EMBEDDED RESOLUTION OF SINGULARITIES IN RIGID ANALYTIC GEOMETRY.

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Abstract. We give a rigid analytic version of Hironaka's Embedded Resolution of Singularities over an algebraically closed field of characteristic zero, complete with respect to a non-archimedean norm. This resolution is local with respect to the Grothendieck topology. The proof uses Hironaka's original result, together with an application of our analytization functor.

0. Introduction and preliminaries.

0.1. Introduction

0.1.1. Introduction In this work we prove the analogue version of Hironaka's Embedded Resolution of Singularities in the frame work of rigid analytic geometry. We will work over a fixed algebraically closed field K, of characteristic zero, endowed with a complete non-archimedean norm. The requirement on the characteristic could be dropped, if a version of Hironaka's Theorem would be available in characteristic p.

Our main theorem **(3.2.3)** states that given an hypersurface in an affinoid manifold, we can find a finite affinoid covering of the embedding space and maps above each admissible open, which are a composition of finitely many blowing up maps with 'nice' centers, such that the inverse image of this hypersurface under these maps has normal crossings. We show that this theorem then implies a Desingularisation Theorem **(3.2.5)** in the following sense: Given any integral rigid analytic variety, there exists an admissible affinoid covering of this variety and above each admissible open a finite sequence of blowing up maps after which the space becomes regular.

We haven't bothered to give a fully global version of this theorem, but contented ourselves with a version which is local with respect to the Grothendieck topology, i.e., modulo an admissible affinoid covering. Nonetheless, using the more recent uniform versions of Embedded Resolution of Singularities, such as [**BM 2**] , one can modify the present proof to obtain a global version of Embedded Resolution of Singularities; see the remark following **(3.2.3)**. However, for the applications we have in mind the local version is more than sufficient. The main application appeared already in the papers [**Sch 2**] and [**GS**] on the Uniformization of rigid subanalytic sets (see also [**Sch 3**] and [**Sch 5**] for some more applications). In **(3.2.6)** we give an extension of this Uniformization Theorem to the non-smooth case, by using our Desingularization Theorem. A first but incomplete version of Embedded Resolution had already appeared in our Ph.D. Thesis [**Sch 0**] .

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Let us briefly sketch an outline of the proof of our main theorem. Our main tool is the principle of *analytization* of certain (algebraic) schemes over K. An analytization of a scheme $\mathbf{X}$ over K is essentially a morphism $(\eta, \eta^{\#}) : (X, \mathcal{O}_X) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}})$ of locally ringed spaces over K, where X is a rigid analytic variety, such that this morphism is universal with respect to morphisms of locally ringed spaces from a rigid analytic variety to $\mathbf{X}$, for a precise definition we refer to definition **(1.1.1)**. By the universal property, it is straightforward to give also the definition of the analytization of a map between schemes, both of which admit an analytization, so that we actually obtain a functor from the category of *analytic schemes* (see **(1.3.3)** for a definition) to the category of rigid analytic varieties.

We want to mention that the construction is a generalization of the analytization of a scheme of finite type over K, which for instance is described in the excellent book [**BGR**] . Our analytization is so to speak the relative version of this, since we construct the analytization of any scheme of finite type over an affinoid algebra (and even a slightly larger class, see **(1.3.4)**). In particular, if A is an affinoid algebra, then $\mathrm{Sp}\, A$ is the analytization of $\mathbf{Spec} A$. A notational remark: all schemes and their maps will appear in bold face to distinguish them from rigid analytic varieties and maps. A main property of an analytization $\eta : X \to \mathbf{X}$, is that η induces a bijection between the points of X and the closed points of $\mathbf{X}$ and that the completion of the local morphism in a (closed) point is an isomorphism. Therefore most properties of $\mathbf{X}$ are carried over on, or, vice versa, are determined by X, such as, for example, reducedness, regularity and normality. (See **(1.3.5)**).

We also need a result on the maximal ideals of an algebra of finite type over an affinoid algebra. We prove a Weak Nullstellensatz stating that maximal ideals 'come from points', in other words that their residue field equals K, see **(0.2.2)**. A result no longer true if one were to replace strictly convergent power series by formal power series. We thank the referee for pointing out Lemma **(0.2.1)** to us, thus simplifying the original proof of the Nullstellensatz.

We would like to mention that U. Köpf has independently developed the theory of analytization of schemes of finite type over affinoid algebras in her Ph.D Thesis [**Köp**] , a fact which was brought to our attention by Bosch only after we had already written down the first draft of this work. She uses a different starting point and a slightly different approach, but basically the same results are obtained. In the second part of her work she then gives a GAGA principle for the analytization of proper schemes of finite type over an affinoid algebra. Since, however, her work is not readily available, we decided to present the details of the analytization functor, in order to remain self contained.

In the second chapter, we investigate the analytization of a blowing up map. (For the definition and elementary properties of blowing up maps in rigid analytic geometry, we refer to our paper [**Sch 4**] .) Essentially we show that blowing up commutes with the analytization functor, see **(2.2.2)**. We like to draw the attention to proposition **(2.2.1)**, used to prove the above result. This proposition provides a partial inverse to the analytization functor. We show that in a restricted case, we can attach to a map of rigid analytic varieties which both are analytizations of affine schemes, a map of the corresponding schemes. In other words, we can 'algebraize' this analytic map.

The last section then contains the proof of our main theorem. The proof heavily relies on Hironaka's Theorem, in that we will apply it to an algebraic situation derived from our data and then take the analytization of this algebraic resolution.

Let us be a bit more specific. Given are the affinoid manifold $M = \operatorname{Sp} A$ (i.e., A is an affinoid algebra which is a regular domain) and the hypersurface H in it. The morphism $\eta : M \to \mathbf{M} = \mathbf{Spec}A$ is an analytization morphism. Take a point $x \in M$ and let $\mathbf{x} = \eta(x)$, with $\mathfrak{m}$ the corresponding maximal ideal of A. Apply HIRONAKA's Embedded Resolution (see **(3.1.2)**) to the excellent local ring $A_{\mathfrak{m}}$ in order to find a morphism $\mathbf{h} : \tilde{\mathbf{X}} \to \mathbf{Spec}(A_{\mathfrak{m}})$, which is a finite sequence of blowing up maps with 'nice' centers, rendering the inverse image of the (local germ of the) hypersurface to a normal crossings situation. It should be observed that at this point, we cannot yet apply the analytization functor, since $\mathbf{Spec}(A_{\mathfrak{m}})$ is not an analytic scheme, i.e., an analytization does not exist. But since everything is local in the Zariski topology, we can find a small enough neighborhood of $\mathbf{x}$ over which the map $\mathbf{h}$ can be extended and such that all its main properties remain. This (Zariski) open in $\mathbf{Spec}A$ now admits an analytization. So, using that analytization and blowing up commute and that analytization preserves the necessary properties, we have found a Zariski open U around each point $x \in M$ and a map $h : \tilde{U} \to U$ of the type described above, so that the inverse image of the hypersurface H has normal crossings. Since each covering by Zariski opens is admissible, we are done by taking an admissible affinoid covering of each open U and then selecting a finite subcovering of the collection of all admissible affinoids involved.

We would like to thank M. VAN DER PUT and S. BOSCH for some useful discussions we had with them.

0.1.2. Conventions. Throughout this paper will be fixed an algebraically closed field K endowed with a complete non-archimedean norm. We adopt the notation and the terminology from [**BGR**] for rigid analytic geometry over K. In particular, let X be a rigid analytic variety. We will denote its structure sheaf by $\mathcal{O}_X$. Let $i : Y \to X$ be a closed immersion of rigid analytic varieties. Then we call Y a *closed analytic subvariety* of X. Let $i^{\#} : \mathcal{O}_X \to i_*(\mathcal{O}_Y)$ denote the corresponding surjective homomorphism of $\mathcal{O}_X$-modules. The kernel $\mathcal{I} = \ker(i^{\#})$ is a coherent $\mathcal{O}_X$-ideal and we call it the *$\mathcal{O}_X$-ideal defining* Y, or alternatively, we say that Y is the *closed analytic subvariety of X associated* to the $\mathcal{O}_X$-ideal $\mathcal{I}$.

The underlying set $|i(Y)|$ of the image $i(Y)$ is an *analytic subset* of X. By abuse of notation, we will sometimes consider Y itself as an analytic subset of X, especially when we consider the admissible open given by $X \setminus Y$, where the correct notation should be $X \setminus |i(Y)|$. Note that on an analytic subset Y of X, we can define many structures of a closed analytic subvariety. Namely one for each coherent $\mathcal{O}_X$-ideal $\mathcal{I}$, such that $\mathrm{V}(\mathcal{I}) = Y$. Recall that

$$\mathrm{V}(\mathcal{I}) = \{\, x \in X \mid \mathcal{I}_x \neq \mathcal{O}_{X,x} \,\},$$

and we call this analytic subset the *zero-set* of $\mathcal{I}$. (Any analytic subset is realized in such way). In particular there is exactly one structure of a reduced analytic subvariety on Y, given by the coherent $\mathcal{O}_X$-ideal $\mathfrak{id}(Y)$, which is a radical ideal.

0.1.3. Definition. Given a map $f : Y \to X$ and a coherent sheaf of $\mathcal{O}_X$-ideals $\mathcal{I}$, we call the *inverse image ideal sheaf* of $\mathcal{I}$, the image of the canonical map $f^*\mathcal{I} \to \mathcal{O}_Y$, and we denote this coherent sheaf of $\mathcal{O}_Y$-modules by $f^{-1}(\mathcal{I})\mathcal{O}_Y$, or, when no confusion can arise, simply by $\mathcal{I}\mathcal{O}_Y$.

If Z is the closed analytic subvariety of X defined by $\mathcal{I}$, then we define $f^{-1}(Z)$ to be the closed analytic subvariety of Y associated to $\mathcal{I}\mathcal{O}_Y$. In other words, we have

that $f^{-1}(Z) = Z \times_X Y$. Of course, if Z is only considered as an analytic subset of X, we mean by $f^{-1}(Z)$ only the closed analytic subset, which is the set-theoretical inverse image of Z.

In particular, if both X and Y are affinoid, with corresponding affinoid algebra A, respectively B, and if $\mathfrak{a}$ is the ideal of A corresponding to $\mathcal{I}$, then $\mathfrak{a}B$ corresponds to $\mathcal{I}\mathcal{O}_Y$.

If $y \in Y$, then we will sometimes denote the stalk of $f^{-1}(\mathcal{I})\mathcal{O}_Y$ at y by $\mathcal{I}\mathcal{O}_{Y,y}$, in stead of the more cumbersome $(f^{-1}(\mathcal{I})\mathcal{O}_Y)_y$ or $\mathcal{I}_x\mathcal{O}_{Y,y}$, where $x = f(y)$.

The next result will be used below. It's proof is fairly easy, but by lack of reference we give it nevertheless.

0.1.4. Lemma. *Let $X = \operatorname{Sp} A$ be an affinoid variety and let $\mathfrak{a}$ be an ideal in A. Let U be an admissible open of X, contained in $X \setminus \mathrm{V}(\mathfrak{a})$. Then $\mathfrak{a}\mathcal{O}_X(U) = \mathcal{O}_X(U)$.*

Proof. See [**Sch 4**, Lemma 0.4] . ■

0.2. Weak Nullstellensatz for $K\langle X\rangle[Y]$

0.2.1. Lemma. *Let A be a domain with field of fractions F, such that F does not equal the localisation A_f for any element $f \in A$. If $\mathfrak{m}$ is a maximal ideal in $A[Y]$ with $Y = (Y_1, \ldots, Y_n)$, then $\mathfrak{m}F[Y] = 1$.*

Proof. Suppose not, so that $\mathfrak{m} \cap A = (0)$. Hence the field $L = A[Y]/\mathfrak{m}$ contains F. As L is finitely generated over F, it follows that it is a finite field extension of F. Let $P_i \in A[T]$ be a minimal polynomial of Y_i (viewed as an element of L) over F, for $i = range1n$, and let f be the product of the leading coefficients of these P_i. Hence L is integral over A_f. However, since L is a field, it follows that also A_f is a field and whence equal to F, contradiction. ■

0.2.2. Corollary (Weak Nullstellensatz). *Let K be a not necessarily algebraically closed field which is endowed with a complete non-archimedean norm. Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ be finite sets of variables. Then any maximal ideal $\mathfrak{m}$ of $K\langle X\rangle[Y]$ is algebraic, i.e., $K\langle X\rangle[Y]/\mathfrak{m}$ is a finite field extension of K.*

Remark. Hence, in particular, if K is algebraically closed, each maximal ideal $\mathfrak{m}$ is of the form

$$\mathfrak{m} = (X_1 - x_1, \ldots, X_n - x_n, Y_1 - y_1, \ldots, Y_m - y_m),$$

with $x_i, y_i \in K$ and $|x_i| \leq 1$.

Proof. We will give a proof by induction on the number n of X-variables. If there are none, the statement is nothing but Hilberts Nullstellensatz for polynomial rings over a field K.

So assume $n \geq 1$ and the theorem proven for a smaller number of X-variables. Let L denote the fraction field of $K\langle X\rangle$. We will consider two different cases.

Case 1. Suppose first that $\mathfrak{m}L[Y] = 1$. Hence there exists a non-zero element $\rho(X)$ in $\mathfrak{m} \cap K\langle X\rangle$. By the Noether Normalization Theorem for affinoid algebras, we know that after a change of variables, there exists a finite injective map $K\langle X'\rangle \hookrightarrow$

$K\langle X\rangle/(\rho)$ ([**BGR**, 6.1.2. Corollary 2]), where $X' = (X_1, \ldots, X_{n-1})$. Let $\mathfrak{m}' = \mathfrak{m} \cap K\langle X'\rangle[Y]$, then also the map

$$K\langle X'\rangle[Y]/\mathfrak{m}' \hookrightarrow K\langle X\rangle[Y]/\mathfrak{m}$$

is finite and injective. But since the latter is a field, the former has also to be a field, which by induction must be finite over K. This establishes the first case.

Case 2. We may now assume that $\mathfrak{m}L[Y] \neq 1$. By **(0.2.1)**, this implies that for some $f \in K\langle X\rangle$ we would have that $K\langle Y\rangle_f = L$. However this is ruled out by the following argument. Let $g \in K\langle X\rangle$ be an irreducible element not dividing f. Using that $K\langle X\rangle$ is a UFD, we see that $1/g$ does not belong to the localisation $K\langle X\rangle_f$. ■

1. Analytization

1.1. Definition of Analytization

1.1.1. Definition. Let $(\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}})$ be a scheme over K. We call a rigid analytic variety X an *analytization* of $\mathbf{X}$, if there exists a morphism of locally ringed spaces over K,

$$(\eta, \eta^{\#}) : (X, \mathcal{O}_X) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}})$$

such that, given any rigid analytic variety $(Y, \mathcal{O}_Y)$, and, given any morphism

$$(\theta, \theta^{\#}) : (Y, \mathcal{O}_Y) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}),$$

of locally ringed spaces over K, there exists a unique map of rigid analytic varieties

$$(\varphi, \varphi^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

making following diagram commute

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{(\theta, \theta^{\#})} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \\ {\scriptstyle (\varphi, \varphi^{\#})}\downarrow & & \| \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta, \eta^{\#})]{} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \end{array}$$

Note that, since an analytization is defined by a universal property, we have that, if an analytization exists, then it must be unique (up to a unique isomorphism). We will denote this analytization by

$$\mathbf{X}^{\mathrm{an}} = X.$$

The morphism $(\eta, \eta^{\#})$ will sometimes be referred to as the *analytizing* morphism.

Let us now look at morphisms. Suppose $\mathbf{X}$ and $\mathbf{Y}$ are two K-schemes which have an analytization $\eta : \mathbf{X}^{\mathrm{an}} \to \mathbf{X}$, respectively $\zeta : \mathbf{Y}^{\mathrm{an}} \to \mathbf{Y}$ (for sake of simplicity, we will sometimes not write the corresponding map of sheaves). Let

$$\mathbf{f} : \mathbf{X} \to \mathbf{Y}$$

be a map of schemes (over K). Then there exists a unique map $\mathbf{X}^{\mathrm{an}} \to \mathbf{Y}^{\mathrm{an}}$, denoted by $\mathbf{f}^{\mathrm{an}}$, such that following diagram commutes

$$\begin{array}{ccc} (\mathbf{X}^{\mathrm{an}}, \mathcal{O}_{\mathbf{X}^{\mathrm{an}}}) & \xrightarrow{(\eta,\eta^{\#})} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \\ {\scriptstyle (\mathbf{f}^{\mathrm{an}},\mathbf{f}^{\mathrm{an}\#})}\downarrow & & \downarrow{\scriptstyle (\mathbf{f},\mathbf{f}^{\#})} \\ (\mathbf{Y}^{\mathrm{an}}, \mathcal{O}_{\mathbf{Y}^{\mathrm{an}}}) & \xrightarrow[(\zeta,\zeta^{\#})]{} & (\mathbf{Y}, \boldsymbol{\mathcal{O}}_{\mathbf{Y}}) \end{array}$$

This follows immediately from the definition of $\mathbf{Y}^{\mathrm{an}}$ applied to the composite map $\mathbf{f} \circ \eta : \mathbf{X}^{\mathrm{an}} \to \mathbf{X} \to \mathbf{Y}$.

If $\mathbf{g} : \mathbf{Y} \to \mathbf{Z}$ is a second map of schemes, where $\mathbf{Z}$ is a K-scheme which also admits an analytization, then one checks that

$$(\mathbf{g} \circ \mathbf{f})^{\mathrm{an}} = \mathbf{g}^{\mathrm{an}} \circ \mathbf{f}^{\mathrm{an}}.$$

Note that, if $\mathbf{f}$ is injective, then so is $\mathbf{f}^{\mathrm{an}}$, provided we know that ζ is injective (which will be the case in all the situations we know that an analytization exists).

1.1.2. Lemma. *Let $\mathbf{X}$ be a scheme over K. In order to check whether a given morphism of locally ringed spaces $(\eta, \eta^{\#}) : (X, \mathcal{O}_X) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}})$, where X is a rigid analytic variety, is an analytization of $\mathbf{X}$, it is enough to check the universal property in definition* **(1.1.1)** *only for Y affinoid.*

Proof. Assume that the universal property has been checked for every affinoid variety and let Y be an arbitrary rigid analytic variety, such that there exists a morphism

$$(\theta, \theta^{\#}) : (Y, \mathcal{O}_Y) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}),$$

of locally ringed spaces over K. Let $\{Y_i\}_i$ be an admissible affinoid covering and let $(\theta_i, {\theta_i}^{\#})$ be the restriction of $(\theta, \theta^{\#})$ to $(Y_i, \mathcal{O}_{Y_i})$. By our hypothesis we can find unique maps of rigid analytic varieties

$$(\varphi_i, \varphi_i^{\#}) : (Y_i, \mathcal{O}_{Y_i}) \to (X, \mathcal{O}_X)$$

making the following diagram commute

$$\begin{array}{ccc} (Y_i, \mathcal{O}_{Y_i}) & \xrightarrow{(\theta_i,{\theta_i}^{\#})} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \\ {\scriptstyle (\varphi_i,{\varphi_i}^{\#})}\downarrow & & \| \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta,\eta^{\#})]{} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \end{array} \tag{1}$$

The uniqueness of the $(\varphi_i, \varphi_i^{\#})$, ensures us that they agree on $Y_i \cap Y_j$, for all $i \neq j$. Indeed, let $\{U_k\}_k$ be an admissible affinoid covering of $Y_i \cap Y_j$. Then both $\varphi_i|_{U_k}$ and $\varphi_j|_{U_k}$ make the following diagram commute

$$\begin{array}{ccc} (U_k, \mathcal{O}_{U_k}) & \xrightarrow{(\theta,\theta^{\#})|_{U_k}} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \\ \downarrow & & \| \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta,\eta^{\#})]{} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \end{array} \tag{2}$$

By our hypothesis, we have that there exists only one map $U_k \to X$ making the above diagram (2) commute. Hence $\varphi_i|_{U_k}$ and $\varphi_j|_{U_k}$ must be equal. From this our claim follows directly.

Therefore, we can paste the φ_i together (see [**BGR**, 9.3.3. Proposition 1]) to obtain a map of rigid analytic varieties

$$(\varphi, \varphi^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

making following diagram commute

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{(\theta, \theta^{\#})} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \\ {\scriptstyle (\varphi, \varphi^{\#})}\downarrow & & \| \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta, \eta^{\#})]{} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \end{array} \tag{3}$$

The uniqueness of $(\varphi, \varphi^{\#})$, follows from the fact that any $(\varphi, \varphi^{\#})$ making (3) commute, when restricted to $(Y_i, \mathcal{O}_{Y_i})$, is a solution to the commutativity of (1), and therefore must coincide with $(\varphi_i, \varphi_i^{\#})$. ■

1.1.3. Lemma. *Let* $\mathbf{X}$ *be a scheme over* K*, which admits an analytization*

$$(\eta, \eta^{\#}) : (\mathbf{X}^{\mathrm{an}}, \mathcal{O}_{\mathbf{X}^{\mathrm{an}}}) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}).$$

Let $\mathbf{U}$ *be an open of* $\mathbf{X}$*. Then the restriction*

$$(\eta, \eta^{\#})|_{\eta^{-1}(\mathbf{U})} : (\eta^{-1}(\mathbf{U}), \mathcal{O}_{\eta^{-1}(\mathbf{U})}) \to (\mathbf{U}, \boldsymbol{\mathcal{O}}_{\mathbf{U}})$$

of $(\eta, \eta^{\#})$ *is an analytization of* $\mathbf{U}$*.*

Proof. Let us simplify notation by putting $X = \mathbf{X}^{\mathrm{an}}$ and $U = \eta^{-1}(\mathbf{U})$ and let $\zeta = \eta|_U$. First of all, note that by definition of an analytization, η is continuous, so that U is an admissible open in X, and hence in particular is a rigid analytic variety.

Let

$$(\theta, \theta^{\#}) : (Y, \mathcal{O}_Y) \to (\mathbf{U}, \boldsymbol{\mathcal{O}}_{\mathbf{U}}),$$

be a morphism of locally ringed spaces over K, where Y is a rigid analytic variety. Since X is the analytization of $\mathbf{X}$, there exists a unique map of rigid analytic varieties

$$(\varphi, \varphi^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

making following diagram commute

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{(\theta, \theta^{\#})} & (\mathbf{U}, \boldsymbol{\mathcal{O}}_{\mathbf{U}}) \\ {\scriptstyle (\varphi, \varphi^{\#})}\downarrow & & \downarrow \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta, \eta^{\#})]{} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \end{array}$$

The commutativity of above diagram implies that $\varphi(Y) \subset U$, so that φ can be considered as a map $Y \to U$. This map is necessarily unique, since the original map was. ■

1.2. Construction of an Analytization for Affine Schemes

1.2.0. Let A be an affinoid algebra and B a finitely generated A-algebra. Let us denote by $\mathbf{X} = \mathbf{Spec}(B)$ the affine scheme associated to B. In this section we want to give a construction of a rigid analytic variety X and a map of locally ringed spaces $(\eta, \eta^{\#}) : (X, \mathcal{O}_X) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}})$, which in the next section will be proved to be the analytization of $\mathbf{X}$.

1.2.1. Construction. Since B is finitely generated over A, there exists a finite set of variables $T = (T_1, \ldots, T_s)$ and an ideal I of $A[T]$, such that

$$B = \frac{A[T]}{I}. \tag{1}$$

Choose $\pi \in K$, such that $|\pi| < 1$ and define, for all i, the following affinoid algebras

$$B_i \stackrel{\text{def}}{=} \frac{A\langle \pi^i T\rangle}{IA\langle \pi^i T\rangle}.$$

This gives rise to a sequence of affinoid algebras

$$B \to \ldots B_2 \to B_1 \to B_0.$$

Let us denote by $X_i = \mathrm{Sp}(B_i)$. Hence, using [**BGR**, 7.2.2. Corollary 6] , we have an ascending chain of affinoid subdomains

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots.$$

We can paste these together (see loc. cit.) in order to obtain a rigid analytic variety X. We have that

$$X = \bigcup_i X_i \tag{2}$$

and $\{X_i\}_i$ is an admissible affinoid covering of X.

1.2.2. Claim. *The points of X are in one-one correspondence with the closed points of $\mathbf{X}$, i.e. with the maximal ideals of B. (Recall our convention that K is algebraically closed.)*

Proof. Let us define a map (of sets)

$$\eta : X \to \mathbf{X}$$

as follows. Let x be a point of X, say $x \in X_i$. Let $\mathfrak{m}_i$ be the corresponding maximal ideal of B_i. Then we define $\eta(x)$ as the point of $\mathbf{X}$ corresponding to $\mathfrak{m} = \mathfrak{m}_i \cap B$. It is easy to see that this definition of $\eta(x)$ does not depend on the particular i we chose. From the inclusions $K \hookrightarrow B/\mathfrak{m} \hookrightarrow B_i/\mathfrak{m}_i = K$, we conclude that $\mathfrak{m}$ is even a maximal ideal of B, or, in other words, that $\eta(x)$ is a closed point of $\mathbf{X}$. To prove that this map is a bijection from X to the set of closed points of $\mathbf{X}$, we construct its inverse as follows.

Let $\mathfrak{m}$ be a maximal ideal of B. Let $A = K\langle X\rangle/\mathfrak{a}$ be a representation of A, where $X = (X_1, \ldots, X_n)$. Using the Weak Nullstellensatz **(0.2.2)**, we can write

$$\mathfrak{m} = (X_1 - x_1, \ldots, X_n - x_n, T_1 - t_1, \ldots, T_s - t_s)B$$

where x_j and t_j are elements of K with $|x_j| \leq 1$, for all j. Choose i big enough, such that $\left|\pi^i t_j\right| \leq 1$, for all j. Then $\mathfrak{m}B_i$ remains a maximal ideal in B_i, so corresponds to a point x of $X_i \subset X$. Again we have that the assignment of x to $\mathfrak{m}$ is independent of the choice of i. Moreover, $\mathfrak{m}B_i \cap B = \mathfrak{m}$, so that $\mathfrak{m}$ corresponds to the point $\eta(x)$. ■

1.2.3. Claim. *The map η is a continuous morphism of topological spaces.*

Proof. Let $\mathfrak{a}$ be an ideal in B and let $\mathbf{V} = \mathbf{V}(\mathfrak{a})$ be the corresponding zero-set of $\mathfrak{a}$ in $\mathbf{X}$. We can define a coherent $\mathcal{O}_X$-ideal $\mathcal{I}$, by setting $\mathcal{I}(X_i) = \mathfrak{a}B_i$. The reader should check that this uniquely defines $\mathcal{I}$ and that, moreover, $\mathcal{I}$ is coherent. It is now an easy exercise to show that

$$\eta^{-1}(\mathbf{V}) = \mathrm{V}(\mathcal{I}).$$

Hence the inverse image of a closed subset of $\mathbf{X}$ under η is an analytic subset of X. Therefore, the inverse image of a Zariski-open of $\mathbf{X}$ under η is a Zariski-open subset of X and hence is admissible open. This proves the continuity of η. ■

1.2.4. Claim. *There exists a map $\eta^{\#} : \boldsymbol{\mathcal{O}}_{\mathbf{X}} \to \eta_* \mathcal{O}_X$ of $\boldsymbol{\mathcal{O}}_{\mathbf{X}}$-sheaves.*

Proof. Let $\eta_i : X_i \to \mathbf{X}$ denote the restriction of η to X_i. It is enough to construct maps of $\boldsymbol{\mathcal{O}}_{\mathbf{X}}$-sheaves,

$$\eta_i^{\#} : \boldsymbol{\mathcal{O}}_{\mathbf{X}} \to (\eta_i)_* \mathcal{O}_{X_i} \tag{3}$$

which are compatible with each other. Without proof we state that it is enough to construct natural maps

$$\eta_i^{\#}(\mathbf{U}) : \boldsymbol{\mathcal{O}}_{\mathbf{X}}(\mathbf{U}) \to \mathcal{O}_{X_i}(\eta_i^{-1}(\mathbf{U}))$$

for $\mathbf{U}$ of the form $\mathbf{U} = \mathbf{X} \setminus \mathbf{V}(f)$, where $f \in B$. Hence $U = \{\, x \in X \mid f(x) \neq 0 \,\}$ is the inverse image of $\mathbf{U}$ under η. Let $U_i = U \cap X_i$ so that $U_i = \eta_i^{-1}(\mathbf{U})$. In other words, we need a natural map

$$\eta_i^{\#}(\mathbf{U}) : \boldsymbol{\mathcal{O}}_{\mathbf{X}}(\mathbf{U}) \to \mathcal{O}_{X_i}(U_i). \tag{4}$$

But $\boldsymbol{\mathcal{O}}_{\mathbf{X}}(\mathbf{U}) = B_f$ (the localization of B at the multiplicative set of powers of f). Since $U_i \subset X_i$ is an admissible open, we have the natural restriction map

$$\mathcal{O}_{X_i}(X_i) = B_i \to \mathcal{O}_{X_i}(U_i).$$

By **(0.1.4)**, we know that f, considered as an element of $\mathcal{O}_{X_i}(U_i)$ via the composite map $B \to B_i \to \mathcal{O}_{X_i}(U_i)$, is invertible. Hence we obtain from this composite map, a map

$$B_f \to \mathcal{O}_{X_i}(U_i).$$

This is our desired map of (4). We leave it as an exercise to the reader to verify that these maps define a map $\eta_i^{\#}$ of $\boldsymbol{\mathcal{O}}_{\mathbf{X}}$-sheaves as in (3), and hence a map $\eta^{\#}$ as claimed. ■

1.2.5. Claim. *The map*

$$(\eta, \eta^{\#}) : (X, \mathcal{O}_X) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}})$$

is a morphism of locally ringed spaces over K.

Proof. The only new thing to be proved is that the induced maps on the stalks are local. Let therefore $x \in X$ be a point, say $x \in X_i$. Let $\mathfrak{m}_i$ denote the corresponding maximal ideal of B_i. From **(1.2.2)**, we get that $\mathfrak{m} = \mathfrak{m}_i \cap B$ is a maximal ideal of B, corresponding to $\eta(x) = \mathbf{x}$ and $\mathfrak{m}B_i = \mathfrak{m}_i$. We have now a sequence of natural (local) maps

$$\boldsymbol{\mathcal{O}}_{\mathbf{X},\mathbf{x}} = B_{\mathfrak{m}} \to (B_i)_{\mathfrak{m}_i} \xrightarrow{\epsilon} \mathcal{O}_{X_i,x} = \mathcal{O}_{X,x}$$

where the map ϵ is given by [**BGR**, 7.3.2. Proposition 3] . The composition is exactly the map $\eta_x^{\#}$, and hence the latter is local. ■

1.2.6. Claim. *The map*

$$(\eta, \eta^{\#}) : (X, \mathcal{O}_X) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}})$$

is a locally formal isomorphism.

Remark. We say in general that a morphism of locally ringed spaces

$$(\alpha, \alpha^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a *locally formal isomorphism*, if, for every point $x \in X$, we have that the completion

$$\widehat{\alpha}_x^{\#} : \widehat{\mathcal{O}}_{Y,\alpha(x)} \to \widehat{\mathcal{O}}_{X,x}$$

of the local map $\alpha_x^{\#}$ is an isomorphism.

Proof. Let x be a point in X, say $x \in X_i$ and let $\mathfrak{m}_i$ denote the corresponding maximal ideal in B_i of x. Let $\mathfrak{m} = \mathfrak{m}_i \cap B$, so that $\mathfrak{m}$ is the maximal ideal of B corresponding to $\eta(x) = \mathbf{x}$ and $\mathfrak{m}B_i = \mathfrak{m}_i$. The local ring at $\mathbf{x}$ is equal to $B_{\mathfrak{m}}$ and by [**BGR**, 7.3.2. Proposition 3] , the completion of the local ring at x is isomorphic with $\widehat{(B_i)}_{\mathfrak{m}_i}$. Hence we must show that the natural map

$$\widehat{\eta}_x^{\#} : \widehat{B}_{\mathfrak{m}} \to \widehat{(B_i)}_{\mathfrak{m}_i} \tag{5}$$

is an isomorphism. However, we have that $\mathfrak{m}\widehat{(B_i)}_{\mathfrak{m}_i} = \mathfrak{m}_i\widehat{(B_i)}_{\mathfrak{m}_i}$. Therefore, if we tensor (5) with $K = \widehat{B}_{\mathfrak{m}}/\mathfrak{m}\widehat{B}_{\mathfrak{m}}$ over $\widehat{B}_{\mathfrak{m}}$, we obtain an isomorphism. By [**Mats**, Theorem 8.4] , we deduce that $\widehat{\eta}_x^{\#}$ is surjective. Let us now show injectivity. Since the map $A[Y] \to A\langle Y\rangle$ is flat, the same holds for $B \to B_i$, for all i. Hence also $\widehat{\eta}_x^{\#}$ is flat and since it is local, it is faithfully flat and hence injective. ■

Remark. Note that by faithfully flat descent we get that the local maps $\eta_x^{\#}$ are flat.

It is not clear at first whether the association $\mathbf{X} \rightsquigarrow X$ is well-defined, since the construction of X depended on the representation (1) of B and the choice of π. However, as a consequence of the following theorem, we will get that X is in fact independent of these choices.

1.3. Analytization of an Analytic Scheme

1.3.1. Proposition. *Let A be an affinoid algebra and B a finitely generated A-algebra. Let $\mathbf{X} = \mathbf{Spec}(B)$ be the corresponding scheme over K and let X and $(\eta, \eta^{\#})$ be as constructed in* **(1.2)**. *Then*

$$(\eta, \eta^{\#}) : (X, \mathcal{O}_X) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}})$$

is the analytization of $\mathbf{X}$.

Proof. Let Y be a rigid analytic variety and let

$$(\theta, \theta^{\#}) : (Y, \mathcal{O}_Y) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \tag{1}$$

be a map of locally ringed spaces (over K). We have to prove that there exists a unique factorization of $(\theta, \theta^{\#})$ over $(X, \mathcal{O}_X)$. In other words, we have to show that there exists a unique map of rigid analytic varieties

$$(\varphi, \varphi^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

such that the following diagram commutes

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{(\theta,\theta^{\#})} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \\ {\scriptstyle (\varphi,\varphi^{\#})}\downarrow & & \| \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta,\eta^{\#})]{} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \end{array} \tag{2}$$

By lemma **(1.1.2)**, we can assume that $Y = \operatorname{Sp} C$ is affinoid. From (1), we get a K-algebra morphism

$$f \stackrel{\text{def}}{=} \theta^{\#}(\mathbf{X}) : B = \boldsymbol{\mathcal{O}}_{\mathbf{X}}(\mathbf{X}) \to \mathcal{O}_Y(Y) = C.$$

Let

$$B = \frac{A[T]}{I},$$

be the representation (1) of **(1.2.1)**, with the aid of which we constructed X. Choose i big enough, such that, for all j, we have that $|f(T_j)| < \left|1/\pi^i\right|$. Therefore, we can factor f over a map

$$g : B_i = \frac{A\langle \pi^i T \rangle}{IA\langle \pi^i T \rangle} \to C$$

by sending T_j to $f(T_j)$. This is well-defined by [**BGR**, 6.1.1. Proposition 4] , since by our choice of i, the elements $f(\pi^i T_j)$ are power-bounded elements. This map g gives rise to a map of affinoid varieties $\varphi' : Y = \operatorname{Sp} C \to X_i = \operatorname{Sp}(B_i)$. We claim that the composite map $\varphi : Y \to X_i \hookrightarrow X$ is the desired map.

Let us first prove that (2) is commutative. Let $y \in Y$ be a point and let $\mathbf{x} = \theta(y)$. Let $\mathfrak{m}$ be the maximal ideal in C corresponding to y and let $\mathfrak{p}$ be the prime ideal of B corresponding to $\mathbf{x}$. We claim that

$$\mathfrak{p} = \mathfrak{m} \cap B \tag{3}$$

via the map $f : B \to C$. Indeed, since $(\theta, \theta^{\#})$ is a morphism of locally ringed spaces, we must have a local morphism

$$\theta_y^{\#} : B_{\mathfrak{p}} \to \mathcal{O}_{Y,y}.$$

Since $C_{\mathfrak{m}}$ and $\mathcal{O}_{Y,y}$ have the same completion, we get a local map

$$B_{\mathfrak{p}} \to \widehat{C}_{\mathfrak{m}}. \tag{4}$$

This means that $\mathfrak{p}B_{\mathfrak{p}} = \mathfrak{m}\widehat{C}_{\mathfrak{m}} \cap B_{\mathfrak{p}}$, from which our claim (3) follows readily.

Since $\mathfrak{m} \cap B$ corresponds to $\eta\varphi(y)$, we get by (3) that $\mathfrak{p}$ is a maximal ideal and that $\mathbf{x} = \eta(\varphi(y))$. This proves the commutativity of diagram (2) considered only as maps of sets. To prove the commutativity as a diagram of morphisms of locally ringed spaces, we only need to check commutativity on the stalks in each point. Therefore, suppose that $x \in X_i$ and let $\mathfrak{N} = \mathfrak{m} \cap B_i$ be the maximal ideal of B_i corresponding to $x \in X_i$. Consider following diagram

$$\begin{array}{ccccc} B_{\mathfrak{p}} & \longrightarrow & \mathcal{O}_{Y,y} & \xrightarrow{\ i\ } & \widehat{C}_{\mathfrak{m}} \\ \| & & \uparrow & & \uparrow \\ B_{\mathfrak{p}} & \xrightarrow{\eta_x^{\#}} & \mathcal{O}_{X,x} & \longrightarrow & \widehat{(B_i)}_{\mathfrak{N}}. \end{array} \tag{5}$$

The outer diagram is commutative by construction and the second inner diagram since all maps are natural. Since i is injective, we conclude that also the first diagram has to be commutative, which is exactly what we needed to show.

Next, we have to show that φ is uniquely determined by (2). Hence let

$$\psi : Y \to X$$

be another map of rigid analytic varieties making (2) commutative. Let $y \in Y$ be a point and set $x = \varphi(y)$ and $x' = \psi(y)$. By the commutativity of diagram (2), we have that $\eta(x) = \theta(y) = \eta(x')$, and hence, since η is a injection, we obtain that $x = x'$. Hence, as a map of sets, φ and ψ agree.

Let us keep notation as above and suppose that $x = \varphi(y) \in X_i$. Let as before $\mathfrak{m}$ correspond to y and $\mathfrak{p}$ to $\mathbf{x} = \theta(y)$ and let $\mathfrak{N} = \mathfrak{m} \cap B_i$ be the maximal ideal of B_i corresponding to $x \in X_i$. Using the commutativity of (2), we get two local maps

$$\varphi_y^{\#}, \psi_y^{\#} : \mathcal{O}_{X,x} = \mathcal{O}_{X_i,x} \to \mathcal{O}_{Y,y}$$

making the first inner diagram of (5) commute. Since the completion

$$\widehat{\eta}_x^{\#} : \widehat{B}_{\mathfrak{p}} \to \widehat{(B_i)}_{\mathfrak{N}}$$

of $\eta_x^{\#}$ is an isomorphism by **(1.2.6)**, we get, from the commutativity of this diagram, that

$$\widehat{\varphi}_x^{\#} = \widehat{\psi}_x^{\#}.$$

Since this holds for all points $x \in X$, one deduces that $\varphi = \psi$. ■

1.3.2. Examples.

(1) Suppose that in **(1.2)** B is already affinoid, so that we can take the trivial representation $B = A$. We get that all $B_i = B$ and hence $X = \operatorname{Sp} B$ is the analytization of $\mathbf{Spec}(B)$.

(2) Suppose that in **(1.2)** $A = K$, in other words, that $\mathbf{X}$ is of finite type over K. Then in [**BGR**, 9.3.4. Example 2] the authors construct a rigid analytic variety $\mathbf{X}^{\mathrm{an}}$, which they call the *associated rigid analytic variety* of $\mathbf{X}$. Their construction is exactly the same as ours in this special case, justifying our notation and proving that the in loc. cit. described associated rigid analytic variety is exactly the analytization of $\mathbf{X}$. Therefore, we also obtain a better proof of the uniqueness of their construction.

1.3.3. Definition. We call a scheme $\mathbf{X}$ over K an *analytic* scheme, if $\mathbf{X}$ admits an open affine covering $\{\mathbf{X}_i\}_i$, where, for each i, the scheme $\mathbf{X}_i$ is of finite type over an affinoid algebra A_i.

1.3.4. Theorem. *Each analytic scheme* $\mathbf{X}$ *has an analytization.*

Proof. Let $\{\mathbf{X}_i\}_i$ be an open affine covering of $\mathbf{X}$, with each $\mathbf{X}_i$ of finite type over some affinoid algebra. From **(1.3.1)**, we know that, for each i, we have an analytization

$$(\eta_i, \eta_i^{\#}) : (\mathbf{X}_i^{\mathrm{an}}, \mathcal{O}_{\mathbf{X}_i^{\mathrm{an}}}) \to (\mathbf{X}_i, \mathcal{O}_{\mathbf{X}_i,}).$$

To simplify our notations, let us denote by $X_i = \mathbf{X}_i^{\mathrm{an}}$ and set $\mathbf{X}_{ij} = \mathbf{X}_i \cap \mathbf{X}_j$. From **(1.1.3)**, we get that

$$\eta_i^{-1}(\mathbf{X}_{ij}) = \mathbf{X}_{ij}^{\mathrm{an}} = \eta_j^{-1}(\mathbf{X}_{ij}).$$

Hence we can paste the X_i together along these common open subsets, in order to obtain a rigid analytic variety X. Likewise, we would like to paste the η_i together, in order to obtain a morphism $(\eta, \eta^{\#})$ of locally ringed spaces. One should be a bit careful with this, since X and $\mathbf{X}$ are of a different nature. However, the usual proofs for pasting morphisms in the rigid analytic case (see [**BGR**, 9.3.3. Proposition 1]) or the algebraic geometric case, are carried over without any surprises, so we will not go into details.

We claim that the above constructed rigid analytic variety X is the wanted analytization (with analytizing map η). We will just give the outlines of the proof, since most details are tedious but straightforward.

Let $(\theta, \theta^{\#}) : (Y, \mathcal{O}_Y) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}})$ be a morphism of locally ringed spaces, where Y is a rigid analytic variety. Let $Y_i = \theta^{-1}(\mathbf{X}_i)$ and θ_i the restriction of θ to Y_i. Hence the Y_i are admissible opens of Y and the collection $\{Y_i\}_i$ is an admissible covering of Y. Since X_i is the analytization of $\mathbf{X}_i$, there exists, for each i, a unique morphism of locally ringed spaces

$$(\varphi_i, \varphi_i^{\#}) : (Y_i, \mathcal{O}_{Y_i}) \to (X_i, \mathcal{O}_{X_i})$$

such that the following diagram commutes

$$\begin{array}{ccc} (Y_i, \mathcal{O}_{Y_i}) & \xrightarrow{(\theta_i, {\theta_i}^{\#})} & (\mathbf{X_i}, \boldsymbol{\mathcal{O}}_{\mathbf{X_i}}) \\ {\scriptstyle (\varphi_i, {\varphi_i}^{\#})}\downarrow & & \| \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta_i, {\eta_i}^{\#})]{} & (\mathbf{X_i}, \boldsymbol{\mathcal{O}}_{\mathbf{X_i}}) \end{array} \tag{3}$$

Using **(1.1.3)**, one sees that φ_i and φ_j have to agree on $Y_i \cap Y_j$. Hence we can paste them together to obtain a map $\varphi : Y \to X$, such that

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{(\theta, \theta^{\#})} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \\ {\scriptstyle (\varphi_i, {\varphi_i}^{\#})}\downarrow & & \| \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta_i, {\eta_i}^{\#})]{} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \end{array} \tag{4}$$

commutes. The uniqueness of φ follows from the uniqueness of the φ_i and the fact that any φ making (4) commute, when restricted to Y_i, renders (3) commutative and hence has to be equal to φ_i. ■

1.3.5. Corollary. *Let* $\mathbf{X}$ *be an analytic scheme and* $\eta : X = \mathbf{X}^{\mathrm{an}} \to \mathbf{X}$ *be its analytization. Then* η *is a locally formal isomorphism and the set of (closed) points of* X *is in one-one correspondence, through* η*, with the set of closed points of* $\mathbf{X}$*. The local map in each (closed) point is flat.*

Moreover, $\mathbf{X}$ *is reduced, regular or normal, if and only if,* X *is reduced, regular or normal, respectively.*

Proof. From the proof of **(1.3.4)** and **(1.2.6)** the first three statements follow immediately. For the last statement, recall that an affinoid algebra is a G-ring (Grothendieck ring, see [**Mats**, §32] for a definition) and therefore also each finitely generated algebra over an affinoid algebra is a G-ring. The last assertion is now clear by Theorem 32.2 of loc. cit. and [**BGR**, 7.3.2. Proposition 8] . ■

2. Blowing Up and Analytization.

2.1. Sheaves and Analytization

2.1.1. Definition. We want to recall the following definitions of inverse image and direct image of a sheaf. Let us just give the definitions in the cases we are interested in. Let $\mathbf{X}$ be an analytic scheme and let $\eta : X \to \mathbf{X}$ be the analytization of $\mathbf{X}$. Let $\boldsymbol{\mathcal{F}}$ be a sheaf on $\mathbf{X}$. Recall that the *inverse image sheaf* $\eta^{-1}(\boldsymbol{\mathcal{F}})$ is defined as the sheafification of the presheaf

$$V \mapsto \varinjlim_{\eta(V)\subset \mathbf{U}} \boldsymbol{\mathcal{F}}(\mathbf{U}).$$

If, moreover, $\boldsymbol{\mathcal{F}}$ is an $\boldsymbol{\mathcal{O}}_{\mathbf{X}}$-module, then this inverse image sheaf $\eta^{-1}(\boldsymbol{\mathcal{F}})$ is an $\eta^{-1}(\boldsymbol{\mathcal{O}}_{\mathbf{X}})$-module. Hence we can form the tensor product with $\mathcal{O}_X$ to obtain the $\mathcal{O}_X$-module

$$\eta^*(\boldsymbol{\mathcal{F}}) = \eta^{-1}(\boldsymbol{\mathcal{F}}) \otimes_{\eta^{-1}(\boldsymbol{\mathcal{O}}_{\mathbf{X}})} \mathcal{O}_X$$

called the *inverse image* of $\boldsymbol{\mathcal{F}}$.

Let $\mathcal{G}$ be an $\mathcal{O}_X$-module. Then one defines the *direct image* sheaf $\eta_*(\mathcal{G})$, given by the rule

$$\eta_*(\mathcal{G})(\mathbf{U}) = \mathcal{G}(\eta^{-1}(\mathbf{U})),$$

where $\mathbf{U}$ is an open in $\mathbf{X}$. This is an $\boldsymbol{\mathcal{O}}_{\mathbf{X}}$-module.

2.1.2. Lemma. *Let* $\eta : X \to \mathbf{X}$ *be the analytization of the analytic scheme* $\mathbf{X}$*. Let* $\boldsymbol{\mathcal{F}}$ *be an* $\boldsymbol{\mathcal{O}}_{\mathbf{X}}$*-module. Let* x *be a point of* X *and let* $\mathbf{x} = \eta(x)$*. Then the following holds.*

(1) *We have that*
   (i) $(\eta^{-1}(\boldsymbol{\mathcal{F}}))_x \cong \boldsymbol{\mathcal{F}}_{\mathbf{x}}$,
   (ii) $\eta^*(\boldsymbol{\mathcal{F}})_x \cong \boldsymbol{\mathcal{F}}_{\mathbf{x}} \otimes_{\boldsymbol{\mathcal{O}}_{\mathbf{X},\mathbf{x}}} \mathcal{O}_{X,x}$.
(2) *If* $\boldsymbol{\mathcal{F}}$ *is coherent, then so is* $\eta^*(\boldsymbol{\mathcal{F}})$.
(3) *We have that* $\boldsymbol{\mathcal{F}}$ *is invertible, if and only if,* $\eta^*(\boldsymbol{\mathcal{F}})$ *is.*
(4) *If* $\boldsymbol{\mathcal{F}}$ *is an* $\boldsymbol{\mathcal{O}}_{\mathbf{X}}$*-ideal, then* $\eta^*(\boldsymbol{\mathcal{F}})$ *is an* $\mathcal{O}_X$*-ideal. Moreover,*

$$\eta^*(\boldsymbol{\mathcal{F}})_x = \boldsymbol{\mathcal{F}}_{\mathbf{x}}\mathcal{O}_{X,x}.$$

Proof. (1) is easy and left to the reader as an exercise. (2) follows along the same lines as in the algebraic geometric case (see for instance [**Ha**, Chapter II, Proposition 5.8] . (3) follows from the second isomorphism of (1) and the fact that $\boldsymbol{\mathcal{O}}_{\mathbf{X},\mathbf{x}} \to \mathcal{O}_{X,x}$ is flat by **(1.3.5)**. Finally, (4) follows from the fact that, for each admissible open U of X, the map

$$\eta^{-1}(\boldsymbol{\mathcal{O}}_{\mathbf{X}})(U) \to \mathcal{O}_X(U)$$

is flat by using (1) and **(1.3.5)**. ■

Remark. Note that the direct image of a coherent sheaf is in general not coherent anymore, as can be easily seen by the next example. Let A be an affinoid algebra and $X = \operatorname{Sp} A$ and $\mathbf{X} = \mathbf{Spec}(A)$ and let $\eta : X \to \mathbf{X}$ be the analytization map. Then $\eta_*(\mathcal{O}_X)$ is in general not a coherent $\boldsymbol{\mathcal{O}}_{\mathbf{X}}$-module (neither an $\boldsymbol{\mathcal{O}}_{\mathbf{X}}$-ideal). For instance, if $U = X \setminus \mathrm{V}(f)$, where $f \in A$, then $\mathcal{O}_X(U)$ is not a finite A_f-module.

2.1.3. Corollary. *Let $\mathbf{X}$ be an analytic scheme and let $\eta : X \to \mathbf{X}$ be its analytization. Let $\boldsymbol{\mathcal{I}}$ be a coherent $\boldsymbol{\mathcal{O}}_{\mathbf{X}}$-ideal and let $\mathbf{Z}$ be the closed subscheme of $\mathbf{X}$ defined by it. Let Z be the closed analytic subvariety of X defined by $\eta^*\boldsymbol{\mathcal{I}}$. Then the following holds.*

(1) *As analytic subsets, we have that $Z = \eta^{-1}(\mathbf{Z})$.*
(2) *Z is the analytization of $\mathbf{Z}$.*

Remark. Therefore, in the sequel, we will mean by $\eta^{-1}(\mathbf{Z})$ the rigid analytic variety with closed analytic subvariety structure given by $\eta^*\boldsymbol{\mathcal{I}}$.

Proof. (1) is easy, using (4) of **(2.1.2)**. In order to prove (2), let us first define a morphism

$$(\zeta, \zeta^{\#}) : (Z, \mathcal{O}_Z) \to (\mathbf{Z}, \boldsymbol{\mathcal{O}}_{\mathbf{Z}}) \tag{3}$$

of locally ringed spaces. Let $\zeta = \eta|_Z$, so that by (1), we have a continuous map $\zeta : Z \to \mathbf{Z}$. Next we have to define a map $\zeta^{\#}$ of $\boldsymbol{\mathcal{O}}_{\mathbf{Z}}$-sheaves. Let $\mathbf{U}$ be an open of $\mathbf{Z}$. Hence, there exists an open $\mathbf{W}$ of $\mathbf{X}$, such that $\mathbf{U} = \mathbf{W} \cap \mathbf{Z}$. By definition, we have that

$$\boldsymbol{\mathcal{O}}_{\mathbf{Z}}(\mathbf{U}) = \frac{\boldsymbol{\mathcal{O}}_{\mathbf{X}}(\mathbf{W})}{\boldsymbol{\mathcal{I}}(\mathbf{W})}. \tag{4}$$

If we denote by $W = \eta^{-1}(\mathbf{W})$, then by (1) again, we have that $\zeta^{-1}(\mathbf{U}) = W \cap Z$. Furthermore, we get by **(2.1.2.(4))** that

$$(\eta^*\boldsymbol{\mathcal{I}})(W) = \boldsymbol{\mathcal{I}}(\mathbf{W})\mathcal{O}_X(W).$$

Hence we obtain that

$$\mathcal{O}_Z(\zeta^{-1}(\mathbf{U})) = \frac{\mathcal{O}_X(W)}{\boldsymbol{\mathcal{I}}(\mathbf{W})\mathcal{O}_X(W)}. \tag{5}$$

Therefore, we define, with aide of (4) and (5), the map

$$\zeta^{\#}(\mathbf{U}) : \boldsymbol{\mathcal{O}}_{\mathbf{Z}}(\mathbf{U}) \to \mathcal{O}_Z(\zeta^{-1}(\mathbf{U}))$$

as the base change

$$\frac{\boldsymbol{\mathcal{O}}_{\mathbf{X}}(\mathbf{W})}{\boldsymbol{\mathcal{I}}(\mathbf{W})} \to \frac{\mathcal{O}_X(W)}{\boldsymbol{\mathcal{I}}(\mathbf{W})\mathcal{O}_X(W)}$$

of $\eta^{\#}(\mathbf{W}) : \boldsymbol{\mathcal{O}}_{\mathbf{X}}(\mathbf{W}) \to \mathcal{O}_X(W)$. This defines the wanted map in (3) and the reader should check that it is indeed a morphism of locally ringed spaces.

Let Y be an arbitrary rigid analytic variety and let

$$(\theta, \theta^{\#}) : (Y, \mathcal{O}_Y) \to (\mathbf{Z}, \boldsymbol{\mathcal{O}}_{\mathbf{Z}}),$$

be a morphism of locally ringed spaces. After composing this with the closed immersion $\mathbf{Z} \hookrightarrow \mathbf{X}$, we get from the fact that X is the analytization of $\mathbf{X}$, a unique map $\varphi : Y \to X$ of rigid analytic varieties, making following diagram commute

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{(\theta,\theta^{\#})} & (\mathbf{Z}, \boldsymbol{\mathcal{O}}_{\mathbf{Z}}) \\ {\scriptstyle (\varphi,\varphi^{\#})}\downarrow & & \downarrow \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta,\eta^{\#})]{} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \end{array} \tag{6}$$

Clearly, from the commutativity of the above diagram and (1), we obtain that $\varphi(Y) \subset Z$. We want to proof that φ even as a map of rigid analytic varieties factors through Z. Therefore, we need to show that $(\eta^*\boldsymbol{\mathcal{I}})\mathcal{O}_Y = 0$. We can check this on the stalks. So let y be a point of Y and let $x = \varphi(y)$ and $\mathbf{x} = \theta(y) = \eta(x)$. From (1) of **(2.1.2)**, we get that

$$\begin{aligned} (\eta^*\boldsymbol{\mathcal{I}})\mathcal{O}_{Y,y} &= (\eta^*\boldsymbol{\mathcal{I}})_x\mathcal{O}_{Y,y} \\ &= \boldsymbol{\mathcal{I}}_{\mathbf{x}}\mathcal{O}_{Y,y} = 0, \end{aligned}$$

where the vanishing of the latter follows from the fact that the composed map $Y \to \mathbf{X}$ factors over $\mathbf{Z}$, by the commutativity of (6), and clearly $\boldsymbol{\mathcal{I}}_{\mathbf{x}}\boldsymbol{\mathcal{O}}_{\mathbf{Z},\mathbf{x}} = 0$.

Hence, φ becomes a map $Y \to Z$ which renders following diagram commutative

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{(\theta,\theta^{\#})} & (\mathbf{Z}, \boldsymbol{\mathcal{O}}_{\mathbf{Z}}) \\ {\scriptstyle (\varphi,\varphi^{\#})}\downarrow & & \| \\ (Z, \mathcal{O}_Z) & \xrightarrow[(\zeta,\zeta^{\#})]{} & (\mathbf{Z}, \boldsymbol{\mathcal{O}}_{\mathbf{Z}}) \end{array}$$

The uniqueness of this map is easily verified. ■

2.2. Analytization of a Blowing Up

2.2.1. Proposition. *Let* $\mathbf{X}$ *be an affine analytic scheme and let*

$$(\eta, \eta^{\#}) : (X, \mathcal{O}_X) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}})$$

be its analytization. Let $Y = \operatorname{Sp} A$ *be an affinoid variety and let* $\mathbf{Y} = \mathbf{Spec}(A)$*. Let*

$$(\zeta, \zeta^{\#}) : (Y, \mathcal{O}_Y) \to (\mathbf{Y}, \boldsymbol{\mathcal{O}}_{\mathbf{Y}})$$

be the analytization of $\mathbf{Y}$*. Let* $\theta : Y \to X$ *be a map of rigid analytic varieties. Then there exists a unique map of schemes* $\boldsymbol{\theta} : \mathbf{Y} \to \mathbf{X}$*, making following diagram commute*

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{(\zeta, \zeta^{\#})} & (\mathbf{Y}, \boldsymbol{\mathcal{O}}_{\mathbf{Y}}) \\ {\scriptstyle (\theta, \theta^{\#})}\downarrow & & \downarrow{\scriptstyle (\boldsymbol{\theta}, \boldsymbol{\theta}^{\#})} \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta, \eta^{\#})]{} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \end{array} \tag{1}$$

Moreover, θ *equals the analytization* $\boldsymbol{\theta}^{\mathrm{an}}$ *of* $\boldsymbol{\theta}$*.*

Proof. Let $\mathbf{X} = \mathbf{Spec}(B)$, where B is a finitely generated S-algebra and S is an affinoid algebra. The map η induces a morphism of algebras

$$\eta^{\#}(\mathbf{X}) : B = \boldsymbol{\mathcal{O}}_{\mathbf{X}}(\mathbf{X}) \to \mathcal{O}_X(X),$$

whereas the map θ induces a morphism of algebras

$$\theta^{\#}(X) : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y) = A.$$

Composing these two morphisms gives a map $f : B \to A$, which induces a map of schemes

$$\boldsymbol{\theta} : \mathbf{Y} \to \mathbf{X}.$$

Let us prove that this map meets the requirements of the statement. So, first of all, we have to show that (1) commutes. Let y be a point of Y and let $\mathfrak{N}$ be the corresponding maximal ideal of A. Hence $\zeta(y)$ corresponds also to this maximal ideal and therefore, $\boldsymbol{\theta}\zeta(y)$ corresponds to the prime ideal $\mathfrak{N} \cap B$ of B.

On the other hand, take a representation $B = S[T]/I$ as in (1) of **(1.2.1)** and define the affinoid algebras B_i as in loc. cit., so that $X = \bigcup X_i$, where $X_i = \operatorname{Sp} B_i$. Suppose that $\theta(y) \in X_i$. Let $\mathfrak{m}_i$ denote the maximal ideal of B_i corresponding to $\theta(y)$ and let $\mathfrak{m} = \mathfrak{m}_i \cap B$. Hence $\mathfrak{m}$ is the maximal ideal of B corresponding to $\eta\theta(y)$. So we need to prove that $\mathfrak{m} = \mathfrak{N} \cap B$. Since $\mathfrak{m}$ is maximal, we only need to show that $\mathfrak{m} \subset \mathfrak{N}$. Consider the completion

$$\widehat{\theta}_y^{\#} : \widehat{(B_i)}_{\mathfrak{m}_i} \to \widehat{A}_{\mathfrak{N}}$$

of the local map $\theta_y^{\#}$. Composed with the local map $B_{\mathfrak{m}} \to \widehat{(B_i)}_{\mathfrak{m}_i}$, this yields a local map

$$B_{\mathfrak{m}} \to \widehat{A}_{\mathfrak{N}}.$$

This proves that $\mathfrak{m} \subset \mathfrak{N}$, as we had to show. The uniqueness of $\boldsymbol{\theta}$ is clear from the fact that η and ζ are locally formal isomorphisms which are bijections between the sets of closed points. Moreover, diagram (1) proves that θ must be the analytization of $\boldsymbol{\theta}$. ■

2.2.2. Theorem. *Let* $\mathbf{X}$ *be an analytic scheme and let*

$$(\eta, \eta^{\#}) : (X, \mathcal{O}_X) \to (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}})$$

be its analytization. Let $\mathbf{Z}$ *be a closed subscheme of* $\mathbf{X}$*. Let*

$$\boldsymbol{\pi} : \tilde{\mathbf{X}} \to \mathbf{X}$$

be the blowing up of $\mathbf{X}$ *with center* $\mathbf{Z}$*. Then* $\tilde{\mathbf{X}}$ *is also an analytic scheme. Moreover, let*

$$(\tilde{\eta}, \tilde{\eta}^{\#}) : (\tilde{X}, \mathcal{O}_{\tilde{X}}) \to (\tilde{\mathbf{X}}, \boldsymbol{\mathcal{O}}_{\tilde{\mathbf{X}}})$$

denote its analytization. Then the map

$$\pi = \boldsymbol{\pi}^{\mathrm{an}} : \tilde{X} \to X$$

is the blowing up of X *with center* $Z = \mathbf{Z}^{\mathrm{an}} (= \eta^{-1}(\mathbf{Z}))$*.*

Proof. Since the blowing up map $\boldsymbol{\pi}$ is proper by [**Ha**, Chapter II, Proposition 7.10] , we have that $\tilde{\mathbf{X}}$ is of finite type over $\mathbf{X}$, and hence is also an analytic scheme.

Let $\boldsymbol{\mathcal{I}}$ denote the coherent $\boldsymbol{\mathcal{O}}_{\mathbf{X}}$-ideal defining $\mathbf{Z}$. Let $\mathcal{I} = \eta^*(\boldsymbol{\mathcal{I}})$. Then $\mathcal{I}$ is the coherent $\mathcal{O}_X$-ideal defining Z, by **(2.1.3)**. One verifies easily, using **(2.1.2.(4))** that

$$\tilde{\eta}^*(\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{O}}_{\tilde{\mathbf{X}}}) \cong \mathcal{I}\mathcal{O}_{\tilde{X}}$$

by checking this on all the stalks. Hence by **(2.1.2.(3))**, we get that this last sheaf is invertible.

Let $Y = \operatorname{Sp} A$ be an affinoid variety and let $f : Y \to X$ be a map of rigid analytic varieties, such that $\mathcal{I}\mathcal{O}_Y$ is invertible. To complete the proof, we have to show that there exists a unique map $g : Y \to \tilde{X}$ making following diagram commute

$$\begin{array}{ccc} Y & \xrightarrow{\ g\ } & \tilde{X} \\ \| & & \downarrow \pi \\ Y & \xrightarrow[f]{} & X \end{array} \tag{1}$$

Let $\mathbf{Y} = \mathbf{Spec}(A)$ and let

$$(\zeta, \zeta^{\#}) : (Y, \mathcal{O}_Y) \to (\mathbf{Y}, \boldsymbol{\mathcal{O}}_{\mathbf{Y}})$$

be the analytization of $\mathbf{Y}$.

Case 1. Assume that $\mathbf{X} = \mathbf{Spec}(B)$, where B is finitely generated over an affinoid algebra.

By **(2.2.1)**, there exists a unique map

$$\mathbf{f} : \mathbf{Y} = \mathbf{Spec}(A) \to \mathbf{X}$$

making the following diagram commute

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{(\zeta, \zeta^{\#})} & (\mathbf{Y}, \boldsymbol{\mathcal{O}}_{\mathbf{Y}}) \\ {\scriptstyle (f, f^{\#})}\downarrow & & \downarrow{\scriptstyle (\mathbf{f}, \mathbf{f}^{\#})} \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta, \eta^{\#})]{} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \end{array} \tag{2}$$

By **(2.1.2.(4))**, one sees that

$$\zeta^*(\boldsymbol{\mathcal{I}\mathcal{O}_Y}) = \mathcal{I}\mathcal{O}_Y. \tag{3}$$

Hence by **(2.1.2.(3))**, we get that $\boldsymbol{\mathcal{I}\mathcal{O}_Y}$ is invertible. Therefore, by the universal property defining blowing up in algebraic geometry, there exists a unique map $\mathbf{g} : \mathbf{Y} \to \tilde{\mathbf{X}}$, making following diagram commute

$$\begin{array}{ccc} \mathbf{Y} & \xrightarrow{\mathbf{g}} & \tilde{\mathbf{X}} \\ \| & & \downarrow \boldsymbol{\pi} \\ \mathbf{Y} & \xrightarrow[\mathbf{f}]{} & \mathbf{X} \end{array} \tag{4}$$

If we set $g = \mathbf{g}^{\mathrm{an}}$, then the analytization of diagram (4) is exactly diagram (1), proving its commutativity.

We need to prove that this map g is unique. Hence let $h : Y \to \tilde{X}$ be another map making diagram (1) commute. Let $\{\tilde{\mathbf{X}}_i\}_i$ be an open affine covering of $\tilde{\mathbf{X}}$. Let $\tilde{X}_i = \tilde{\eta}^{-1}(\tilde{\mathbf{X}}_i)$. Hence, by **(1.1.3)** we have that the restriction $\tilde{\eta} : \tilde{X}_i \to \tilde{\mathbf{X}}_i$ is the analytization of $\tilde{\mathbf{X}}_i$ (where we will no longer distinguish in notation between a map and its restriction). It is always possible to find an admissible affinoid covering $\{Y_i = \operatorname{Sp} A_i\}_i$ of Y, such that $h(Y_i) \subset \tilde{X}_i$. Let $\mathbf{Y}_i = \mathbf{Spec}(A_i)$ and let $\zeta_i : Y_i \to \mathbf{Y}_i$ denote the analytization of $\mathbf{Y}_i$. By proposition **(2.2.1)** we can find unique maps $\mathbf{h}_i : \mathbf{Y}_i \to \tilde{\mathbf{X}}_i$, making following diagram commute

$$\begin{array}{ccc} (Y_i, \mathcal{O}_{Y_i}) & \xrightarrow{(\zeta_i, {\zeta_i}^{\#})} & (\mathbf{Y_i}, \boldsymbol{\mathcal{O}_{\mathbf{Y_i}}}) \\ {\scriptstyle (h, h^{\#})}\downarrow & & \downarrow{\scriptstyle (\mathbf{h}_i, {\mathbf{h}_i}^{\#})} \\ (\tilde{X}_i, \mathcal{O}_{\tilde{X}_i}) & \xrightarrow[(\tilde{\eta}, \tilde{\eta}^{\#})]{} & (\tilde{\mathbf{X}}_\mathbf{i}, \boldsymbol{\mathcal{O}_{\tilde{\mathbf{X}}_\mathbf{i}}}) \end{array} \tag{5}$$

Moreover, also by **(2.2.1)**, we find unique maps $\boldsymbol{\alpha}_i : \mathbf{Y}_i \to \mathbf{Y}$ making the following diagram commute

$$\begin{array}{ccc} (Y_i, \mathcal{O}_{Y_i}) & \xrightarrow{(\zeta_i, {\zeta_i}^{\#})} & (\mathbf{Y_i}, \boldsymbol{\mathcal{O}_{\mathbf{Y_i}}}) \\ \downarrow & & \downarrow{\scriptstyle (\boldsymbol{\alpha}_i, {\boldsymbol{\alpha}_i}^{\#})} \\ (Y, \mathcal{O}_Y) & \xrightarrow[(\zeta, \zeta^{\#})]{} & (\mathbf{Y}, \boldsymbol{\mathcal{O}_{\mathbf{Y}}}) \end{array} \tag{6}$$

Claim A. *The ideal $\boldsymbol{\mathcal{I}\mathcal{O}_{\mathbf{Y}_i}}$ is invertible.*

Suppose we proved this, then, by the universal property of a blowing up applied to the composite map $\mathbf{f}\boldsymbol{\alpha}_i$, there exists a unique map $\mathbf{t}_i : \mathbf{Y}_i \to \tilde{\mathbf{X}}$, such that following diagram commutes

$$\begin{array}{ccc} \mathbf{Y}_i & \xrightarrow{\mathbf{t}_i} & \tilde{\mathbf{X}} \\ \| & & \downarrow \boldsymbol{\pi} \\ \mathbf{Y}_i & \xrightarrow[\mathbf{f}\boldsymbol{\alpha}_i]{} & \mathbf{X} \end{array} \tag{7}$$

Clearly, the map $\mathbf{g}\boldsymbol{\alpha}_i$ renders (7) commutative, hence we get that

$$\mathbf{t}_i = \mathbf{g}\boldsymbol{\alpha}_i. \tag{8}$$

Claim B. *We have that*

$$\mathbf{f}\boldsymbol{\alpha}_i = \boldsymbol{\pi}\mathbf{h}_i.$$

Assuming the claim, we get that also $\mathbf{h}_i$ renders (7) commutative and hence must be equal to $\mathbf{t}_i$. Together with (8), we therefore get that

$$\mathbf{h}_i = \mathbf{g}\boldsymbol{\alpha}_i.$$

Taking the analytization of these maps, we obtain that $h|_{Y_i} = g|_{Y_i}$, proving the uniqueness of g.

So, the only thing which remains to be done is proving both claims A and B. Claim A follows from the identity

$$\zeta_i^*(\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{O}}_{\mathbf{Y}_i}) = \mathcal{I}\mathcal{O}_{Y_i},$$

which can be derived from (3). By assumption the latter ideal is invertible, so that we are done by **(2.1.2.(3))**.

To prove claim B, let us first show that the composition of both maps with ζ_i are equal. By (6) and then using (2) we get that

$$\begin{aligned}\mathbf{f}\boldsymbol{\alpha}_i\zeta_i &= \mathbf{f}\zeta \\ &= \eta f.\end{aligned}$$

On the other hand, by (5), the definition of $\pi = \boldsymbol{\pi}^{\mathrm{an}}$ and our assumption on h, we get that

$$\begin{aligned}\boldsymbol{\pi}\mathbf{h}_i\zeta_i &= \boldsymbol{\pi}\tilde{\eta}h \\ &= \eta\pi h \\ &= \eta f.\end{aligned}$$

Hence, both $\mathbf{f}\boldsymbol{\alpha}_i$ and $\boldsymbol{\pi}\mathbf{h}_i$ are solutions to the commutativity of the following diagram

$$\begin{array}{ccc} (Y_i, \mathcal{O}_{Y_i}) & \xrightarrow{(\zeta_i, \zeta_i{}^{\#})} & (\mathbf{Y_i}, \boldsymbol{\mathcal{O}}_{\mathbf{Y_i}}) \\ {\scriptstyle (f, f^{\#})}\downarrow & & \downarrow \\ (X, \mathcal{O}_X) & \xrightarrow[(\eta, \eta^{\#})]{} & (\mathbf{X}, \boldsymbol{\mathcal{O}}_{\mathbf{X}}) \end{array}$$

Since by **(2.2.1)** there exists a unique solution to this commutativity, both maps must be the same, therefore establishing our claim and finishing the proof in this case.

Case 2. Let $\mathbf{X}$ be an arbitrary analytic scheme and let $\{\mathbf{X}_i\}_i$ be an affine open covering of $\mathbf{X}$. Let $X_i = \eta^{-1}(\mathbf{X}_i)$. Then from **(1.1.3)** we know that X_i is the analytization of $\mathbf{X}_i$. From algebraic geometry we know that the restriction

$$\boldsymbol{\pi}_i : \boldsymbol{\pi}^{-1}(\mathbf{X}_i) \to \mathbf{X}_i$$

of $\boldsymbol{\pi}$ to $\boldsymbol{\pi}^{-1}(\mathbf{X}_i)$, is the blowing up of $\mathbf{X}_i$ with center $\mathbf{Z} \cap \mathbf{X}_i$. Let $\tilde{X}_i$ denote the analytization of $\boldsymbol{\pi}^{-1}(\mathbf{X}_i)$ and let

$$\pi_i : \tilde{X}_i \to X_i \tag{4}$$

denote the analytization of $\boldsymbol{\pi}_i$. Hence from case 1, we obtain that (4) is the blowing up of X_i with center the analytization of $\mathbf{Z} \cap \mathbf{X}_i$, which equals $Z \cap X_i$ by **(2.1.3)**. Using **(1.1.3)**, we have that

$$\tilde{X}_i = \tilde{\eta}^{-1}(\boldsymbol{\pi}^{-1}(\mathbf{X}_i)).$$

In other words, we get that

$$\pi_i = \pi|_{\tilde{X}_i}.$$

We are now done by lemma [**Sch 4**, Proposition 1.4.4] . ■

2.2.3. Corollary. *Let X be a rigid analytic variety, Z a closed analytic subvariety and $\pi : \tilde{X} \to X$ the blowing up of X with center Z. If X is reduced, then so is $\tilde{X}$. If X and Z are both manifolds, then so is $\tilde{X}$.*

Proof. The questions being local, we may assume that $X = \operatorname{Sp} A$ is affinoid. Let $\mathbf{X} = \mathbf{Spec}A$ and $\mathbf{Z} = \mathbf{V}I$, where I is the ideal defining Z. Hence by **(2.1.3)**, X and Z are the analytizations of $\mathbf{X}$ and $\mathbf{Z}$ respectively. By **(1.3.5)**, $\mathbf{X}$ is reduced (respectively, $\mathbf{X}$ and $\mathbf{Z}$ are regular), if X is (respectively, X and Z are). Let $\tilde{\pi} : \tilde{\mathbf{X}} \to \mathbf{X}$ be the blowing up of $\mathbf{X}$ with center $\mathbf{Z}$. Then it is well-known that $\tilde{\mathbf{X}}$ is reduced (respectively, regular). Since by the previous theorem **(2.2.2)**, $\tilde{X}$ is the analytization of $\tilde{\mathbf{X}}$ we are done by using **(1.3.5)** once more. ■

3. Embedded Resolution of Singularities.

3.1. Hironaka's Embedded Resolution of Singularities

3.1.1. Definition. Let A be a noetherian regular ring (i.e. a ring all of whose localizations are regular local rings) and $f \in A$, with $f \neq 0$. Let $\mathfrak{p}$ be a prime ideal of A, then we say that f has *normal crossings* at $\mathfrak{p}$, if there exist a regular system of parameters $\{\xi_1, \ldots, \xi_d\}$ of $A_{\mathfrak{p}}$, a unit $u \in A_{\mathfrak{p}}$ and integers $N_i \in \mathbb{N}$, such that we can write f, considered as an element of $A_{\mathfrak{p}}$, as

$$f = u\xi_1^{N_1} \cdots \xi_d^{N_d}.$$

We will say that f has *normal crossings* in A (or in $\mathbf{Spec}(A)$), if it has normal crossings in each prime ideal of A. More general, if $\mathbf{X}$ is a regular integral (or, smooth) scheme, $\mathbf{V} \subset \mathbf{X}$ a closed subset of codimension one (a *hypersurface*, for short) and $\mathbf{x} \in \mathbf{X}$, then we say that $\mathbf{V}$ has *normal crossings* in $\mathbf{x}$, if f has normal crossings in the local ring $\boldsymbol{\mathcal{O}}_{\mathbf{X},\mathbf{x}}$ of $\mathbf{X}$ at $\mathbf{x}$, where f is a local equation of $\mathbf{V}$ at $\mathbf{x}$. We say that $\mathbf{V}$ has *normal crossings* in $\mathbf{X}$, if it has normal crossings in each point of $\mathbf{X}$.

Remark. Let A be a noetherian regular ring and $f \in A$, with $f \neq 0$. One verifies that the locus of points $\mathfrak{p} \in \mathbf{Spec}(A)$, such that f has normal crossings at $\mathfrak{p}$ is Zariski open (see for instance [**BM**]) and the same is true if we work in the maximal spectrum (i.e. only consider maximal ideals).

3.1.2. Theorem (Hironaka's Embedded Resolution of Singularities). *Let A be an excellent regular local ring which contains a field of characteristic zero, $\mathbf{X}$ a regular integral scheme of finite type over $\mathbf{Spec}(A)$ and $\mathbf{V}$ a hypersurface of $\mathbf{X}$. Then there exist a regular integral scheme $\tilde{\mathbf{X}}$ of finite type over $\mathbf{Spec}(A)$ and a map $\mathbf{h} : \tilde{\mathbf{X}} \to \mathbf{X}$, such that*

(i) $\mathbf{h}$ *is a composition of finitely many blowing up maps with respect to smooth centers of codimension at least two,*
(ii) $\mathbf{h}^{-1}(\mathbf{V})$ *has normal crossings in* $\tilde{\mathbf{X}}$.

Proof. See [**Hi**, p.146 Corollary 3 and p.161 Remark] . For an explanation of (i), see the remark following **(3.2.3)** below. ■

3.2. Embedded Resolution of Singularities in Rigid Analytic Geometry

3.2.1. Definition. Let $M = \operatorname{Sp} A$ be an affinoid variety. We call M an *affinoid manifold* if A is a regular domain. In other words, M is irreducible and for each $x \in M$, the local ring $\mathcal{O}_{M,x}$ is regular. Indeed, if $\mathfrak{m}_x$ denotes the maximal ideal of A corresponding to x, then we know from [**BGR**, 7.3.2. Proposition 8] that $A_{\mathfrak{m}_x}$ is regular if and only if $\mathcal{O}_{M,x}$ is regular.

Let M be a rigid analytic variety, then we will call M a *rigid analytic manifold*, if it is quasi-compact and the local ring $\mathcal{O}_{M,x}$ at each point $x \in M$ is regular. We sometimes might express this also by saying that M is *regular*. So, in particular, M admits a finite admissible affinoid covering $\mathcal{X} = \{X_i\}_i$, such that each X_i is an affinoid manifold. Indeed, just take a finite admissible affinoid covering $\mathcal{X} = \{X_i\}_i$ of M. Since by [**BGR**, 9.1.4. p.346] each connected component of X_i is an admissible affinoid open in X_i, we may already assume that all the X_i are connected. But each point of $X_i = \operatorname{Sp} A_i$ is regular, proving that A_i is a regular ring, hence by applying [**Kap**, Theorem 168] and using that X_i is connected, we conclude that A_i is a regular domain and hence each X_i is an affinoid manifold.

3.2.2. Lemma. *Let $\mathbf{X}$ be a smooth analytic scheme and let $\eta : X \to \mathbf{X}$ be its analytization. Let $\mathbf{V}$ be a hypersurface in $\mathbf{X}$ (with its reduced induced subscheme structure) and let $V = \eta^{-1}(\mathbf{X})$ be the analytization of $\mathbf{V}$. Let $x \in X$ be a point and let $\mathbf{x} = \eta(x)$ be the corresponding point in $\mathbf{X}$. If $\mathbf{V}$ has normal crossings at $\mathbf{x}$, then V has normal crossings at x.*

Remark. Note that by **(1.3.5)**, we know that X is regular, so that it makes sense to talk about normal crossings at a point of X. Also by loc. cit., we get that dimensions and codimensions are preserved under analytization, since all the local maps are flat. In other words, V is again a hypersurface.

Proof. Let $(\xi_1, \ldots, \xi_d)$ be a regular system of parameters in $\boldsymbol{\mathcal{O}}_{\mathbf{X},\mathbf{x}}$, where d is the dimension of $\mathbf{X}$. Since the local map

$$\eta_x : \boldsymbol{\mathcal{O}}_{\mathbf{X},\mathbf{x}} \to \mathcal{O}_{X,x}$$

is an analytic isomorphism, we get that the maximal ideal of $\boldsymbol{\mathcal{O}}_{\mathbf{X},\mathbf{x}}$ generates the maximal ideal in $\mathcal{O}_{X,x}$. Therefore $(\xi_1, \ldots, \xi_d)$ is also a regular system of parameters in $\mathcal{O}_{X,x}$. From this our claim follows immediately. ■

3.2.3. Theorem (Embedded Resolution in Rigid Analytic Geometry). *Suppose that K is of characteristic zero. Let $M = \operatorname{Sp} A$ be an affinoid manifold and $f \in A$, with $f \neq 0$.*

Then there exists a finite admissible affinoid covering $\mathcal{X} = \{X_i\}_{i\in I}$ of M, and, for each $i \in I$, a rigid analytic manifold $\tilde{X}_i$ and a map $h_i : \tilde{X}_i \to X_i$ of rigid analytic varieties, such that

(i) *h_i is a composition of finitely many blowing up maps with respect to regular centers of codimension at least two,*
(ii) *$h_i^{-1}(\mathrm{V}(f) \cap X_i)$ has normal crossings in $\tilde{X}_i$.*

Remark. Let us explain what we mean by (i). Fix some $i \in I$ and let us, for the sake of convenience, drop the indices i, so that we can write $h : \tilde{X} \to X$. Saying that h is of the type as described in (i) means the following. There exists rigid analytic varieties Y_j, for $j = 0, \ldots, k$, where $Y_0 = X$ and $Y_k = \tilde{X}$, and maps $\pi_j : Y_{j+1} \to Y_j$, for $j = 0, \ldots, k-1$, such that each π_j is the blowing up of Y_j with center Z_j, which is smooth and of codimension at least two in Y_j, such that

$$h = \pi_{k-1} \circ \cdots \circ \pi_0.$$

By **(2.2.3)** all Y_k are rigid analytic manifolds, so, in particular, so is $\tilde{X}$ is.

Below we will consider the strict transform of a closed analytic subspace H of X under such a map h. With this we mean the consecutive strict transforms under the π_j, i.e., for each $j = 0, \ldots, k-1$, let $W_{j+1} \subset Y_{j+1}$ denote the strict transform of $W_j \subset Y_j$ under π_j, where $W_0 = H$. In other words, W_{j+1} is the blowing up of W_j with center $W_j \cap H$. We then will call $\tilde{H} = W_k$ the strict transform of H under h. Note that the strict transform $\tilde{H}$ 'survives', i.e. is not the empty space, if and only if, neither of the W_j is fully contained in the center of blowing up Z_j. Consequently, if the subspace H we started with was irreducible, the same holds for all strict transforms by [**Sch 4**, Corollary 3.2.3] , and moreover, each blowing up map is surjective and therefore so is their composition $h|_{\tilde{H}}$. In particular h is surjective.

Proof. Let $\mathbf{M} = \mathbf{Spec}(A)$ and let

$$\eta : M \to \mathbf{M}$$

denote the analytization of $\mathbf{M}$. Let $x \in M$ be a point and let $\mathbf{x} = \eta(x)$ be the corresponding point in $\mathbf{M}$. Let $\mathfrak{m}$ be the maximal ideal of A corresponding to x (and $\mathbf{x}$). Let $\mathbf{X} = \mathbf{Spec}(A_{\mathfrak{m}})$. Let $\mathbf{T}$ be the hypersurface of $\mathbf{X}$ defined by f and let $\mathbf{V}$ be the hypersurface defined by the same f, but now as a closed subscheme of $\mathbf{M}$.

By **(3.1.2)**, we can find a regular integral scheme $\tilde{\mathbf{X}}$ and a map $\mathbf{h} : \tilde{\mathbf{X}} \to \mathbf{X}$, such that

(i) $\mathbf{h}$ is a composition of finitely many blowing up maps with respect to smooth centers of codimension at least two,
(ii) $\mathbf{h}^{-1}(\mathbf{T})$ has normal crossings in $\tilde{\mathbf{X}}$.

Therefore, we can find an $s \notin \mathfrak{m}$, such that $\mathbf{h}$ can be *extended* to $\mathbf{Y} = \mathbf{Spec}(A_s)$. By this we mean that there exists a scheme $\tilde{\mathbf{Y}}$ and a map $\mathbf{g} : \tilde{\mathbf{Y}} \to \mathbf{Y}$ such that $\mathbf{g}$ is a composition of finitely many blowing up maps with respective centers extensions

of the centers of $\mathbf{h}$. Moreover, the canonical map $\boldsymbol{\alpha} : \mathbf{X} \to \mathbf{Y}$ gives rise to a map $\tilde{\boldsymbol{\alpha}} : \tilde{\mathbf{X}} \to \tilde{\mathbf{Y}}$ such that the following diagram

$$\begin{array}{ccc} \tilde{\mathbf{X}} & \xrightarrow{\mathbf{h}} & \mathbf{X} \\ {\scriptstyle\tilde{\boldsymbol{\alpha}}}\downarrow & & \downarrow{\scriptstyle\boldsymbol{\alpha}} \\ \tilde{\mathbf{Y}} & \xrightarrow[\mathbf{g}]{} & \mathbf{Y} \end{array}$$

is the strict transform diagram of $\boldsymbol{\alpha}$ under $\mathbf{h}$. Indeed, this follows from the fact that $\boldsymbol{\alpha}$ is flat. We leave the details to the reader.

Since the regular locus is open, we can even choose s in such manner that all the centers of $\mathbf{g}$ are smooth. In particular, we get that also $\tilde{\mathbf{Y}}$ is smooth.

From the remark after **(3.1.1)**, we know that the locus of points of $\tilde{\mathbf{Y}}$ at which $\mathbf{g}^{-1}(\mathbf{V} \cap \mathbf{Y})$ has no normal crossings is a closed subset of $\tilde{\mathbf{Y}}$. Hence its image under $\mathbf{g}$ is a closed subset in $\mathbf{Y}$, since $\mathbf{g}$ is proper. Therefore, we can choose s in such manner that $\mathbf{g}^{-1}(\mathbf{V} \cap \mathbf{Y})$ has normal crossings everywhere.

By construction $\mathbf{Y}$ is an analytic scheme and, moreover, by **(1.1.3)**, we know that $Y = \eta^{-1}(\mathbf{Y})$ is its analytization. Let $\tilde{Y}$ be the analytization of $\tilde{\mathbf{Y}}$ and let $g : \tilde{Y} \to Y$ be the analytization of $\mathbf{g}$. By theorem **(2.2.2)** we know that g is a composition of finitely many blowing up maps with respect to regular centers of codimension at least two. Moreover, by **(3.2.2)**, we get that $g^{-1}(V \cap Y)$ has normal crossings.

To summarize, we found, for each point $x \in M$, a Zariski-open $Y^{(x)}$, containing x, and maps $g^{(x)} : \tilde{Y}^{(x)} \to Y^{(x)}$, such that (i) and (ii) of our statement hold for these maps. Since each covering by Zariski open subsets is admissible, we can take an admissible affinoid covering of each $Y^{(x)}$, so that the union of all these admissible affinoid coverings forms an admissible affinoid covering of M. Therefore, already finitely many of these cover M, say $\mathcal{X} = \{X_i\}_i$. For each i, there exists an $x \in M$, such that $X_i \subset Y^{(x)}$. Hence, if we set $\tilde{X}_i = {g^{(x)}}^{-1}(X_i)$ and

$$h_i = g^{(x)}\Big|_{\tilde{X}_i} : \tilde{X}_i \to X_i,$$

then, by [**Sch 4**, Proposition 1.4.4] , the h_i and the $\tilde{X}_i$ meet the requirements of our statement. ■

Remark. With a little extra effort, one can prove a global version of this theorem in the sense that only a single map $h\colon \tilde{X} \to X$ with the properties (i) and (ii) is needed. Namely, in stead of applying HIRONAKA's Theorem to the hypersurface given by $f = 0$ in each $\mathbf{X} = \mathbf{Spec}(A_{\mathfrak{m}})$, use [**BM 2**] . The latter gives a canonically defined blowing up process $g^{(x)}$ on the various $Y^{(x)}$ (notation as above), which therefore patch together to form a global map h.

3.2.4. Corollary. *Suppose that K is of characteristic zero. Let X be a rigid analytic manifold and H an irreducible hypersurface in X.*

Then there exists an admissible affinoid covering $\mathcal{X} = \{X_i\}_{i\in I}$ of X, and, for each $i \in I$, a rigid analytic manifold $\tilde{X}_i$ and a map $h_i : \tilde{X}_i \to X_i$ of rigid analytic varieties, such that

(i) *h_i is a composition of finitely many blowing up maps with respect to regular centers of codimension at least two,*

(ii) *the strict transform of $H \cap X_i$ under h_i is a rigid analytic manifold.*

Proof. By taking an admissible affinoid covering, we may assume that X is affinoid. By **(3.2.3)** we can find a (finite) admissible affinoid covering $\mathcal{X} = \{X_i\}_{i\in I}$ of X, and, for each $i \in I$, a rigid analytic manifold $\tilde{X}_i$ and a map $h_i : \tilde{X}_i \to X_i$ of rigid analytic varieties, such that h_i satisfies condition (i) of loc. cit. and, moreover, the inverse image $h_i^{-1}(H \cap X_i)$ has normal crossings in $\tilde{X}_i$.

In other words, in each point $x \in \tilde{X}_i$, there exists a regular system of parameters $\xi = (\xi_1, \ldots, \xi_t)$, such that the local equation of $h_i^{-1}(X \cap X_i)$ is given by a monomial

$$\xi_1^{e_1} \cdots \xi_t^{e_t} = 0.$$

But the strict transform of $H \cap X_i$ under h_i is an irreducible component of this inverse image. Indeed, the strict transform is again irreducible and reduced by **(2.2.3)** and the remark before the proof of **(3.2.3)**, and an analytic subvariety of $\tilde{X}_i$ of codimension one by [**Sch 4**, Corollary 3.2.3. and Proposition 3.1.2] . Hence the local equation of the strict transform must be given by $\xi_i = 0$, for some i, and hence is regular. ■

Remark. We call the resolution of singularities of H as above an *embedded resolution.* As observed in the remark before the proof of **(3.2.3)**, the strict transform of each $H \cap X_i$ under h_i is obtained by a sequence $\tilde{h}_i$ of blowing up maps, which are derived from the ones in h_i by restricting the centers. In particular, the centers used in $\tilde{h}_i$ are of codimension at least one. However, they might fail to be regular.

3.2.5. Theorem (Resolution of Singularities). *Suppose that K is of characteristic zero. Let X be an integral (=irreducible and reduced) rigid analytic variety.*

Then there exists an admissible affinoid covering $\mathcal{X} = \{X_i\}_{i\in I}$ of X, and, for each $i \in I$, a rigid analytic manifold $\tilde{X}_i$ and a map $h_i : \tilde{X}_i \to X_i$ of rigid analytic varieties which is a composition of finitely many blowing up maps with respect to centers of codimension at least one. In particular, each h_i is surjective.

Proof. Since the statement is local with respect to the Grothendieck topology, we may assume that $X = \operatorname{Sp} A$ is affinoid, with A a domain. We then can embed X in an $W = \operatorname{Sp}(K\langle X\rangle)$, for some variables X. In other words, we can assume from the start that X is embedded in a rigid analytic manifold W and we will prove the theorem under this additional assumption by induction on the codimension d of X in W.

If $d = 1$ (i.e., X is a hypersurface in W), then we are done by **(3.2.4)** and the remark following it.

For general codimension $d > 1$, take any analytic hypersurface V of W, containing X. By **(3.2.4)** applied to V, we can find an 'embedded resolution' for V. Again, since everything is local, we can assume without loss of generality that we have found an analytic manifold $\tilde{W}$ and a map $h : \tilde{W} \to W$ of which the centers are smooth and of codimension at least two and such that the strict transform $\tilde{V}$ under this map is smooth as well. Again, looking at each stage in the blowing up process, there is no loss in generality if we assume that h is given by one blowing up with (smooth) center Z of codimension at least two (in W). There are two cases to be considered.

Case 1. X is contained in Z. Since Z is smooth, we are done by induction on the codimension, since the codimension of X in Z has become smaller than d.

Case 2. X is not contained in Z. Hence the strict transform $\tilde{X}$ of X under h is given by blowing up X with center $Z \cap X$, which is of codimension at least one in X. Again we are done by induction on the codimension, applied this time to the pair $\tilde{X} \subset \tilde{V}$.

As for the last statement on the surjectivity, this has already been observed in the remark before the proof of **(3.2.3)**. ■

3.2.6. Corollary. *Suppose that K has characteristic zero. Let X be a quasi-compact integral rigid analytic variety and let Σ be a subanalytic subset of X. Then there exist finitely many maps $h_i \colon \tilde{X}_i \to X$ of rigid analytic varieties, for $i = 1, \ldots, s$, with each h_i a finite composition of local blowing up maps, such that*

(i) *$h_i^{-1}(\Sigma)$ is (globally) semianalytic in $\tilde{X}_i$, for all $i = 1, \ldots, s$;*
(ii) *the union of all $\operatorname{Im} h_i$ equals X.*

Proof. For the definition of subanalytic and semianalytic sets, see [**GS**] . By **(3.2.5)**, there exists a finite admissible affinoid covering X_i and maps $h_i \colon \tilde{X}_i \to X_i$ which are compositions of finitely many blowing up maps, such that each $\tilde{X}_i$ is a quasi-compact rigid analytic manifold. Taking a finite admissible covering Y_{ij} on each of these $\tilde{X}_i$ and applying the Uniformization Theorem [**GS**, Theorem 3.1] to the subanalytic sets $h_i^{-1}(\Sigma) \cap Y_{ij}$, we obtain the required maps. ■

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