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Projective Dimension and the Singular Locus

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Abstract

For a Noetherian local ring, the prime ideals in the singular locus completely determine the category of finitely generated modules up to direct summands, extensions and syzygies. From this some simple homological criteria are derived for testing whether an arbitrary module has finite projective dimension.

Key Words: projective dimension, singular locus, syzygies, Betti numbers

I. Introduction

In this paper, we address the problem of determining whether a certain module Ω over a Noetherian ring R has finite projective dimension. By the results of Jensen and others, the exact value of this projective dimension, when finite, might depend on one's model of set theory (see for instance [5, §5]). Here, we will content ourselves with proving its finiteness. In fact, we actually will give criteria for Ω to have finite flat dimension (that is to say, admitting a finite flat resolution). It is well-known that this is equivalent with having finite projective dimension (see for instance [5, Proposition 5.6]). Moreover, for Noetherian local rings, flat dimension is at most the dimension of the ring ([1, Theorem 2.4]).

Since this is essentially a local issue, I will in the remainder of this introduction assume that R is moreover local with maximal ideal $\mathfrak{m}$ and residue field k. If Ω is finitely generated, then the vanishing of $\operatorname{Tor}_n^R(\Omega, k)$ suffices to conclude that Ω has flat (whence projective) dimension at most $n-1$. The Local Flatness Criterion (see for instance [7, Theorem 22.3]) can be used to extend this to certain non-finitely generated modules, albeit chiefly in case $n = 1$. However, for an arbitrary R-module Ω, the vanishing of a single *Tor* module will not suffice. In fact, k-rigidity fails in this generality. That is to say, the vanishing of $\operatorname{Tor}_n^R(\Omega, k)$ does not necessarily entail the vanishing of the higher *Tor* modules $\operatorname{Tor}_m^R(\Omega, k)$, with $m > n$. However, I will prove in Theorem V.4 that k-rigidity holds whenever n is at least the dimension of R, under the additional assumption that R is Cohen-Macaulay. Although the proof uses the degeneration of spectral sequences and does not work without the Cohen-Macaulay assumption, it is conceivable that this latter assumption can actually be removed from the statement.

Nonetheless, this restriction on the asymptotical behavior of the Betti numbers does not yet solve our original problem. To this end, we need to show that $\operatorname{Tor}_n^R(\Omega, \cdot)$ is the zero functor, for some $n \geq 1$, and in fact, it suffices to show this for the restriction of $\operatorname{Tor}_n^R(\Omega, \cdot)$ to the category $\mathtt{mod}_R$ of finitely generated R-modules. In homological algebra, one can distinguish three rules of inference for the vanishing of a functor: (1) if the functor is additive, vanishing is preserved under direct summands; (2) if the functor is exact in the middle, then vanishing on the outer two modules in a short exact sequence entails the vanishing on the inner module; (3) if we have a collection of derived functors, then vanishing is transferred between two consecutive derivatives by taking (co-)syzygies.* In conclusion, we need to understand the structure of the category $\mathtt{mod}_R$ up to the formation of direct summand, extension and syzygy. This will be formalized in this paper by the notion of *syzygycal net* (see Section VI for details). If we want to study a *Tor* functor in a single dimension, we should not use the third inference rule; the corresponding weaker notion is that of a *net*. In other words, a net is a subclass of $\mathtt{mod}_R$ closed under direct summands and extensions. The key observation is that $\mathtt{mod}_R$, as a net, can be built up from a relative small collection of (cyclic) modules: it suffices to take all R/I where I is either a prime ideal in the singular locus of R or otherwise a parameter ideal (an ideal generated by as many elements as its height). If R is moreover Cohen-Macaulay, then any parameter ideal is generated by a regular sequence and therefore has finite projective dimension. Therefore, the following result (Theorem VI.8 in the text) is immediate under this additional assumption, whereas if R is not Cohen-Macaulay, a more detailed study of Koszul homology is required.

Main Theorem. *Let R be a Noetherian ring. Any finitely generated R-module can be built up from cyclic modules of the form $R/\mathfrak{p}$ with $\mathfrak{p}$ a prime ideal in the singular locus of R, by taking direct summands, extensions and syzygies.*

It follows from a careful analysis of the way in which a finitely generated R-module is obtained from the singular locus by the three inference rules, that, for some $a > 0$, the vanishing of $\mathrm{Tor}_n^R(\Omega, R/\mathfrak{p})$ for all $n = a, \dots, a+d$ and all $\mathfrak{p}$ in the singular locus of R, implies that Ω has finite projective dimension. Using k-rigidity in high dimension (as explained above), it suffices to show this for a single value $n \geq d$ (in the non-local case, we need $n > d$) under the additional Cohen-Macaulay assumption. In fact, we can improve this even further to obtain the following result.

Corollary. *Let R be a d-dimensional local Cohen-Macaulay ring with residue field k. Let Ω be an arbitrary R-module. If the Betti*

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*There is in fact a fourth rule, deformation by means of regular elements, which is implicitly used in the proofs of Theorems V.4 and VI.10.

number $\beta_a^R(\mathfrak{p};\Omega)$ *vanishes, that is to say, if* $\operatorname{Tor}_a^R(\Omega, R/\mathfrak{p})_{\mathfrak{p}} = 0$*, for some* $a \geq d$ *and for all* $\mathfrak{p}$ *in the singular locus of* R*, then* Ω *has finite projective dimension.*

In particular, if R *is Cohen-Macaulay with an isolated singularity, then the vanishing of the single* Tor *module* $\operatorname{Tor}_a^R(\Omega, k)$*, for some* $a \geq d$*, implies that* Ω *has finite projective dimension.*

Without the Cohen-Macaulay assumption, we cannot formulate such a criterion using the vanishing of *Tor* in just a single dimension. In stead, we need to require that we have vanishing for all n in some interval of length $d+1$. Presumably, this is not an optimal result and it would be interesting to reduce the size of such a test interval; of course, if k-rigidity in high dimensions holds, then we can reduce this again to a single value for n.

II. Nets

II.1. Definition. Let R be a ring. The collection of all (isomorphism classes of) finitely generated R-modules will be denoted by $\mathtt{mod}_R$. Let $\mathcal{N} \subset \mathtt{mod}_R$ be a class of finitely generated R-modules. We say that $\mathcal{N}$ is a *net*, if $\mathcal{N}$ is closed under extensions and direct summands. In other words, if the following two conditions are satisfied.

(`Net`) If we have a short exact sequence of finitely generated R-modules

$$0 \to K \to M \to N \to 0$$

for which K and N belong to $\mathcal{N}$, then so does M.

(`DirSum`) If the above sequence is split exact (that is to say, $M \cong K \oplus N$) and M belongs to $\mathcal{N}$, then so do both K and N.

In particular, if R belongs to a net $\mathcal{N}$, then so does any finitely generated free module, and, more generally, any finitely generated projective module. The intersection of an arbitrary number of nets is again a net. Therefore, for each class of finitely generated R-modules $\mathcal{K}$, there exists a smallest net containing it. We will denote this net by $\mathtt{net}(\mathcal{K})$ and call it the *net generated by* $\mathcal{K}$.

Recall that a subset V of $\operatorname{Spec} R$ is called *stable under specialization*, if $\mathfrak{p} \in V$ and $\mathfrak{p} \subset \mathfrak{q} \in \operatorname{Spec} R$ imply $\mathfrak{q} \in V$. In particular, a Zariski closed subset is stable under specialization.

For an arbitrary subset V of $\operatorname{Spec} R$, let us denote by $\mathtt{supp}(V)$ the collection of all finitely generated R-modules M which support in V, that is to say, M belongs to $\mathtt{supp}(V)$, if $M_\mathfrak{p} \neq 0$ implies $\mathfrak{p} \in V$, for every prime ideal $\mathfrak{p}$ of R. Recall that for a finitely generated R-module M, the support $\operatorname{Supp} M$ is equal to the Zariski closed set defined by the annihilator, $\operatorname{Ann}_R(M)$, of M.

II.2. Lemma. *Let R be a Noetherian ring and let V be a subset of* $\operatorname{Spec} R$*. If V is stable under specialization, then $\mathtt{supp}(V)$ is a net, and, moreover, as such, it is generated by all cyclic modules of the form $R/\mathfrak{p}$ with $\mathfrak{p} \in V$.*

Proof. Let $\mathcal{N}$ be the net generated by all cyclic modules $R/\mathfrak{p}$ with $\mathfrak{p} \in V$. We need to show that $\mathtt{supp}(V) = \mathcal{N}$. Let M be a finitely generated R-module with $\operatorname{Supp} M \subset V$. There is a filtration by finitely generated R-modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each M_{i+1}/M_i is isomorphic to some $R/\mathfrak{p}$, with $\mathfrak{p} \in \operatorname{Supp} M$ (see for instance [7, Theorem 6.4]). By an inductive use of rule (Net), it follows that $M \in \mathcal{N}$.

Conversely, we need to show that if $M \in \mathcal{N}$, then its support lies inside V. This is trivial for the modules $R/\mathfrak{p}$ with $\mathfrak{p} \in V$, since the support of $R/\mathfrak{p}$ consists of the Zariski closed set defined by $\mathfrak{p}$ and since V is stable under specialization. If the support of a module lies in V, then so does the support of any of its direct summands. Therefore, by an inductive argument, it suffices to prove that if

$$0 \to K \to M \to N \to 0$$

is exact with the support of K and N contained in V, then so is the support of M. However, this is clear, since always $\operatorname{Supp} M = \operatorname{Supp} K \cup \operatorname{Supp} N$. □

II.3. Example (Nets and their generators). Let us apply the lemma to various choices of V. The resulting nets are then generated by all cyclic modules of the form $R/\mathfrak{p}$ with $\mathfrak{p} \in V$.

1. Let V be $\operatorname{Spec} R$. This is trivially stable under specialization and $\mathtt{supp}(V) = \mathtt{mod}_R$.
2. Fix some $h \geq 0$ and let V be the subset of $\operatorname{Spec} R$ consisting of all prime ideals $\mathfrak{p}$ of height at least h. This is clearly stable under specialization and $\mathtt{supp}(V)$ is the net $\mathcal{I}_h$ consisting of all finitely generated R-modules whose annihilator has height at least h.
3. Fix some $h \geq 0$ and let V be the subset of $\operatorname{Spec} R$ consisting of all prime ideals $\mathfrak{p}$ of depth at least h. Again this is stable under specialization and $\mathtt{supp}(V)$ is the net $\mathcal{G}_h$ consisting of all finitely generated R-modules of grade at least h.
4. Fix some $h \geq 0$ and let V be the subset of $\operatorname{Spec} R$ consisting of all prime ideals $\mathfrak{p}$ for which $R/\mathfrak{p}$ has dimension at most h. Again this is stable under specialization and $\mathtt{supp}(V)$ is the net $\mathcal{D}_h$ consisting of all finitely generated R-modules of dimension at most h.
5. Let V be the subset of $\operatorname{Spec} R$ consisting of all maximal ideals. This is trivially stable under specialization and $\mathtt{supp}(V)$ consists of all finitely generated R-modules of finite length.

The usefulness of nets becomes apparent by the following result.

II.4. Proposition. *Let R be a ring and let $\mathbb{F}$ be an additive functor from the category of R-modules to an abelian category. Suppose $\mathbb{F}$ is exact in the middle. Let $\mathcal{K}$ be a collection of finitely generated R-modules such that $\mathbb{F}(K) = 0$, for each $K \in \mathcal{K}$. Then $\mathbb{F}(M) = 0$, for each $M \in \mathtt{net}(\mathcal{K})$.*

Proof. Recall that a (covariant) functor is called *exact in the middle*, if any short exact sequence

$$0 \to K \to M \to N \to 0$$

transforms into an exact sequence

$$\mathbb{F}(K) \to \mathbb{F}(M) \to \mathbb{F}(N).$$

For a contravariant functor the definition is the same apart from reversing the arrows. The statement is now immediate using induction on the number of times the rules (`Net`) and (`DirSum`) are used. □

II.5. Remark. For R an arbitrary ring and Ω an arbitrary R-module, the following are additive functors which are exact in the middle: $\operatorname{Tor}_i^R(\Omega, \cdot)$, $\operatorname{Ext}^i_R(\Omega, \cdot)$ and $\operatorname{Ext}^i_R(\cdot, \Omega)$, for any i, and, more generally, so is any derived functor of a left or right exact functor.

II.6. Corollary. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k. Let Ω be an arbitrary R-module. If $\operatorname{Tor}_n^R(\Omega, k) = 0$, for some $n \in \mathbb{N}$, then $\operatorname{Tor}_n^R(\Omega, M) = 0$, for every R-module M of finite length.*

Proof. Follows immediately from (5) in Example II.3 in combination with Proposition II.4 and Remark II.5. □

Of course, the same is true for any other additive functor which is exact in the middle. For some more applications of Corollary II.6, see [8].

III. Singular Locus

The *singular locus* of a Noetherian ring R is the collection of all prime ideals $\mathfrak{p}$ of R for which $R_\mathfrak{p}$ is not regular. We will denote it by $\operatorname{Sing} R$. In this paper, an ideal I is called a *parameter* ideal, if it is generated by as many elements as its height (the reader should be aware that this is not always the standard usage of the term). With the phrase *I generates locally one of its minimal primes*, we will mean that for some minimal prime $\mathfrak{p}$ of I, we have that $IR_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. Note that if I is moreover a parameter ideal, then such a minimal prime is necessarily in the regular locus of R. The key result of this paper is the following theorem.

III.1. Theorem. *Let R be a Noetherian ring. Then $\mathtt{mod}_R$, as a net, is generated by all cyclic modules of the form R/I, with I either a prime ideal in the singular locus of R or else a parameter ideal which generates locally one of its minimal primes.*

Proof. Let $\mathcal{N}$ be the net generated by all R/I with I either in the singular locus of R or else a parameter ideal which generates locally one of its minimal primes. By (1) in Example II.3 (or by a simple induction on the number of generators), $\mathtt{mod}_R$ is generated as a net by all cyclic modules. Therefore, towards a contradiction, we may assume that at least one $R/\mathfrak{a}$ does not belong to $\mathcal{N}$. Let $\mathfrak{a}$ be maximal among all such ideals. Let $\mathfrak{p}$ be a minimal prime of $\mathfrak{a}$. If $\mathfrak{a} \neq \mathfrak{p}$, then we have an exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{a} \to R/\mathfrak{b} \to 0$$

with $\mathfrak{a} \subsetneq \mathfrak{b}$. However, by maximality both $R/\mathfrak{p}$ an $R/\mathfrak{b}$ belong to $\mathcal{N}$. By rule (Net), then so does $R/\mathfrak{a}$, contradiction. Therefore $\mathfrak{a} = \mathfrak{p}$. By our assumption, we must have that $\mathfrak{p} \notin \operatorname{Sing} R$, that is to say, that $R_\mathfrak{p}$ is regular. I claim that there exists a parameter ideal I inside $\mathfrak{p}$ of height $\operatorname{ht}\mathfrak{p}$, such that $IR_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. This is an easy exercise in prime avoidance, but for sake of completeness, I will give a proof by induction on the height h of $\mathfrak{p}$. The case $h = 0$ is trivial, since $R_\mathfrak{p}$ is then a field. Suppose $h > 0$. In particular, $\mathfrak{p}$ is not contained in any minimal prime of R. By prime avoidance, we can find $x_1 \in \mathfrak{p}$, not in $\mathfrak{p}^2$ nor in any minimal prime $\mathfrak{p}_{0j}$ of R. In particular, x_1R has height one. If $h = 1$ we are done, since regularity implies that $x_1R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. Otherwise, if $h > 1$, we have that $\mathfrak{p}$ is neither a minimal prime $\mathfrak{p}_{1j}$ of x_1R nor can it be equal to $x_1R + \mathfrak{p}^2$, by NAKAYAMA's Lemma. By prime avoidance, we can find $x_2 \in R$, not in $x_1R + \mathfrak{p}^2$ nor in any $\mathfrak{p}_{0j}$ or $\mathfrak{p}_{1j}$. It follows that $(x_1, x_2)R$ has height 2. Continuing this way, we find $x_1, \dots, x_h \in \mathfrak{p}$, such that their images in $\mathfrak{p}R_\mathfrak{p}/\mathfrak{p}^2R_\mathfrak{p}$ are linearly independent and such that $(x_1, \dots, x_h)R$ has height h. Since $R_\mathfrak{p}$ is regular, the first condition implies that $(x_1, \dots, x_h)R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$ (see for instance [7, Theorem 14.2]).

Since $\mathfrak{p}$ is a minimal prime of I, we can find $s \in R$, such that $\mathfrak{p} = (I :_R s)$. Since $IR_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, one checks that $s \notin \mathfrak{p}$. We have an equality

$$I = \mathfrak{p} \cap (I + sR).$$

Indeed, one inclusion is immediate, so take $z \in \mathfrak{p} \cap (I + sR)$. Hence we can write $z = i + sa$, with $i \in I$ and $a \in R$. Since $sa = z - i \in \mathfrak{p}$ and since $s \notin \mathfrak{p}$, we get $a \in \mathfrak{p}$. Since $\mathfrak{p} = (I : s)$, we get that sa belongs to I, whence so does z, as claimed.

In general, if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in a ring R, then we have an exact sequence

$$0 \to R/(\mathfrak{a} \cap \mathfrak{b}) \xrightarrow{s} (R/\mathfrak{a}) \oplus (R/\mathfrak{b}) \xrightarrow{t} R/(\mathfrak{a} + \mathfrak{b}) \to 0,$$

where s sends an element $x + (\mathfrak{a} \cap \mathfrak{b})$ to the pair $(x + \mathfrak{a}, x + \mathfrak{b})$ and t sends a pair $(x + \mathfrak{a}, y + \mathfrak{b})$ to the element $(x - y) + (\mathfrak{a} + \mathfrak{b})$. Applied to the present situation with $\mathfrak{a} = \mathfrak{p}$ and $\mathfrak{b} = I + sR$, we get an exact sequence

$$0 \to R/I \to R/\mathfrak{p} \oplus R/(I + sR) \to R/(\mathfrak{p} + sR) \to 0. \qquad (1)$$

By maximality, $R/(\mathfrak{p} + sR)$ belongs to $\mathcal{N}$ and by construction so does R/I. Using the rules (`Net`) and (`DirSum`), it then follows from (1) that also $R/\mathfrak{p}$ belongs to $\mathcal{N}$, contradiction. □

III.2. Remark. If R is moreover Cohen-Macaulay, then any parameter ideal is generated by an R-regular sequence. Under this additional assumption, fix some $h \in \mathbb{N}$. Consider the net $\mathcal{G}_h$ consisting of all finitely generated R-modules of grade at least h, as discussed in (3) in Example II.3. By Lemma II.2, the net $\mathcal{G}_h$ is generated by all $R/\mathfrak{p}$ with $\mathfrak{p}$ a prime ideal of depth at least h. Analyzing the above proof, we see that the collection of all R/I already generate $\mathcal{G}_h$, where I is either an ideal generated by an R-regular sequence of length at least h or a prime ideal in the singular locus of R of depth at least h.

However, in the presence of embedded associated primes, funny things can happen: it is very well possible that $\mathfrak{p}$ belongs to the regular locus of R but its depth is strictly less than its height. If R is not Cohen-Macaulay, then the nets $\mathcal{G}_h$ are of lesser use. Instead, one can use the nets $\mathcal{I}_h$, introduced in (2) in Example II.3, consisting of all finitely generated R-modules whose annihilator has height at least h. The argument in the proof of the Theorem shows that this net is generated by all R/I, with I an ideal of height at least h which is either a parameter ideal locally generating one of its minimal primes or else a prime ideal in the singular locus of R.

The following example shows some of the complications that arise in the absence of the Cohen-Macaulay property.

III.3. Example. Let k be a field and let

$$R := (k[X, Y, Z]/I)_{\mathfrak{m}}$$

where I is the ideal generated by X^2, XY and XZ and where $\mathfrak{m}$ is the maximal ideal generated by X, Y and Z. One checks that R has dimension 2 and depth 0, so that it is not Cohen-Macaulay. Moreover, $\operatorname{Sing} R = \{\mathfrak{m}\}$. In this case, we cannot expect that we can strengthen the assertion in the Theorem by taking only parameter ideals generated by a regular sequence. Indeed, in this case, there are no such, except for the zero ideal. However, the net generated by $k = R/\mathfrak{m}$ and R is not $\mathtt{mod}_R$. Indeed, suppose the contrary. Let $\mathfrak{g} = XR$ and $\mathfrak{p} = (X,Y)R$. Consider the exact sequence

$$0 \to R_{\mathfrak{p}} \to R_{\mathfrak{g}} \to \Omega \to 0$$

Since $R_{\mathfrak{g}}$ is flat and $R_{\mathfrak{p}} \otimes k = 0$, we get after tensoring this sequence with k that $\operatorname{Tor}_1^R(\Omega, k) = 0$. Therefore, if $\mathtt{mod}_R$ would be equal to $\mathtt{net}(k, R)$, then by Corollary II.6, we would have that $\operatorname{Tor}_1^R(\Omega, \cdot)$ is identically zero on $\mathtt{mod}_R$. Consequently, Ω would be flat. However, tensoring the above sequence with R/YR yields $\operatorname{Tor}_1^R(\Omega, R/YR) \cong R_{\mathfrak{p}}/YR_{\mathfrak{p}}$ and the latter is isomorphic to $k(\mathfrak{p})$, showing that Ω is not flat. Note that $R_{\mathfrak{p}} \cong k(Z)[Y]_{(Y)}$ is a DVR, although $\mathfrak{p}$ has depth 0.

III.4. Corollary. *Let R be a d-dimensional Cohen-Macaulay ring and let Ω be an R-module. If $\operatorname{Tor}_a^R(\Omega, R/\mathfrak{p}) = 0$, for some $a \geq d$ and all $\mathfrak{p} \in \operatorname{Sing} R$, then Ω has finite projective dimension.*

Proof. Since this is a local problem, we may assume without loss of generality that R is local with maximal ideal $\mathfrak{m}$. Moreover, as the conclusion holds trivially for regular local rings, we may assume that R is not regular. Let I be a parameter ideal which, moreover, generates locally one of its minimal primes. In particular, this last condition forces that minimal prime to be in the regular locus. Therefore, any such parameter ideal has height h at most $d-1$. Moreover, since R is Cohen-Macaulay, I is generated by an R-regular sequence of length h by [7, Theorem 17.4]. It is well-known (see for instance [2, Corollary 1.6.14]) that R/I has projective dimension h. Therefore, we get that

$$\operatorname{Tor}_a^R(\Omega, R/I) = 0.$$

By Theorem III.1 and Proposition II.4, we conclude that $\operatorname{Tor}_a^R(\Omega, \cdot)$

vanishes identically on $\mathtt{mod}_R$, showing that Ω has flat dimension (at most $a-1$). Since any flat module has finite projective dimension, the claim follows. □

There is a similar criterion for finite injective dimension.

> *If* $\operatorname{Ext}^a_R(R/\mathfrak{p},\Omega)=0$, *for some* $a \geq d$ *and all* $\mathfrak{p} \in \operatorname{Sing} R$, *then* Ω *has finite injective dimension.*

Indeed, the question is again local and by [7, §18 Lemma 1], it suffices to show that $\operatorname{Ext}^a_R(\cdot,\Omega)$ vanishes on $\mathtt{mod}_R$ to conclude that Ω has finite injective dimension at most $a-1$.

IV. Big Cohen-Macaulay Modules

We apply the results from the previous section to obtain a flatness criterion for balanced big Cohen-Macaulay modules over a local Cohen-Macaulay ring. Recall that an arbitrary module Ω over a Noetherian local ring $(R,\mathfrak{m})$ is called a *big Cohen-Macaulay module*, if there exists a system of parameters $(x_1,\dots,x_d)$ in R, such that $(x_1,\dots,x_d)$ is a Ω-regular sequence. We call Ω moreover *balanced*, if this is true for every system of parameters. Note that if R itself is Cohen-Macaulay, then Ω is a (balanced) big Cohen-Macaulay if (every) some maximal R-regular sequence is Ω-regular. In particular, any flat R-module is a balanced big Cohen-Macaulay module. For a regular local ring, the converse also holds; see [3, p. 77], [4, Proof of Theorem 9.1] or the argument in the proof below. The following is a generalization to local Cohen-Macaulay rings.

IV.1. Theorem. *Let* $(R,\mathfrak{m})$ *be a local Cohen-Macaulay ring with an isolated singularity. Let* k *denote the residue field of* R *and* d *its dimension. For an* R*-module* Ω*, the following are equivalent.*

1. Ω *is flat;*
2. Ω *is a balanced big Cohen-Macaulay module of finite projective dimension;*
3. Ω *is a balanced big Cohen-Macaulay module with* $\operatorname{Tor}^R_1(\Omega,k)=0$.

In fact, for the equivalence of (1) *and* (2)*, we do not need to assume that R has an isolated singularity.*

Proof. Assume first that R is only Cohen-Macaulay. It follows that any system of parameters is an R-regular sequence. As being a regular sequence is preserved by flatness, (1) implies (2). To prove that (2) implies (1), one can use essentially the same argument as in the parenthetical remark in [3, p. 77] or in the proof of [4, Theorem 9.1]. For sake of convenience, I repeat this argument here. Let n be the maximum of all $i \geq 1$ for which $\mathrm{Tor}_i^R(\Omega, \cdot)$ is not identically zero. Note that n is finite, since Ω has finite projective dimension. By (1) in Example II.3, there is some prime ideal $\mathfrak{p}$ of R for which $\mathrm{Tor}_n^R(\Omega, R/\mathfrak{p}) \neq 0$. Let $(x_1, \dots, x_h)$ be a maximal regular sequence in $\mathfrak{p}$. Since R is Cohen-Macaulay, h is the height of $\mathfrak{p}$, so that $\mathfrak{p}$ is a minimal prime of $R/(x_1, \dots, x_h)R$. Therefore, we have a short exact sequence

$$0 \to R/\mathfrak{p} \to R/(x_1, \dots, x_h)R \to C \to 0$$

for some finitely generated R-module C. The *Tor* long exact sequence gives an exact sequence

$$\mathrm{Tor}_{n+1}^R(\Omega, C) \to \mathrm{Tor}_n^R(\Omega, R/\mathfrak{p}) \to \mathrm{Tor}_n^R(\Omega, R/(x_1, \dots, x_h)R)$$

The left most module in this sequence is zero by maximality of n whereas the right most is zero since $(x_1, \dots, x_h)$ is also Ω-regular. Therefore, $\mathrm{Tor}_n^R(\Omega, R/\mathfrak{p}) = 0$, contradiction.

So remains to show the equivalence of (3) with the first two conditions under the additional assumption that R has an isolated singularity. Clearly (1) implies (3). Therefore, assume that (3) holds and we seek to show that then so does (2). We only need to show that Ω has finite projective dimension. Let $(x_1, \dots, x_d)$ be a maximal R-regular sequence. By assumption, $(x_1, \dots, x_d)$ is also Ω-regular. In particular, we have, for every $j > 0$, that

$$\mathrm{Tor}_j^R(\Omega, R/(x_1, \dots, x_d)R) = 0. \tag{2}$$

Since $(x_1, \dots, x_d)R$ is $\mathfrak{m}$-primary, we can find an ascending chain of ideals $\mathfrak{a}_i$ with $\mathfrak{a}_0 = (x_1, \dots, x_d)R$ and $\mathfrak{a}_m = \mathfrak{m}$, such that $\mathfrak{a}_{i+1}/\mathfrak{a}_i \cong k$, for all i. Let us show by lexicographical induction on the pair (j, i)

that all $\mathrm{Tor}_j^R(\Omega, R/\mathfrak{a}_i) = 0$, for $j \geq 1$ and $i = 0, \dots, m$. When $j = 1$, this follows from our assumption that $\mathrm{Tor}_1^R(\Omega, k) = 0$ and Corollary II.6, since each $\mathfrak{a}_i$ is $\mathfrak{m}$-primary. Therefore, let $j > 1$. Equality (2) proves the case $i = 0$, hence we may also assume that $i > 0$. By construction, we have an exact sequence

$$0 \to k \to R/\mathfrak{a}_{i-1} \to R/\mathfrak{a}_i \to 0.$$

From the *Tor* long exact sequence we get that

$$\mathrm{Tor}_j^R(\Omega, R/\mathfrak{a}_{i-1}) \to \mathrm{Tor}_j^R(\Omega, R/\mathfrak{a}_i) \to \mathrm{Tor}_{j-1}^R(\Omega, k).$$

By induction on i, the first of these modules is zero and by induction on j, so is the last. This proves the claim. In particular, we showed that $\mathrm{Tor}_j^R(\Omega, k)$ vanishes, for all $j \geq 1$. By Corollary III.4 we get that Ω has finite projective dimension, as required. □

Using KUNZ's Theorem, we get the following criterion for regularity. Recall that for a ring of prime characteristic p, the Frobenius endomorphism is defined by $\mathbf{F}_p \colon x \mapsto x^p$. Let us write $R^{\mathbf{F}_p}$ for R viewed as an R-module via $\mathbf{F}_p$.

IV.2. Corollary. *Let R be a reduced Cohen-Macaulay ring of prime characteristic p. Then R is regular if, and only if, $R^{\mathbf{F}_p}$ has finite projective dimension.*

Proof. One direction is of course just SERRE's homological characterization of regularity. Therefore assume R is a reduced Cohen-Macaulay ring such that $R^{\mathbf{F}_p}$ has finite projective dimension over R. Since everything is preserved under localization, we may assume that R is local. Clearly, if $(x_1, \dots, x_n)$ is R-regular, then the same is true for $(x_1^p, \dots, x_n^p)$, showing that $R^{\mathbf{F}_p}$ is a balanced big Cohen-Macaulay module. Theorem IV.1 then yields that $R^{\mathbf{F}_p}$ is flat over R. By KUNZ's Theorem (see for instance [6, Theorem107]), it follows that R is regular. □

Note that if R is moreover local, then R is regular if, and only if, $\mathrm{Tor}_1^R(R^{\mathbf{F}_p}, k) = 0$, where k is the residue field of R. Indeed, by the Local Flatness Criterion ([7, Theorem 22.3]) the vanishing of $\mathrm{Tor}_1^R(R^{\mathbf{F}_p}, k)$ implies that $R \to R^{\mathbf{F}_p}$ (that is to say, the homomorphism $\mathbf{F}_p$) is flat and KUNZ's Theorem then shows that R is regular. For some other, related consequences of Theorem IV.1, see [9].

V. Asymptotic behavior of Betti numbers

V.1. Definition. Let R be a Noetherian ring and let Ω be an arbitrary R-module. The *n-th Betti number* of Ω at the prime $\mathfrak{p}$, is the (possibly infinite) dimension of the $k(\mathfrak{p})$-vector space

$$\operatorname{Tor}_n^{R_\mathfrak{p}}(\Omega_\mathfrak{p}, k(\mathfrak{p})) = \operatorname{Tor}_n^R(\Omega, R/\mathfrak{p})_\mathfrak{p},$$

where $k(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$, that is to say, $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. We denote n-th Betti number by $\beta_n^R(\mathfrak{p};\Omega)$, or simply, by $\beta_n(\mathfrak{p};\Omega)$ if the ring is understood.

The following result gives a restriction on the asymptotic behavior of the Betti numbers.

V.2. Theorem (k-rigidity in high dimensions). *Let $(R,\mathfrak{m})$ be a d-dimension local Cohen-Macaulay ring with residue field k and let Ω be an arbitrary R-module. If $\operatorname{Tor}_a^R(\Omega,k)=0$, for some $a \geq d$, then $\operatorname{Tor}_m^R(\Omega,k)=0$, for all $m \geq a$.*

Proof. We will induct on the dimension d of R. Suppose first that $d=0$. If $a=0$, so that $0=\operatorname{Tor}_0^R(\Omega,k)=\Omega\otimes_R k$, whence $\Omega=\mathfrak{m}\Omega$, we get that $\Omega=0$, as $\mathfrak{m}$ is nilpotent. Suppose next that $a=1$. By Corollary II.6, it follows that $\operatorname{Tor}_1^R(\Omega,M)=0$, for every R-module M of finite length. However, since R is Artinian, every finitely generated R-module has finite length. Therefore, Ω is in fact flat, and the claim holds trivially. For $a \geq 2$, we can take syzygies to reduce to the case $a=1$.

Next, suppose that $d=1$. Again, by taking syzygies, we may reduce to the case that $a=1$. Since R is Cohen-Macaulay, we can find an R-regular element $x\in\mathfrak{m}$, so that the standard spectral sequence

$$\operatorname{Tor}_p^{R/xR}(\operatorname{Tor}_q^R(\Omega,R/xR),k) \implies \operatorname{Tor}_{p+q}^R(\Omega,k)$$

degenerates into an exact sequence

$$\begin{aligned}\operatorname{Tor}_{m-1}^{R/xR}(\operatorname{Ann}_\Omega(x),k) \to \operatorname{Tor}_m^R(\Omega,k) \to \operatorname{Tor}_m^{R/xR}(\Omega/x\Omega,k) \to \dots \\ \to \operatorname{Ann}_\Omega(x)\otimes k \to \operatorname{Tor}_1^R(\Omega,k) \to \operatorname{Tor}_1^{R/xR}(\Omega/x\Omega,k) \to 0, \quad (3)\end{aligned}$$

where $\operatorname{Ann}_\Omega(x)$ denotes the submodule of all elements in Ω annihilated by x. Since by assumption $\operatorname{Tor}_1^R(\Omega,k)$ vanishes, so does $\operatorname{Tor}_1^{R/xR}(\Omega/x\Omega,k)$ by (3). Therefore, we obtain by the above zero dimensional case that all $\operatorname{Tor}_m^{R/xR}(\Omega/x\Omega,k)$ vanish, for $m \geq 1$. Using (3) once we more, we get that $\operatorname{Ann}_\Omega(x)\otimes k$ vanishes. Again by the zero dimensional case we obtain that all $\operatorname{Tor}_m^{R/xR}(\operatorname{Ann}_\Omega(x),k)$ vanish. Therefore, so do all $\operatorname{Tor}_m^R(\Omega,k)$, by (3).

Finally, assume $d \geq 2$ and let $x \in \mathfrak{m}$ be R-regular. Let

$$0 \to \Pi \to \Phi \to \Omega \to 0$$

be exact, with Φ free. In particular, we obtain, for all $p \geq 1$, that

$$\operatorname{Tor}_{p+1}^R(\Omega,k) \cong \operatorname{Tor}_p^R(\Pi,k). \quad (4)$$

Moreover, since Π is a submodule of a free module, we have that x is also Π-regular, so that

$$\operatorname{Tor}_m^R(\Pi,k) \cong \operatorname{Tor}_m^{R/xR}(\Pi/x\Pi,k), \quad (5)$$

for all $m \geq 1$. By assumption $\operatorname{Tor}_a^R(\Omega,k) = 0$, where $a \geq d \geq 2$, so that by (4) and (5), we have that $\operatorname{Tor}_{a-1}^{R/xR}(\Pi/x\Pi,k) = 0$. Therefore, $\operatorname{Tor}_m^{R/xR}(\Pi/x\Pi,k) = 0$, for all $m \geq a-1$, by our induction hypothesis. By (4) and (5) again, it follows that $\operatorname{Tor}_p^R(\Omega,k) = 0$, for all $p \geq a$. □

V.3. Example. In low dimensions, however, k-rigidity fails, even for regular local rings, as the following example shows. Let $(R,\mathfrak{m})$ be a regular local ring of dimension $d \geq 2$. Let E be the injective hull of the residue field of R. It is well-known that all $\beta_n^R(\mathfrak{m};E) = 0$, except when $n = d$, in which case the Betti number is one (see for instance [2, Exercise 3.3.26]).

I do not know of any counterexample to the Theorem without the Cohen-Macaulay condition.

Our next goal is to replace in Corollary III.4 the requirement that $\operatorname{Tor}_n^R(\Omega, R/\mathfrak{p})$ vanishes, by the weaker condition that $(\operatorname{Tor}_n^R(\Omega, R/\mathfrak{p}))_{\mathfrak{p}}$ vanishes, that is to say, that $\beta_n^R(\mathfrak{p};\Omega) = 0$.

V.4. Theorem. *Let R be a d-dimensional Cohen-Macaulay ring and let Ω be an R-module. If for some $a \geq d$, we have that $\beta_a^R(\mathfrak{p};\Omega) = 0$, for all $\mathfrak{p}$ in the singular locus of R, then Ω has finite projective dimension.*

Proof. By localizing, we may assume from the start that R is a d-dimensional local Cohen-Macaulay ring with residue field k and maximal ideal $\mathfrak{m}$. Moreover, the statement is trivial if R is regular, so that we may assume that $\mathfrak{m}$ lies in the singular locus. By Theorem V.2, all $\beta_m^R(\mathfrak{p};\Omega) = 0$, for $m \geq a$ and $\mathfrak{p} \in \operatorname{Sing} R$. Note that each $R_{\mathfrak{p}}$ is Cohen-Macaulay of dimension at most d.

As in the proof of Corollary III.4, for I a parameter ideal of height at most $d-1$, we have that

$$\operatorname{Tor}_m^R(\Omega, R/I) = 0 \tag{6}$$

for all $m \geq d$, since any such ideal is generated by an R-regular sequence.

I claim that $\operatorname{Tor}_m^R(\Omega, R/\mathfrak{p}) = 0$, for all prime ideals $\mathfrak{p}$ which are in the singular locus $\operatorname{Sing} R$ of R and all $m \geq a$. Assuming the claim, the assertion then follows from Corollary III.4. To prove the claim, we will perform a downward induction on the height h of $\mathfrak{p}$. By assumption, the case $\mathfrak{p} = \mathfrak{m}$, that is to say, $h = d$, holds, so that we may assume $h < d$. By Remark III.2, the net $\mathcal{I}_{h+1}$, introduced in (3) of Example II.3, is generated by all R/I, with I a parameter ideal of height $e \geq h+1$ which generates locally one of its minimal primes, together with all $R/\mathfrak{q}$, with $\mathfrak{q}$ a prime ideal in the singular locus of R of height at least $h+1$. Since R is not regular, e is at most $d-1$.

By induction on h, we have that $\operatorname{Tor}_m^R(\Omega, R/\mathfrak{q}) = 0$, for all $m \geq a$ and all prime ideals $\mathfrak{q} \in \operatorname{Sing} R$ of height at least $h+1$. In view of (6) this implies by Remark III.2 and Proposition II.4 that $\operatorname{Tor}_m^R(\Omega, \cdot)$ vanishes on the whole net $\mathcal{I}_{h+1}$, for all $m \geq a$. Take a height h prime $\mathfrak{p}$ in the singular locus of R, if any. For an arbitrary $x \notin \mathfrak{p}$, we have an exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \to R/\mathfrak{a} \to 0. \tag{7}$$

where $\mathfrak{a} := xR + \mathfrak{p}$. In particular, since $\mathfrak{a}$ has height $h+1$, we get that

$$\mathrm{Tor}_m^R(\Omega, R/\mathfrak{a}) = 0, \tag{8}$$

for all $m \geq a$. Fix some $m \geq a$. From the long exact sequence obtained from (7) by tensoring with Ω, we get from (8), that

$$0 \to \mathrm{Tor}_m^R(\Omega, R/\mathfrak{p}) \xrightarrow{x} \mathrm{Tor}_m^R(\Omega, R/\mathfrak{p}) \to 0$$

is exact (note that at this point, it is crucial that we do not just have vanishing in dimension a). In particular, x is not a zero-divisor on $\mathrm{Tor}_m^R(\Omega, R/\mathfrak{p})$. By assumption

$$(\mathrm{Tor}_m^R(\Omega, R/\mathfrak{p}))_{\mathfrak{p}} = \mathrm{Tor}_m^{R_{\mathfrak{p}}}(\Omega_{\mathfrak{p}}, k(\mathfrak{p})) = 0.$$

It follows from these two observations that $\mathrm{Tor}_m^R(\Omega, R/\mathfrak{p}) = 0$ by Lemma V.5 below. □

V.5. Lemma. *Let Λ be an R-module and let $\mathfrak{p}$ be a prime ideal of R. If $\Lambda_{\mathfrak{p}} = 0$ and x is Λ-regular, for every $x \notin \mathfrak{p}$, then $\Lambda = 0$.*

Proof. Pick any $\tau \in \Lambda$. Since $\Lambda_{\mathfrak{p}} = 0$, there is some $x \notin \mathfrak{p}$, such that $x\tau = 0$. By assumption, x is Λ-regular, showing that $\tau = 0$. □

VI. Syzygycal Nets

The goal of this section is to establish a similar result as Corollary III.4 without the Cohen-Macaulay assumption. To this end, it is useful to view the results from Section III in an alternative way.

VI.1. Definition. Let R be a ring and $\mathcal{N}$ a net. We call $\mathcal{N}$ *syzygycal*, if it is closed under syzygies and co-syzygies. More precisely, if the following condition is satisfied

`(Syz)` Given an exact sequence of finitely generated R-modules

$$0 \to K \to F \to M \to 0, \tag{9}$$

with F free, then $M \in \mathcal{N}$ if, and only if, $K \in \mathcal{N}$.

Since any net contains the zero module (by rule (`DirSum`)) and since R is a syzygy of the zero module, any syzygycal net contains R. Therefore, any syzygycal net contains all finitely generated projective modules. Therefore, by rule (`Syz`), any syzygycal net contains all finitely generated modules of finite projective dimension. In particular, if R is regular, then the only syzygycal net is $\mathtt{mod}_R$ itself.

VI.2. Definition (Meanders). We call the smallest syzygycal net $\mathcal{N}$ generated by some class $\mathcal{K}$ of finitely generated R-modules, the *syzygycal net generated by* $\mathcal{K}$. To measure how far we have 'meandered' from the generators $\mathcal{K}$ by invoking rule (`Syz`), we introduce the notion of a $\mathcal{K}$-*meander* of a member M of $\mathcal{N}$. A $\mathcal{K}$-meander (or simply, *meander*, if $\mathcal{K}$ is clear from the context) will be an interval in $\mathbb{Z}$ of the form $[-a, b]$ with $a, b \in \mathbb{N}$. In the following recursive definition of a meander of M (with respect to $\mathcal{K}$), it is understood that M is derived from $\mathcal{K}$ using the three rules for syzygycal nets and for each of the intermediate modules, at least one meander has already been defined.

1. If $M \in \mathcal{K}$ or M is projective, then $[0, 0]$ is a meander of M.
2. If M is the direct summand of some $N \in \mathcal{N}$, then any meander of N is also a meander of M.
3. Suppose we have an exact sequence

$$0 \to K \to M \to N \to 0$$

with $K, N \in \mathcal{N}$. If J_K is a meander of K and J_N a meander of N, then $J_K \cup J_N$ is a meander of M.
4. Suppose we have an exact sequence

$$0 \to N \to F \to M \to 0$$

with F free and $N \in \mathcal{M}$. If $[-a, b]$ is a meander of N, then $[-a, b + 1]$ is a meander of M.
5. Suppose we have an exact sequence

$$0 \to M \to F \to N \to 0$$

with F free and $N \in \mathcal{M}$. If $[-a, b]$ is a meander of N, then

$[-a-1,b]$ is a meander of M.

This concludes the recursive definition of a meander of M. Of course, a module might have infinitely many meanders, but it will only have finitely many meanders which are minimal with respect to inclusion. Indeed, if $[-a,b]$ is a meander of M, then a moment's reflection shows that M can have at most $a+b$ distinct minimal meanders. Note that if M lies already in the net generated by $\mathcal{K}$, that is to say, is derived form $\mathcal{K}$ using only rules (`Net`) or (`DirSum`), then $[0,0]$ is the unique minimal meander of M.

We will tacitly use the following fact, whose easy proof is left to the reader. Let $\mathcal{K} \subset \mathcal{N}$ be subclasses of finitely generated R-modules. Suppose $\mathcal{N}$ lies in the syzygical net generated by $\mathcal{K}$ and every $N \in \mathcal{N}$ has a $\mathcal{K}$-meander contained in $[-a,b]$. Then any $M \in \mathtt{net}(\mathcal{N})$ has also a $\mathcal{K}$-meander contained in $[-a,b]$ (note that in any case, $\mathtt{net}(\mathcal{N})$ is contained in the syzygycal net generated by $\mathcal{K}$).

VI.3. Definition (Kernels). Let $\mathbb{F}$ be a right exact (covariant) functor on the category of R-modules. We denote its left derived functors by $\mathrm{L}_n\mathbb{F}$. Let M be an arbitrary finitely generated R-module. We define the $\mathbb{F}$-*kernel*, $\mathrm{Ker}_{\mathbb{F}}(M)$, of M as the collection of all $n \in \mathbb{N}$ for which $\mathrm{L}_n\mathbb{F}(M) = 0$.

For instance, let $(R,\mathfrak{m})$ be a Noetherian local with residue field k ring and let $\mathbb{F}$ be the functor $\cdot \otimes_R k$. If M is a finitely generated R-module of projective dimension d, then $\mathrm{Ker}_{\mathbb{F}}(M) = [d,\infty)$ (if $d=\infty$, this means that $\mathrm{Ker}_{\mathbb{F}}(M) = \emptyset$). If $\mathcal{K}$ is a class of finitely generated R-modules, then $\mathrm{Ker}_{\mathbb{F}}(\mathcal{K})$ is by definition the intersection of all $\mathrm{Ker}_{\mathbb{F}}(K)$ with $K \in \mathcal{K}$. A similar definition can be made for left exact functors and for contravariant functors.

VI.4. Theorem. *Let R be a ring and $\mathbb{F}$ a right exact covariant functor for which $\mathrm{Ker}_{\mathbb{F}}(R) = \mathbb{N} \setminus \{0\}$. Let $\mathcal{K}$ be a class of finitely generated R-modules and let $\mathcal{N}$ be the syzygycal net generated by $\mathcal{K}$. Let $M \in \mathcal{N}$ and let $[-a,b]$ be a meander of M. If for some non-zero $u,v \in \mathbb{N}$, the interval $[u,v]$ lies in $\mathrm{Ker}_{\mathbb{F}}(\mathcal{K})$, then $[u+b, v-a]$ lies in $\mathrm{Ker}_{\mathbb{F}}(M)$.*

Proof. By a recursive argument, it suffices to show this for M and

$[-a, b]$ given by one of the five formation rules in Definition VI.2.

In case (1), the assertion is just our assumption that all positive integers belong to $\mathrm{Ker}_{\mathbb{F}}(R)$. In case (2), let M be a direct summand of some N in $\mathcal{N}$. Any right exact functor, whence also its derived functors, are additive, so that $\mathrm{Ker}_{\mathbb{F}}(N) \subset \mathrm{Ker}_{\mathbb{F}}(M)$, and the assertion holds, since each meander of N is also a meander of M. In case (3), let

$$0 \to K \to M \to N \to 0$$

be an exact sequence, where $K, N \in \mathcal{N}$ have respective meanders $[-a_K, b_K]$ and $[-a_N, b_N]$ and where $[-a, b]$ is obtained as the union of these two meanders, that is to say,

$$a = \max\{a_K, a_N\} \qquad b = \max\{b_K, b_N\}$$

By induction, $[u + b_K, v - a_K]$ lies in $\mathrm{Ker}_{\mathbb{F}}(K)$ and $[u + b_N, v - a_N]$ lies in $\mathrm{Ker}_{\mathbb{F}}(N)$. From the long exact sequence

$$\mathrm{L}_n\mathbb{F}(K) \to \mathrm{L}_n\mathbb{F}(M) \to \mathrm{L}_n\mathbb{F}(N).$$

it is clear that $\mathrm{L}_n\mathbb{F}(M) = 0$, for all n in

$$[u + b_K, v - a_K] \cap [u + b_N, v - a_N] = [u + b, v - a].$$

In case (4), consider an exact sequence

$$0 \to N \to F \to M \to 0$$

with F free and $N \in \mathcal{M}$ with meander $[-a, b-1]$. By induction, we have that $[u + b - 1, v - a]$ lies in $\mathrm{Ker}_{\mathbb{F}}(N)$. From the long exact sequence of derived functors, we get isomorphisms

$$\mathrm{L}_{n+1}\mathbb{F}(M) \cong \mathrm{L}_n\mathbb{F}(N) \tag{10}$$

for all $n \geq 1$. Therefore, since everything gets shifted up by one, we have that $[u + b, v - a + 1]$ lies in $\mathrm{Ker}_{\mathbb{F}}(M)$ and the assertion holds. In case (5), the same reasoning holds, where this time we have to shift everything down by one. □

We turn now to the generalization of Theorem III.1 and Corollary III.4. Let us first reinterpret the result of Theorem III.1 in our new terminology.

VI.5. Corollary. *Let R be a Cohen-Macaulay ring. The syzygycal net generated by the singular locus* $\operatorname{Sing} R$ *of R, is* $\mathtt{mod}_R$.

More precisely, if $\mathcal{S}$ is the class of all $R/\mathfrak{p}$ with $\mathfrak{p} \in \operatorname{Sing} R$, then any finitely generated R-module M has an $\mathcal{S}$-meander contained in $[0, d]$*, where d is the dimension of R. If R is moreover local but not regular, then M has an $\mathcal{S}$-meander contained in* $[0, d-1]$.

Proof. Let $\mathcal{N}$ be the syzygycal generated by $\mathcal{S}$. If I is a height e parameter ideal which generates locally one of its minimal primes, then I is generated by a regular sequence of length e, since R is Cohen-Macaulay. The Koszul complex of this sequence is a free resolution of R/I by [2, Corollary 1.6.14]. Therefore, R/I belongs to $\mathcal{N}$ with ($\mathcal{S}$-)meander $[0, e]$. In general $e \leq d$, and if R is local but not regular, then I cannot have height d, so that $e \leq d - 1$. The statement now follows from Theorem III.1. □

To extend this to the non-Cohen-Macaulay case, we need a lemma about homology of complexes.

VI.6. Lemma. *Let R be a Noetherian ring and $\mathcal{N}$ a syzygycal net. Let $\mathbf{F}_\bullet$ be a finite complex of finitely generated free R-modules. If for all $i > 0$, the homology modules $\mathrm{H}_i(\mathbf{F}_\bullet)$ belong to $\mathcal{N}$, then so does* $\mathrm{H}_0(\mathbf{F}_\bullet)$.

Proof. Let $\mathbf{F}_\bullet$ be a finite free complex of length e, that is to say, a complex of the form

$$F_e \xrightarrow{f_e} F_{e-1} \xrightarrow{f_{e-1}} \dots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0$$

with each F_i a finitely generated free R-module. Let $H_i := \mathrm{H}_i(\mathbf{F}_\bullet)$, for $i = 0, \dots, e$. Let K_i and Z_{i-1} denote respectively the kernel and the image of f_i. Put $K_0 := F_0$ and $Z_e := 0$. By assumption, all $H_i = K_i/Z_i$ for $i = 1, \dots, e$, belong to $\mathcal{N}$. We prove by downward induction on i that Z_i and K_i belong to $\mathcal{N}$, where the case $i = e$ holds by assumption (note that $K_e = H_e$). When we have reached $i = 0$, we then conclude by rule (`Syz`) that also $H_0 = F_0/Z_0$ belongs to $\mathcal{N}$ and we are done.

Therefore, suppose the claim proven for $i + 1$, with $0 \leq i < e$.

From the exact sequence

$$0 \to K_{i+1} \to F_{i+1} \to Z_i \to 0$$

and induction, the cosyzygy Z_i belongs to $\mathcal{N}$ by rule ($\mathtt{Syz}$). If $i = 0$, we are done, since $K_0 = F_0$ by definition. If $i > 0$, the exact sequence

$$0 \to Z_i \to K_i \to H_i \to 0$$

our assumption and rule ($\mathtt{Net}$) then show that K_i belongs to $\mathcal{N}$. □

VI.7. Remark. In fact, the above proof shows that $\mathrm{H}_0(\mathbf{F}_\bullet)$ belongs to the syzygycal net generated by all $\mathrm{H}_i(\mathbf{F}_\bullet)$ with $i > 0$ and as such, it has a meander $[0, e+1]$, where e is the length of $\mathbf{F}_\bullet$, since we took $e+1$ times a cosyzygy.

VI.8. Theorem. *Let R be a d-dimensional Noetherian ring. The syzygycal net generated by the singular locus* $\operatorname{Sing} R$ *of R, is* $\mathtt{mod}_R$.

More precisely, if $\mathcal{S}$ is the class of all $R/\mathfrak{p}$ with $\mathfrak{p} \in \operatorname{Sing} R$, then any finitely generated R-module M has an $\mathcal{S}$-meander contained in $[0, d+1]$*, where d is the dimension of R. If R is moreover local but not regular, then M has an $\mathcal{S}$-meander contained in* $[0, d]$.

Proof. By Theorem III.1, the net $\mathtt{mod}_R$ is generated by the cyclic modules of the form R/I, with $I \in \operatorname{Sing} R$ or I a parameter ideal which generates locally one of its minimal primes. Therefore, it suffices to show that any cyclic module of the form R/I with $I = (x_1, \dots, x_h)R$ a parameter ideal as above, belongs to the syzygycal net generated by $\mathcal{S}$, with the indicated meander. Let $\mathbf{K}_\bullet(I)$ be the Koszul complex of $(x_1, \dots, x_h)$. If $\mathfrak{p}$ is a prime containing I but not in the singular locus of R, then the image of $(x_1, \dots, x_h)$ in $R_\mathfrak{p}$ is an $R_\mathfrak{p}$-regular sequence, since $R_\mathfrak{p}$ is in particular Cohen-Macaulay. Therefore, $\mathbf{K}_\bullet(I)$ becomes acyclic after localization at $\mathfrak{p}$. It follows that the homology modules $\mathrm{H}_i(\mathbf{K}_\bullet(I))$ for $i > 0$ all have support inside $\operatorname{Sing} R$. Therefore, by Lemma II.2 and the fact that $\operatorname{Sing} R$ is stable under specialization, each $\mathrm{H}_i(\mathbf{K}_\bullet(I))$, for $i > 0$, belongs to the net generated by $\mathcal{S}$. By Lemma VI.6, we then get that $\mathrm{H}_0(\mathbf{K}_\bullet(I)) = R/I$ belongs to the syzygycal net generated by $\mathcal{S}$. By Remark VI.7, it has an $\mathcal{S}$-meander $[0, h+1]$.

Since always $h \leq d$, the first assertion of the final statement is clear. Suppose finally that R is moreover local but not regular. Since the parameter ideal I is assumed to locally generate one of its minimal primes, $h < d$ (lest the maximal ideal of R is generated by d elements) and the last assertion follows. □

VI.9. Theorem. *Let R be a d-dimensional Noetherian ring and let Ω be an R-module. If there is some $a \geq 1$, such that $\operatorname{Tor}_{a+j}^R(\Omega, R/\mathfrak{p}) = 0$ (respectively, $\operatorname{Ext}_R^{a+j}(R/\mathfrak{p}, \Omega) = 0$), for all $j = 0, \dots, d+1$ and all $\mathfrak{p} \in \operatorname{Sing} R$, then Ω has finite flat (respectively, finite injective) dimension (at most $a+d$).*

Moreover, if R is local but not regular, then we only have to check the vanishing of the Tor *modules in the range $j = 0, \dots, d$.*

Proof. Let $\mathbb{F} = \Omega \otimes_R \cdot$ in the first case and $\operatorname{Hom}_R(\cdot, \Omega)$ in the second case. In either case, $\operatorname{Ker}_{\mathbb{F}}(R) = [1, \infty)$, so that Theorem VI.4 applies. Let $\mathcal{S}$ consist of all $R/\mathfrak{p}$ with $\mathfrak{p} \in \operatorname{Sing} R$. By assumption, $[a, a+d+1] \subset \operatorname{Ker}_{\mathbb{F}}(\mathcal{S})$. Let M be an arbitrary finitely generated R-module. By Theorem VI.8, the interval $[0, d+1]$ contains an $\mathcal{S}$-meander of M. Therefore, by Theorem VI.4, the interval $[a+d+1, a+d+1]$ lies in $\operatorname{Ker}_{\mathbb{F}}(M)$. This shows that in the first case $\operatorname{Tor}_{a+d+1}^R(\Omega, \cdot)$ vanishes on each finitely generated R-module, showing that Ω has flat dimension at most $a+d$. In the second case, the vanishing of $\operatorname{Ext}_R^{a+d+1}(\cdot, \Omega)$ on $\mathtt{mod}_R$ implies that Ω has injective dimension at most $a+d$, by [7, §18 Lemma 1]. The final assertion now follows from the last statement in Theorem VI.8. □

The final result is a local criterion similar to Theorem V.4, involving only Betti numbers. Since we do not know whether k-rigidity in high dimensions holds for non-Cohen-Macaulay rings, we can only state the following weaker version of Theorem V.4.

VI.10. Theorem. *Let R be a Noetherian ring and let Ω be an R-module. If for some $a \geq 1$, we have that $\beta_{a+j}^R(\mathfrak{p}; \Omega) = 0$, for all $\mathfrak{p}$ in the singular locus of R and all $j = 0, \dots, \operatorname{ht} \mathfrak{p}$, then Ω has finite projective dimension (and, in fact, flat dimension at most $a-1$).*

Proof. By localizing, we may assume from the start that R is a Noetherian local ring with residue field k and maximal ideal $\mathfrak{m}$. We will prove the result by induction on the dimension d of R. If $d = 0$, then either R is a field and there is nothing to prove, or we have that $\operatorname{Tor}_a^R(\Omega, k) = 0$. By Corollary II.6, we get that $\operatorname{Tor}_a^R(\Omega, M) = 0$, for every R-module M of finite length. Since R is Artinian, every finitely generated R-module has finite length and hence we showed that Ω has finite flat dimension at most $a - 1$. So let $d > 0$ and assume that the result is proven for all lower dimensional Noetherian local rings. If R is regular, there is nothing to prove, so we may assume that $\mathfrak{m}$ lies in the singular locus of R. Let $\mathfrak{p}$ be a prime ideal different from $\mathfrak{m}$. Since the singular locus of $R_\mathfrak{p}$ is contained in the singular locus of R, induction on the dimension yields that $\Omega_\mathfrak{p}$ has finite flat dimension at most $a - 1$. In particular, $\beta_j(\mathfrak{p}; \Omega) = 0$, for all $j \geq a$ and all prime ideals $\mathfrak{p}$ different from $\mathfrak{m}$.

I claim that $\operatorname{Tor}_j^R(\Omega, R/\mathfrak{p}) = 0$, for all prime ideals $\mathfrak{p}$ of R and for all $j \in [a, a + d - h]$, where h is the height of $\mathfrak{p}$. Assuming the claim, we get that $\operatorname{Tor}_a^R(\Omega, R/\mathfrak{p})$ vanishes for all prime ideals $\mathfrak{p}$, proving that Ω has flat dimension at most $a - 1$ by (1) in Example II.3. To prove the claim, we perform a downward induction on the height h of the prime ideal $\mathfrak{p}$. Since the case $h = d$ is covered by the hypothesis (recall that R is singular), we may assume that $h < d$. By (2) in Example II.3 and our induction hypothesis on h, we get that $\operatorname{Tor}_j^R(\Omega, R/\mathfrak{a}) = 0$, for all $\mathfrak{a}$ of height at least $h + 1$ and all $j \in [a, a + d - h + 1]$.

Let x be an arbitrary element of R not in $\mathfrak{p}$, and put $\mathfrak{a} := xR + \mathfrak{p}$. Tensoring the short exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \to R/\mathfrak{a} \to 0$$

with Ω, yields a long exact sequence

$$\operatorname{Tor}_{j+1}^R(\Omega, R/\mathfrak{a}) \to \operatorname{Tor}_j^R(\Omega, R/\mathfrak{p}) \xrightarrow{x} \operatorname{Tor}_j^R(\Omega, R/\mathfrak{p})$$

Since $\mathfrak{a}$ has height $h + 1$, the left most module vanishes for all j in the range $[a, a + d - h]$. Therefore, x is not a zero-divisor on $\operatorname{Tor}_j^R(\Omega, R/\mathfrak{p})$, for all $j \in [a, a + d - h]$. Since $\beta_j(\mathfrak{p}; \Omega) = 0$, for all $j \geq a$, we get by Lemma V.5 that $\operatorname{Tor}_j^R(\Omega, R/\mathfrak{p}) = 0$, for $j \in [a, a + d - h]$, as claimed. □

Note that in this proof, we did not use the theory of syzygycal nets. Moreover, the range in which the Betti numbers are required to vanish is smaller than the one given in Theorem VI.9. Together with k-rigidity in high dimensions (Theorem V.2), this also provides in the Cohen-Macaulay case an alternative proof for Theorem V.4.

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