension of the form $R \subseteq R(\xi)$ (by faithful flatness), so that at any time we may make such an extension if needed. There is nothing to show if $m = 0$, so we induct on $m > 0$. By [29, Theorem 19.6], the depth of R must be positive. By Lemma 7.3, we may assume after making a Nagata extension, that some minimal generator x of $\mathfrak{m}$ is R-regular. Put $S := R/xR$, so that its embedding dimension is $m - 1$. For each $i > 1$, we have an isomorphism

$$\operatorname{Tor}_i^R(S, k) \cong \operatorname{Tor}_{i-1}^R(S, \mathfrak{m}) \cong \operatorname{Tor}_{i-1}^S(S, \mathfrak{m}/x\mathfrak{m}) = 0$$

since x is R-regular, whence also $\mathfrak{m}$-regular. This implies that the complex

$$0 \to S^{a_n} \to S^{a_{n-1}} \to \cdots \to S^{a_1}$$

is acyclic, that is to say, is a finite free resolution of $\mathfrak{m} \otimes S = \mathfrak{m}/x\mathfrak{m}$. I claim that k is a direct summand of $\mathfrak{m}/x\mathfrak{m}$. Assuming the claim, Lemma 7.10 then yields that k admits a finite free resolution as an S-module. Therefore, by our induction hypothesis, S is pseudo-regular, whence has depth $m - 1$. It follows from (6) that R has depth m, showing that it is pseudo-regular.

To prove the claim, choose $x_2, \ldots, x_m \in \mathfrak{m}$ so that $(x, x_2, \ldots, x_m)R = \mathfrak{m}$. Let H be the R-submodule of $\mathfrak{m}/x\mathfrak{m}$ generated by the image of x. Hence $H \cong k$ and we want to show that H is a direct summand of $\mathfrak{m}/x\mathfrak{m}$. Let N be the submodule generated by the images of the $x_2, \ldots, x_m$ in $\mathfrak{m}/x\mathfrak{m}$, so that $\mathfrak{m}/x\mathfrak{m} = H + N$. Let $a \in \mathfrak{m}$ and suppose its image in $\mathfrak{m}/x\mathfrak{m}$ lies in $H \cap N$. It follows that we can write a in two different ways, namely as $a = a_1 x = a_2 x_2 + \cdots + a_m x_m + rx$ with $a_i \in R$ and $r \in \mathfrak{m}$. By Nakayama's lemma, we therefore must have $a_1 \equiv r \equiv 0 \mod \mathfrak{m}$, that it so say, $a \in x\mathfrak{m}$. In other words, we showed that $H \cap N = 0$ and hence that $\mathfrak{m}/x\mathfrak{m} \cong H \oplus N$, as required. □

7.12 Remark. Under the assumptions of the theorem, k has projective dimension equal to the geometric dimension of R and $\operatorname{Eul}(k) = 0$ (use the Koszul complex to calculate both numbers). The Koszul complex is minimal and therefore $\operatorname{Tor}_i^R(k, k)$ has dimension equal to $\binom{n}{i}$ for all i.

7.13 Remark. Using a similar argument, one can show that R is pseudo-Cohen-Macaulay if and only if there exists a generic sequence $\mathbf{x}$ such that $R/\mathbf{x}R$ has a finite free resolution (which then can be chosen to be the Koszul complex $K_\bullet(\mathbf{x})$ of $\mathbf{x}$). For a related result, see Proposition 8.11 below.

To not confuse with our present terminology we deviate from [7] or [18, §5] by calling a ring *Bertin-Serre regular*, if every finitely generated ideal has finite projective dimension. If R is moreover coherent, then it is shown that any finitely generated ideal admits a finite free resolution. Applied to the maximal ideal, we get immediately from Theorem 7.11:

7.14 Corollary. *A coherent Bertin-Serre regular local ring of finite embedding dimension is pseudo-regular.* □

For the converse, we have the following:

7.15 Corollary. *Let $(R, \mathfrak{m})$ be a pseudo-regular local ring of geometric dimension d, and let be M an R-module. If M is finitely $d{+}1$-presented, then M has finite projective dimension (at most d).*

Proof. By Lemma 7.9, there exists a minimal exact sequence $F_\bullet$ with $n = d + 1$, and the i-th Betti number of M is the rank of F_i. However, k has projective dimension d by Remark 7.12, and hence $\beta_{d+1}(M) = 0$, showing that $F_{d+1} = 0$. □

7.16 Corollary. *Let R be a pseudo-regular local ring of geometric dimension one. If R is coherent, then it is Bertin-Serre regular.*

Proof. Let I be a finitely generated ideal. Since R is coherent, it is finitely presented. Hence R/I is finitely 2-presented, and therefore has finite projective dimension by Corollary 7.15. □

We cannot expect for this result to also hold if the geometric dimension d is strictly bigger than one, since a coherent Bertin-Serre regular ring is *Cohen-Macaulay in the sense of [20]* and therefore admits a regular sequence of length d (that is to say, in such a ring, naive depth always equals depth). To obtain a converse, we require a stronger coherence condition:

7.17 Theorem. *A local ring of finite embedding dimension is coherent and Bertin-Serre regular if and only if it is pseudo-regular and every finitely generated ideal has finite presentation type.*

Proof. If R is coherent and Bertin-Serre regular, then any finitely generated ideal has a finite free resolution by [17], whence has in particular finite presentation type. Moreover, R is pseudo-regular by Corollary 7.14. To prove the converse, let I be a finitely generated ideal. By assumption, I, whence also R/I, is finitely n-presented, and therefore has finite projective dimension by Corollary 7.15 applied with n sufficiently large. □

In [53], Soublin calls a ring R *uniformly coherent*[8] if there exists a function $\alpha \colon \mathbb{N} \to \mathbb{N}$ such that any morphism $R^n \to R$ has a kernel generated by at most $\alpha(n)$ elements.

[8] This is quite a strong hypothesis, even for Noetherian rings, for which it forces, among other things, that the dimension is at most two.

7.18 Theorem. *Let R be a uniformly coherent local ring of finite embedding dimension. Then every finitely generated ideal of R has finite presentation type. In particular, R is pseudo-regular if and only if it is Bertin-Serre regular.*

Proof. By [53] or [3, Corollary 2.3], the countable direct product $R^{\mathbb{N}}$ is coherent. Since a finitely generated submodule of a finitely generated free R-module embeds in $R^{\mathbb{N}}$, it is finitely presented. Applied to the syzygies of a finitely generated ideal I, we see that I has finite presentation type. The second assertion then follows from Theorem 7.17. □

Pseudo-Cohen-Macaulay local rings

Recall that we called R pseudo-Cohen-Macaulay, if its depth equals its geometric dimension.

7.19 Theorem. *A pseudo-Cohen-Macaulay local ring is cata-Cohen-Macaulay.*

Proof. Let R be a pseudo-Cohen-Macaulay local ring of geometric dimension d and let $\mathbf{x}$ be a generic sequence. Since R has depth d, the grade of $\mathfrak{n} := \mathbf{x}R$ is d, implying that all $H_i(\mathbf{x}; R)$ vanish, for $i > 0$. For $i = 1$, this yields that $\mathbf{x}$ is quasi-regular by [8, Ch. X, §9, Théorème 1]. Hence R is cata-Cohen-Macaulay by Theorem 6.9. □

The converse is in general false: R can be cata-Cohen-Macaulay without being pseudo-Cohen-Macaulay; an example is provided by the depth zero cata-regular ring in 6.3. On the other hand, neither is it the case that in a pseudo-Cohen-Macaulay local ring R every R-regular element is $\widehat{R}$-regular. For instance, R could be a non-separated domain, in which case any non-zero element in the ideal of infinitesimals is R-regular, but zero in $\widehat{R}$. This also gives an example of an R-regular element which is not part of a generic subset. From the proof of [29, Theorem 16.3], it follows that if R is separated and cata-Cohen-Macaulay, then every generic element is R-regular. In particular, we showed that if R has geometric dimension one, then R is cata-Cohen-Macaulay if and only if R is pseudo-Cohen-Macaulay.

7.20 Example. Let $R_w := A/(\xi^2, \xi\zeta^w)A$ where $A := k[[\xi, \zeta]]$. It follows that all R_w have depth zero and dimension one. Hence their ultraproduct $R_\natural$ has depth zero and ultra-dimension one. The cataproduct $R_\sharp$ is isomorphic to $k_\natural[[\xi, \zeta]]/\xi^2 k_\natural[[\xi, \zeta]]$, where $k_\natural$ is the ultrapower of k. This is a one-dimensional Cohen-Macaulay local ring. Hence $R_\natural$ is cata-Cohen-Macaulay and has geometric dimension one. In conclusion, $R_\natural$ is isodimensional and cata-Cohen-Macaulay, but not pseudo-Cohen-Macaulay.

7.21 Corollary. *A local ring of finite embedding dimension is pseudo-regular if and only if it is pseudo-Cohen-Macaulay and has multiplicity one.*

Proof. The direct implication follows from Proposition 7.7 and Corollary 6.10. Conversely, if R has multiplicity one and is pseudo-Cohen-Macaulay, then it is cata-Cohen-Macaulay by Theorem 7.19, whence cata-regular by Corollary 6.10, and the result now follows from Proposition 7.7. □

7.22 Corollary. *Let R be a pseudo-Cohen-Macaulay local ring R of geometric dimension two. If R is either a domain or separated, then any generic sequence is R-regular.*

Proof. Let (x, y) be a generic sequence in R. If R is a domain, then x is R-regular. Let us show that the same holds if R is separated. Since R has depth two by assumption, $H_2(x, y; R) = 0$. This means that whenever $ax + by = 0$ for some $a, b \in R$ then $(a, b) = r(y, -x)$ for some $r \in R$. In particular, if $a \in \operatorname{Ann}_R(x)$, then $(a, 0) = r(y, -x)$ for some $r \in R$, showing that $a \in y \operatorname{Ann}_R(x)$. In other words, $\operatorname{Ann}_R(x) = y \operatorname{Ann}_R(x)$ so that by induction $\operatorname{Ann}_R(x) = y^n \operatorname{Ann}_R(x)$ whence $\operatorname{Ann}_R(x) \subseteq \mathfrak{I}_R = 0$. This concludes the proof that x is R-regular. Using once more the above characterization of $H_2 = 0$, we see that in either case, y is R/xR-regular, whence (x, y) is R-regular. □

We can generalize Proposition 5.11 substantially under an additional Cohen-Macaulay assumption.

7.23 Proposition. *Let R be a local ring of finite embedding dimension and let M be an R-module of finite length. If R is pseudo-Cohen-Macaulay, then $\operatorname{Tor}_i^R(\widehat{R}, M)$ vanishes for all $i > 0$.*

Proof. Since M has finite length, its annihilator is $\mathfrak{m}$-primary, and hence contains a generic sequence by Corollary 3.13. Since $R \to R(\xi)$ is faithfully flat, the vanishing of the Tor's is unaffected by such an extension. Hence, after some Nagata extension, we may assume, using Remark 7.4, that R admits an R-regular, generic sequence $\mathbf{x}$ contained in the annihilator of M. Since $\widehat{R}$ is Cohen-Macaulay by Theorem 7.19, the sequence $\mathbf{x}$ is also $\widehat{R}$-regular. By a well-known deformation property of Tor modules, we get

$$\operatorname{Tor}_i^R(\widehat{R}, M) \cong \operatorname{Tor}_i^{R/\mathbf{x}R}(\widehat{R}/\mathbf{x}\widehat{R}, M)$$

for all $i > 0$. Vanishing now follows since $R/\mathbf{x}R \cong \widehat{R}/\mathbf{x}\widehat{R}$ by Lemma 2.8. □

Given a module M over a local ring R of finite embedding dimension, we define its *geometric dimension* to be the geometric dimension of $R/\operatorname{Ann}_R(M)$, and we denote it $\operatorname{gdim}(M)$. Since the notions of grade and depth also extend to modules, we may call a finitely generated R-module M *pseudo-Cohen-Macaulay*, if its geometric dimension equals its depth.

7.24 Corollary. *Let R be a local ring of finite embedding dimension and let M be a finitely generated R-module. If both R and M are pseudo-Cohen-Macaulay, then $\operatorname{Tor}_i^R(\widehat{R}, M) = 0$, for all $i > 0$, and $M \otimes \widehat{R}$ is a Cohen-Macaulay module.*

Proof. We induct on the geometric dimension e of M. If $e = 0$, then M is a finitely generated module over the Artinian local ring $R/\operatorname{Ann}_R(M)$, whence has finite length, and the result follows from Proposition 7.23. So assume $e > 0$. As far as proving the vanishing is concerned, we may always, by faithfully flat descent, take a Nagata extension of R. Hence, by the module analogue of Lemma 7.3 (the proof of which is left to the reader), we may assume, after possibly taking a Nagata extension, that x is an M-regular element. From the exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

we get, by tensoring with $\widehat{R}$, part of a long exact sequence

$$0 = \operatorname{Tor}_{i+1}^R(\widehat{R}, M/xM) \to \operatorname{Tor}_i^R(\widehat{R}, M) \xrightarrow{x} \operatorname{Tor}_i^R(\widehat{R}, M) \to \operatorname{Tor}_{i+1}^R(\widehat{R}, M/xM) = 0$$

where the two outer modules are zero by induction. Fix i and put $T := \operatorname{Tor}_i^R(\widehat{R}, M)$. Since $T = xT$, we have $T = \mathfrak{I}_R T$. As $\widehat{R}$ is Noetherian, $\mathfrak{I}_R \widehat{R}$ vanishes, whence so does $\mathfrak{I}_R T$, since T is the homology of a complex of modules over $\widehat{R}$. This shows $T = 0$, completing our proof of the first assertion.

To prove that $\widehat{M} := M \otimes \widehat{R}$ is Cohen-Macaulay, we induct once more on the geometric dimension e of M, where the case $e = 0$ is trivial since then $M = \widehat{M}$. Once more, faithfully flat descent allows us to take a Nagata extension if necessary, and so we may assume that there exists an element $x \in R$ which is R-regular and M-regular. Since M/xM is again pseudo-Cohen-Macaulay (as its geometric dimension and depth have both decreased by one), $\operatorname{Tor}_1^R(\widehat{R}, M/xM) = 0$ by the first part. Hence tensoring the exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

yields an exact sequence

$$0 \to \widehat{M} \xrightarrow{x} \widehat{M} \to \widehat{M}/x\widehat{M} \to 0$$

By induction, $\widehat{M}/x\widehat{M}$ is Cohen-Macaulay, and whence so is $\widehat{M}$ by the latter exact sequence. □

7.25 Example. In [37], a class of local rings was introduced which extends the class of Cohen-Macaulay local rings. More precisely, for each $d, e \geq 0$, let $CM_{d,e}$ be the class of all local rings R such that there exists an R-regular sequence of length d and such that the minimal length of a homomorphic image $R/\mathbf{x}R$ is e, where $\mathbf{x}$ is an arbitrary tuple in R of length d. The latter condition implies that R has geometric dimension at most d, and the former that its depth is at least d. It follows that R is pseudo-Cohen-Macaulay of geometric dimension d. Let $\mathbf{x}$ be an arbitrary tuple of length d. Suppose $R/\mathbf{x}R$ is Artinian of length l (by assumption $l \geq e$). Hence $R/\mathbf{x}R \cong \widehat{R}/\mathbf{x}\widehat{R}$ and $\mathbf{x}$ is generic in R. Moreover, $\mathbf{x}$ is $\widehat{R}$-regular, since $\widehat{R}$ is Cohen-Macaulay. It follows that the ideal $\mathbf{x}\widehat{R}$ has multiplicity l. For a general choice of system of parameters $\mathbf{y}$ in $\widehat{R}$, the ideal $\mathbf{y}\widehat{R}$ is a reduction of $\mathfrak{m}\widehat{R}$ ([29, Theorem 14.14]), so that the multiplicity of $\mathbf{y}\widehat{R}$ is equal to $\operatorname{mult}(\widehat{R})$. By assumption, the minimal value of the multiplicity of an ideal generated by a d-tuple from R is e. Since these form a general subset of all d-tuples in $\widehat{R}$, we showed that $\widehat{R}$ has multiplicity e, whence so does R by Remark 3.5. In fact, we have the following characterization of these classes:

7.26 Theorem. *A local ring R is pseudo-Cohen-Macaulay of geometric dimension d and multiplicity e if and only if $R(\xi)$ belongs to the class $CM_{d,e}$ for some (d-)tuple of indeterminates ξ.*

Proof. Since R and $R(\xi)$ are easily seen to have the same multiplicity (by comparing their completions), one direction follows from the previous discussion. On the other hand, suppose R is pseudo-Cohen-Macaulay of geometric dimension d and multiplicity e. By the same argument as above, we may choose a generic sequence $\mathbf{x}$ in R such that $\mathbf{x}\widehat{R}$ is a reduction of $\mathfrak{m}\widehat{R}$, whence has multiplicity e. It follows from Lemma 2.8 that $R/\mathbf{x}R$ has length e and by a similar argument that this is the least possible length. In order to construct an R-regular sequence, we have to go to an extension $R(\xi)$ by Lemma 7.3 and this extension is then in the class $CM_{d,e}$. □

In particular, by Corollary 7.21, a local ring R is pseudo-regular if and only if $R(\xi)$ belongs to $CM_{d,1}$ for some d and some d-tuple of indeterminates ξ. Moreover, by Proposition 7.5, an ultra-Noetherian local ring belongs to $CM_{d,e}$ if and only if it is pseudo-Cohen-Macaulay of geometric dimension d and multiplicity e.

Let R be a local ring of finite embedding dimension. We say that R is *pseudo-Gorenstein*, if it is pseudo-Cohen-Macaulay and there exists a generic sequence $\mathbf{x}$ such that $R/\mathbf{x}R$ is an Artinian Gorenstein ring.

7.27 Proposition. *A pseudo-Cohen-Macaulay local ring is pseudo-Gorenstein if and only if it is cata-Gorenstein.*

In fact, let $(R,\mathfrak{m})$ be a pseudo-Cohen-Macaulay local ring of geometric dimension d and let k be its residue field. If R is pseudo-Gorenstein, then $\operatorname{Ext}^i_R(k,R)=0$, for all $i\neq d$ and $\operatorname{Ext}^d_R(k,R)\cong k$. Conversely, if $\operatorname{Ext}^i_R(k,R)$ vanishes for some $i>d$ or if $\operatorname{Ext}^d_R(k,R)\cong k$, then R is pseudo-Gorenstein.

Proof. Let $\mathbf{x}$ be a generic sequence in R. By Lemma 7.2, the extension $R(\xi)$ is again pseudo-Cohen-Macaulay and $\mathbf{x}$ is generic in $R(\xi)$. Since

$$R/\mathbf{x}R \to (R/\mathbf{x}R)(\xi) \cong R(\xi)/\mathbf{x}R(\xi)$$

is faithfully flat and unramified, the former is Gorenstein if and only if the latter is. Since the Ext-functors commute with faithfully flat base change, we may replace R by $R(\xi)$ everywhere and assume by Lemma 7.3 that $\mathbf{x}$ is a regular sequence.

In particular, R is pseudo-Gorenstein if and only if $R/\mathbf{x}R\cong\widehat{R}/\mathbf{x}\widehat{R}$ is Gorenstein if and only if $\widehat{R}$ is Gorenstein, since $\mathbf{x}$ is $\widehat{R}$-regular. This already proves the first assertion. Since $\mathbf{x}$ is R-regular, we have

$$\operatorname{Ext}^i_R(k,R)\cong\operatorname{Ext}^{i-d}_{R/\mathbf{x}R}(k,R/\mathbf{x}R) \tag{9}$$

where we let $\operatorname{Ext}^j_R(\cdot,\cdot)$ be the zero functor for negative j (see for instance [9, Lemma 3.1.16] and the proof of (3) $\Leftrightarrow$ (1) of [29, Theorem 16.6]). The final assertion now follows from [29, Theorem 18.1] applied to the Artinian local ring $R/\mathbf{x}R$. □

It follows that if R is pseudo-Gorenstein, then $R/\mathbf{x}R$ is Gorenstein for every generic sequence $\mathbf{x}$.

8. Ultra-singularities

We now compare the 'cata' and 'pseudo' versions from the previous two sections with their 'ultra' counterparts. Throughout this section, unless mentioned explicitly, $R_\natural$

is an ultra-Noetherian local ring with maximal ideal $\mathfrak{m}_\natural$ and residue field $k_\natural$, realized as the ultraproduct of Noetherian local rings $(R_w, \mathfrak{m}_w)$ of bounded embedding dimension and residue field k_w. Recall (Lemma 5.6) that the cataproduct $R_\sharp$ of the R_w is the separated quotient as well as the completion of $R_\natural$, and it is in particular Noetherian.

8.1 Theorem. *For an ultra-Noetherian local ring $R_\natural$, the following are equivalent:*

8.1.1. *$R_\natural$ is pseudo-regular;*
8.1.2. *$R_\natural$ is ultra-regular;*
8.1.3. *$R_\natural$ is cata-regular and isodimensional.*

Proof. Let $R_\natural$ be the ultraproduct of Noetherian local rings R_w of bounded embedding dimension. If $R_\natural$ is pseudo-regular, then it is isodimensional by Theorem 4.4 and therefore cata-regular by Theorem 5.19. Moreover, by Łos' Theorem, almost all R_w are regular since embedding dimension and depth are first-order definable. This shows that $R_\natural$ is ultra-regular, and the converse follows along the same lines. Finally, if $R_\natural$ is cata-regular and isodimensional, then it is pseudo-regular, again by Theorem 4.4. □

The same proof also shows that $R_\natural$ is ultra-regular if and only if it is not ultra-singular. In view of Lemma 5.6, we may rephrase the theorem as follows:

8.2 Corollary. *Let R_w be Noetherian local rings of the same dimension and parameter degree and let $R_\sharp$ be their cataproduct. Then almost all R_w are regular if and only if $R_\sharp$ is.* □

8.3 Corollary. *Any localization of an ultra-regular local ring at a finitely related prime ideal is ultra-regular.*

Proof. Let $R_\natural$ be an ultra-regular local ring, given as the ultraproduct of d-dimensional regular local rings R_w, and let $\mathfrak{p} \in \text{FR-Spec}(R_\natural)$. By Proposition 5.1, there exist prime ideals $\mathfrak{p}_w \subseteq R_w$ whose ultraproduct is equal to $\mathfrak{p}$. Since almost each $(R_w)_{\mathfrak{p}_w}$ is regular of dimension at most d, their ultraproduct $(R_\natural)_\mathfrak{p}$ is ultra-regular (of geometric dimension at most d). □

We conclude our discussion of ultra-regular rings with an ultraproduct version of Corollary 5.15.

8.4 Corollary. *The canonical embedding $R \to R_\natural$ of an excellent local ring in its ultrapower has ultra-regular fibers at finitely related prime ideals: for every $\mathfrak{p} \in \text{FR-Spec}(R_\natural)$, the fiber ring $(R_\natural/\mathfrak{g}R_\natural)_\mathfrak{p}$ is ultra-regular, where $\mathfrak{g} = \mathfrak{p} \cap R$.*

Proof. To show that $(R_\natural/\mathfrak{g}R_\natural)_\mathfrak{p}$ is ultra-regular, we may replace R by $R/\mathfrak{g}$, since $R_\natural/\mathfrak{g}R_\natural$ is the ultrapower of $R/\mathfrak{g}$, and assume without loss of generality that R is a domain and $\mathfrak{p} \cap R = (0)$. By Corollary 5.3, the localization $(R_\natural)_\mathfrak{p}$ has finite embedding dimension, and $\mathfrak{p}$ is the ultraproduct of prime ideals $\mathfrak{p}_w \in \operatorname{Spec}(R)$. Since R is an excellent domain, its singular locus is a proper, closed subset, say, defined by a non-zero ideal $I \subseteq R$. If almost all $\mathfrak{p}_w$ would belong to this singular locus, then they would almost all contain I, whence so would $\mathfrak{p}$, contradicting that $\mathfrak{p} \cap R = (0)$. Hence almost all $\mathfrak{p}_w$ are in the regular locus, and the result now follows from the proof of Corollary 8.3. □

8.5. Ultra-Cohen-Macaulay local rings

Recall that $R_\natural$ is called *ultra-Cohen-Macaulay* if almost all R_w are Cohen-Macaulay. We can characterize this property in terms of the fundamental inequalities (1).

8.6 Theorem. *For an ultra-Noetherian local ring $R_\natural$, the following are equivalent*

8.6.1. *$R_\natural$ is ultra-Cohen-Macaulay;*
8.6.2. *the depth of $R_\natural$ equals its ultra-dimension.*

In particular, $R_\natural$ is pseudo-Cohen-Macaulay if and only if it is ultra-Cohen-Macaulay and isodimensional.

Proof. The first assertion follows immediately from the fact that depth is first-order. The second assertion is now also clear, since a pseudo-Cohen-Macaulay must be isodimensional by Theorem 5.19. □

8.7 Remark. Note that unlike in the regular case, isodimensionality together with being cata-Cohen-Macaulay is not sufficient for being pseudo-Cohen-Macaulay, as example 7.20 shows.

Also note that ultra-Cohen-Macaulay does not imply pseudo-Cohen-Macaulay nor even cata-Cohen-Macaulay. Namely, let $(R, \mathfrak{m})$ be a non-Cohen-Macaulay local ring and let $R_\natural$ and $R_\sharp$ be the respective ultraproduct and cataproduct of the $R/\mathfrak{m}^n$. Corollaries 5.10 and 5.15 together imply that $R_\sharp$ is not Cohen-Macaulay. Hence $R_\natural$ is not cata-Cohen-Macaulay, although it is clearly ultra-Cohen-Macaulay (there is no contradiction with the above theorem, since $R_\natural$ is not isodimensional).

8.8 Corollary. *The cataproduct of Cohen-Macaulay local rings having the same dimension and the same multiplicity, is again Cohen-Macaulay.*

Proof. Let $R_\natural$ and $R_\sharp$ be the respective ultraproduct and cataproduct of Noetherian local rings R_w of the same multiplicity and the same dimension. If almost all R_w are Cohen-Macaulay, then $R_\natural$ is isodimensional by Remark 5.25. Therefore, $R_\natural$ is pseudo-Cohen-Macaulay by Theorem 8.6, and hence $R_\sharp$ is Cohen-Macaulay by Theorem 7.19. □

Let us call an ultra-module $M_\natural$, that is to say, an ultraproduct of R_w-modules M_w, *ultra-Cohen-Macaulay*, if almost all M_w are Cohen-Macaulay. Although such a module need not be finitely generated, we have:

8.9 Lemma. *For each w, let M_w be a finitely generated module over R_w, and let $M_\natural$ be their ultraproduct. If almost all R_w are Cohen-Macaulay, of the same dimension and multiplicity, then $M_\natural$ is finitely generated and pseudo-Cohen-Macaulay if and only if almost all M_w are Cohen-Macaulay of the same multiplicity.*

Proof. If almost all M_w are Cohen-Macaulay of multiplicity l, then there exists, by [9, Theorem 4.6.10], an R_w-regular and M_w-regular sequence $\mathbf{x}_w$ such that $M_w/\mathbf{x}_w M_w$ has length l. Since each sequence can have length at most d, almost all have the same length $s \leq d$. The ultraproduct $M_\natural/\mathbf{x}_\natural M_\natural$, too, has length l by Proposition 5.13, where $\mathbf{x}_\natural$ is the ultraproduct of the $\mathbf{x}_w$. In particular, $M_\natural$ is finitely generated. Moreover, $\mathbf{x}_\natural$ is

$M_\natural$-regular, showing that $M_\natural$ has depth at least s. On the other hand, since $M_\natural/\mathbf{x}_\natural M_\natural$ has finite length, the geometric dimension of $M_\natural$ is at most s. This proves that $M_\natural$ is pseudo-Cohen-Macaulay.

Conversely, assume $M_\natural$ is pseudo-Cohen-Macaulay and finitely generated. As depth is first-order, by the (module version of) Proposition 7.5, there exists an $M_\natural$-regular sequence $\mathbf{x}_\natural$ such that $M_\natural/\mathbf{x}_\natural M_\natural$ has geometric dimension zero. As $M_\natural$ is finitely generated, $M_\natural/\mathbf{x}_\natural M_\natural$ has finite length, say, l. Letting $\mathbf{x}_w$ be tuples in R_w having as ultraproduct $\mathbf{x}_\natural$, the ultraproduct of the $M_w/\mathbf{x}_w M_w$ is equal to $M_\natural/\mathbf{x}_\natural M_\natural$, and hence almost all $M_w/\mathbf{x}_w M_w$ have length l by Proposition 5.13. Moreover, almost each $\mathbf{x}_w$ is M_w-regular, showing that M_w is Cohen-Macaulay, of multiplicity at most l, by another application of [9, Theorem 4.6.10]. □

Let us call the *multiplicity type* $\epsilon(R)$ of a Cohen-Macaulay local ring R the supremum of the multiplicities of its indecomposable, maximal Cohen-Macaulay modules. Hence, R has bounded multiplicity type if and only if $\epsilon(R) < \infty$.

8.10 Corollary. *For some $d, e \in \mathbb{N}$ and for each w, let R_w be a d-dimensional Cohen-Macaulay local ring of multiplicity e, and let $R_\sharp$ be their cataproduct. If, for some ϵ, almost all R_w have multiplicity type at most ϵ, then so does $R_\sharp$.*

Proof. Let M be an indecomposable, maximal, whence d-dimensional, Cohen-Macaulay $R_\sharp$-module. Since M is finitely generated, say by m elements, there exist m-generated R_w-modules M_w whose cataproduct is equal to M. Since $M = M_\natural/\mathfrak{I}_R M_\natural$ is Cohen-Macaulay, $M_\natural$ is pseudo-Cohen-Macaulay. Since the ultraproduct $M_\natural$ of the M_w is then also finitely generated, almost all M_w are maximal Cohen-Macaulay modules of the same multiplicity as M, by Lemma 8.9. By Łos' Theorem, almost all M_w must be indecomposable, since M, whence $M_\natural$, is. By assumption, therefore, almost all M_w, whence also M, has multiplicity at most ϵ. □

The next result, which is some type of coherence property for ultra-Cohen-Macaulay local rings, will be used in §11 to deduce some uniform bounds on Betti numbers. Recall that the *i-th Betti number* $\beta_i(M)$ of a module over a local ring R with residue field k is the (possibly infinite) dimension of $\operatorname{Tor}_i^R(M, k)$; for the notion of finite presentation type, see §7.8.

8.11 Proposition. *If $R_\natural$ is an isodimensional, ultra-Cohen-Macaulay local ring, then every finitely generated pseudo-Cohen-Macaulay $R_\natural$-module (e.g., every $R_\natural$-module of finite length) has finite presentation type. More precisely, for any given e, if almost each M_w is a Cohen-Macaulay R_w-module of multiplicity e, then, for each n, almost all M_w have the same n-th Betti number as their ultraproduct $M_\natural$ and as their cataproduct $M_\sharp$.*

Proof. In view of Lemma 8.9, it suffices to prove the second assertion. We will show, by induction on n, that

$$\beta_n(M_\natural) = \beta_n(M_\sharp) = \beta_n(M_w)$$

for almost all w. The case $n = 0$ follows from Proposition 5.13, since $M_\natural$ is finitely generated by Lemma 8.9. So assume $n \geq 1$.

Let

$$F_{n,w} \to F_{n-1,w} \to \cdots \to F_{1,w} \to M_w \to 0$$

be a minimal finite free resolution of M_w, with each $F_{i,w}$ a finite free R_w-module of rank $r_{i,w} := \beta_i(M_w)$ (see §7.8). Taking ultraproducts, we get by Łos' Theorem a minimal resolution

$$F_{n,\natural} \to F_{n-1,\natural} \to \cdots \to F_{1,\natural} \to M_\natural \to 0 \tag{10}$$

By induction and Lemma 7.9, we get $F_{i,\natural} \cong R_\natural^{r_i}$, for $i < n$, where r_i is the common value of almost all $\beta_i(M_w)$. Theorem 8.6 implies that $R_\natural$ is pseudo-Cohen-Macaulay, and hence by Corollary 7.24, all $\operatorname{Tor}_i^{R_\natural}(R_\sharp, M_\natural)$ vanish for $i > 0$. Therefore, if we tensor (10) with $R_\sharp$, we get again a minimal resolution

$$F_{n,\sharp} \to R_\sharp^{r_{n-1}} \to \cdots \to R_\sharp^{r_1} \to M_\sharp \to 0$$

Since $R_\sharp$ is Noetherian and the resolution is minimal, $r_i = \beta_i(M_\sharp)$ for $i < n$, and the last module in this resolution, $F_{n,\sharp}$, is generated by $r_n := \beta_n(M_\sharp)$ elements. Tensoring with the common residue field $k_\natural$ of $R_\natural$ and $R_\sharp$, we get

$$k_\natural^{r_n} \cong F_{n,\sharp}/\mathfrak{m}_\natural F_{n,\sharp} \cong F_{n,\natural}/\mathfrak{m}_\natural F_{n,\natural}.$$

Since the latter module is the ultraproduct of the $F_{n,w}/\mathfrak{m}_w F_{n,w} \cong k_w^{r_{n,w}}$, where k_w is the residue field of R_w, we get $r_{n,w} = r_n$ for almost all w, as we wanted to show. □

8.12 Theorem. *A pseudo-Cohen-Macaulay ultra-Noetherian local ring $R_\natural$ is cata-Gorenstein if and only if it is ultra-Gorenstein; and it is a cata-'complete intersection' if and only if it is an ultra-'complete intersection'.*

In particular, if R_w are Cohen-Macaulay local rings having the same dimension and multiplicity, then their cataproduct $R_\sharp$ is respectively Gorenstein or a complete intersection if and only if so are almost all R_w.

Proof. The second assertion follows from the first in view of Theorem 5.23 and Theorem 8.6. We already observed that $R_\natural$ is isodimensional, so that $R_\sharp$ and almost all R_w have the same dimension, d, say. Hence if $\mathbf{x}_\natural$ is a generic sequence in $R_\natural$, realized as an ultraproduct of tuples $\mathbf{x}_w$ in R_w, then almost each $\mathbf{x}_w$ is a system of parameters in R_w, whence R_w-regular. Therefore, almost all R_w are Gorenstein if and only if so are almost all $R_w/\mathbf{x}_w R_w$. This in turn is equivalent with $R_\natural/\mathbf{x}_\natural R_\natural$ being Gorenstein by Łos' Theorem (using that these are Artinian local rings; see [36] for more details). Since $R_\natural/\mathbf{x}_\natural R_\natural \cong R_\sharp/\mathbf{x}_\natural R_\sharp$, the latter is then equivalent with $R_\sharp$ being Gorenstein.

By Proposition 8.11, we have a minimal free resolution of $R_\sharp$-modules

$$R_\sharp^r \to R_\sharp^m \to R_\sharp \to k_\natural \to 0$$

where $r = \beta_2(k_\natural) = \beta_2(k_w)$ and $m = \beta_1(k_\natural) = \beta_1(k_w)$, for almost all w. Moreover, $R_\sharp$ has the same dimension d as almost all R_w by Theorem 8.6. By [9, Theorem 2.3.3], therefore, $R_\sharp$ is a complete intersection if and only if $r = m(m+2)/2 - d$, if and only if almost all R_w are complete intersections. □

Lefschetz Hulls

In [4], we showed that every Noetherian local ring R of equal characteristic zero (that is to say, containing the rationals) admits an ultra-Noetherian faithfully flat extension $\mathfrak{D}(R)$ which is *Lefschetz*, meaning that $\mathfrak{D}(R)$ is the ultraproduct of Noetherian local rings R_w of prime characteristic. In fact, the R_w may be chosen to be complete with algebraically closed residue field. We call $\mathfrak{D}(R)$ a *Lefschetz hull* of R. Although the construction can be made more functorial, it still depends on a choice of a cardinal number larger than the cardinality of R. However, in case R is of finite type over an uncountable[9] algebraically closed field of characteristic zero, there is a canonical choice for $\mathfrak{D}(R)$, called the *non-standard hull* of R and denoted R_∞; see [40, 49] for details. In view of our characterizations of pseudo-singularities in this section, the following result is immediate from [4, Theorem 5.2]:

8.13 Theorem. *A Noetherian local ring R of equal characteristic zero with Lefschetz hull $\mathfrak{D}(R)$ is Cohen-Macaulay (respectively, Gorenstein or regular) if and only if $\mathfrak{D}(R)$ is pseudo-Cohen-Macaulay (respectively, pseudo-Gorenstein or pseudo-regular).*

9. Cata-normalizations

An extremely useful fact in commutative algebra is the existence of Noether normalizations: any finitely generated algebra over a field or any complete Noetherian local domain admits a regular subring over which it is module-finite. This result is not hard to show in equal characteristic, so that we will adopt this additional assumption in this section to formulate an analogue for local rings of finite embedding dimension. In the sequel, let $(R, \mathfrak{m})$ be an equicharacteristic local ring with residue field k and let $\pi\colon R \to k$ denote the induced surjection.

Weak coefficient fields

A subfield κ of R is called a *weak coefficient field* of R if the restriction of π to κ induces an algebraic extension $\pi(\kappa) \subseteq k$. If this extension is an isomorphism, then we call κ a *coefficient field* of R (in the literature one also encounters the notion of a *quasi-coefficient* defined as a weak coefficient field κ for which the induced extension $\pi(\kappa) \subseteq k$ is also separable). The next result is well-known, but its proof is included for convenience.

9.1 Lemma. *Let $(R, \mathfrak{m})$ be an equicharacteristic local ring. For any subfield κ_0 of R, we can find a weak coefficient field κ of R containing κ_0.*

If, moreover, R has characteristic zero and is Henselian, then we can choose κ to be a coefficient field.

Proof. Let κ be maximal among all subfields of R containing κ_0 (such a field exists by Zorn's lemma). We need to show that the extension $\pi(\kappa) \subseteq k$ is algebraic, where k is the residue field of R and $\pi\colon R \to k$ the residue map. To this end, take an arbitrary

[9] Strictly speaking, of cardinality equal to 2^λ, for some infinite cardinal λ.

element $u \in k \setminus \pi(\kappa)$. Let $a \in R$ be such that $\pi(a) = u$. It follows that $a \notin \kappa$. By maximality, the subring $\kappa[a]$ of R generated by a must contain a non-zero non-invertible element (lest $\kappa(a)$ be a larger subfield of R). This means that $P(a) \in \mathfrak{m}$, for some non-zero $P \in \kappa[\xi]$. Hence taking reductions, we get $P^\pi(u) = 0$ in k, where P^π is the polynomial obtained from P by applying π to each of its coefficients. Since P^π is not identical zero, u is algebraic over $\pi(\kappa)$.

To prove the last assertion, assume by way of contradiction that R has characteristic zero and is Henselian, but that $\pi(\kappa)$ is strictly contained in k. Take $u \in k \setminus \pi(\kappa)$. Let p be a minimal equation of u over $\pi(\kappa)$ and let $P \in \kappa[\xi]$ be such that its image P^π is equal to p. Since u is a single root of p, Hensel's Lemma yields the existence of a root $a \in R$ of P with $\pi(a) = u$. However, this implies that the field $\kappa[\xi]/P\kappa[\xi]$ embeds in R via the assignment $\xi \mapsto a$, contradicting the maximality of κ. □

A local homomorphism $A \to R$ is called *cata-integral* (respectively, *cata-finite*, *cata-injective*, *cata-surjective*, *cata-flat*) if its completion $\widehat{A} \to \widehat{R}$ is integral (respectively, finite, injective, surjective, flat). Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension.

Cata-normalization

A *cata-normalization* of R is a cata-integral local homomorphism $\theta\colon (A, \mathfrak{p}) \to (R, \mathfrak{m})$ such that A is a (Noetherian) regular local ring and $\mathfrak{p}R$ is $\mathfrak{m}$-primary. We say that a cata-normalization θ is *Cohen*, if $\mathfrak{p}R = \mathfrak{m}$, and *Noether* if θ is injective.

9.2 Theorem. *An equicharacteristic local ring of finite embedding dimension admits a cata-normalization, which can be chosen to be either Cohen or Noether.*

Proof. Let $(R, \mathfrak{m})$ be an equicharacteristic local ring of finite embedding dimension. By Lemma 9.1, there exists a weak coefficient field κ of R. Choose a tuple $\mathbf{x} := (x_1, \dots, x_n)$ generating some $\mathfrak{m}$-primary ideal. Let A be the localization of the polynomial ring $\kappa[\xi]$ at the ideal generated by the indeterminates $\xi = (\xi_1, \dots, \xi_n)$. Let $\theta\colon A \to R$ be the (unique) κ-algebra homomorphism which sends ξ_i to x_i, for each i. To show that θ is a cata-normalization, we only need to show that its completion is integral, since the other conditions are immediate. Therefore, without loss of generality, we may already assume that A and R are complete, so that both rings are now Noetherian. Let $\pi\colon R \to k$ be the residue map and let l be a finite extension of $\pi(\kappa)$ contained in k. Put $B_l := \pi^{-1}(l)$. Since $\kappa + \mathfrak{m} \subseteq B_l$, one checks easily that B_l is a local ring with maximal ideal $\mathfrak{m}$. The local homomorphism $A \to B_l$ induces a finite extension of residue fields. Therefore, since A is complete and B_l is separated, B_l is finitely generated as an A-module by [29, Theorem 8.4]. Since k is the union of all its finite extensions l containing $\pi(\kappa)$, so is R the union of all the B_l, showing that R is integral over A.

It is clear that if we choose $\mathbf{x}$ so that it generates $\mathfrak{m}$, then θ is Cohen. Assume next that $\mathbf{x}$ is a generic sequence. In particular, $\widehat{R}$ has dimension n by Theorem 3.4. Since $\widehat{A}$ is an n-dimensional domain and $\widehat{A} \to \widehat{R}$ is integral, this map must be injective. But then so must $A \to R$ be, that is to say, θ is Noether. □

9.3 Remark. If I is a finitely generated ideal of R, then we can always choose a cata-normalization $A \to R$ with the additional property that there is some ideal $J \subseteq A$ with $JR = I$. Simply choose $\mathbf{x}$ so that it contains a set of generators of I.

9.4 Remark. From the above proof it is also clear that if R admits a coefficient field, then we can choose the cata-normalization $A \to R$ to be cata-finite.

9.5 Theorem. *An equicharacteristic local ring R of finite embedding dimension is cata-Cohen-Macaulay if and only if there exists a cata-flat, cata-normalization $A \to R$.*

Proof. If $\widehat{A} \to \widehat{R}$ is flat with A regular, then $\widehat{R}$ is Cohen-Macaulay by [29, Corollary to Theorem 23.3], since the closed fiber has dimension zero. This proves one direction. To prove the converse implication, assume that $\widehat{R}$ is Cohen-Macaulay. Let $A \to R$ be any Noether cata-normalization. Since $\widehat{A} \to \widehat{R}$ is a local homomorphism of Noetherian local rings of the same dimension, with closed fiber having dimension zero, it is flat by [29, Theorem 23.1], because $\widehat{A}$ is regular and $\widehat{R}$ is Cohen-Macaulay. □

From the proof it follows that any Noether cata-normalization of a cata-Cohen-Macaulay local ring is cata-flat. We conclude this section with an instance of true Noether Normalization:

9.6 Theorem. *If $R_\natural$ is an ultraproduct of equicharacteristic complete d-dimensional Noetherian local rings, then $R_\natural$ is isodimensional if and only if there exists an ultra-regular local subring $S_\natural \subseteq R_\natural$ such that $R_\natural$ is module-finite over it.*

Proof. Let us show that the if-direction holds for any ultra-Noetherian local ring of finite embedding dimension. Let $S_\natural \subseteq R_\natural$ be a finite extension with $S_\natural$ ultra-regular, realized as the ultraproduct of regular local rings $S_w \subseteq R_w$. By Proposition 5.13, if $R_\natural$ is generated by at most N elements over $S_\natural$, then almost each R_w is generated by at most N elements over S_w. If $\mathbf{y}_w$ is a regular system of parameters in S_w, then its image is a system of parameters in R_w. Since $R_w/\mathbf{y}_w R_w$ has vector space dimension at most N over the residue field of S_w, its length is at most N, showing that the R_w have bounded parameter degree. Hence, $R_\natural$ is isodimensional by Theorem 5.23.

Conversely, assume $R_\natural$ is as in the statement, so that in particular its geometric dimension is d. By Theorem 5.23, almost all R_w have parameter degree ρ, for some $\rho < \infty$. By [46, Corollary 3.8], almost each R_w is a module-finite extension of a regular subring S_w, generated as an S_w-module by at most ρ elements. Let $S_\natural$ be the ultra-regular local ring given as the ultraproduct of the S_w. Another application of Proposition 5.13 yields that $R_\natural$ is generated by at most ρ elements over $S_\natural$. □

9.7 Example. The equicharacteristic condition is necessary as the following example shows. Fix a prime number p and an indeterminate ξ, and let $\mathbb{Z}_p$ denote the ring of p-adic integers. Put $R_w := \mathbb{Z}_p[\xi]/(\xi^{2w+1} - p^2)\mathbb{Z}_p[\xi]$ and let $R_\natural$ be the ultraproduct of the R_w. Each R_w is a one-dimensional complete local Cohen-Macaulay domain with multiplicity (=parameter degree) two. Hence $R_\natural$ is isodimensional (indeed, the cataproduct $R_\sharp \cong (\mathbb{Z}_{p\sharp}/p^2\mathbb{Z}_{p\sharp})[[\xi]]$ is also one-dimensional, where $\mathbb{Z}_{p\sharp}$ is the catapower of $\mathbb{Z}_p$).

Suppose there is an ultra-regular subring $S_\natural \subseteq R_\natural$ such that $R_\natural$ is generated as an $S_\natural$-module by N elements. Hence by Łos' Theorem, there is a regular subring

$S_w \subseteq R_w$, such that R_w is generated as an S_w-module by N elements, for almost all w. This, however, contradicts [46, Proposition 3.5 and Example 3.2], where it is shown that the least number of generators for any regular subring of R_w must be equal to the length of R_w/pR_w (the so-called equi-parameter degree of R_w), that is to say, must be at least $2w+1$.

By varying the prime p as well (say, by letting p_w be an enumeration of all prime numbers and replacing p by p_w in the definition of R_w), we can construct a similar counterexample $R_\natural$ which itself is equicharacteristic zero. This latter ring also shows the extent to which the Ax-Kochen-Ershov theorem ([5, 15, 16]) holds. Namely, let $V_\natural$ be the ultraproduct of the $\mathbb{Z}_{p_w}$, so that $V_\natural$ is in particular ultra-regular. By Ax-Kochen-Ershov, $V_\natural$ is also the ultraproduct of the $\mathbb{F}_{p_w}[[t]]$ where $\mathbb{F}_{p_w}$ is the p_w-element field and t a single indeterminate. Put $R'_w := \mathbb{F}_{p_w}[[t,\xi]]/(\xi^{2w+1}-t^2)\mathbb{F}_{p_w}[[t,\xi]]$ and let $R'_\natural$ be their ultraproduct (so that $R'_\natural$ and R'_w are the equicharacteristic analogues of $R_\natural$ and R_w). Both $R_\natural$ and $R'_\natural$ contain $V_\natural$ as a subring in a natural way, but neither extension is finite. However, there is a second embedding of $V_\natural$ into $R'_\natural$ making the latter a finite extension. Namely, $V_\natural$ is also isomorphic with the subring given as the ultraproduct of the $\mathbb{F}_{p_w}[[\xi]]$. Under this identification, $R'_\natural$ is isomorphic to $V_\natural[t]/(t^2-\alpha)V_\natural[t]$, where α is the ultraproduct of the ξ^{2w+1}. In conclusion, $R_\natural$ and $R'_\natural$ cannot be isomorphic (note, however, that their cataproducts are isomorphic, to $F_\natural[[t,\xi]]/t^2F_\natural[[t,\xi]]$, where $F_\natural$ is the ultraproduct of the $\mathbb{F}_{p_w}$).

9.8 Remark. Using [46, Proposition 3.5], we can use the same argument to show that if $R_\natural$ is an ultraproduct of complete d-dimensional Noetherian local rings of mixed characteristic and of bounded equi-parameter degree, then $R_\natural$ admits an ultra-regular local subring $S_\natural$ over which it is module-finite. Recall that the *equi-parameter* degree of a Noetherian local ring A of mixed characteristic p is the least possible length of a homomorphic image A/I modulo a parameter ideal $I \subseteq A$ containing p.

9.9 Corollary. *If $S_\natural \subseteq R_\natural$ is a local module-finite extension of ultra-Noetherian local rings with $S_\natural$ ultra-regular, then $R_\natural$ is pseudo-Cohen-Macaulay if and only if it is flat over $S_\natural$.*

Proof. Let $(S_w,\mathfrak{n}_w)$ and R_w be Noetherian local rings with ultraproduct equal to $(S_\natural,\mathfrak{n}_\natural)$ and $R_\natural$ respectively. By Łos' Theorem, almost all $S_w \subseteq R_w$ are finite extensions with S_w regular. Suppose first that $R_\natural$ is pseudo-Cohen-Macaulay, whence ultra-Cohen-Macaulay by Theorem 8.6. Hence almost all R_w are Cohen-Macaulay, whence flat over S_w. To show that $R_\natural$ is flat over $S_\natural$, it suffices by [29, Theorem 7.8(3)] to show that $\operatorname{Tor}_1^{S_\natural}(R_\natural, S_\natural/I_\natural)$ vanishes for all finitely generated ideals $I_\natural$ of $S_\natural$. Choose $I_w \subseteq S_w$ whose ultraproduct equals $I_\natural$. Since ultraproducts commute with homology, $\operatorname{Tor}_1^{S_\natural}(R_\natural, S_\natural/I_\natural)$ is the ultraproduct of the $\operatorname{Tor}_1^{S_w}(R_w, S_w/I_w)$. Since the latter are zero by flatness, so is the former.

Conversely, suppose $S_\natural \subseteq R_\natural$ is flat. In particular, $R_\natural$ isodimensional by (the proof of) Theorem 9.6. By the same argument as above, the vanishing of $\operatorname{Tor}_1^{S_\natural}(R_\natural, S_\natural/\mathfrak{n}_\natural)$ implies the vanishing of almost all $\operatorname{Tor}_1^{S_w}(R_w, S_w/\mathfrak{n}_w)$. By the local flatness criterion, this implies that almost all R_w are flat over S_w, whence are Cohen-Macaulay. Hence

$R_\natural$ is ultra-Cohen-Macaulay, and therefore pseudo-Cohen-Macaulay by Theorem 8.6. □

10. Homological theorems

In this section, we prove for local rings of finite embedding dimension the counterparts of the homological theorems from commutative algebra, under the the assumption that the completion is equicharacteristic. We start with an immediate corollary of the definitions:

10.1 Corollary (Monomial Theorem)**.** *Let R be a local ring of geometric dimension d and let $\mathbf{x}$ be a generic sequence in R. Suppose R has either equal characteristic or otherwise is infinitely ramified (see §2.9). If $\nu_0, \dots, \nu_s \in \mathbb{N}^d$ are multi-indices such that ν_0 does not belong to the semigroup generated by $\nu_1, \dots, \nu_s$, then $\mathbf{x}^{\nu_0}$ does not lie in the ideal in R generated by $\mathbf{x}^{\nu_1}, \dots \mathbf{x}^{\nu_s}$.*

Proof. If the contrary were true, then the same ideal membership holds in the completion $\widehat{R}$. However, by Proposition 3.9, the image of $\mathbf{x}$ in $\widehat{R}$ is a system of parameters, thus violating the usual Monomial Theorem (see for instance [21]), since $\widehat{R}$ is equicharacteristic. □

A special instance of the assertion (which is often already referred to as the Monomial Theorem) is the fact that for any generic sequence $(x_1, \dots, x_d)$ in R, we have

$$(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \dots, x_d^{t+1})R \tag{11}$$

for all t. In the Noetherian setup, the latter result suffices to show the so-called Direct Summand Theorem (see for instance [9, Lemma 9.2.2]). However, it is not clear how to derive this in the present setup (presently, I can only get a weaker version, which I omit here).

Next we have a look at the Hochster-Roberts theorem. Although one can formulate a more general version, we will only give the result for local homomorphisms $R \to S$ which are *locally of finite type*, meaning that S is a localization of some finitely generated R-algebra. Note that the class of local rings of finite embedding dimension is closed under such algebras: if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is locally of finite type, then so is $R/\mathfrak{m} \to S/\mathfrak{m}S$. In particular, $S/\mathfrak{m}S$ is Noetherian, and $\mathfrak{n}(S/\mathfrak{m}S)$ is finitely generated. Hence if $\mathfrak{m}$ is finitely generated, then so is $\mathfrak{n}$.

10.2 Theorem (Hochster-Roberts)**.** *Let $R \to S$ be a local homomorphism between local rings of finite embedding dimension. Suppose R has equal characteristic or is infinitely ramified. If $R \to S$ is cyclically pure and locally of finite type, and if S is cata-regular, then R is cata-Cohen-Macaulay.*

Proof. It suffices to show that $\widehat{R} \to \widehat{S}$ is cyclically pure, for then the classical Hochster-Roberts theorem shows that $\widehat{R}$ is Cohen-Macaulay by [25, Theorem 2.3], since $\widehat{S}$ is regular and equicharacteristic. To prove cyclical purity, we need to show that $I = I\widehat{S} \cap \widehat{R}$ for each ideal I in $\widehat{R}$. Since any ideal is an intersection of $\mathfrak{m}\widehat{R}$-primary ideals, it suffices to show this for I an $\mathfrak{m}\widehat{R}$-primary ideal, where $\mathfrak{m}$ is the maximal ideal of R.

By Lemma 2.8, any such ideal is extended from R, that is to say of the form $I = \mathfrak{a}\widehat{R}$ with $\mathfrak{a}$ an $\mathfrak{m}$-primary ideal of R. Since $S/\mathfrak{a}S$ is locally of finite type over the Artinian local ring $R/\mathfrak{a}$, it is Noetherian. Therefore, $\mathfrak{a}S$ is closed, so that $I\widehat{S}\cap S = \mathfrak{a}\widehat{S}\cap S = \mathfrak{a}S$ by Lemma 2.4. Hence, in the composition

$$\widehat{R}/I \cong R/\mathfrak{a} \to S/\mathfrak{a}S \to \widehat{S}/I\widehat{S}$$

all maps are injective, as the first is an isomorphism by Lemma 2.4 and the second is injective by assumption. This proves that $I\widehat{S} \cap \widehat{R} = I$, as required. □

10.3 Remark. Combining this result with Theorems 6.9, 8.1 and 9.5, and Corollary 7.14 yields Corollary 1.1 from the introduction. In the theorem, less than cyclical purity is required; it suffices that $R \to S$ is *pure-closed*, in the sense that $IS \cap R = I$ for every closed (equivalently, every $\mathfrak{m}$-primary) ideal $I \subseteq R$. Furthermore, we may weaken the condition that $R \to S$ is locally of finite type to requiring that its closed fiber $S/\mathfrak{m}S$ is Noetherian. In order to apply the techniques from §13 and deduce an asymptotic version of the Hochster-Roberts theorem in mixed characteristic, we would like to prove a stronger result: namely, under an additional isodimensionality assumption, may we conclude that R is pseudo-Cohen-Macaulay?

To obtain other homological properties, we follow Hochster's treatment [21], by generalizing the notion of big Cohen-Macaulay modules. In fact, as in the Noetherian case, we can even put a ring structure on the latter:

Big Cohen-Macaulay algebras

We call an R-algebra B a *big Cohen-Macaulay algebra* if some generic sequence of R is B-regular; we call B a *balanced big Cohen-Macaulay algebra* if every generic sequence is B-regular.

10.4 Theorem. *Let R be a local ring of finite embedding dimension. If R has equal characteristic or is infinitely unramified, then it admits a balanced big Cohen-Macaulay algebra.*

Proof. By the work of Hochster and Huneke ([23, 25]) or the more canonical construction of [4, §7] (note that the algebras in the latter paper are local), any equicharacteristic Noetherian local ring admits a balanced big Cohen-Macaulay algebra. This applies in particular to the completion $\widehat{R}$ as it is always equicharacteristic by the discussion in §2.9. So remains to show that any balanced big Cohen-Macaulay $\widehat{R}$-algebra B is a balanced big Cohen-Macaulay R-algebra. However, this is clear for if $\mathbf{x}$ is a generic sequence, then it is a system of parameters in $\widehat{R}$ by Proposition 3.9, whence B-regular. □

10.5 Remark. We may drop the requirement on the characteristic when R has geometric dimension at most three, since in that case, regardless of characteristic, $\widehat{R}$ admits a balanced big Cohen-Macaulay algebra by [22]. In particular, all the homological theorems below also hold under this assumption.

10.6 Remark. In fact, we may choose balanced big Cohen-Macaulay algebras in a weakly functorial way in the following sense. We will call a local homomorphism $R \to S$ *cata-permissible*, if $\widehat{R} \to \widehat{S}$ is permissible in the sense of [28, §9] or [4, §7.9]. In that case, we may choose a balanced big Cohen-Macaulay $\widehat{R}$-algebra B (whence a balanced big Cohen-Macaulay R-algebra), a balanced big Cohen-Macaulay $\widehat{S}$-algebra B' (whence a balanced big Cohen-Macaulay S-algebra) and a homomorphism $B \to B'$ extending $\widehat{R} \to \widehat{S}$, whence also extending $R \to S$. Recall from the cited sources that any local algebra is permissible over an equidimensional and universally catenary Noetherian local ring (e.g., a complete local domain).

10.7 Proposition. *If R is pseudo-regular with residue field k and if B is a balanced big Cohen-Macaulay R-algebra, then all $\operatorname{Tor}_i^R(B,k)$ vanish for $i > 0$, and $IB \cap R$ is equal to the closure of I for each ideal $I \subseteq R$.*

Proof. It is not hard to verify that $B \otimes_R S$ is a balanced big Cohen-Macaulay S-algebra, for $S := R(\xi)$ and ξ a tuple of indeterminates. Since $R \to S$ is faithfully flat, we may pass from R to S and therefore assume in view of Remark 7.4 that the maximal ideal of R is generated by a regular sequence $\mathbf{x}$. Since $\mathbf{x}$ is also B-regular and $k = R/\mathbf{x}R$,

$$\operatorname{Tor}_i^R(B,k) \cong \operatorname{Tor}_i^{R/\mathbf{x}R}(B/\mathbf{x}B,k) = 0$$

for all $i > 0$. Therefore, for any $\mathfrak{m}$-primary ideal $\mathfrak{n}$, we get $\operatorname{Tor}_1^{R/\mathfrak{n}}(B/\mathfrak{n}B,k) = 0$ by [44, Lemma 2.1]. Since $R/\mathfrak{n}$ is Artinian, $B/\mathfrak{n}B$ is faithfully flat by the Local Flatness Criterion, and hence in particular $\mathfrak{n} = \mathfrak{n}B \cap R$. The last assertion then follows since any closed ideal is the intersection of all $\mathfrak{m}$-primary ideals containing it. □

Using an argument similar to the one in the proof of Corollary 7.24, one can show that under the above assumptions, each $\operatorname{Tor}_i^R(B,M)$ vanishes, for $i > 0$ and M a finitely generated pseudo-Cohen-Macaulay module: for the Artinian case, induct on the length of M, and for the general case, on the depth of M, using that $\mathfrak{I}_R B = 0$ by construction; details are left for the reader. Before stating the next result, we need to introduce some terminology. We will follow the treatment in [9, §9.4] and refer to this source for more details. Let $F_\bullet$ be a complex

$$0 \to F_s \xrightarrow{\varphi_s} F_{s-1} \xrightarrow{\varphi_{s-1}} \dots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \to 0$$

with each F_i a finitely generated free R-module. We call s the length of $F_\bullet$ and we call the cokernel of φ_1 the *cokernel* of the complex. For each $1 \leq n \leq s$, we will define the n-th *Fitting* ideal $I_n(F_\bullet)$ of $F_\bullet$ as follows. Fix $1 \leq n \leq s$ and put

$$r := \sum_{i=n}^{s} (-1)^{i-n} \operatorname{rank} F_i.$$

Let Γ be a matrix representing the morphism φ_n (by choosing bases for F_n and F_{n-1}) and let $I_n(F_\bullet)$ be the ideal in R generated by all $r \times r$-minors of Γ. One shows that this is independent from the choices made.

We say that $F_\bullet$ is *acyclic* if all $H_i(F_\bullet)$ vanish, for $i > 0$; if also $H_0(F_\bullet)$ vanishes (that is to say, if the cokernel of $F_\bullet$ is zero), then we say that $F_\bullet$ is *exact*. In particular, if $F_\bullet$ is acyclic, then it is a finite free resolution of its cokernel.

10.8 Theorem. *Let $(R, \mathfrak{m})$ be an equicharacteristic or an infinitely ramified local ring of finite embedding dimension. Let $F_\bullet$ be a finite complex of finitely generated free R-modules of length s and let M be its cokernel. If the geometric codimension of $I_n(F_\bullet)$ is at least n for each $n = 1, \dots, s$, then the geometric codimension of $\operatorname{Ann}_R(\mu)$ is at most s, for any non-zero minimal generator μ of M.*

Proof. Let d and e be the geometric dimension of R and $R/\operatorname{Ann}_R(\mu)$ respectively. In view of Proposition 3.15, we need to show that $d - e \leq s$, and we do this by induction on e. Assume first that $e = 0$, so that $\operatorname{Ann}_R(\mu)$ is $\mathfrak{m}$-primary. By Theorem 10.4, there exists a balanced big Cohen-Macaulay R-algebra B. By Proposition 3.15, we can find part of a generic sequence of length n in $I_n(F_\bullet)$, which is therefore B-regular. Hence each $I_n(F_\bullet)B$ has grade at least n, and the Buchsbaum-Eisenbud Acyclicity criterion ([9, Theorem 9.1.6]) then yields that the complex $F_\bullet \otimes_R B$ is acyclic. Since B, whence each module in $F_\bullet \otimes_R B$, has depth d, and since $M \otimes_R B$ is the cokernel of $F_\bullet \otimes_R B$, the depth of $M \otimes_R B$ is at least $d - s$ by [9, Proposition 9.1.2(e)]. By Nakayama's lemma, the image of μ in $M/\mathfrak{m}M$ is non-zero, which implies that $\mu \otimes 1$ is non-zero in $M \otimes_R B$. Since the annihilator of $\mu \otimes 1$ contains $\operatorname{Ann}_R(\mu)$, it is $\mathfrak{m}$-primary. It follows that $M \otimes_R B$ has depth zero, and hence that $d - s \leq 0$.

Assume now that $e > 0$. The threshold primes of R and $\operatorname{Ann}_R(\mu)$ are all different from $\mathfrak{m}$, and so are the threshold primes of those $I_n(F_\bullet)$ that are not $\mathfrak{m}$-primary. By prime avoidance, we may therefore choose $x \in \mathfrak{m}$ outside all these finitely many threshold primes. Put $R_n := R/I_n(F_\bullet)$ and $S := R/xR$. We want to apply the induction hypothesis to the complex $F_\bullet \otimes_R S$ and the image of μ in $M \otimes_R S$. By Corollary 3.12, the geometric dimension of S and $R_0 \otimes_R S$ are $d - 1$ and $e - 1$ respectively, and the geometric dimension of $S/I_n(F_\bullet \otimes_R S) \cong R_n \otimes_R S$ is at most $d - n - 1$ (this is trivially true if $I_n(F_\bullet)$ is $\mathfrak{m}$-primary and follows from Lemma 3.8 in the remaining case). Since $S/\operatorname{Ann}_S(\mu)$ is a residue ring of $R_0 \otimes_R S$, its geometric dimension is at most $e - 1$, so that our induction hypothesis applies, yielding $d - 1 - (e - 1) \leq \operatorname{gcodim}(\operatorname{Ann}_S(\mu)) \leq s$. □

We can now generalize the new intersection theorems due to Evans-Griffith and Peskine-Szpiro-Roberts.

10.9 Corollary. *Let $(R, \mathfrak{m})$ be an equicharacteristic or an infinitely ramified local ring of finite embedding dimension. Let $F_\bullet$ be a finite complex of finitely generated free R-modules of length s and let M be its cokernel.*

10.9.1. *If $F_\bullet$ is acyclic when localized at any closed prime ideal of R different from $\mathfrak{m}$ and there exists a non-zero minimal generator of M whose annihilator is $\mathfrak{m}$-primary, then $\operatorname{gdim}(R) \leq s$.*

10.9.2. *If $F_\bullet$ is exact when localized at any closed prime ideal of R different from $\mathfrak{m}$ and $s < \operatorname{gdim}(R)$, then $F_\bullet$ is exact.*

Proof. To prove (10.9.1), assume $s < d := \operatorname{gdim}(R)$. We reach the desired contradiction from Theorem 10.8, if we can show that $R/I_n(F_\bullet)$ has geometric dimension at most $d - n$, for all $n = 1, \dots, s$. Fix n and let $I := I_n(F_\bullet)$. There is nothing to show if I is $\mathfrak{m}$-primary, so that we may exclude this case. By Remark 9.3, we can choose a cata-normalization $A_0 \to R$ and an ideal $J \subseteq A_0$ such that $JR = I$ (note that I is finitely generated by construction). Let A be the image of A_0 in R, so that $A \subseteq R$ is also cata-integral and cata-injective (although Noetherian, A will, in general, no longer be regular). Since $\widehat{A} \to \widehat{R}$ is integral and injective, $\widehat{A}$, whence also A, has dimension d. Suppose JA has height h and let $\mathfrak{q}$ be a minimal prime of JA of height h. By [29, Theorem 9.3], we can find a prime ideal $\mathfrak{P}$ in $\widehat{R}$ lying above $\mathfrak{q}$. Let $\mathfrak{p} := \mathfrak{P} \cap R$ (which is therefore closed by Corollary 2.7). Note that since I is not $\mathfrak{m}$-primary, $h < d$, and therefore $\mathfrak{p} \neq \mathfrak{m}$. By assumption, $(F_\bullet)_\mathfrak{p}$ is acyclic, so that the grade of $IR_\mathfrak{p}$ is at least n by the Buchsbaum-Eisenbud Acyclicity criterion ([9, Theorem 9.1.6]). By [9, Proposition 9.1.2(g)], the grade of $JA_\mathfrak{q}$ is therefore also at least n. In particular, $A_\mathfrak{q}$ has depth at least n, showing that $n \leq h$. This in turn implies that A/JA has dimension at most $d - n$. Since $\widehat{A}/J\widehat{A} \to \widehat{R}/I\widehat{R}$ is integral, the dimension of the first ring is at least that of the second ring. Hence we showed that $\widehat{R}/I\widehat{R}$ has dimension at most $d - n$. By Lemma 2.4 and Theorem 3.4, this in turn implies that R/I has geometric dimension at most $d - n$, as required.

The second assertion follows from the first by a standard argument (see for instance the proof of [9, Corollary 9.4.3]). Namely, it implies that the cokernel M of $F_\bullet$ has finite length. The only way that this does not contradict (10.9.1) is that $M = 0$ (by Nakayama's Lemma). This in turn implies that we can split of the last term in $F_\bullet$ and then an inductive argument on s finishes the proof. □

We can translate these results to more familiar versions of the homological theorems.

10.10 Theorem (Superheight)**.** *Let $R \to S$ be a local homomorphism of equicharacteristic or infinitely ramified local rings of finite embedding dimension and let M be an R-module admitting a finite free resolution $F_\bullet$ of length s. If $M \otimes_R S$ has finite length, then S has geometric dimension at most s.*

Proof. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the respective maximal ideals of R and S. Let $\mathfrak{q}$ be an ideal in S different from $\mathfrak{n}$ and put $\mathfrak{p} := \mathfrak{q} \cap R$. Since the localization of $M \otimes_R S$ at $\mathfrak{q}$ is zero, we get

$$M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \otimes_{k(\mathfrak{p})} S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = 0,$$

where $k(\mathfrak{p})$ is the residue field of $\mathfrak{p}$. Since $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is non-zero, $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ must be zero, and therefore $M_\mathfrak{p} = 0$, by Nakayama's Lemma. Hence $(F_\bullet)_\mathfrak{p}$ is exact whence split exact. Therefore, this remains so after tensoring with $S_\mathfrak{q}$. In other words, the conditions of (10.9.2) are met for the complex $F_\bullet \otimes_R S$ over the ring S, showing that S must have geometric dimension at most s. □

10.11 Theorem (Intersection Theorem)**.** *Let R be an equicharacteristic or an infinitely ramified local ring of finite embedding dimension and let M, N be R-modules. If M has a finite free resolution of length s, then $\operatorname{gdim}(N) \leq s + \operatorname{gdim}(M \otimes_R N)$.*

Proof. Assume first that $M \otimes_R N$ has finite length and let $S := R/\operatorname{Ann}_R(N)$. It follows that $M \otimes_R S$ has finite length, so that the geometric dimension of S is at most s by Theorem 10.10. For the general case, we induct on the geometric dimension of $M \otimes N$. Using Proposition 3.9, one can find $x \in R$ such that it is part of a generic subset of both $R/\operatorname{Ann}_R(N)$ and $R/\operatorname{Ann}_R(M \otimes N)$. It follows that the geometric dimensions of N/xN and $M \otimes N/xN$ both have dropped by one, so that we are done by induction. $\square$

10.12 Theorem (Canonical Element Theorem)**.** *Let $(R, \mathfrak{m})$ be an equicharacteristic or an infinitely ramified local ring of finite embedding dimension. Let $F_\bullet$ be a free resolution of the residue field k of R and let $\mathbf{x}$ be a generic sequence in R. If γ is a complex morphism from the Koszul complex $K_\bullet(\mathbf{x})$ to $F_\bullet$, extending the natural map $\gamma_0\colon K_0(\mathbf{x}) = R/\mathbf{x}R \to k$, then the morphism $\gamma_d\colon K_d(\mathbf{x}) \to F_d$ is non-zero, where d is the geometric dimension of R.*

Proof. Suppose γ_d is zero. Let B be a local balanced big Cohen-Macaulay algebra for R and let $y \in B$ be such that its image in $B/\mathbf{x}B$ is a non-zero socle element. Define $\psi_0\colon R \to B$ by sending 1 to y. Since $\mathbf{x}$ is B-regular, the Koszul complex $K_\bullet(\mathbf{x}; B) := K_\bullet(\mathbf{x}) \otimes B$ is acyclic. It follows that ψ_0 extends to a morphism of complexes $\psi\colon F_\bullet \to K_\bullet(\mathbf{x}; B)$. Let $\alpha := \psi \circ \gamma$ be the composition $K_\bullet(\mathbf{x}) \to K_\bullet(\mathbf{x}; B)$. In particular $\alpha_0(1) = y$ and $\alpha_d = 0$. On the other hand, α_0 induces by tensoring a morphism of complexes $\beta := 1 \otimes \alpha_0\colon K_\bullet(\mathbf{x}) \to K_\bullet(\mathbf{x}) \otimes B = K_\bullet(\mathbf{x}; B)$. Since $K_\bullet(\mathbf{x}; B)$ is acyclic, α and β differ by a homotopy σ. In particular, $\beta_d = \beta_d - \alpha_d = \sigma_{d-1} \circ \delta_d$, where $\delta_d\colon K_d(\mathbf{x}) = R \to K_{d-1}(\mathbf{x}) = R^d$ is the left most map in the Koszul complex. Since the image of δ_d lies in $\mathbf{x}R^d$, we get $y = \beta_d(1) = \sigma_{d-1} \circ \delta_d(1) \in \mathbf{x}B$, contradicting our choice of y. $\square$

To formulate the next result, which extends a result of Eisenbud and Evans in [13], recall that for an R-module M and an element $z \in M$, the *order ideal* of z is the ideal $\mathcal{O}_M(z)$ consisting of all images $\alpha(z)$ for $\alpha \in \operatorname{Hom}_R(M, R)$. Moreover, if R is a domain with field of fractions K, then the *rank* of M is defined as the dimension of the vector space $M \otimes_R K$.

10.13 Theorem (Generalized Principal Ideal Theorem)**.** *Let $(R, \mathfrak{m})$ be an equicharacteristic or an infinitely ramified local domain of finite embedding dimension, and let M be a finitely generated R-module. If $z \in \mathfrak{m}M$, then the geometric codimension of $\mathcal{O}_M(z)$ is at most the rank of M.*

Proof. Let h be the geometric codimension of $\mathcal{O}_M(z)$, let r be the rank of M, and let d be the geometric dimension of R. By definition, there exists a generic sequence $(x_1, \dots, x_d)$ with $x_i \in \mathcal{O}_M(z)$, for $i = 1, \dots, h$. Replacing M by $M \oplus R^{d-h}$ and z by the element $(z, x_{h+1}, \dots, x_d)$, so that both the rank of M and the geometric codimension of $\mathcal{O}_M(z)$ increase by $d - h$, we may assume that $\mathcal{O}_M(z)$ contains a generic sequence $\mathbf{x}$. Let $\mathbf{y}$ be a finite tuple generating $\mathfrak{m}$. As explained in the proof of [9, Theorem 9.3.2], the canonical homomorphism $R/\mathbf{x}R \to R/\mathbf{y}R$ induces a morphism of Koszul complexes $\alpha\colon K_\bullet(\mathbf{x}) \to K_\bullet(\mathbf{y})$. Let $F_\bullet$ be a free resolution of the residue field $R/\mathbf{y}R$ of R and $\beta\colon K_\bullet(\mathbf{y}) \to F_\bullet$ be an induced morphism of complexes. By Theorem 10.12, applied to the composition $\beta \circ \alpha$, we get in degree d a non-zero morphism $\beta_d \circ \alpha_d$, showing in particular that α_d is non-zero as well. Since α_d is just the d-th exterior power of α_1, the rank of α_1 is at least d. On the other hand, α_1 factors by construction through $\operatorname{Hom}_R(M, R)$, whence has rank at most r, yielding the desired inequality $d \leq r$ (see [9, Theorem 9.3.2] for more details). $\square$

11. Uniform bounds on Betti numbers

In the next two sections, we apply the previous theory to derive uniformity results for Noetherian local rings. In this section, we study Betti numbers. Recall that given a module M over a local ring R with residue field k, its *n-th Betti number* $\beta_n(M)$ is defined as the vector space dimension of $\operatorname{Tor}_n^R(M,k) \cong \operatorname{Ext}_R^n(M,k)$. It is equal to the rank of the n-th module in a minimal free resolution of M (provided such a resolution exists), and by Nakayama's Lemma, it is then also equal to the least number of generators of an n-th syzygy of M. One usually studies the behavior of these Betti numbers for a fixed module as n goes to infinity. In contrast, we will study their behavior for fixed n as we vary the module.

11.1 Theorem. *For each quadruple (d,e,l,n) of non-negative integers, there exists a bound $\Delta(d,e,l,n)$ with the following property. If R is a d-dimensional local Cohen-Macaulay ring of multiplicity e, and M is a Cohen-Macaulay R-module of multiplicity at most l, then*

$$\beta_n(M) \leq \Delta(d,e,l,n).$$

Proof. Suppose not, so that for some quadruple (d,e,l,n), we cannot define such an upper bound. This means that for every w, we can find a d-dimensional Cohen-Macaulay local ring R_w of multiplicity e, and a Cohen-Macaulay R_w-module M_w of multiplicity at most l, such that $\beta_n(M_w) \geq w$. By Theorem 5.23, the ultraproduct $R_\natural$ is isodimensional, and by Lemma 8.9, the ultraproduct $M_\natural$ is finitely generated and pseudo-Cohen-Macaulay. Since the cataproduct $M_\sharp$ is therefore finitely generated over the (Noetherian) cataproduct $R_\sharp$, its n-th Betti number $\beta_n(M_\sharp)$ is finite, and by Proposition 8.11, equal to almost all $\beta_n(M_w)$, contradiction. □

Theorem 11.1 applied to the residue field of R yields Corollary 1.2 from the introduction. We can also reformulate the previous theorem in terms of universal resolutions:

11.2 Corollary. *For each triple (d,e,l), there exists a countably generated $\mathbb{Z}$-algebra Z and a complex $\mathcal{F}_\bullet$ of finite free Z-modules, with the following property. If R is a d-dimensional local Cohen-Macaulay ring of multiplicity e, and M a finitely generated Cohen-Macaulay module of multiplicity at most l, then there exists a homomorphism $Z \to R$, such that for any n and any R-module N, we have*

$$\operatorname{Tor}_n^R(M,N) \cong H_n(\mathcal{F}_\bullet \otimes_Z N) \qquad \text{and} \qquad \operatorname{Ext}_R^n(M,N) \cong H^n(\operatorname{Hom}_Z(\mathcal{F}_\bullet, N)).$$

If we impose furthermore that R is regular (whence $e = 1$) or, more generally, that M has finite projective dimension, then we may take Z to be a finitely generated $\mathbb{Z}$-algebra and $\mathcal{F}_\bullet$ a complex of length d.

Proof. For each n, let $\delta_n := \Delta(d,e,l,n)$ be the bound given by Theorem 11.1, and let Ξ_n be a tuple of indeterminates viewed as a $\delta_{n-1} \times \delta_n$-matrix. Let Z be the polynomial ring over $\mathbb{Z}$ generated by all indeterminates Ξ_n modulo the relations $\Xi_n \cdot \Xi_{n+1} = 0$, expressing that the product of two consecutive matrices is zero. We then define the complex $\mathcal{F}_\bullet$ by letting its n-th term be Z^{δ_n}, and its n-th differential the matrix Ξ_n.

By construction, $\mathcal{F}_\bullet$ is a free complex. Now, given R and M as in the statement, Theorem 11.1 implies that we may assign to each entry in Ξ_n, a value in R so that under the induced map $Z \to R$, the complex $\mathcal{F}_\bullet \otimes_Z R$ becomes a free resolution of M. The statement now follows from the definition of Tor and Ext. □

The n-th *Bass number* $\mu_n(M)$ of a finitely generated R-module M is the vector space dimension of $\mathrm{Ext}^n_R(k, M)$, where k is the residue field of R. The q-th Bass number, with q equal to the depth of M, is also called the *type* of M.

11.3 Corollary. *The type (respectively, for each n, the n-th Bass number) of a finitely generated module M over a local Cohen-Macaulay ring R is bounded above by a function (in n) depending only on the dimension and multiplicity of R, and on the minimal number of generators of M.*

Proof. Since the depth of M is at most the dimension of R, it suffices to prove the claim for a fixed n. By Corollary 11.2, there is a resolution $F_\bullet$ of k by finite free R-modules F_n whose ranks $\beta_n(k)$ are bounded by the dimension and multiplicity of R. Since $\mathrm{Ext}^n_R(k, M)$ is the n-th cohomology of $\mathrm{Hom}_R(F_\bullet, M)$, its length $\mu_n(M)$ is at most the number of generators of $\mathrm{Hom}_R(F_n, M) \cong M^{\beta_n(k)}$, and the claim follows. □

Let us extend some definitions from [51]. We will call a homomorphism $R \to S$ of Noetherian local rings *formally etale* (or a *scalar extension*), if it is faithfully flat and unramified (=the maximal ideal of R extends to the maximal ideal of S). Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be Noetherian local rings, and let M be a finitely generated R-module and N a finitely generated S-module. We define the *jet distance* between M and N as the real number

$$d(M, N) := e^{-\alpha}$$

where α is the (possibly infinite) supremum of all j such that there exists an Artinian local ring T, together with formally etale extensions $R/\mathfrak{m}^j \to T$ and $S/\mathfrak{n}^j \to T$, yielding $M \otimes_R T \cong N \otimes_S T$. As shown in [51] (where the distance is only defined between rings), the jet distance is a (quasi-)metric, and, roughly speaking, up to a formally etale base change, limits in this metric space can be calculated by means of cataproducts.

11.4 Theorem. *For each quadruple of positive integers (d, e, l, n), there exists a bound $\varepsilon := \varepsilon(d, e, l, n) > 0$ such that if R and S are d-dimensional local Cohen-Macaulay rings of multiplicity e, and M and N are finitely generated Cohen-Macaulay modules of multiplicity at most l over R and S respectively, with $d(M, N) \leq \varepsilon$, then $\beta_n(N) = \beta_n(M)$.*

Proof. Suppose no such bound exists for the pair (d, e, l, n), resulting in a counterexample for each w, given by d-dimensional Cohen-Macaulay local rings $(R_w, \mathfrak{m}_w)$ and $(S_w, \mathfrak{n}_w)$ of multiplicity e, and finitely generated Cohen-Macaulay modules M_w and N_w of multiplicity at most e over R_w and S_w respectively, such that $d(M_w, N_w) \leq e^{-w}$, but $\beta_n(M_w) \neq \beta_n(N_w)$. Since Betti numbers are preserved under formally etale extensions, the techniques in [51] allows us to reduce to the case that the distance condition means that

$$R_w/\mathfrak{m}_w^w \cong S_w/\mathfrak{n}_w^w \quad \text{and} \quad M_w/\mathfrak{m}_w^w M_w \cong N_w/\mathfrak{n}_w^w N_w \tag{12}$$

Let $M_\natural$ and $N_\natural$ be the respective ultraproducts of the R_w, S_w, M_w, and N_w. By Corollary 8.8, the respective ultraproducts $R_\natural$ and $S_\natural$ are pseudo-Cohen-Macaulay local rings, and by Lemma 8.9, the respective ultraproducts $M_\natural$ and $N_\natural$ are finitely generated pseudo-Cohen-Macaulay modules over $R_\natural$ and $S_\natural$ respectively. Moreover, by Łos' Theorem and modding out infinitesimals, we get from (12) that the respective cataproducts $R_\sharp$ and $S_\sharp$ are isomorphic, as are the respective cataproducts $M_\sharp$ and $N_\sharp$. By Proposition 8.11, we therefore get for almost all w, the following contradictory equalities

$$\beta_n(M_w) = \beta_n(M_\sharp) = \beta_n(N_\sharp) = \beta_n(N_w).$$

□

11.5 Corollary. *Given a local Cohen-Macaulay ring R, there exists, for each $n \in \mathbb{N}$, a bound $\delta := \delta(n) > 0$ such that if M and N are maximal Cohen-Macaulay modules with $\mathrm{d}(M,N) \leq \delta$, then $\beta_n(N) = \beta_n(M)$.*

Proof. If $d(M,N) < 1$, then M and N have the same minimal number of generators m. In view of Theorem 11.4, it suffices to show that the multiplicity of M and N are uniformly bounded in terms of m. Let $\tilde{e}$ and $\tilde{q}$ be respectively the maximum multiplicity of $R/\mathfrak{p}$ and the maximal length of $R_\mathfrak{p}$, where $\mathfrak{p}$ runs over the finitely many d-dimensional prime ideals of R. Since we have a surjective map $R^m \to M$, tensoring with one of these d-dimensional prime ideals $\mathfrak{p}$ shows that the length of $M_\mathfrak{p}$ is at most $m\tilde{q}$. The bound on the multiplicity now follows from [9, Corollary 4.6.8]. □

11.6 Theorem. *If an equicharacteristic Cohen-Macaulay local ring with uncountable algebraically closed residue field has bounded multiplicity type (respectively, finite representation type), then so does its completion.*

In particular, to establish the Brauer-Thrall conjecture for an equicharacteristic Cohen-Macaulay local ring with uncountable algebraically closed residue field, it suffices to prove it for its completion.

Proof. Let k be the residue field of R, and let $\widehat{R}$ and $R_\sharp$ be the respective completion and catapower (with respect to a countable index set) of R. By Corollary 8.10, in either case does $R_\sharp$ have bounded multiplicity type. By Corollary 5.16, the cataproduct $R_\sharp$ is obtained by taking the completion of the base change $\widehat{R} \otimes_k k_\natural$, where $k_\natural$ is the ultrapower of k. Since k is algebraically closed and uncountable, $k_\natural \cong k$. In particular, the base change $M \otimes_{\widehat{R}} R_\sharp$ of any indecomposable maximal Cohen-Macaulay $\widehat{R}$-module M remains an indecomposable maximal Cohen-Macaulay $R_\sharp$-module. Since the multiplicity of $M \otimes_{\widehat{R}} R_\sharp$ is at most $\epsilon(R_\sharp)$, so is the multiplicity of M by faithfully flat descent, proving already that $\widehat{R}$ has bounded multiplicity type. Assume that R has in fact finite representation type. As in the proof of Corollary 8.10, we can find indecomposable maximal Cohen-Macaulay R-modules M_w with cataproduct equal to $M \otimes_{\widehat{R}} R_\sharp$. Since by assumption there are only finitely many indecomposable maximal Cohen-Macaulay R-modules, almost all M_w are equal to one of these, say N, and $M \otimes_{\widehat{R}} R_\sharp$ is just the catapower of N, that is to say, equal to $N \otimes_R R_\sharp$. By faithfully flat descent, $M = N \otimes_R \widehat{R}$. Note that we in fact proved that for a Cohen-Macaulay local ring of finite presentation type, any indecomposable maximal Cohen-Macaulay

$\widehat{R}$-module is obtained by base change from an indecomposable maximal Cohen-Macaulay R-module.

To prove the last assertion, assume that R has finite multiplicity type $\epsilon(R)$. By what we just proved, $\widehat{R}$ too has finite multiplicity type. If the Brauer-Thrall conjecture holds for $\widehat{R}$, then it has finite representation type. Let $N_1, \dots, N_s$ be all its indecomposable maximal Cohen-Macaulay modules. Let M be an indecomposable maximal Cohen-Macaulay R-module. Since $\widehat{M} = M \otimes \widehat{R}$ is then a maximal Cohen-Macaulay $\widehat{R}$-module, as are all of its direct summands, it is of the form

$$\widehat{M} \cong N_1^{e_1} \oplus \cdots \oplus N_s^{e_s}. \tag{13}$$

Since M, whence also $\widehat{M}$, has multiplicity at most $\epsilon(R)$, each e_i is at most $\epsilon(R)$. Hence there are only finitely many possible decompositions (13), proving that there are then also only finitely many possibilities for M. □

Proofs of Corollaries 1.4, 1.5 and 1.6

Assume that no such bound exists, so that we can find d-dimensional Cohen-Macaulay local rings R_w of multiplicity e, all of whose indecomposable maximal Cohen-Macaulay modules have multiplicity at most ϵ, but there are at least w many. Let $R_\sharp$ be their cataproduct, which therefore is a d-dimensional Cohen-Macaulay local ring of multiplicity e and multiplicity type at most ϵ. By assumption, it has only finitely many indecomposable, maximal Cohen-Macaulay modules. However, by choosing for each w some indecomposable maximal Cohen-Macaulay R_w-module, and taking their cataproduct, we get an indecomposable maximal Cohen-Macaulay module by Corollary 7.24, and we get infinitely many non-isomorphic ones in this fashion, contradiction.

The second corollary follows immediately from the definitions and Theorem 11.1. To prove the third, let e be the multiplicity of R/I. Since $I = xR$ for some regular element $x \in R$, the residue ring R/I is Cohen-Macaulay and has projective dimension one. Hence $\beta_1(R/I) = 1$ and $\beta_2(R/I) = 0$. Choose some $\varepsilon > 0$ as given by Theorem 11.4 such that $d(R/I, M) \leq \varepsilon$ implies that R/I and M have the same zeroth, first and second Betti number, for M a Cohen-Macaulay module of multiplicity at most e. Note that from $\beta_0(M) = \beta_0(R/I) = 1$ it follows that M is of the form R/J, so that in the statement, we did not even need to assume that M was cyclic. Choose a such that $e^{-a} \leq \varepsilon$. In particular, $d(R/I, R/J) \leq \varepsilon$, and therefore $\beta_1(R/J) = 1$, yielding that J is cyclic, and $\beta_2(R/J) = 0$, yielding that it is invertible. □

In terms of the *Poincare series* of a module M, defined as

$$P_R(M;t) := \sum_n \beta_n(M)t^n,$$

our results yield:

11.7 Corollary. *Over a fixed local Cohen-Macaulay ring, the Poincare series is a continuous map from the metric space of Cohen-Macaulay modules of multiplicity at most e (respectively, from the space of all maximal Cohen-Macaulay modules), to $\mathbb{Z}[[t]]$ with its t-adic topology.*

Proof. For each n, we can choose by Theorem 11.4 (respectively, by Corollary 11.5), an $\varepsilon > 0$ such that $d(M,N) \leq \varepsilon$ implies that the first n Betti numbers of M and N are the same, for M and N Cohen-Macaulay modules of multiplicity at most e (respectively, maximal Cohen-Macaulay modules). Hence $P_R(M;t) \equiv P_R(N;t) \mod t^n\mathbb{Z}[[t]]$. □

Although we did not formulate it here, we may even extend this result by also varying the base ring over all local Cohen-Macaulay rings of a fixed dimension and multiplicity; see [51, §8]. Applied to a regular local ring, we immediately get:

11.8 Corollary. *Let R be a regular local ring. For each e, there exists $\delta := \delta(e) > 0$ such that if M and N are Cohen-Macaulay modules of multiplicity at most e and $d(M,N) \leq \delta$, then $P_R(M;t) = P_R(N;t)$.* □

12. Uniform arithmetic

In this section, we prove several uniform bounds, and show that the existence of such bounds is often equivalent with a certain ring theoretic property. We start with examining the domain property. It is not true in general that the catapower of a domain is again a domain: let R be the local ring at the origin of the plane curve over a field k given by $f := \xi^2 - \zeta^2 - \zeta^3$. The catapower of R is $k_\natural[[\xi,\zeta]]/fk_\natural[[\xi,\zeta]]$, where $k_\natural$ is the ultrapower of k, and this is not a domain (since $1+\zeta$ has a square root in $k_\natural[[\xi,\zeta]]$). Clearly, the problem is that R is not *analytically irreducible*, that is to say, not a cata-domain.

Before we give a necessary and sufficient condition for a catapower to be a domain, let us introduce some terminology which makes for a smoother presentation of our results. Put $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. By an *$n$-ary numerical* function, we mean a map from $f\colon \bar{\mathbb{N}}^n \to \bar{\mathbb{N}}$, with the property that $f(s_1,\dots,s_n) = \infty$ if and only if one of the entries s_i is equal to ∞. Moreover, we will always assume that a numerical function f is non-decreasing in any of its arguments, that is to say, if $s_i \leq t_i$ for $i = 1,\dots,n$, then $f(s_1,\dots,s_n) \leq f(t_1,\dots,t_n)$. To indicate that a numerical function depends on a ring R, we will write the ring as a subscript.

Recall that R has *bounded multiplication* if there exists a binary numerical function μ_R (called a *uniformity function*) such that

$$\operatorname{ord}(xy) \leq \mu_R(\operatorname{ord}(x), \operatorname{ord}(y))$$

for all $x,y \in R$ (see §2.1 for the definition of order). In view of our definition of numerical function, the ideal of infinitesimals in a local ring with bounded multiplication is a prime ideal, and hence the separated quotient is a domain.

12.1 Theorem. *Let $(R,\mathfrak{m})$ be a Noetherian local ring. The following are equivalent:*

12.1.1. *R is analytically irreducible;*
12.1.2. *R has bounded multiplication;*
12.1.3. *some (equivalently, any) catapower $R_\sharp$ of R is a domain.*

Proof. The implication (12.1.2) ⇒ (12.1.1) is clear from the above discussion, since having bounded multiplication is easily seen to be preserved under completions. In order to prove (12.1.1) ⇒ (12.1.3), assume R is analytically irreducible and let $R_\sharp$ be its catapower. Since $\widehat{R}$ has the same catapower by Corollary 5.8, we may moreover assume that R is a complete Noetherian local ring. If R is normal, then so is $R_\sharp$ by Corollary 5.15, and hence again a domain. For the general case, let S be the normalization of R, so that $R \subseteq S$ is a finite extension. Since R is complete, S is again local. By Proposition 5.17, we get an extension $R_\sharp \subseteq S_\sharp$. Since we argued that $S_\sharp$ is a domain, the same therefore is true for $R_\sharp$.

Remains to show (12.1.3) ⇒ (12.1.2). By way of contradiction, suppose no bound exists for the pair (a, b). In other words, we can find $x_n, y_n \in R$ such that $\operatorname{ord}(x_n) = a$, $\operatorname{ord}(y_n) = b$ and $x_n y_n \in \mathfrak{m}^n$. Letting $x_\natural$ and $y_\natural$ be their respective ultraproducts, we get $\operatorname{ord}(x_\natural) = a$, $\operatorname{ord}(y_\natural) = b$ and $x_\natural y_\natural \in \mathfrak{I}_{R_\natural}$. Since $\mathfrak{I}_{R_\natural}$ is by assumption prime, $x_\natural$ or $y_\natural$ lies in $\mathfrak{I}_{R_\natural}$, neither of which is possible. □

12.2 Remark. The equivalence of (12.1.1) and (12.1.2) is well-known and is usually proven by a valuation argument. By [54, Theorem 3.4] and [27, Proposition 2.2] these conditions are also equivalent with the existence of a *linear* uniformity function: $\mu_R(a, b) := k_R \max\{a, b\}$, for some $k := k_R \in \mathbb{N}$, in which case we say that R has *k-bounded multiplication*. For a further result along these lines, see [32, Proposition 5.6].

By the same argument proving implication (12.1.3) ⇒ (12.1.2), we get:

12.3 Corollary. *Let R_n be Noetherian local rings of bounded embedding dimension. If (almost) all R_n have bounded multiplication with respect to the same uniformity function $\mu = \mu_{R_n}$, then so do their ultraproduct $R_\natural$ and cataproduct $R_\sharp$. In particular, $R_\sharp$ is a domain.* □

Note that the converse is not true. For instance, if R is a complete Noetherian local domain, then the cataproduct of the $R/\mathfrak{m}^n$ is a domain by Corollaries 5.10 and 12.3. If instead of order, we use degree (see §3.16 for the definition), we get the following analogue of bounded multiplication, this time in terms of a bound whose dependence on the ring is only through its embedding dimension.

12.4 Theorem. *There exists a ternary numerical function ω with the following property. For every Noetherian local ring R and any two elements $x, y \in R$, we have an inequality*

$$\deg(xy) \leq \omega(\operatorname{embdim}(R), \deg(x), \deg(y)).$$

Proof. Towards a contradiction, suppose such a function cannot be defined on the triple (m, a, b). This means that for each n, we can find a Noetherian local ring R_n of embedding dimension m and elements $x_n, y_n \in R_n$ such that $\deg(x_n) = a$, $\deg(y_n) = b$ and $\deg(x_n y_n) \geq n$. Let $R_\natural$, $x_\natural$ and $y_\natural$ be the respective ultraproduct of the R_n, the x_n and the y_n. Let d be the ultra-dimension of $R_\natural$, so that almost all R_n have dimension d. By Corollary 3.17, almost each R_n has parameter degree at most a and hence $R_\natural$ is isodimensional by Theorem 5.23. Hence $x_\natural$ and $y_\natural$ are both generic by Corollary 5.26, and hence so is their product $x_\natural y_\natural$ by Corollary 3.12. However, this contradicts Corollary 5.26 as the $x_n y_n$ have unbounded degree. □

From the exact sequence

$$R/xR \xrightarrow{y} R/xyR \to R/yR \to 0$$

where the first map is induced by multiplication by y, we see that $\deg(xy) \leq \deg(x) + \deg(y)$ for all x, y in a one-dimensional Noetherian local ring (in fact, if R is Cohen-Macaulay, then the first map is injective and we even have equality). I do not know what happens in higher dimensions.

12.5. Order versus degree

We next investigate the relationship between order and degree. If R is Cohen-Macaulay and x is R-regular, then the degree of R is just the multiplicity of R/xR. By [29, Theorem 14.9], we get $\operatorname{ord}(x) \leq \deg(x)/\operatorname{mult}(R)$. In particular, $\operatorname{ord}(x) \leq \deg(x)$, and this latter inequality could very well always be true (see also §12.19 below). At any rate, we have:

12.6 Corollary. *There exists a binary numerical function π with the following property. For every Noetherian local ring R and every element $x \in R$, we have an inequality*

$$\operatorname{ord}(x) \leq \pi(\operatorname{embdim}(R), \deg(x)).$$

Proof. Suppose for some pair (m, a), we have for each n, a counterexample $x_n \in \mathfrak{m}_n^n$ of degree a in the Noetherian local ring $(R_n, \mathfrak{m}_n)$ of embedding dimension m. Let $x_\natural \in R_\natural$ be the ultraproduct so that by Theorem 5.23, the degree of $x_\natural$ is a, yet $x_\natural \in \mathfrak{I}_{R_\natural}$, contradicting Corollary 3.17. □

Applying Corollary 12.6 to a product and then using Theorem 12.4, we get the existence of a ternary numerical function η such that for any Noetherian local ring R and elements $x, y \in R$, we have

$$\operatorname{ord}(xy) \leq \eta(\operatorname{embdim}(R), \deg(x), \deg(y)) \tag{14}$$

For Noetherian local rings that are analytically irreducible, order and degree are mutually bounded, and in fact, we have the following more precise result:

12.7 Theorem. *There exists a quaternary numerical function ζ with the following property. For every d-dimensional Noetherian local domain $(R, \mathfrak{m})$ of parameter degree at most e and k-bounded multiplication, and for every $x \in R$, we have an inequality*

$$\deg(x) \leq \zeta(d, e, k, \operatorname{ord}(x)).$$

Proof. It suffices to show that there exists a function β such that if $\operatorname{ord}(x) < a$ for some $x \in R$ and some $a \in \mathbb{N}$, then $\deg(x) < \beta(d, e, k, a)$. Suppose no such bound exists for the quadruple (d, e, k, a). Hence, for each n, we can find a d-dimensional Noetherian local domain $(R_n, \mathfrak{m}_n)$ of parameter degree at most e and k-bounded multiplication, and an element $x_n \notin \mathfrak{m}_n^a$ whose degree is at least n. Let $(R_\natural, \mathfrak{m}_\natural)$ and $x_\natural$ be the respective ultraproduct of the $(R_n, \mathfrak{m}_n)$ and the x_n. By Theorem 5.23, the geometric dimension of $R_\natural$ is d. Since the R_w/x_wR_w have dimension $d-1$, but unbounded parameter degree, the same theorem shows that the geometric dimension of $R_\natural/x_\natural R_\natural$ is strictly bigger than its ultra-dimension $d-1$, whence is also equal to d. In particular, $x_\natural$ is not generic. Since the cataproduct $R_\sharp$ is a domain by Corollary 12.3, we get $x_\natural \in \mathfrak{I}_{R_\natural}$ by Corollary 3.12. However, by Łos' Theorem, $x_\natural \notin \mathfrak{m}_\natural^a$, a contradiction. □

Whereas order is a filtering function (see §2), inducing the $\mathfrak{m}$-adic filtration on R, this is no longer true for degree. For instance, let R be the local ring at the origin of the curve with equation $\xi\zeta + \xi^3 + \zeta^3 = 0$. Then both ξ and ζ have degree three, but their sum $\xi + \zeta$ has degree two. As we will see below in §12.19, on regular local rings, degree is filtering. Can one characterize in general rings for which degree is filtering? Is, for every n, the set of elements having degree at least n always a finite union of ideals? In other words, as far as its properties are concerned, degree is still a mysterious function. However, its main use in this paper is to characterize properties via its asymptotic behavior, as we will now discuss.

12.8. Characterizations through uniform behavior

Recall that a Noetherian local ring is *analytically unramified* if its completion is reduced. Any reduced excellent local ring is analytically unramified ([29, Theorem 32.2]).

12.9 Corollary. *A Noetherian local ring R is analytically unramified if and only if there exists a numerical function ν_R, such that for every $x \in R$, we have an inequality*

$$\operatorname{ord}(x^2) \leq \nu_R(\operatorname{ord}(x)).$$

Proof. Since order remains unaffected by completion, we may assume that R is moreover complete. Suppose R is reduced. It suffices to show that there exists a function ν_R such that $x^2 \in \mathfrak{m}^{\nu_R(b)}$ implies $x \in \mathfrak{m}^b$. By way of contradiction, suppose this is false for b. Hence, we can find $x_n \in R$ such that $x_n^2 \in \mathfrak{m}^n$, but $x_n \notin \mathfrak{m}^b$. Let $R_\natural$ be the ultrapower of R and let $x_\natural$ be the ultraproduct of the x_n. By Łos' Theorem, $x_\natural^2 \in \mathfrak{I}_{R_\natural}$ and $x_\natural \notin \mathfrak{m}^b R_\natural$. However, $\mathfrak{I}_{R_\natural}$ is radical by Corollary 5.15. Hence $x_\natural^2 \in \mathfrak{I}_{R_\natural}$ implies $x_\natural \in \mathfrak{I}_{R_\natural}$, contradiction.

Conversely, let the function ν_R be as asserted. If x^2 is zero, then its order is infinite. The only way that this can be bounded by $\nu_R(\operatorname{ord}(x))$, is for x to have infinite order too, meaning that $x = 0$. This shows that R is reduced. □

By a similar argument, one easily shows that if all R_n have bounded squares (in the sense of the corollary) with respect to the same function $\nu = \nu_{R_n}$, then their cataproduct is reduced. If R is analytically irreducible, then the results of Hübl-Swanson (see Remark 12.2) imply that we may take $\nu_R(b)$ of the form $k_R b$ for some k_R and all b. I do not know whether this is still true in general. Similarly, for the bounds we are about to prove, is their still some vestige of linearity?

12.10 Corollary. *A Noetherian local ring R is analytically irreducible if and only if there exists a numerical function ξ_R such that for every $x \in R$, we have an inequality*

$$\deg(x) \leq \xi_R(\operatorname{ord}(x)).$$

Proof. In view of Remark 12.2, the direct implication follows from an application of Theorem 12.7. As for the converse, suppose degree is bounded in terms of order. Since both order and degree remain the same after passing to the completion, we may moreover assume R is complete. Since a non-zero element has finite order, it has finite degree whence is generic. This shows that there are no non-zero prime ideals of maximal dimension, which in turn forces the zero ideal to be a prime ideal. □

Tweaking (14) slightly (for a fixed ring R), we can characterize the following property. Recall that a Noetherian local ring R is called *unmixed*, if each associated prime $\mathfrak{p}$ of its completion $\widehat{R}$ has the same dimension as R; if the above is only true for minimal primes of $\widehat{R}$, then we say that R is *quasi-unmixed* (also called *formally equidimensional*).

12.11 Lemma. *If a Noetherian local ring is (quasi-)unmixed, then so is its catapower.*

Proof. By Corollary 5.8, we may assume R is a complete (quasi-)unmixed Noetherian local ring. Let us first show that the catapower $R_\sharp$ is quasi-unmixed. In any case, R and $R_\sharp$ have the same dimension, say d. Since $R_\sharp$ is complete by Lemma 5.6, we need to show that every minimal prime $\mathfrak{q} \subseteq R_\sharp$ has dimension d. Since R is complete, it is of the form S/I for some complete regular local ring S and some ideal $I \subseteq S$. By Corollary 5.15, the catapower $S_\sharp$ of S is regular, whence a domain. Let $\mathfrak{p} := \mathfrak{q} \cap R$ and $\mathfrak{P} := \mathfrak{p} \cap S$. By flatness, $\mathfrak{p}$ is a minimal prime of R by [29, Theorem 15.1], whence has dimension d, as R is equidimensional.

Since $S \to S_\sharp$ is flat, $S_\sharp/\mathfrak{P}S_\sharp$ is equidimensional by [29, Theorem 31.5]. Since $S_\sharp/IS_\sharp \cong R_\sharp$, we get $S_\sharp/\mathfrak{P}S_\sharp \cong R_\sharp/\mathfrak{p}R_\sharp$. Since $\mathfrak{q}$ is necessarily a minimal prime of $\mathfrak{p}R_\sharp$, equidimensionality yields that $R_\sharp/\mathfrak{q}$ and $R_\sharp/\mathfrak{p}R_\sharp$ have the same dimension. Since $R_\sharp/\mathfrak{p}R_\sharp$ is the catapower of $R/\mathfrak{p}$, this dimension is just d, showing that $\mathfrak{q}$ is a d-dimensional prime.

Assume next that R is unmixed. Since R has no embedded primes, it satisfies Serre's condition (S_1), whence so does $R_\sharp$ by Corollary 5.15 and [29, Theorem 23.9]. Since we already know that $R_\sharp$ is quasi-unmixed, it is in fact unmixed. □

12.12 Theorem. *A Noetherian local ring R is unmixed if and only if there is a binary numerical function χ_R such that for every $x, y \in R$, we have an inequality*

$$\operatorname{ord}(xy) \leq \chi_R(\deg(x), \operatorname{ord}(y)). \tag{15}$$

Proof. Assume first that R is unmixed. Since degree and order remain the same when we pass to the completion, we may assume R is complete. By Lemma 12.11, the catapower $R_\sharp$ is then also unmixed. By way of contradiction, assume that for some pair (a, b), we can find elements $x_n, y_n \in R$ with $\deg(x_n) \leq a$ and $\operatorname{ord}(y_n) \leq b$, such that $x_n y_n \in \mathfrak{m}^n$. Hence, in the ultrapower $R_\natural$ of R, the ultraproduct $x_\natural$ of the x_n has degree at most a and the ultraproduct $y_\natural$ of the y_n has order at most b, but $x_\natural y_\natural \in \mathfrak{I}_{R_\natural}$. Since $x_\natural$ has finite degree, it is generic and hence its image in $R_\sharp$ lies outside any prime of maximal dimension. Since $R_\sharp$ is unmixed, $x_\natural$ is therefore $R_\natural$-regular and hence $y_\natural = 0$ in $R_\sharp$, contradicting that its order is at most b.

Conversely, assume a function χ with the proscribed properties exists and let x be a generic element, say, of degree a. We have to show that x is R-regular. If not, then

$xy = 0$ for some non-zero y, say, of order b. However, the order of xy is bounded by $\chi(a,b)$, contradiction. $\square$

By the same argument, one easily proves that the cataproduct of Noetherian local rings R_n of bounded embedding dimension is unmixed, provided almost each R_n satisfies the hypothesis of the statement with respect to the same uniformity function $\chi = \chi_{R_n}$. In order to characterize quasi-unmixedness, we have to introduce one more invariant. Given a Noetherian local ring R, we define its *nilpotency degree* to be the least t such that $\mathfrak{n}^t = 0$, where $\mathfrak{n}$ is the nilradical of R. Hence R is reduced if and only if its nilpotency degree is one.

12.13 Proposition. *A Noetherian local ring R of nilpotency degree at most t is quasi-unmixed if and only if there exists a binary numerical function θ_R such that for every $x, y \in R$, we have an inequality*

$$\operatorname{ord}((xy)^t) \leq \theta_R(\deg(x), \operatorname{ord}(y^t)).$$

Proof. Again, we may pass to the completion of R, since all invariants remain unchanged under completion, and assume from the start that R is complete. Suppose that θ_R has the above property. To show that R is quasi-unmixed, which in the complete case is just being equidimensional, we need to show that any generic element x lies outside any minimal prime of R. A moment's reflection shows that this is equivalent with showing that x is R_{red}-regular. Hence, towards a contradiction, assume $y \in R$ is a non-nilpotent element in R such that xy is nilpotent. By definition of t, this means $y^t \neq 0$, but $(xy)^t = 0$. However, the order of $(xy)^t$ is bounded by the finite number $\theta_R(\deg(x), \operatorname{ord}(y^t))$, contradiction.

Conversely, assume R is equidimensional, but no function θ_R can be defined for some pair (a, b). Hence we can find counterexamples $x_n \in R$ of degree a and $y_n \in R$ such that $y_n^t \notin \mathfrak{m}^{b+1}$, but $(x_n y_n)^t \in \mathfrak{m}^n$. Let $x_\natural$, $y_\natural$ and $R_\natural$ be the respective ultraproducts, so that $x_\natural$ is generic by Corollary 5.26, and $y_\natural^t \notin \mathfrak{m}^{b+1} R_\natural$, but $(x_\natural y_\natural)^t \in \mathfrak{I}_{R_\natural}$ by Łos' Theorem. However, by Lemma 12.11, the cataproduct $R_\sharp$ is again equidimensional (note that $R_\sharp$ is complete), and therefore, $x_\natural$, being generic in $R_\sharp$, is $(R_\sharp)_{\text{red}}$-regular. Hence $(x_\natural y_\natural)^t = 0$ in $R_\sharp$ yields that $y_\natural$ is nilpotent in $R_\sharp$. Let $\mathfrak{n}$ be the nilradical of R. Since $R_\sharp/\mathfrak{n}R_\sharp$ is the catapower of $R_{\text{red}} = R/\mathfrak{n}$ by Corollary 5.7, it is reduced by Corollary 5.15. This proves that the nilradical of $R_\sharp$ is just $\mathfrak{n}R_\sharp$ and hence in particular, $R_\sharp$ has nilpotency degree t too. Therefore, $y_\natural^t = 0$, contradicting that $y_\natural^t \notin \mathfrak{m}^{b+1} R_\sharp$. $\square$

12.14 Theorem. *A d-dimensional Noetherian local ring R is Cohen-Macaulay if and only if there exists a binary numerical function δ_R such that for all d-tuples $\mathbf{x} := (x_1, \dots, x_d)$ and $(y_1, \dots, y_d)$ with $\mathbf{x}$ a system of parameters, we have an inequality*

$$\operatorname{ord}_R(x_1 y_1 + \cdots + x_d y_d) \leq \delta_R(\ell(R/\mathbf{x}R), \operatorname{ord}_{R/(x_1,\dots,x_{d-1})R}(y_d)). \tag{16}$$

Moreover, the function δ_R only depends on the dimension and the multiplicity of R.

Proof. Assume first a function δ_R with the asserted properties exists. In order to prove that R is Cohen-Macaulay, we take a system of parameters $(z_1, \dots, z_d)$ and show that it is R-regular. Fix some i and suppose $a_1z_1 + \cdots + a_iz_i = 0$. We need to show that $a_i \in I := (z_1, \ldots, z_{i-1})R$. Fix some k and define x_j and y_j as follows. If $j = 1, \ldots, d-i$, then $x_j := z^k_{i+j}$ and $y_j := 0$; if $j = d-i+1, \ldots, d$, then $x_j := z_{i+j-d}$ and $y_j := a_{i+j-d}$. In other words, we have

$$\begin{aligned} \mathbf{x} &:= (x_1, \ldots, x_d) = (z^k_{i+1}, \ldots, z^k_d, z_1, \ldots, z_i) \\ \mathbf{y} &:= (y_1, \ldots, y_d) = (0, \ldots, 0, a_1, \ldots, a_i) \\ x_1y_1 &+ \cdots + x_dy_d = a_1z_1 + \cdots + a_iz_i = 0. \end{aligned} \tag{17}$$

Apply (16) to these two tuples $\mathbf{x}$ and $\mathbf{y}$. Since $\mathbf{x}$ is again a system of parameters, $\ell(R/\mathbf{x}R)$ is finite. Hence, by the last equation in (17), the order of $y_d = a_i$ in $R/(x_1, \ldots, x_{d-1})R$ must be infinite, that is to say,

$$a_i \in (x_1, \ldots, x_{d-1})R = I + (z^k_{i+1}, \ldots, z^k_d)R.$$

Since this holds for all k, Krull's intersection theorem yields $a_i \in I$.

To prove the converse, suppose R is Cohen-Macaulay, but $\delta_R(a, b)$ is undefined for some pair (a, b). This means that there exists for each n, a system of parameters $\mathbf{x}_n := (x_{1n}, \ldots, x_{dn})$ such that $R/\mathbf{x}_nR$ has length a, and a d-tuple $\mathbf{y}_n := (y_{1n}, \ldots, y_{dn})$, such that

$$\mathrm{ord}_{R/(x_{1n},\ldots,x_{d-1,n})R}(y_{dn}) = b$$

and $x_{1n}y_{1n} + \cdots + x_{dn}y_{dn}$ has order at least n. Let $\mathbf{x}_\natural := (x_{1\natural}, \ldots, x_{d\natural})$ and $y_{i\natural}$ be the respective ultraproducts of the $\mathbf{x}_n$ and y_{in} inside the ultrapower $R_\natural$ of R. By Łos' Theorem, the order of $y_{d\natural}$ in $R_\natural/(x_{1\natural}, \ldots, x_{d-1\natural})R_\natural$ is b, the length of $R_\natural/\mathbf{x}_\natural R_\natural$ is a, and the sum $x_{1\natural}y_{1\natural} + \cdots + x_{d\natural}y_{d\natural}$ is an infinitesimal. In particular, the image of $\mathbf{x}_\natural$ in the catapower $R_\sharp$ is a system of parameters, whence $R_\sharp$-regular, since $R_\sharp$ is Cohen-Macaulay by Corollary 5.15. Since $x_{1\natural}y_{1\natural} + \cdots + x_{d\natural}y_{d\natural} = 0$ in $R_\sharp$, regularity forces $y_{d\natural}$ to be in the ideal $(x_{1\natural}, \ldots, x_{d-1\natural})R_\sharp$, contradicting that its order in $R_\sharp/(x_{1\natural}, \ldots, x_{d-1\natural})R_\sharp$ is finite.

To prove the final statement, observe that for fixed dimension d and multiplicity e, we may modify the above proof by taking each counterexample $\mathbf{x}_n$ and $\mathbf{y}_n$ in some d-dimensional local Cohen-Macaulay ring R_n of multiplicity e. Indeed, by Corollary 8.8, the cataproduct $R_\sharp$ of the R_n is again Cohen-Macaulay so that we can copy the above argument. □

One can view the previous result as a quantitative version of the unmixedness theorem. Namely, we can rewrite condition (16) as follows: for any $d-1$-tuple $\mathbf{z}$ and any $x, y \in R$, if $\mathbf{z}$ is part of a system of parameters, then

$$\mathrm{ord}_{R/\mathbf{z}R}(xy) \leq \delta_R(\deg_{R/\mathbf{z}R}(x), \mathrm{ord}_{R/\mathbf{z}R}(y)). \tag{18}$$

Comparing this with (15), we can now rephrase Theorem 12.14 using the following terminology: by a *curve*, we mean a one-dimensional subscheme C of $X := \mathrm{Spec}(R)$; we call a curve C a *complete intersection in* X if it is of the form $\mathrm{Spec}(R/I)$ with I an ideal generated by $\dim R - 1$ elements; we call C *unmixed*, if its coordinate ring is (note that this is equivalent with C being Cohen-Macaulay).

12.15 Corollary. *A Noetherian local ring R is Cohen-Macaulay if and only if every complete intersection curve C in $\operatorname{Spec}(R)$ is unmixed with respect to a uniformity function $\chi = \chi_C$ (as given by Theorem 12.12) independent from C.* □

We can depart from other criteria for Cohen-Macaulayness to get some more uniformity characterizations. For instance, we could use the criterion proven in [50, Corollary 5.2.11] that R is Cohen-Macaulay if and only if every system of parameters $\mathbf{x} := (x_1, \dots, x_d)$ is *independent*, in the sense that a relation $x_1y_1 + \cdots + x_dy_d = 0$ implies that all y_i lie in $\mathbf{x}R$. Thus, we get the following modified form of (16): *a d-dimensional Noetherian local ring R is Cohen-Macaulay if and only if there exists a binary numerical function δ'_R such that for every two d-tuples $\mathbf{x} := (x_1, \dots, x_d)$ and $(y_1, \dots, y_d)$, we have an inequality*

$$\operatorname{ord}_R(x_1y_1 + \cdots + x_dy_d) \leq \delta'_R(\ell(R/\mathbf{x}R), \operatorname{ord}_{R/\mathbf{x}R}(y_d)).$$

Next, we characterize normality:

12.16 Theorem. *A Noetherian local ring R is normal if and only if there exists a binary numerical function ε_R such that for all $x, y, z \in R$, we have an inequality*

$$\min_k \{\operatorname{ord}_{R/z^kR}(xy^k)\} \leq \varepsilon_R(\operatorname{ord}(x), \operatorname{ord}_{R/zR}(y)). \tag{19}$$

Proof. Suppose R is normal, but ε_R cannot be defined for a pair (a, b). Hence, for each n, there exist elements $x_n, y_n, z_n \in R$ such that x_n has order a and y_n has order b modulo z_nR, but $\operatorname{ord}_{R/z_n^kR}(x_ny_n^k) \geq n$ for all k. Let $x_\natural, y_\natural, z_\natural \in R_\natural$ be the respective ultraproducts of $x_n, y_n, z_n \in R$. In particular, $x_\natural$ is non-zero in the catapower $R_\sharp$ and $y_\natural \notin z_\natural R_\sharp$. On the other hand, since $x_\natural y_\natural^k \in z_\natural^k R_\sharp$ for all k, a well-known criterion shows that $y_\natural$ lies in the integral closure of $z_\natural R_\sharp$. Since $R_\sharp$ is normal by Corollary 5.15, any principal ideal is integrally closed, so that $y_\natural \in z_\natural R_\sharp$, contradiction.

Conversely, assume a numerical function ε_R exists with the proscribed properties. Taking $z = 0$ in (19), we see that R is a domain by Theorem 12.1. Suppose y/z is an element in the field of fractions of R which is integral over R. We want to show that $y/z \in R$. Since y is then in the integral closure of zR, there exists a non-zero x such that $xy^k \in z^kR$ for all k. The left hand side in (19) is therefore infinite, whence so must the right hand side be, forcing $y \in zR$. □

In our last two examples, we show how also tight closure conditions fit in our present program of characterizing properties by certain uniform behavior. We will adopt the usual tight closure notation of writing $I^{[q]}$ as an abbreviation for the ideal $(w_1^q, \dots, w_n^q)R$, where $I := (w_1, \dots, w_n)R$ is some ideal and q is some power of the prime characteristic p of R. An element $y \in R$ lies in the *tight closure* I^* of I, if there exists $c \in R$ outside all minimal prime ideals, such that $cy^q \in I^{[q]}$ for all powers q of p. We say that R is *F-rational* if some parameter ideal is tightly closed, in which case every parameter ideal is tightly closed (recall that a *parameter ideal* is an ideal generated by a system of parameters). On the other hand, if every ideal is tightly closed, then we call R *weakly F-regular*.

12.17 Theorem. *An excellent local ring R of characteristic p is pseudo-rational if and only if there exists a ternary numerical function φ_R such that for all elements $x, y \in R$ and every (equivalently, some) parameter ideal I, we have an inequality*

$$\min_q \{\operatorname{ord}_{R/I^{[q]}}(xy^q)\} \leq \varphi_R(\deg(x), \ell(R/I), \operatorname{ord}_{R/I}(y)) \qquad (20)$$

where q runs over all powers of p.

Proof. We will use Smith's tight closure characterization [52] that R is pseudo-rational if and only if it is F-rational. Assume first that R is pseudo-rational whence F-rational, but a numerical function φ_R cannot be defined on the triple (a, b, c). Hence there exist for each n, elements $x_n, y_n \in R$ and a parameter ideal I_n in R such that x_n has degree a and R/I_n is an Artinian local ring of length b in which y_n has order c, but $\operatorname{ord}_{R/I_n^{[q]}}(xy^q) \geq n$ for all powers q of p. Let $x_\natural, y_\natural, I_\natural$ be the respective ultraproducts of the x_n, y_n, I_n and let $R_\sharp$ be the catapower of R. Let J be a parameter ideal in R. Hence $JR_\sharp$ is a parameter ideal in $R_\sharp$. Since $R \to R_\sharp$ is regular by Corollary 5.15 and since J is tightly closed, so is $JR_\sharp$ by [28, Theorem 131.2] or [24], showing that $R_\sharp$ is F-rational.

Since a pseudo-rational local ring is a domain, $x_\natural$ is generic in $R_\sharp$ and $I_\natural R_\sharp$ is a parameter ideal in $R_\sharp$. Moreover, $y_\natural \notin I_\natural R_\sharp$, but $x_\natural y_\natural^q \in I_\natural^{[q]} R_\sharp$ for all q. By definition of tight closure, $y_\natural \in (I_\natural R_\sharp)^*$. In particular, every parameter ideal, including $I_\natural R_\sharp$, is tightly closed and hence $y_\natural \in I_\natural R_\sharp$, contradiction.

Conversely, assume φ_R satisfies (20) for some parameter ideal I. To verify that R is F-rational, let $y \in I^*$. Hence, for some $x \in R$ not in any minimal prime, $xy^q \in I^{[q]}$ for all q. The left hand side of (20) is therefore infinite whence so is the right hand side. Since x is generic, whence has finite degree, the third argument must be infinite, that is to say, $y \in I$. □

12.18 Theorem. *A Noetherian local ring $(R, \mathfrak{m})$ of characteristic p is weakly F-regular if and only if there exists a ternary numerical function ψ_R such that for all elements $x, y \in R$ and all $\mathfrak{m}$-primary ideals I, we have an inequality*

$$\min_q \{\operatorname{ord}_{R/I^{[q]}}(xy^q)\} \leq \psi_R(\deg(x), \ell(R/I), \operatorname{ord}_{R/I}(y)) \qquad (21)$$

where q runs over all powers of p.

Proof. Note that for R to be weakly F-regular, it suffices that every $\mathfrak{m}$-primary ideal is tightly closed, since by Krull's Intersection Theorem, any ideal is an intersection of $\mathfrak{m}$-primary ideals. Moreover, if R is weakly F-regular, then so is its catapower $R_\sharp$ by [24, Theorem 7.3] in conjunction with Corollary 5.15. In view of these facts, the proof is now almost identical to the one for Theorem 12.17; details are left to the reader. □

12.19. Epilogue: characterization of regularity

Let me make a few further observations, although they do no longer relate to our proof method. If R is regular, then in fact $\operatorname{ord}(xy) = \operatorname{ord}(x) + \operatorname{ord}(y)$. However, the latter condition does not characterize regularity, but only the strictly weaker condition that the associated graded ring $\operatorname{gr}(R)$ is a domain. The following condition, however, does characterize regularity: *a Noetherian local ring R is regular if and only if* $\operatorname{ord}(x) = \deg(x)$ *for all* $x \in R$. Indeed, if R is regular and $\operatorname{ord}(x) = a$, then by judiciously choosing a regular system of parameters $(x_1, \dots, x_d)$, we can ensure that x still has order a in $V := R/(x_1, \dots, x_{d-1})R$. Since V is a discrete valuation ring with uniformizing parameter x_d, one checks that $\ell(V/xV) = a$. Since $\deg(x) \leq \ell(R/(x, x_1, \dots, x_{d-1})R) = a$, we get $\deg(x) \leq \operatorname{ord}(x)$. The other inequality follows from our discussion in §12.5.

Conversely, if order and degree agree, then in particular there exists an element of degree one, and hence a system of parameters $\mathbf{x}$ such that $R/\mathbf{x}R$ has length one, whence is a field, showing that $\mathbf{x}$ is a regular system of parameters. □

13. Asymptotic homological conjectures in mixed characteristic

In [39, 47], we derived asymptotic versions of the homological conjectures for local rings of mixed characteristic p, where by *asymptotic*, we mean that the residual characteristic p must be large with respect to the complexity of the data. In the above papers, complexity was primarily given in terms of the degrees of the polynomials defining the data. In this paper, we phrase complexity in terms of (natural) invariants of the ring and the data only.

Improved New Intersection Theorem

To not have to repeat each time the conditions from this theorem, we make the following definition: given a finite complex $F_\bullet$ of finitely generated free R-modules, a *finite free complex*, for short, we say that its *rank* is at most r, if all F_i have rank at most r; and we say that its *INIT-degree* is at most l, if each $H_i(F_\bullet)$, for $i > 0$, has length at most l, and $H_0(F_\bullet)$ has a minimal generator generating a submodule of length at most l. Recall that the *length* of $F_\bullet$ is the largest n such that $F_n \neq 0$.

13.1 Theorem (Asymptotic Improved New Intersection Theorem)**.** *For each triple of non-negative integers (m, r, l), there exists a bound $\kappa(m, r, l)$ with the following property. Let R be a Noetherian local ring of mixed characteristic p and of embedding dimension at most m. If $F_\bullet$ is a finite free complex of rank at most r and INIT-degree at most l, then its length is at least the dimension of R, provided $p \geq \kappa(m, r, l)$.*

Proof. Since the dimension of R is at most m, there is nothing to show for complexes of length m or higher. Suppose the result is false for some triple (m, r, l). This means that for infinitely many distinct prime numbers p_w, we can find a d_w-dimensional Noetherian local ring $(R_w, \mathfrak{m}_w)$ of mixed characteristic p_w and embedding dimension at most m, and we can find a finite free complex $F_{\bullet w}$ of rank at most r, of length $s_w \leq m$, and of INIT-degree at most l, such that $s_w < d_w$. Choose a non-principal ultrafilter and let $(R_\natural, \mathfrak{m}_\natural)$ be the ultraproduct of the $(R_w, \mathfrak{m}_w)$. Since $s_w < d_w \leq m$, their respective ultraproducts satisfy $s < d \leq m$. By Theorem 5.19, the geometric dimension of $R_\natural$ is at least d. Let $F_{\bullet\natural}$ be the ultraproduct of the complexes $F_{\bullet w}$. Since the ranks are at most r, each module in $F_{\bullet\natural}$ is a free $R_\natural$-module of rank at most r. Since ultraproducts commute with homology, and preserve uniformly bounded length by Proposition 5.13, the higher homology groups $H_i(F_{\bullet\natural})$ have finite length (at most l). Furthermore, by assumption, we can find a minimal generator μ_w of $H_0(F_{\bullet w})$ generating a submodule of length at most l. Hence the ultraproduct $\mu_\natural$ of the μ_w is by Łos' Theorem a minimal generator of $H_0(F_{\bullet\natural})$, generating a submodule of length at most l. In conclusion, $F_{\bullet\natural}$ has INIT-degree at most l. In particular, $F_{\bullet\natural}$ is acyclic when localized at a non-maximal prime ideal, and hence (10.9.1) from Corollary 10.9 applies, yielding that $s \geq \operatorname{gdim}(R_\natural) \geq d$, contradiction. □

We can even give an asymptotic version of Theorem 10.8, albeit in terms of some less natural bounds.

13.2 Theorem. *For each triple of non-negative integers (m, r, l), there exists a bound $\sigma(m, r, l)$ with the following property. Let $(R, \mathfrak{m})$ be a Noetherian local ring of mixed characteristic p and of embedding dimension at most m, and let $F_\bullet$ be a finite free complex of rank at most r. Let M be the cokernel of $F_\bullet$, and let μ be a non-zero minimal generator of M. Assume each $R/I_k(F_\bullet)$ has dimension at most $\dim R - k$ and parameter degree at most l, for $k \geq 1$, and $R/\operatorname{Ann}_R(\mu)$ has parameter degree at most l.*

If $p \geq \sigma(m, r, l)$, then the length of the complex $F_\bullet$ is at least the codimension of $\operatorname{Ann}_R(\mu)$.

Proof. Suppose the result is false for some triple (m, r, l). This means that for infinitely many distinct prime numbers p_w, we can find a d_w-dimensional mixed characteristic Noetherian local ring $(R_w, \mathfrak{m}_w)$ whose residue field has characteristic p_w and whose embedding dimension is at most m, and we can find a finite free complex $F_{\bullet w}$ of length s_w and of rank at most r, and a non-zero minimal generator μ_w of its cokernel M_w such that $R_w/I_k(F_{\bullet w})$ has dimension at most $d_w - k$ and parameter degree at most l, for all $k = 1, \dots, s_w$, and such that $R_w/\operatorname{Ann}_{R_w}(\mu_w)$ has parameter degree at most l, but dimension strictly less than $d_w - s_w$. Choose a non-principal ultrafilter and let $(R_\natural, \mathfrak{m}_\natural)$ be the ultraproduct of the $(R_w, \mathfrak{m}_w)$. Since $s_w \leq d_w \leq m$, their respective ultraproducts satisfy $s \leq d \leq m$. By Theorem 5.19, the geometric dimension of $R_\natural$ is at least d. Let $F_{\bullet\natural}$ and μ be the ultraproduct of the complexes $F_{\bullet w}$ and the minimal generators μ_w respectively. Since the ranks are at most r, each module in $F_{\bullet\natural}$ will be a free $R_\natural$-module of rank at most r. By Theorem 5.23, the geometric dimension of $R_\natural/I_k(F_{\bullet\natural})$ is at most $d-k$, for all $k = 1, \dots, s$. Also by Łos' Theorem, μ is a minimal generator of the cokernel of $F_{\bullet\natural}$ and $R_\natural/\operatorname{Ann}_{R_\natural}(\mu)$, being the ultraproduct of the $R_w/\operatorname{Ann}_{R_w}(\mu_w)$, has geometric dimension strictly less than $d - s$ by Theorem 5.23. However, this is in contradiction with Theorem 10.8, which yields that $R_\natural/\operatorname{Ann}_{R_\natural}(\mu)$ has geometric dimension at least $d - s$. □

Using the same techniques, we can deduce from Theorem 10.12 the following asymptotic version (details are left to the reader).

13.3 Theorem (Asymptotic Canonical Element Theorem)**.** *For each triple of non-negative integers (m, r, l), there exists a bound $\rho(m, r, l)$ with the following property. Let R be a d-dimensional Noetherian local ring of mixed characteristic p and embedding dimension at most m, and let $F_\bullet$ be a free resolution of the residue field k of R, of rank at most r.*

If $\mathbf{x}$ *is a system of parameters in* R *such that* $R/\mathbf{x}R$ *has length at most* l *and if the morphism of complexes* $\gamma\colon K_\bullet(\mathbf{x}) \to F_\bullet$ *extends the natural homomorphism* $R/\mathbf{x}R \to k$*, then* $\gamma_d \neq 0$*, provided* $p \geq \rho(m, r, l)$. □

13.4 Remark. Perhaps it is not entirely justified to call this theorem a 'canonical element theorem', since it does not necessarily produce a canonical element in local cohomology like it does in the equicharacteristic case. This is due to the fact that we can not apply the theorem to the various 'powers' of a system of parameters as in the discussion in [9, p. 346-347] without having to raise the bound $\rho(n, r, l)$. In particular, the above result does not imply an asymptotic version of the Direct Summand conjecture.

Ramification

Instead of requiring that the residual characteristic is large in the above asymptotic results, we can also require the ramification to be large, as we will now explain. For the proofs, we only need to apply the corresponding versions in §10 for infinitely ramified local rings of finite embedding dimension. The main observation is the following immediate corollary of Łos' Theorem:

13.5 Lemma. *Let* R_w *be Noetherian local rings of mixed characteristic* p *and embedding dimension* m*. If for each* n*, almost all* R_w *have ramification index at least* n*, then their ultraproduct* $R_\natural$ *is infinitely ramified and hence their cataproduct* $R_\sharp$ *has equal characteristic* p. □

13.6 Theorem. *For each triple of non-negative integers* (m, r, l)*, there exists a non-negative integer* $\kappa(m, r, l)$ *with the following properties. Let* $(R, \mathfrak{m})$ *be a* d*-dimensional mixed characteristic Noetherian local ring of embedding dimension at most* m*, and let* $F_\bullet$ *be a finite free complex of rank at most* r*. If the ramification index of* R *is at least* $\kappa(m, r, l)$*, then the following are true:*

13.6.1. *If* $F_\bullet$ *has INIT-degree at most* l*, then the length of* $F_\bullet$ *is at least* d*.*

13.6.2. *If each* $R/I_k(F_\bullet)$ *has dimension at most* $d - k$ *and parameter degree at most* l*, for* $k \geq 1$*, and if* μ *is a non-zero minimal generator of the cokernel of* $F_\bullet$ *such that* $R/\operatorname{Ann}_R(\mu)$ *has parameter degree at most* l*, then the length of* $F_\bullet$ *is at least the codimension of* $\operatorname{Ann}_R(\mu)$*.*

13.6.3. *If* $F_\bullet$ *is a free resolution of* $R/\mathfrak{m}$*, if* $\mathbf{x}$ *is a system of parameters in* R *such that* $R/\mathbf{x}R$ *has length at most* l *and if the morphism of complexes* $\gamma\colon K_\bullet(\mathbf{x}) \to F_\bullet$ *extends the natural homomorphism* $R/\mathbf{x}R \to R/\mathfrak{m}$*, then* $\gamma_d \neq 0$*.*

Proof. Suppose first that such a bound for a triple (m, r, l) cannot be found in a fixed residual characteristic p. In other words, we can find mixed characteristic p Noetherian local rings R_w, whose embedding dimension is at most m, and whose ramification index is at least w, satisfying the negation of one of the above properties. By Lemma 13.5, their cataproduct is equicharacteristic and the proof follows by the previous discussion; details are left to the reader. To make this bound independent from p as well, we use the corresponding bounds from the previous theorems. □

Monomial Theorem

By the same process as above, we can derive some asymptotic version of the Monomial Theorem from Corollary 10.1. Unfortunately, the bounds will also depend on the monomials involved, and hence does not lead to an asymptotic version of the Direct Summand conjecture. More precisely, given $\nu_0, \dots, \nu_s \in \mathbb{N}^d$ with ν_0 not a positive linear combination of the ν_i and given l, m, there is a bound N depending on these data, such that for every mixed characteristic p Noetherian local ring R of embedding dimension at most m and dimension d, and for every system of parameters $\mathbf{x} := (x_1, \dots, x_d)$ in R such that $R/\mathbf{x}R$ has length at most l, if either p or the ramification index of R is at least N, then $\mathbf{x}^{\nu_0}$ does not belong to the ideal in R generated by the $\mathbf{x}^{\nu_i}$.

In particular, for fixed m and l, we get a bound N_t, for each $t \geq 1$, such that (11) holds, whenever $\mathbf{x}$ and R satisfy the assumptions from the previous paragraph. To derive from this an asymptotic version of the Direct Summand conjecture, we need to show that the N_t can be chosen independently from t. To derive this conclusion, we would like to establish the following result. Let $(R_\natural, \mathfrak{m}_\natural)$ be an isodimensional ultra-Noetherian local ring, say the ultraproduct of d-dimensional Noetherian local rings $(R_w, \mathfrak{m}_w)$ of bounded embedding dimension and parameter degree. Let $H^d_\infty(R_\natural)$ be the ultraproduct of the local cohomology groups $H^d_{\mathfrak{m}_w}(R_w)$. There is a natural map $H^d_{\mathfrak{m}_\natural}(R_\natural) \to H^d_\infty(R_\natural)$.

13.7 Conjecture. *The canonical map $H^d_{\mathfrak{m}_\natural}(R_\natural) \to H^d_\infty(R_\natural)$ is injective.*

Without proof, I state that the conjecture is true when $R_\natural$ is ultra-Cohen-Macaulay. Let us show how this conjecture implies that the N_t can be chosen to be independent from t, thus yielding a true asymptotic version of the Monomial Theorem (whence also of the Direct Summand Theorem) in mixed characteristic. Indeed, assume the conjecture and let $(x_{1\natural}, \dots, x_{d\natural})$ be a generic sequence in $R_\natural$ and choose $x_{iw} \in R_w$ so that their ultraproduct is $x_{i\natural}$. Since the (image of the) element $1/(x_{1\natural} \cdots x_{d\natural})$ in the top local cohomology module $H^d_{\mathfrak{m}_\natural}(R_\natural)$ is non-zero by Corollary 10.1—here we realize $H^\bullet_{\mathfrak{m}_\natural}(R_\natural)$ as the cohomology of the Čech complex associated to $(x_{1\natural}, \dots, x_{d\natural})$—its image in $H^d_\infty(R_\natural)$ is therefore also non-zero, whence almost each $1/(x_{1w} \cdots x_{dw})$ is non-zero in $H^d_{\mathfrak{m}_w}(R_w)$. Hence (11) is valid for almost each $(x_{1w}, \dots, x_{dw})$ and all t.

Towards a proof of the full Improved New Intersection Theorem

Although our methods can in principle only prove asymptotic versions, a better understanding of the bounds can in certain cases lead to a complete solution of the conjecture. To formulate such a result, let us say that a numerical function f *grows sub-linearly* if there exists some $0 \leq \alpha < 1$ such that $f(n)/n^\alpha$ remains bounded when n goes to infinity.

13.8 Theorem. *Suppose that for each pair (m, r) the numerical function $f_{m,r}(l) := \kappa(m, r, l)$ grows sub-linearly, where κ is the numerical function given in* (13.6.1)*, then the Improved New Intersection Theorem holds.*

Proof. Let $\mathcal{I}_{m,r,l}$ be the collection of counterexamples with invariants (m, r, l), that is to say, all mixed characteristic Noetherian local rings R of embedding dimension at most m, admitting a finite free complex $F_\bullet$ of rank at most r and INIT-degree at most l, such that the length of $F_\bullet$ is strictly less than the dimension of R. We have to show that $\mathcal{I}_{m,r,l}$ is empty for all (m, r, l), so by way of contradiction, assume it is not for the triple (m, r, l). For each n, let $f(n)$ be the supremum of the ramification indexes of counterexamples in $\mathcal{I}_{m,r,n}$ (and equal to 0 if there is no counterexample). By Theorem 13.6, this supremum is always finite. By assumption, f grows sub-linearly, so that for some positive real numbers c and $\alpha < 1$, we have $f(n) \leq cn^\alpha$, for all n. In particular, for n larger than the $(1-\alpha)$-th root of $\frac{cl^\alpha}{f(l)}$, we have

$$f(ln) < nf(l). \tag{22}$$

Let $(R, \mathfrak{m})$ be a counterexample in $\mathcal{I}_{m,r,l}$ of ramification index $f(l)$, witnessed by the finite free complex $F_\bullet$ of length strictly less than the dimension of R. Since the completion of R will be again a counterexample in $\mathcal{I}_{m,r,l}$ of the same ramification index, we may assume R is complete, whence by Cohen's structure theorem of the form $R = V[[\xi]]/I$ for some discrete valuation ring V, some tuple of indeterminates ξ, and some ideal $I \subseteq V[[\xi]]$. Let $n \gg 0$ so that (22) holds, and let $W := V[t]/(t^n - \pi)V[t]$, where π is a uniformizing parameter of V. Let $S := W[[\xi]]/IW[[\xi]]$, so that $R \to S$ is faithfully flat and S has the same dimension and embedding dimension as R. By construction, its ramification index is equal to $nf(l)$. By faithful flatness, $F_\bullet \otimes_R S$ is a finite free complex of length strictly less than the dimension of S, with homology equal to $\mathrm{H}_\bullet(F_\bullet) \otimes_R S$. I claim that if H is an R-module of length a, then $H \otimes_R S$ has length na. Assuming this claim, it follows that S belongs to $\mathcal{I}_{m,r,nl}$, and hence its ramification is by definition at most $f(ln)$, contradicting (22).

The claim is easily reduced by induction to the case $a = 1$, that is to say, when H is equal to the residue field $R/\mathfrak{m} = V/\pi V = k$. In that case, $H \otimes_R S = S/\mathfrak{m}S = W/\pi W$, and this is isomorphic to $k[t]/t^n k[t]$, a module of length n. □

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