A LOCAL FLATNESS CRITERION FOR COMPLETE MODULES

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ABSTRACT. We prove various extensions of the Local Flatness Criterion over a Noetherian local ring R with residue field k. For instance, if Ω is a complete R-module of finite projective dimension, then Ω is flat if and only if $\operatorname{Tor}_n^R(\Omega, k) = 0$ for all $n = 1, \dots, \operatorname{depth}(R)$. In low dimensions, we have the following criteria. If R is one-dimensional and reduced, then Ω is flat if and only if $\operatorname{Tor}_1^R(\Omega, k) = 0$. If R is two-dimensional, then in order for Ω to be flat, it suffices that it is separated, that its projective dimension is finite and that $\operatorname{Tor}_1^R(\Omega, k) = 0$.

Many of these criteria have global counterparts and in particular, it is shown that the $\mathfrak{a}$-adic completion of a flat module of finite projective dimension over an arbitrary Noetherian ring is again flat.

1. INTRODUCTION

Flatness is an important and often useful property, and, consequently, one wants general criteria to detect it. The literature on the subject is extensive, and many sorts of criteria have been proposed, sometimes with additional assumptions on the module or the algebra. In this paper, mainly homological criteria will be studied. Preferably, the criteria should require the vanishing of only finitely many homological invariants.

More concretely, let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and Ω an arbitrary R-module (in the text we will also treat non-local rings). The *n-th Betti number* of Ω is by definition the vector space dimension of $\operatorname{Tor}_n^R(\Omega, k)$. The main theme of this paper is how the vanishing of certain Betti numbers of Ω influences its flatness. For instance, if Ω is finitely generated, then the vanishing of the first Betti number already implies flatness. The Local Flatness Criterion extends this to include all modules finitely generated over a Noetherian local R-algebra ([10, Theorem 22.3]; for some generalizations, see [16, Theorem 2.6.3] or Theorem 5.1 below). Nonetheless, many modules are not of this type and similar flatness criteria do not seem to exist for them. It should be pointed out that the vanishing of the first Betti number is in general not sufficient: for instance, let R be a two-dimensional regular local ring and let Ω be the residue field of a height one prime ideal in R; for a separated counterexample, see Example 7.1 below. Nonetheless, some vestige of flatness is preserved: the vanishing of the first Betti number of a non-degenerated R-algebra S implies that $R \to S$ is cyclically pure, that is to say, $I = IS \cap R$ for all ideals $I \subseteq R$ (see [14, Theorem 2.2]). Here a non-zero R-module Ω is called *separated* if the intersection of all $\mathfrak{m}^n\Omega$ is zero; and *non-degenerated*, if $\Omega \neq \mathfrak{m}\Omega$. More consequences of the vanishing of the first Betti number are treated in §6.

What if we require that some (preferably finitely many) Betti numbers vanish? This is of course a necessary condition, but unfortunately, even the vanishing of all Betti numbers is not sufficient in general (see Examples 7.2 and 7.3). The reason for this failure is twofold. Firstly, a flat module over a Noetherian local ring has necessarily finite projective

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dimension. Therefore, whatever flatness criterion we propose, it should entail at least finitude of projective dimension. This question, in terms of vanishing of certain *Tor*-modules, has been studied in detail in [13] and some of it is reproduced here in the final section. It should be noted that, depending on the size of the singular locus, these tests may require the vanishing of infinitely many *Tor*-modules. To not confound the issue, finitude of projective dimension will be taken as part of the hypotheses. Nonetheless, even under the assumption that the module has finite projective dimension, the vanishing of all Betti numbers does not always entail its flatness (see Example 7.3).

The second obstruction is more serious and comes from the fact that a non-finitely generated module need not be separated or might even be degenerated. In particular, Nakayama's Lemma fails in general. An additional complication is that the class of separated modules is not closed under base change. Namely, if Ω is separated, $\Omega/I\Omega$ need not be separated for some ideal I. It turns out that imposing certain separatedness conditions will yield valid 'Betti tests for flatness'.

Results. Let me now quote some of the more important results proved in this paper.

Theorem 7.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and let q be the depth of R. Let Ω be an arbitrary R-module. If Ω has finite projective dimension and all $\operatorname{Tor}_n^R(\Omega, k)$ vanish for $n = 1, \dots, q$, then $\widehat{\Omega}$ is flat.*

In particular, for a complete R-module of finite projective dimension, flatness is equivalent with the vanishing of the first q Betti numbers. The assumptions in Theorem 7.6 also imply that $\widehat{\Omega}$ is $\mathfrak{m}$-adically complete (more precisely, the $\mathfrak{m}$-adic topology coincides with the canonical completion topology). Another corollary is the result from [2] that the completion of a flat module is again flat. In fact, we can prove the following global version, generalizing a result of Enochs in [6] by removing the assumption on A to have finite Krull dimension; note that a flat module over a finite dimensional Noetherian ring has always finite projective dimension by [11, Corollary 3.2.7].

Theorem 5.9. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal in A and Ω an A-module. If Ω is flat and has finite projective dimension, then its $\mathfrak{a}$-adic completion $\widehat{\Omega}^{\mathfrak{a}}$ is also flat.*

Moreover, $\widehat{\Omega}^{\mathfrak{a}}$ is $\mathfrak{a}$-adically complete and we have isomorphisms $\widehat{\Omega}^{\mathfrak{a}}/\mathfrak{a}^m\widehat{\Omega}^{\mathfrak{a}} \cong \Omega/\mathfrak{a}^m\Omega$ for all $m \geq 1$.

Another useful flatness criterion is for modules of projective dimension one.

Theorem 5.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and Ω an arbitrary R-module. If $\operatorname{Tor}_1^R(\Omega, k) = 0$ and Ω has projective dimension one, then Ω is flat.*

In low dimensions, we establish criteria involving only the first Betti number.

Theorem 6.3 and 6.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and let Ω be an R-module of finite projective dimension. If R is one-dimensional, or, if R is two-dimensional and Ω is separated, then the vanishing of $\operatorname{Tor}_1^R(\Omega, k)$ implies that Ω is flat.*

Methods. Our proofs will be based upon the following well-known homological criterion: the flatness of a module Ω is equivalent with

$$\operatorname{Tor}_1^R(\Omega, R/\mathfrak{a}) = 0 \quad \text{for all (prime) ideals } \mathfrak{a} \text{ of } R.$$

In order to reduce the number of homological tests needed, we have to better understand the category of cyclic R-modules 'up to homology'. I introduced in [13] the following formalism to aid us with this problem: a class of finitely generated R-modules is a *net* if it is closed under extensions and direct summands. For our purposes, we need a relative version, and this is discussed in detail in §2. However, to generate the class of all cyclic modules as a net, we need in general infinitely many generators. In §3, we explain how imposing one more 'net-type' generation rule allows us to reduce this to a single generator, to wit, the residue field. Loosely speaking, a net is *formational* if whenever M/xM belongs to it, for some M-regular element x, then so does M. The various separatedness assumptions that occur in our criteria stem from the following use of this formational rule: if $\operatorname{Tor}_n^R(\Omega, M/xM) = 0$ for some n, then from the long exact *Tor*-sequence, multiplication with x on $\operatorname{Tor}_n^R(\Omega, M)$ is surjective, and hence if $\operatorname{Tor}_n^R(\Omega, M)$ is non-degenerated, then it must be zero. Before we derive the flatness criteria in §5 using formational nets, we first need to study some properties of flat modules in §4. In §6 we then investigate some vestiges of flatness for modules whose first Betti number vanishes. The final section contains some examples showing that some of our results are sharp. We also discuss in more detail the hypothesis on the finitude of projective dimension, relying on results from [13].

2. Nets

Let A be a Noetherian ring (always assumed commutative, with unit). For an ideal $\mathfrak{a} \subseteq A$, we let $\mathrm{V}(\mathfrak{a})$ denote the Zariski closed subset of all prime ideals $\mathfrak{p}$ of A containing $\mathfrak{a}$. Let $\mathbf{M}$ be a class of finitely generated A-modules containing the zero module.

2.1. **Definition.** A subclass $\mathbf{N} \subseteq \mathbf{M}$ is called an $\mathbf{M}$-*net*, if $\mathbf{N}$ is closed under extensions and direct summands in $\mathbf{M}$. More precisely, $\mathbf{N}$ is an $\mathbf{M}$-net provided for every short exact sequence

$$0 \to K \to M \to N \to 0$$

of A-modules in $\mathbf{M}$,

- **(`Net`):** if K and N belong to $\mathbf{N}$, then so does M;
- **(`DirSum`):** if this sequence is moreover split exact (that is to say, if $M \cong K \oplus N$) and M belongs to $\mathbf{N}$, then so do both K and N.

Clearly, $\mathbf{M}$ itself is an $\mathbf{M}$-net. The intersection of an arbitrary number of $\mathbf{M}$-nets is again an $\mathbf{M}$-net. Therefore, for each subset $\mathbf{K} \subseteq \mathbf{M}$, there exists a smallest $\mathbf{M}$-net containing it, which we will call the $\mathbf{M}$-*net generated by* $\mathbf{K}$. If the $\mathbf{M}$-net generated by $\mathbf{K}$ is equal to $\mathbf{M}$ itself, then we will simply say that $\mathbf{K}$ is *net-generating* for $\mathbf{M}$.

2.2. *Remark.* One easily verifies that a module $M \in \mathbf{M}$ belongs to the $\mathbf{M}$-net generated by $\mathbf{K}$ if and only if there exist short exact sequences

$$0 \to M_i \to M_{i+1} \to K_i \to 0 \qquad (1)$$

for $i = 0, \dots, n-1$, with M_0 and each K_i a direct summand of a module in $\mathbf{K}$ and with $M = M_n$. In proofs, we can therefore induct on n to show that a certain module belongs to the net generated by a particular class.

2.3. **Proposition.** *Let A be a Noetherian ring and let $\mathfrak{a}$ be an ideal in A. Let $\mathbf{P}_\mathfrak{a}$ be the collection of all modules of the form $A/\mathfrak{p}$ with $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$. Then $\mathbf{P}_\mathfrak{a}$ is net-generating for the class of all cyclic modules of the form A/I with I an ideal containing some power of $\mathfrak{a}$, as well as for the class of all finitely generated A-modules that are annihilated by some power of $\mathfrak{a}$.*

Proof. Let $\mathbf{M}_\mathfrak{a}$ be the class of all finitely generated A-modules that are annihilated by some power of $\mathfrak{a}$ and let $\mathbf{C}_\mathfrak{a}$ be the subclass of all cyclic modules in $\mathbf{M}_\mathfrak{a}$. Towards a contradiction, suppose that the first statement is false and let I be maximal among all ideals containing a power of $\mathfrak{a}$ for which A/I does not lie in the $\mathbf{C}_\mathfrak{a}$-net generated by $\mathbf{P}_\mathfrak{a}$. Clearly, I cannot be a prime ideal. Let $\mathfrak{p}$ be a minimal prime of I. It follows that $\mathfrak{p}$ contains $\mathfrak{a}$ and we have an exact sequence

$$0 \to A/\mathfrak{p} \to A/I \to A/J \to 0 \tag{2}$$

with J properly containing I. By maximality, A/J lies in the $\mathbf{C}_\mathfrak{a}$-net generated by $\mathbf{P}_\mathfrak{a}$. Since $A/\mathfrak{p} \in \mathbf{P}_\mathfrak{a}$, we get a contradiction by means of rule (`Net`).

Assume next that the second assertion is false. Hence there exists $M \in \mathbf{M}_\mathfrak{a}$ not belonging to the $\mathbf{M}_\mathfrak{a}$-net generated by $\mathbf{P}_\mathfrak{a}$. Let N be maximal so that M/N does not belong to the $\mathbf{M}_\mathfrak{a}$-net generated by $\mathbf{P}_\mathfrak{a}$ and choose an element $\mu \in M$ outside N. The image of μ in M/N generates a non-zero cyclic module C belonging to $\mathbf{M}_\mathfrak{a}$ whence to $\mathbf{C}_\mathfrak{a}$. By what we just proved, C lies in the $\mathbf{C}_\mathfrak{a}$-net generated by $\mathbf{P}_\mathfrak{a}$ whence a priori in the $\mathbf{M}_\mathfrak{a}$-net generated by $\mathbf{P}_\mathfrak{a}$. By maximality, so does $M/(N + A\mu)$, so that the exact sequence

$$0 \to C \to M/N \to M/(N + A\mu) \to 0$$

again contradicts rule (`Net`). □

Two special cases are worth mentioning separately, where in the first, we take $\mathfrak{a}$ to be the zero ideal, and in the second, the maximal ideal.

(2.3.1) The class of all cyclic A-modules of the form $A/\mathfrak{p}$ with $\mathfrak{p}$ a prime ideal of A is net-generating for the class of all cyclic A-modules as well as for the class of all finitely generated A-modules.

(2.3.2) The residue field of a Noetherian local ring is net-generating for the class of all modules of finite length.

Application to functors. The main application of nets is through the following easy observation.

2.4. **Proposition.** *Let A be a ring and let $\mathcal{F}$ be an additive functor from a category of A-modules to some Abelian category. Suppose $\mathcal{F}$ is exact in the middle. Let $\mathbf{K} \subseteq \mathbf{M}$ be classes of finitely generated A-modules with $0 \in \mathbf{M}$. If $\mathcal{F}(K) = 0$ for every $K \in \mathbf{K}$, then $\mathcal{F}(M) = 0$ for every M in the $\mathbf{M}$-net generated by $\mathbf{K}$.*

Proof. Recall that a (covariant) functor is called *exact in the middle*, if any short exact sequence

$$0 \to K \to M \to N \to 0$$

transforms into an exact sequence

$$\mathcal{F}(K) \to \mathcal{F}(M) \to \mathcal{F}(N).$$

For a contravariant functor the definition is the same apart from reversing the arrows. The statement is now immediate by an easy inductive argument (see Remark 2.2). □

For A an arbitrary ring and Ω an arbitrary A-module, each of the functors $\operatorname{Tor}_i^A(\Omega, \cdot)$, $\operatorname{Ext}_A^i(\Omega, \cdot)$ or $\operatorname{Ext}_A^i(\cdot, \Omega)$ is exact in the middle, for any i. More generally, any derived functor of a left or right exact functor is exact in the middle. Therefore, Propositions 2.3 and 2.4 yield:

2.5. **Corollary.** *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring with residue field* k*. Let* Ω *be an arbitrary* R*-module and let* $n \geq 1$*. If* $\mathrm{Tor}_n^R(\Omega, k)$ *vanishes, then so does* $\mathrm{Tor}_n^R(\Omega, N)$ *for every* R*-module* N *of finite length.*

More generally, if A *is a Noetherian ring,* $\mathfrak{a}$ *an ideal of* A *and* Ω *an* A*-module, such that* $\mathrm{Tor}_n^A(\Omega, R/\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$*, then* $\mathrm{Tor}_n^A(\Omega, M) = 0$ *for all finitely generated* A*-modules annihilated by some power of* $\mathfrak{a}$*.*

Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A and Ω an A-module. We will denote the $\mathfrak{a}$-*adic completion* of Ω by $\widehat{\Omega}^{\mathfrak{a}}$. Recall that $\widehat{\Omega}^{\mathfrak{a}}$ is given as the inverse limit of all $\Omega/\mathfrak{a}^m\Omega$ endowed with the inverse limit topology and that in this topology $\widehat{\Omega}^{\mathfrak{a}}$ is complete. If Ω is not finitely generated, the inverse limit topology on $\widehat{\Omega}^{\mathfrak{a}}$ might not be the same as the $\mathfrak{a}$-adic one. That is to say, the canonical morphisms $\widehat{\Omega}^{\mathfrak{a}}/\mathfrak{a}^m\widehat{\Omega}^{\mathfrak{a}} \to \Omega/\mathfrak{a}^m\Omega$, although always surjective, might fail to be injective, even when $m = 1$. Put differently, $\mathfrak{a}^m\widehat{\Omega}^{\mathfrak{a}}$ need not be closed in the inverse limit topology. If A is local with maximal ideal $\mathfrak{m}$, then by the *completion* of Ω we mean its $\mathfrak{m}$-adic completion and denote it simply by $\widehat{\Omega}$.

2.6. **Corollary.** *Let* A *be a Noetherian ring and* $\mathfrak{a}$ *an ideal of* A*. Let*

$$0 \to \Pi \to \Lambda \to \Omega \to 0$$

be a short exact sequence of A*-modules. If* $\mathrm{Tor}_1^A(\Omega, A/\mathfrak{p}) = 0$ *for all* $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$*, then*

$$0 \to \widehat{\Pi}^{\mathfrak{a}} \to \widehat{\Lambda}^{\mathfrak{a}} \to \widehat{\Omega}^{\mathfrak{a}} \to 0 \tag{3}$$

is exact.

Proof. By Corollary 2.5, all $\mathrm{Tor}_1^A(\Omega, A/\mathfrak{a}^m)$ vanish, for each $m \geq 1$. Hence, for each m, tensoring with $A/\mathfrak{a}^m$ yields an exact sequence

$$0 \to \Pi/\mathfrak{a}^m\Pi \to \Lambda/\mathfrak{a}^m\Lambda \to \Omega/\mathfrak{a}^m\Omega \to 0.$$

Taking these exact sequences for all m simultaneously, gives an exact sequence of inverse systems. Since the first inverse system is flasque, we have, after taking inverse limits, the desired exact sequence (3) by [8, Proposition 1.6] or [10, Theorem 8.1]. □

3. Formational nets

In this section, we amend the net-generating rules by one additional rule, which will enable us to generate the class of all finitely generated modules from a relatively small subclass (which in the local case can even be taken to be a singleton). Let A be a Noetherian ring and $\mathfrak{a}$ a proper ideal of A. Let $\mathbf{M}$ be a class of finitely generated A-modules containing the zero module and let $\mathbf{N}$ be an $\mathbf{M}$-net.

3.1. **Definition.** We say that $\mathbf{N}$ is $\mathfrak{a}$-*formational*, if the following holds: given an A-module $M \in \mathbf{M}$ and an M-regular element $x \in \mathfrak{a}$, if M/xM belongs to $\mathbf{N}$, then so does M. In other words, the new generating rule is:

> **(`Auto`):** Given an exact sequence
>
> $$0 \to M \xrightarrow{x} M \to N \to 0$$
>
> with $x \in \mathfrak{a}$ and $M \in \mathbf{M}$. If N belongs to $\mathbf{N}$, then so does M.

3.2. *Remark.* If $\mathfrak{a}M = M$ for some $M \in \mathbf{M}$, then M belongs to any $\mathfrak{a}$-formational $\mathbf{M}$-net. Indeed, by Nakayama's Lemma, there is some $x \in \mathfrak{a}$ such that $(1-x)M = 0$. In particular, $M = xM$ and x is M-regular, so that we have an exact sequence (`Auto`) with $N = 0$.

If R is a local ring and $\mathfrak{m}$ its maximal ideal, then we call an $\mathbf{M}$-net *formational*, if it is $\mathfrak{m}$-formational. Given $\mathbf{K} \subseteq \mathbf{M}$, the $\mathfrak{a}$-*formational* $\mathbf{M}$-*net generated by* $\mathbf{K}$ is by definition the smallest $\mathfrak{a}$-formational $\mathbf{M}$-net containing $\mathbf{K}$. If the $\mathfrak{a}$-formational $\mathbf{M}$-net generated by $\mathbf{K}$ is $\mathbf{M}$, then we say that $\mathbf{K}$ is $\mathfrak{a}$-*formationally net-generating* for $\mathbf{M}$. A similar inductive description of the $\mathfrak{a}$-formational $\mathbf{M}$-net generated by $\mathbf{K}$ can be given as in Remark 2.2, but this time the connection between M_i and M_{i+1} is either given by an exact sequence (1) or by an exact sequence of the form

$$0 \to M_{i+1} \xrightarrow{x} M_{i+1} \to M_i \to 0$$

with x an M_{i+1}-regular element inside $\mathfrak{a}$.

3.3. **Theorem.** *Let A be a Noetherian ring and let $\mathfrak{a}$ be a proper ideal. Let $\mathbf{P}_\mathfrak{a}$ be the set of all $A/\mathfrak{p}$ with $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$. Then $\mathbf{P}_\mathfrak{a}$ is $\mathfrak{a}$-formationally net-generating for the class of all cyclic A-modules as well as for the class of all finitely generated A-modules.*

Proof. We will prove the second statement, the first then follows by inspection of our argument. Let $\mathbf{M}$ be the class of all finitely generated A-modules and let $\mathbf{N}$ be the $\mathfrak{a}$-formational $\mathbf{M}$-net generated by $\mathbf{P}_\mathfrak{a}$. Towards a contradiction, assume that there exists some $M \in \mathbf{M}$ not in $\mathbf{N}$. Let N be a maximal submodule of M such that M/N does not belong to $\mathbf{N}$. In particular, $\mathfrak{a}(M/N)$ is not equal to M/N lest M/N belongs to $\mathbf{N}$ by Remark 3.2. We consider two cases.

Case 1. Suppose that M/N has non-zero $\mathfrak{a}$-depth. Hence there exists an $x \in \mathfrak{a}$, such that x is M/N-regular. Therefore we have an exact sequence

$$0 \to M/N \xrightarrow{x} M/N \to M/(N + xM) \to 0.$$

However, N is properly contained in $N + xM$, so that by maximality $M/(N + xM)$ belongs to $\mathbf{N}$, contradicting rule (`Auto`).

Case 2. Assume now that M/N has $\mathfrak{a}$-depth zero, which by [10, Theorem 16.7] is equivalent with $\mathrm{Hom}_A(A/\mathfrak{a}, M/N) \neq 0$. Take a non-zero morphism $A/\mathfrak{a} \to M/N$ and factor this out over its kernel. We obtain an embedding $A/\mathfrak{b} \hookrightarrow M/N$, where $\mathfrak{b}$ is some proper ideal of A containing $\mathfrak{a}$. Let $\mathfrak{p}$ be a minimal prime of $\mathfrak{b}$, so that in particular $\mathfrak{a} \subseteq \mathfrak{p}$. Composing an embedding $A/\mathfrak{p} \hookrightarrow A/\mathfrak{b}$ with $A/\mathfrak{b} \hookrightarrow M/N$ gives rise to an exact sequence

$$0 \to A/\mathfrak{p} \to M/N \to M/N' \to 0$$

where N' is some submodule containing N properly. By maximality, M/N' belongs to $\mathbf{N}$ and by assumption so does $A/\mathfrak{p}$. However, this is in violation of rule (`Net`). □

Note that $\mathbf{P}_\mathfrak{a}$ is finite if $\mathfrak{a}$ is a maximal ideal, so that in particular we get:

3.4. **Corollary.** *If R is a Noetherian local ring, then its residue field is formationally net-generating for the class of all finitely generated R-modules.*

Application to linear functors. A functor $\mathcal{F}$ on a category of A-modules is called *linear* if it preserves multiplication, that is to say, the image under $\mathcal{F}$ of the morphism $M \xrightarrow{x} M$ given by multiplication by $x \in A$, is the multiplication map $\mathcal{F}(M) \xrightarrow{x} \mathcal{F}(M)$. Recall from [4, Proposition 9.1.2] that an arbitrary A-module Ω is said to have $\mathfrak{a}$-*depth zero*, for $\mathfrak{a}$ an ideal in A, if $\mathrm{Hom}_A(A/\mathfrak{a}, \Omega)$ is non-zero, that is to say, if there is some non-zero element $\mu \in \Omega$ such that $\mathfrak{a}\mu = 0$. By [4, Proposition 9.1.4] this is equivalent with Ω having an associated prime containing $\mathfrak{a}$. We call Ω *non-degenerated in the* $\mathfrak{a}$-*adic topology*, if either Ω is zero or otherwise $\mathfrak{a}\Omega \neq \Omega$; and $\mathfrak{a}$-*adically separated*, if the intersection of all $\mathfrak{a}^m\Omega$ is zero. The following proposition is the analogue of Proposition 2.4 for formational nets.

3.5. **Proposition.** *Let A be a Noetherian ring and let $\mathfrak{a}$ be a proper ideal of A. Let $\mathcal{F}$ be an additive linear functor on a category of A-modules which is exact in the middle. Let $\mathbf{K} \subseteq \mathbf{M}$ be classes of finitely generated A-modules with $0 \in \mathbf{M}$. If $\mathcal{F}$ is covariant, assume $\mathcal{F}(M)$ is non-degenerated in the $\mathfrak{a}$-adic topology for every $M \in \mathbf{M}$, and if $\mathcal{F}$ is contravariant, assume that $\mathcal{F}(M)$ is either zero or has zero $\mathfrak{a}$-depth for every $M \in \mathbf{M}$.*

If $\mathcal{F}(K) = 0$ for every $K \in \mathbf{K}$, then $\mathcal{F}(M) = 0$ for every M in the $\mathfrak{a}$-formational $\mathbf{M}$-net generated by $\mathbf{K}$.

Proof. Let us first consider the covariant case. The proof will follow from Proposition 2.4 and an inductive argument, once we have shown the following. Let

$$0 \to M \xrightarrow{x} M \to N \to 0 \tag{4}$$

be an exact sequence with $x \in \mathfrak{a}$ and $\mathcal{F}(N) = 0$. We have to show that $\mathcal{F}(M) = 0$. Our hypotheses on the functor $\mathcal{F}$ imply that we have an exact sequence

$$\mathcal{F}(M) \xrightarrow{x} \mathcal{F}(M) \to \mathcal{F}(N) = 0.$$

Let $\Pi := \mathcal{F}(M)$, so that $x\Pi = \Pi$, whence $\Pi = \mathfrak{a}\Pi$. By assumption Π is non-degenerated in the $\mathfrak{a}$-adic topology, and therefore must be zero.

Similarly, in the contravariant case, we obtain from (4) an exact sequence

$$0 = \mathcal{F}(N) \to \mathcal{F}(M) \xrightarrow{x} \mathcal{F}(M). \tag{5}$$

If $\Pi := \mathcal{F}(M)$ were non-zero, then by assumption, it would have zero $\mathfrak{a}$-depth, so that some non-zero $\pi \in \Pi$ would be annihilated by $\mathfrak{a}$. In particular, $x\pi = 0$, contradicting (5). □

3.6. *Remark.* In the covariant case, we can relax the assumption on $\mathcal{F}$ further: it suffices that $\mathcal{F}(M)$ is non-degenerated in the xA-adic topology for every $M \in \mathbf{M}$ and every $x \in \mathfrak{a}$. In the contravariant case, we only need that for every $M \in \mathbf{M}$ for which $\mathcal{F}(M) \neq 0$, and for every $x \in \mathfrak{a}$, we can find a non-zero element $\mu \in \mathcal{F}(M)$, such that $x\mu = 0$. In other words, we need for each $x \in \mathfrak{a}$ that $\operatorname{Ann}_{\mathcal{F}(M)}(x) \neq 0$. If $\mathfrak{a} = (x_1, \dots, x_s)A$, then $\operatorname{Ann}_{\mathcal{F}(M)}(\mathfrak{a})$ is the intersection of all $\operatorname{Ann}_{\mathcal{F}(M)}(x_i)$, and it might well be that the latter are non-zero but their intersection is zero, so that the above condition is indeed weaker than requiring that $\mathcal{F}(M)$ has zero $\mathfrak{a}$-depth.

3.7. *Remark.* The covariant functors $\operatorname{Tor}_n^A(\Omega, \cdot)$ and $\operatorname{Ext}_A^n(\Omega, \cdot)$ and the contravariant functor $\operatorname{Ext}_A^n(\cdot, \Omega)$ are all linear, for Ω an arbitrary A-module. If Ω is finitely generated, then $\operatorname{Tor}_n^A(\Omega, \cdot)$ takes finitely generated modules to finitely generated modules. In particular, if $(A, \mathfrak{m})$ is a Noetherian local ring, then the image of a finitely generated module under $\operatorname{Tor}_n^A(\Omega, \cdot)$ is separated in the $\mathfrak{m}$-adic topology and hence in any $\mathfrak{a}$-adic topology, for $\mathfrak{a}$ a proper ideal, so that the proposition applies to it. Shortly we will indicate some milder restrictions on Ω for which the theorem will still be applicable.

If Ω has finite length, then so does each $\operatorname{Ext}_A^n(\Omega, M)$ with M finitely generated. Moreover, the functor $\operatorname{Ext}_A^n(\cdot, \Omega)$ maps any module into a module which is either zero or has zero depth, and the proposition applies in either case.

4. Properties of Flat Modules

4.1. **Lemma.** *Let A be a ring and $\mathfrak{a}$ a proper ideal. Let $\Phi := \oplus_E A$ be a free A-module where E is some (possibly infinite) index set and let $\Pi := \prod_E \widehat{A}^{\mathfrak{a}}$ be the corresponding direct product of the $\mathfrak{a}$-adic completion $\widehat{A}^{\mathfrak{a}}$ of A. Then the $\mathfrak{a}$-adic completion of Φ is* *isomorphic to the submodule of* Π *consisting of all sequences for which for every* m*, only finitely many entries are non-zero modulo* $\mathfrak{a}^m\widehat{A}^{\mathfrak{a}}$.

Proof. It is easy to verify that Π is $\mathfrak{a}$-adically complete. Hence $\widehat{\Phi}^{\mathfrak{a}}$ is the $\mathfrak{a}$-adic closure of Φ inside Π, that is to say,

$$\widehat{\Phi}^{\mathfrak{a}} = \bigcap_m \Phi + \mathfrak{a}^m\Pi.$$

The assertion is now clear. □

We leave the proof of the following immediate corollary to the reader.

4.2. **Corollary.** *Let* A *be a Noetherian ring and* Φ *a free* A*-module. For each ideal* $I \subseteq A$*, we have an isomorphism*

$$\widehat{(\Phi/I\Phi)}^{\mathfrak{a}} \cong \widehat{\Phi}^{\mathfrak{a}}/I\widehat{\Phi}^{\mathfrak{a}}.$$

In particular, the $\mathfrak{a}$*-adic topology on* $\widehat{\Phi}^{\mathfrak{a}}$ *is the same as the inverse limit topology and* $\widehat{\Phi}^{\mathfrak{a}}$ *is* $\mathfrak{a}$*-adically complete.* □

4.3. **Proposition.** *Let* A *be a Noetherian ring,* $\mathfrak{a}$ *an ideal of* A *and* Φ *an* A*-module. If* Φ *is free, then its* $\mathfrak{a}$*-adic completion* $\widehat{\Phi}^{\mathfrak{a}}$ *is flat.*

Proof. Since $\widehat{\Phi}^{\mathfrak{a}}$ is an $\widehat{A}^{\mathfrak{a}}$-module and since $A \to \widehat{A}^{\mathfrak{a}}$ is flat, we may assume without loss of generality that A is $\mathfrak{a}$-adically complete. Choose some index set E such that $\Phi = \oplus_E A$ and let Π be the corresponding direct product $\prod_E A$. By Lemma 4.1, we may view $\widehat{\Phi}^{\mathfrak{a}}$ as a submodule of Π. If a is a sequence in Π, then we denote its e-th component by $a(e) \in A$.

To establish flatness, we use the 'equational criterion for flatness' ([10, Theorem 7.6] or [5, Corollary 6.5]). Assume we have a linear form L over A in s variables and an s-tuple $\mathbf{b} := (\mathbf{b}(e))_e \in (\widehat{\Phi}^{\mathfrak{a}})^s$ such that $L(\mathbf{b}) = 0$. Define a map $o\colon E \to \mathbb{N}$ by letting $o(e)$ be the largest m such that all entries of $\mathbf{b}(e) \in A^s$ belong to $\mathfrak{a}^m$. By Lemma 4.1, the level sets $o^{-1}(n)$ of this map are all finite. Let $V \subseteq A^s$ be the submodule generated by all $\mathbf{b}(e)$ with $e \in E$. By Noetherianity, there exists a finite subset $F \subseteq E$ such that V is generated by all $\mathbf{b}(f)$ with $f \in F$. By the Artin-Rees Lemma, there is some c such that

$$V \cap \mathfrak{a}^m A^s \subseteq \mathfrak{a}^{m-c}V \tag{6}$$

for all $m \geq c$. We may choose c large enough so that F lies inside $o^{-1}(\{1,\dots,c-1\})$. Without loss of generality, we may then assume that both sets are in fact equal. Note that by our previous argument, F is still finite.

In particular, $o(e) \geq c$ for all $e \in E \setminus F$, so that $\mathbf{b}(e)$ lies in $\mathfrak{a}^{o(e)-c}V$ by (6). Hence, for each $f \in F$, there exists a sequence $a_f \in \Pi$ such that

$$\mathbf{b} = \sum_{f\in F} \mathbf{b}(f)a_f$$

with $a_f(e) \in \mathfrak{a}^{o(e)-c}$ for all $e \notin F$. By this last condition and Lemma 4.1, each a_f lies in $\widehat{\Phi}^{\mathfrak{a}}$. Since $\mathbf{b}$ is a solution to the linear equation $L = 0$, so is each $\mathbf{b}(f)$ for $f \in F$. Hence we verified the equational flatness criterion. See §5.16 below for an alternative proof. □

4.4. **Artin-Rees like modules.** We say that an A-module Ω is *Artin-Rees like*, if for all ideals I and $\mathfrak{a}$ of A, we can find some c such that

$$I\Omega \cap \mathfrak{a}^m\Omega \subseteq \mathfrak{a}^{m-c}I\Omega$$

for all $m \geq c$.

The classical example of an Artin-Rees like module, due to Artin-Rees ([10, Theorem 8.5]), is a finitely generated module over a Noetherian ring. Using the equational criterion for flatness, any flat module Ω over a Noetherian ring A is Artin-Rees like. Indeed, by the Artin-Rees lemma applied in A, we can find a c such that $I \cap \mathfrak{a}^m \subseteq \mathfrak{a}^{m-c} I$ for all $m \geq c$. In general, if Ω is flat, then $I\Omega \cap J\Omega = (I \cap J)\Omega$ for all ideals I and J (see for instance [10, Theorem 7.4]. Applying this with $J := \mathfrak{a}^m$ then gives the desired result.

4.5. **Universally $\mathfrak{a}$-separated modules.** Let A be a Noetherian ring and $\mathfrak{a}$ an arbitrary proper ideal. We say that an arbitrary A-module Ω is *universally $\mathfrak{a}$-separated*, if $\Omega/I\Omega$ is separated in the $\mathfrak{a}$-adic topology for every ideal I of R. If A is local with maximal ideal $\mathfrak{m}$, then we simply say *universally separated* to mean universally $\mathfrak{m}$-separated. Note that a module Ω is universally $\mathfrak{a}$-separated if and only if $I\Omega$ is closed in the $\mathfrak{a}$-adic topology for every ideal I of A.

Examples of universally separated A-modules are: projective modules; founded modules (see [12]); and, if A is local with maximal ideal $\mathfrak{m}$, finitely generated modules, or more generally, modules which are finitely generated over an A-algebra B with $\mathfrak{m}B$ contained in the Jacobson radical of B. For the first assertion, use that a direct summand of a universally separated module is again universally separated plus the fact that a free module is universally separated; for the second, use that founded implies separated plus the fact that Ω founded over A implies $\Omega/I\Omega$ founded over A/I (see [12, §2]); for the last two, use Krull's Intersection Theorem.

4.6. **Proposition.** *Let A be a Noetherian ring, $\mathfrak{a}$ a proper ideal of A and Ω an A-module. If Ω is $\mathfrak{a}$-adically complete and Artin-Rees like, then it is universally $\mathfrak{a}$-separated. In particular, an $\mathfrak{a}$-adically complete, flat A-module is universally $\mathfrak{a}$-separated.*

Proof. The last statement follows from the first and our discussion in §4.4. So we need to show that for an arbitrary ideal I, the submodule $I\Omega$ is $\mathfrak{a}$-adically closed. In other words, we have to show an equality

$$I\Omega = \bigcap_{m=0}^{\infty} I\Omega + \mathfrak{a}^m\Omega. \tag{7}$$

Choose c so that

$$I\Omega \cap \mathfrak{a}^m\Omega \subseteq \mathfrak{a}^{m-c} I\Omega \tag{8}$$

for all $m \geq c$. Let ω be in the right hand side of (7). For all $m \geq c$, we can write $\omega = \alpha_m + \theta_m$ with $\alpha_m \in I\Omega$ and $\theta_m \in \mathfrak{a}^m\Omega$. Therefore, $\alpha_{m+1} - \alpha_m = \theta_m - \theta_{m+1}$ lies in $I\Omega \cap \mathfrak{a}^m\Omega$. By (8), we have

$$\theta_m - \theta_{m+1} = \beta_m \tag{9}$$

for some $\beta_m \in \mathfrak{a}^{m-c} I\Omega$. It follows form repetitive use of (9) that

$$\omega =: \alpha_c + \beta_c + \beta_{c+1} + \cdots + \beta_m + \theta_{m+1}.$$

Since Ω is $\mathfrak{a}$-adically complete, we get

$$\omega = \alpha_c + \sum_{m \geq c} \beta_m.$$

If $I = (f_1, \dots, f_s)A$, then writing each β_m as a linear combination of the f_i shows that $\omega \in I\Omega$, as required. □

5. Flatness Criteria

We start with a variant of the well-known Local Flatness Criterion–if we take A local and $\mathfrak{a}$ its maximal ideal, then our version agrees with the version stated in [10, Theorem 22.3]. Recall that a module Ω is called $\mathfrak{a}$*-adically ideal-separated*, if for each ideal I, the module $\Omega \otimes_A I$ is $\mathfrak{a}$-adically separated.

5.1. **Theorem** (Local Flatness Criterion)**.** *Let A be a Noetherian ring and let $\mathfrak{a}$ be a proper ideal of A. Let Ω be an arbitrary $\mathfrak{a}$-adically ideal-separated A-module. If $\operatorname{Tor}_1^A(\Omega, A/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{V}(\mathfrak{a})$, then Ω is flat.*

Proof. Set $\mathcal{F} := \operatorname{Tor}_1^A(\Omega, \cdot)$, so that $\mathcal{F}$ is linear and exact in the middle. Using Theorem 3.3 and Proposition 3.5 with respect to the class of cyclic modules, it suffices to show that the non-degeneracy condition in Proposition 3.5 holds for the functor $\mathcal{F}$ restricted to the category of cyclic modules. Indeed, assuming this, we conclude that $\operatorname{Tor}_1^A(\Omega, A/I)$ vanishes for all ideals I, showing that Ω is flat. Therefore, let I be an ideal of A. We need to prove that $\operatorname{Tor}_1^A(\Omega, A/I)$ is non-degenerated in the $\mathfrak{a}$-adic topology. From the short exact sequence

$$0 \to I \to A \to A/I \to 0$$

we obtain an exact sequence

$$0 \to \operatorname{Tor}_1^A(\Omega, A/I) \to \Omega \otimes_A I.$$

By our assumption on Ω, the tensor product $\Omega \otimes_A I$ is $\mathfrak{a}$-adically separated. Therefore, so is $\operatorname{Tor}_1^A(\Omega, A/I)$, as it is a submodule, and hence in particular, it is non-degenerated. □

From the proof it is clear that we can weaken the hypothesis on Ω: it suffices that every submodule of $\Omega \otimes_A I$ is non-degenerated in the $\mathfrak{a}$-adic topology, for every ideal I. In fact, our present techniques are applicable to an even wider class of A-modules, as we will now see.

5.2. **Theorem.** *Let A be a Noetherian ring and let $\mathfrak{a}$ be a proper ideal of A. Let Ω be an arbitrary A-module. Assume that Ω is the quotient of a flat module by a universally $\mathfrak{a}$-separated submodule. If $\operatorname{Tor}_1^A(\Omega, A/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{V}(\mathfrak{a})$, then Ω is flat.*

In particular, if $(R, \mathfrak{m})$ is local with residue field k, then the quotient Ω of a flat module by a universally separated submodule is flat if and only if $\operatorname{Tor}_1^R(\Omega, k) = 0$.

Proof. By the same argument as in the proof of Theorem 5.1, it suffices to show that $\operatorname{Tor}_1^A(\Omega, A/I)$ is $\mathfrak{a}$-adically separated for every ideal I of A. By assumption, we have an exact sequence

$$0 \to \Pi \to \Phi \to \Omega \to 0,$$

with Φ flat and Π universally $\mathfrak{a}$-separated. Tensoring with A/I yields a short exact sequence

$$0 \to \operatorname{Tor}_1^A(\Omega, A/I) \to \Pi/I\Pi. \tag{10}$$

Since $\Pi/I\Pi$ is $\mathfrak{a}$-adically separated, so is $\operatorname{Tor}_1^A(\Omega, A/I)$. □

5.3. **Corollary.** *Let A be a Noetherian ring and let Ω be an arbitrary A-module. If Ω has projective dimension e and if $\operatorname{Tor}_e^A(\Omega, A/\mathfrak{m}) = 0$ for some maximal ideal $\mathfrak{m}$ of A, then Ω has flat dimension at most $e - 1$.*

In particular, if Ω has projective dimension one and $\operatorname{Tor}_1^A(\Omega, A/\mathfrak{m}) = 0$ for some maximal ideal $\mathfrak{m}$, then Ω is flat.

Proof. We induct on the projective dimension e of Ω. If $e = 1$, then Ω is the quotient of a free module by a projective module, so that Theorem 5.2 applies. For $e > 1$, choose a free module Φ and an exact sequence

$$0 \to \Pi \to \Phi \to \Omega \to 0. \tag{11}$$

Therefore, Π has projective dimension $e-1$. From the *Tor* long exact sequence, it follows that $\operatorname{Tor}^A_{e-1}(\Pi, A/\mathfrak{m}) \cong \operatorname{Tor}^A_e(\Omega, A/\mathfrak{m}) = 0$. By induction, Π has flat dimension at most $e-2$. From the exact sequence (11), it then follows that Ω has flat dimension at most $e-1$, as required. ☐

In case R is local and $e = 1$, we get:

5.4. **Theorem.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and Ω an arbitrary R-module. If $\operatorname{Tor}^R_1(\Omega, k) = 0$ and Ω has projective dimension one, then Ω is flat.*

5.5. **Corollary.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k. Suppose we have an R-module morphism $u \colon \Pi \to \Phi$ with Π universally separated and Φ flat. If $u \otimes k \colon \Pi/\mathfrak{m}\Pi \to \Phi/\mathfrak{m}\Phi$ is injective, then so is u. Moreover,* $\operatorname{coker} u$ *and Π are both flat and u is pure.*

Proof. It is well-known that injectivity of $u \otimes k$ implies that u is injective, since Π is separated (see for instance [10, Theorem 22.5]). Let Ω denote the cokernel of u, that is to say, $\Omega = \Phi/u(\Pi)$. The injectivity of $u \otimes k$ implies furthermore that $\operatorname{Tor}^R_1(\Omega, k) = 0$. Since Ω is the quotient of a flat module by an universally separated module, the flatness of Ω follows from Theorem 5.2. Therefore, Π is flat and u is pure. ☐

5.6. **Theorem.** *Let A be a Noetherian ring, $\mathfrak{a}$ a proper ideal of A and Ω an A-module. If Ω has finite projective dimension and $\operatorname{Tor}^A_n(\Omega, A/\mathfrak{p})$ vanishes for all $n \geq 1$ and all $\mathfrak{p} \in \operatorname{V}(\mathfrak{a})$, then the $\mathfrak{a}$-adic completion $\widehat{\Omega}^{\mathfrak{a}}$ of Ω is flat.*

Moreover, we have isomorphisms $\widehat{\Omega}^{\mathfrak{a}}/\mathfrak{a}^m\widehat{\Omega}^{\mathfrak{a}} \cong \Omega/\mathfrak{a}^m\Omega$ for all $m \geq 1$ and $\widehat{\Omega}^{\mathfrak{a}}$ is $\mathfrak{a}$-adically complete.

Proof. Let q be the projective dimension of Ω. We will induct on q. If $q = 0$, then Ω is projective. Since completion commutes with direct summands, it suffices to show that the $\mathfrak{a}$-adic completion of a free module is flat and $\mathfrak{a}$-adically complete, which is what we proved in Corollary 4.2 and Proposition 4.3. So assume $q > 0$ and choose an exact sequence

$$0 \to \Pi \to \Phi \to \Omega \to 0 \tag{12}$$

with Φ a free A-module. It follows that Π has projective dimension $q-1$. By Corollary 2.6, we get an exact sequence

$$0 \to \widehat{\Pi}^{\mathfrak{a}} \to \widehat{\Phi}^{\mathfrak{a}} \to \widehat{\Omega}^{\mathfrak{a}} \to 0. \tag{13}$$

Since $\operatorname{Tor}^A_n(\Pi, A/\mathfrak{p}) = \operatorname{Tor}^A_{n+1}(\Omega, A/\mathfrak{p}) = 0$ for all $n \geq 1$ and all $\mathfrak{p} \in \operatorname{V}(\mathfrak{a})$, induction yields that $\widehat{\Pi}^{\mathfrak{a}}$ is flat and $\mathfrak{a}$-adically complete. Moreover, we have isomorphisms $\widehat{\Pi}^{\mathfrak{a}}/\mathfrak{a}^m\widehat{\Pi}^{\mathfrak{a}} \cong \Pi/\mathfrak{a}^m\Pi$ for all $m \geq 1$. It follows from Proposition 4.6 that $\widehat{\Pi}^{\mathfrak{a}}$ is universally $\mathfrak{a}$-separated. Since $\widehat{\Phi}^{\mathfrak{a}}$ is flat by Proposition 4.3, Theorem 5.2 yields that $\widehat{\Omega}^{\mathfrak{a}}$ is flat.

Tensoring (12) with $A/\mathfrak{a}^m$ and using that $\operatorname{Tor}^A_1(\Omega, A/\mathfrak{a}^m) = 0$ by Corollary 2.5, we get an exact sequence

$$0 \to \Pi/\mathfrak{a}^m\Pi \to \Phi/\mathfrak{a}^m\Phi \to \Omega/\mathfrak{a}^m\Omega \to 0.$$

On the other hand, tensoring (13) with $A/\mathfrak{a}^m$ and using that $\widehat{\Omega}^{\mathfrak{a}}$ is flat gives an exact sequence

$$0 \to \widehat{\Pi}^{\mathfrak{a}}/\mathfrak{a}^m\widehat{\Pi}^{\mathfrak{a}} \to \widehat{\Phi}^{\mathfrak{a}}/\mathfrak{a}^m\widehat{\Phi}^{\mathfrak{a}} \to \widehat{\Omega}^{\mathfrak{a}}/\mathfrak{a}^m\widehat{\Omega}^{\mathfrak{a}} \to 0.$$

By our induction hypothesis, the first modules in both exact sequences are isomorphic, and so are the middle ones. By diagram chasing, the last modules in both sequences are therefore isomorphic too. In particular, the $\mathfrak{a}$-adic topology on $\widehat{\Omega}^{\mathfrak{a}}$ coincides with its inverse limit topology, so that $\widehat{\Omega}^{\mathfrak{a}}$ is also $\mathfrak{a}$-adically complete. □

We state the following important instance of the theorem separately.

5.7. **Corollary.** *Let A be a Noetherian ring, $\mathfrak{a}$ a proper ideal of A and Ω an A-module. If Ω is $\mathfrak{a}$-adically complete and has finite projective dimension, and if $\operatorname{Tor}_n^A(\Omega, A/\mathfrak{p}) = 0$ for all $n \geq 1$ and all $\mathfrak{p} \in \operatorname{V}(\mathfrak{a})$, then Ω is flat.*

5.8. *Remark.* Assume A has finite Krull dimension d. Inspecting the above proof, we see that it suffices that $\operatorname{Tor}_n^A(\Omega, A/\mathfrak{p})$ vanishes for all $1 \leq n \leq d+1$ and all $\mathfrak{p} \in \operatorname{V}(\mathfrak{a})$, since by work of Auslander-Buchsbaum, Bass and Jensen (see for instance [8, p. 44]), the projective dimension of an A-module, when finite, is at most $d+1$. In fact, assume A is moreover local with residue field k and let $\mathfrak{a}$ be equal to the maximal ideal of A. Let $\delta = d$ when A is Cohen-Macaulay, and $\delta = d-1$ otherwise. By a result of Auslander-Buchsbaum ([1, Theorem 2.4]; see also [15, Theorem 4.6]), the flat dimension of Ω is at most δ, so that all $\operatorname{Tor}_n^A(\Omega, k)$ vanish for $n \geq \delta + 1$. Therefore, we only need to check the vanishing of the first δ Betti numbers. We will further improve this bound in Theorem 7.6 below, showing that we may replace δ by the depth of A.

5.9. **Theorem.** *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal in A and Ω an A-module. If Ω is flat and has finite projective dimension, then its $\mathfrak{a}$-adic completion $\widehat{\Omega}^{\mathfrak{a}}$ is also flat.*

Moreover, $\widehat{\Omega}^{\mathfrak{a}}$ is $\mathfrak{a}$-adically complete and we have isomorphisms $\widehat{\Omega}^{\mathfrak{a}}/\mathfrak{a}^m\widehat{\Omega}^{\mathfrak{a}} \cong \Omega/\mathfrak{a}^m\Omega$ for all $m \geq 1$.

Proof. Since all $\operatorname{Tor}_n^A(\Omega, A/\mathfrak{p})$ vanish for $n \geq 1$ and $\mathfrak{p} \in \operatorname{V}(\mathfrak{a})$, Theorem 5.6 yields the flatness of $\widehat{\Omega}^{\mathfrak{a}}$ and the assertion on the topology of $\widehat{\Omega}^{\mathfrak{a}}$. □

This result was proved by Enochs in [6, Proposition 2.3] for Noetherian rings of finite Krull dimension using pure injective resolutions; the result for local rings was already proved by Bartijn-Strooker in [2, Corollaire 3.15] using basic modules. Note that if A has finite Krull dimension, then Ω, being flat, has finite projective dimension by [11, Corollary 3.2.7]. The following corollary generalizes [10, Theorem 22.6].

5.10. **Corollary.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let A be an R-algebra. Let Ω be an A-module which is flat as an R-module. If $a \in A$ is $\Omega/\mathfrak{m}\Omega$-regular, then a is $\widehat{\Omega}$-regular and $\widehat{\Omega}/a\widehat{\Omega}$ is flat over R. If Ω is moreover separated, then a is Ω-regular.*

Proof. By Theorem 5.9, the completion $\widehat{\Omega}$ is flat and complete in the $\mathfrak{m}$-adic topology. Therefore $\widehat{\Omega}$ is universally separated by Proposition 4.6. Let u_a be the endomorphism of $\widehat{\Omega}$ given by multiplication with a. Since $\Omega/\mathfrak{m}\Omega \cong \widehat{\Omega}/\mathfrak{m}\widehat{\Omega}$, our assumption on a implies that $u_a \otimes k$ is injective, where k is the residue field of R. By Corollary 5.5, the endomorphism u_a is pure and injective. In particular, $\widehat{\Omega}/a\widehat{\Omega}$ is flat over R and a is $\widehat{\Omega}$-regular. If Ω is separated, it is a submodule of its completion and it follows that a is Ω-regular. □

5.11. **Definition.** Let A be a ring, $\mathfrak{a}$ an ideal and Ω an A-module. An A-submodule Π of Ω is called $\mathfrak{a}$-*pure*, if $\Pi \cap \mathfrak{a}\Omega = \mathfrak{a}\Pi$.

Equivalently, Π is $\mathfrak{a}$-pure in Ω, if the canonical morphism $\Pi/\mathfrak{a}\Pi \to \Omega/\mathfrak{a}\Omega$ is injective. In particular, being $\mathfrak{a}$-pure is transitive. Assume Ω is moreover flat. Inspecting the long exact sequence for *Tor*, it follows that Π is $\mathfrak{a}$-pure if and only if $\operatorname{Tor}_1^A(\Omega/\Pi, A/\mathfrak{a}) = 0$. In particular, Π is $\mathfrak{a}$-pure for all ideals $\mathfrak{a}$ if and only if Ω/Π is flat, if and only if the inclusion $\Pi \subseteq \Omega$ is pure.

5.12. **Corollary.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let Φ be a flat R-module. Then any finitely generated $\mathfrak{m}$-pure submodule of Φ is free and pure. In particular, any element in $\Phi - \mathfrak{m}\Phi$ has zero annihilator.*

Proof. Let H be a finitely generated and $\mathfrak{m}$-pure submodule of Φ. Put $\Lambda = \Phi/H$. It follows that $\operatorname{Tor}_1^R(\Lambda, k) = 0$. Since H is universally separated, Λ is flat by Theorem 5.2. Therefore, H is pure in Φ and flat, whence free.

The last statement follows by letting H be the module generated by an element ω in $\Phi - \mathfrak{m}\Phi$. An element in $H \cap \mathfrak{m}\Phi$ is of the form $r\omega$ for some $r \in R$. If r is not in $\mathfrak{m}$, then it is a unit, so that $\omega \in \mathfrak{m}\Phi$, contradiction. This shows that H is $\mathfrak{m}$-pure in Φ whence free by the first assertion. As $H \cong R/\operatorname{Ann}_R(\omega)$, we get $\operatorname{Ann}_R(\omega) = 0$, as claimed. □

5.13. **Corollary.** *Let A be a Noetherian ring and $\mathfrak{a}$ a proper ideal of A. Let d be the Krull dimension of A (possibly infinite). Let Ω be an A-module of finite projective dimension. If Ω is universally $\mathfrak{a}$-separated and $\operatorname{Tor}_n^A(\Omega, A/\mathfrak{p}) = 0$ for all $1 \leq n \leq d+1$ and all $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$, then Ω is flat.*

Proof. By Theorem 5.6 and Remark 5.8, the completion $\widehat{\Omega}^{\mathfrak{a}}$ is flat and we have isomorphisms

$$\Omega/I\Omega \cong \widehat{\Omega}^{\mathfrak{a}}/I\widehat{\Omega}^{\mathfrak{a}}$$

for any ideal I containing a power of $\mathfrak{a}$. Since Ω is $\mathfrak{a}$-adically separated, it is a submodule of $\widehat{\Omega}^{\mathfrak{a}}$ and the previous statement then shows that this embedding is I-pure for all I containing some power of $\mathfrak{a}$. Let Π be the quotient module $\widehat{\Omega}^{\mathfrak{a}}/\Omega$. Since $\widehat{\Omega}^{\mathfrak{a}}$ is flat, $\mathfrak{p}$-purity implies that $\operatorname{Tor}_1^A(\Pi, A/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$. Since Ω is universally $\mathfrak{a}$-separated, Theorem 5.2 yields that Π is flat, and therefore, so is Ω. □

The proof even shows that Ω is a pure submodule of its $\mathfrak{a}$-adic completion. By Remark 5.8, we get the following improvement in the local case.

5.14. **Corollary.** *Let R be a d-dimensional Noetherian local ring with residue field k and let δ be equal to d when R is Cohen-Macaulay, and to $d-1$ otherwise. Let Ω be an R-module of finite projective dimension. If Ω is universally separated and $\operatorname{Tor}_n^R(\Omega, k) = 0$ for all $1 \leq n \leq \delta$, then Ω is flat.*

In the next section, we will weaken the separatedness condition for one-dimensional and two-dimensional local rings. As a final corollary, we obtain the following sharpening of Corollary 5.10 for universally separated modules.

5.15. **Corollary.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let A be an R-algebra. Let Ω be an A-module which is universally separated and flat as an R-module. If $a \in A$ is $\Omega/\mathfrak{m}\Omega$-regular, then a is Ω-regular and $\Omega/a\Omega$ is flat over R.*

Proof. It follows from Corollary 5.10 that a is Ω-regular and $\widehat{\Omega}/a\widehat{\Omega}$ is flat. From the proof of Corollary 5.13 it follows that $\Lambda =: \widehat{\Omega}/\Omega$ is flat. Since Ω is universally separated, one checks that the canonical morphism $\Omega/a\Omega \to \widehat{\Omega}/a\widehat{\Omega}$ is injective. By the snake lemma, the cokernel of this inclusion is equal to $\Lambda/a\Lambda$. It follows from the exact sequence

$$0 \to \Lambda \xrightarrow{a} \Lambda \to \Lambda/a\Lambda \to 0$$

that $\Lambda/a\Lambda$ has flat dimension at most one. Since $\widehat{\Omega}/a\widehat{\Omega}$ is flat, so is therefore $\Omega/a\Omega$. □

5.16. **An alternative proof of Proposition 4.3.** As in the proof of the proposition, we may assume that A is $\mathfrak{a}$-adically complete. Let Π be the direct product of copies of A and let Λ be the quotient $\Pi/\widehat{\Phi}^{\mathfrak{a}}$. One checks, using the Artin-Rees lemma, that $\widehat{\Phi}^{\mathfrak{a}}$ is I-pure in Π for any ideal I of A (see Definition 5.11). In particular, $\operatorname{Tor}_1^A(\Lambda, A/\mathfrak{p}) = 0$ for every $\mathfrak{p} \in \operatorname{V}(\mathfrak{a})$. By Lemma 4.2, taking into account that a completion is always separated, it follows that $\widehat{\Phi}^{\mathfrak{a}}$ is universally $\mathfrak{a}$-separated. Therefore, we can apply Theorem 5.2 to conclude that Λ is flat. But then so is $\widehat{\Phi}^{\mathfrak{a}}$. □

6. Vanishing of the first Betti number

In this section, we let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and Ω an arbitrary R-module. We are interested in consequences of the assumption that $\operatorname{Tor}_1^R(\Omega, k)$ vanishes. Recall that a module Ω is called *torsion-free* if any R-regular element is Ω-regular. In particular, if R has depth zero, then any module is vacuously torsion-free.

6.1. **Theorem.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with residue field k and let Ω be an R-module. If Ω is separated and $\operatorname{Tor}_1^R(\Omega, k) = 0$, then Ω is torsion-free.*

Proof. Let x be an R-regular element. Choose $x_1, \dots, x_{d-1} \in \mathfrak{m}$, such that $(x_1, \dots, x_d)$ is a maximal R-regular sequence, where we put $x_d := x$. Since R is Cohen-Macaulay, $\mathfrak{n} := (x_1, \dots, x_d)R$ is primary. Therefore $\operatorname{Tor}_1^R(\Omega, R/\mathfrak{n}) = 0$ by Corollary 2.5. Put $\bar{R} := R/(x_1, \dots, x_{d-1})R$ and $\bar{\Omega} := \Omega/(x_1, \dots, x_{d-1})\Omega$. By [14, Lemma 2.1], we have

$$\operatorname{Tor}_1^{\bar{R}}(\bar{\Omega}, R/\mathfrak{n}) = 0. \tag{14}$$

Since $R/\mathfrak{n} = \bar{R}/x\bar{R}$ and x is $\bar{R}$-regular, (14) implies that x is $\bar{\Omega}$-regular.

If $d = 1$, so that $\bar{R}$ is just R, then x is Ω-regular. Note that we have not used the separatedness condition yet and hence we proved Corollary 6.2 below. So we may assume $d > 1$. Let $x\mu = 0$ for some $\mu \in \Omega$. Since x is $\bar{\Omega}$-regular, the image of μ in $\bar{\Omega}$ is zero, that is to say, $\mu \in (x_1, \dots, x_{d-1})\Omega \subseteq \mathfrak{m}\Omega$. However, replacing $(x_1, \dots, x_{d-1})$ in the above argument by $(x_1^n, x_2^n, \dots, x_{d-1}^n)$ for arbitrary $n \geq 1$, we may conclude by the same argument that $\mu \in \mathfrak{m}^n\Omega$ for all n. Since Ω is separated, $\mu = 0$, as we wanted to show. □

In the course of the previous proof, we derived the following corollary (note that the depth zero case holds trivially).

6.2. **Corollary.** *Let $(R, \mathfrak{m})$ be a one-dimensional Noetherian local ring with residue field k and let Ω be an R-module. If $\operatorname{Tor}_1^R(\Omega, k) = 0$, then Ω is torsion-free.* □

We now turn to some flatness criteria in low dimension. The first result generalizes the well-known fact that flatness is the same as being torsion-free over a discrete valuation ring.

6.3. **Theorem.** *Let $(R, \mathfrak{m})$ be a one-dimensional Noetherian local ring with residue field k and let Ω be an R-module. If $\operatorname{Tor}_1^R(\Omega, k) = 0$ and Ω has finite projective dimension (which already follows from the first hypothesis when R is reduced), then Ω is flat.*

In particular, if R is a one-dimensional local Cohen-Macaulay ring and Ω has finite projective dimension, then Ω is flat if and only if it is torsion-free.

Proof. If R is not Cohen-Macaulay, then Ω has projective dimension at most one by [8, p.44] and the assertion follows from Theorem 5.4 (alternatively, one can use that the flat dimension of Ω, when finite, is zero by [3, Corollary 5.3]). If R is Cohen-Macaulay, then Ω is a so-called *balanced big Cohen-Macaulay module* by Corollary 6.2. Flatness then follows from [13, Theorem IV.1]. Note that if R is reduced, then R has an isolated singularity. By [13, Corollary III.4], the vanishing of $\operatorname{Tor}_1^R(\Omega, k)$ then implies that Ω has finite projective dimension. This proves the first assertion.

One direction in the second assertion is immediate. Hence, assume that Ω is a torsion-free module of finite projective dimension over a one-dimensional local Cohen-Macaulay ring R. Let x be an R-regular element. Since x is Ω-regular, $\operatorname{Tor}_1^R(\Omega, R/xR) = 0$. Choose an exact sequence

$$0 \to k \to R/xR \to C \to 0$$

and tensor with Ω to yield an exact sequence

$$\operatorname{Tor}_2^R(\Omega, C) \to \operatorname{Tor}_1^R(\Omega, k) \to \operatorname{Tor}_1^R(\Omega, R/xR).$$

We already established that the last module in this exact sequence is zero. The first module is also zero because Ω has finite projective dimension, so that its flat dimension is at most one by [1, Theorem 2.4] (see Remark 5.8). Hence $\operatorname{Tor}_1^R(\Omega, k) = 0$, so that Ω is flat by our first assertion. □

6.4. **Corollary.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and let Ω be an arbitrary R-module. If $\operatorname{Tor}_1^R(\Omega, k) = 0$, then Ω is flat along every curve in $\operatorname{Spec} R$.*

Proof. By the latter expression we mean that $\Omega/\mathfrak{a}\Omega$ is flat as an $R/\mathfrak{a}$-module for every radical ideal $\mathfrak{a}$ of R with $\dim(R/\mathfrak{a}) = 1$. Indeed, for $\mathfrak{a}$ of this form, $\operatorname{Tor}_1^R(\Omega, k) = 0$ implies $\operatorname{Tor}_1^{R/\mathfrak{a}}(\Omega/\mathfrak{a}\Omega, k) = 0$ by [14, Lemma 2.1], and we are done by Theorem 6.3. □

In dimension two, we require an additional separatedness condition.

6.5. **Theorem.** *Let $(R, \mathfrak{m})$ be a two-dimensional Noetherian local ring with residue field k and let Ω be an R-module. If Ω is separated, has finite projective dimension and $\operatorname{Tor}_1^R(\Omega, k) = 0$, then Ω is flat.*

Proof. Our proof will be non-uniform, as we need to treat the Cohen-Macaulay differently from the non-Cohen-Macaulay case. Assume first that R is Cohen-Macaulay and let (x, y) be an R-regular sequence. Since Ω is separated, Theorem 6.1 implies that x is Ω-regular. Moreover, since $\operatorname{Tor}_1^{R/xR}(\Omega/x\Omega, k) = 0$ by [14, Lemma 2.1] and since y is R/xR-regular, Corollary 6.2 applied to the one-dimensional local Cohen-Macaulay ring R/xR then yields that y is $\Omega/x\Omega$-regular. Hence (x, y) is Ω-regular and we showed that Ω is a balanced big Cohen-Macaulay module. Flatness now follows from [13, Theorem IV.1].

So assume R is not Cohen-Macaulay. By Theorem 5.6 together with Remark 5.8, the completion $\widehat{\Omega}$ is flat. Since Ω is separated, it is a submodule of its completion and we have an exact sequence

$$0 \to \Omega \to \widehat{\Omega} \to \Lambda \to 0. \tag{15}$$

Since Ω is $\mathfrak{m}$-pure in $\widehat{\Omega}$ and since $\widehat{\Omega}$ is flat, we get $\operatorname{Tor}_1^R(\Lambda, k) = 0$. From the *Tor* long exact sequence, we also get $\operatorname{Tor}_2^R(\Lambda, k) = 0$. Moreover, $\widehat{\Omega}$ being flat, has finite projective dimension. Therefore, so has Λ. From [8, p. 44] it follows that Λ has projective dimension at most two, since R is not Cohen-Macaulay. Therefore, by Corollary 5.3, its flat dimension is at most one. From the exact sequence (15) and the fact that $\widehat{\Omega}$ is flat, it then follows that Ω must be flat. □

In higher dimensions, an even stronger separatedness condition is needed. The following is a strengthening of Corollary 5.13 under the additional Cohen-Macaulay assumption; we will drop this assumption in Corollary 7.8 below.

6.6. **Theorem.** *Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring with residue field k and let Ω be an R-module. If Ω is universally separated, has finite projective dimension and $\operatorname{Tor}_1^R(\Omega, k) = 0$, then Ω is flat.*

Proof. We will use once more [13, Theorem IV.1] to prove flatness. Hence, it suffices to show that if $(x_1, \dots, x_n)$ is R-regular, then it is Ω-regular. We induct on n. Theorem 6.1 yields that x_1 is Ω-regular. By [14, Lemma 2.1], we have $\operatorname{Tor}_1^{R/x_1R}(\Omega/x_1\Omega, k) = 0$. Since $\Omega/x_1\Omega$ is again universally separated, induction yields that $(x_2, \dots, x_n)$ is $\Omega/x_1\Omega$-regular, as we wanted to show. □

Flatificators. We say that an arbitrary R-module Ω *admits a flatificator*, if there is a (unique) smallest ideal $\mathfrak{f}$ such that $\Omega/\mathfrak{f}\Omega$ is flat over $R/\mathfrak{f}$; the ideal $\mathfrak{f}$ is then called the *flatificator* of Ω. In an unpublished paper, I showed that if $(R, \mathfrak{m})$ is a complete Noetherian local ring, A a Noetherian R-algebra such that $\mathfrak{m}A$ lies in the Jacobson radical of A, and Ω a finitely generated A-module, then Ω, viewed as an R-module, has a flatificator. Therefore, the next result can be seen as a partial generalization of the Local Flatness Criterion.

6.7. **Corollary.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and let Ω be an R-module. Let $\mathfrak{n}$ denote the nilradical of R. If Ω admits a flatificator and $\operatorname{Tor}_1^R(\Omega, k) = 0$, then $\Omega/\mathfrak{n}\Omega$ is flat over $R/\mathfrak{n}$. In particular, if R is moreover reduced, then Ω is flat.*

Proof. Let $\mathfrak{f}$ be the flatificator of Ω. Let $\mathfrak{p}$ be an arbitrary prime ideal of R such that $R/\mathfrak{p}$ is one-dimensional, a *one-dimensional prime* for short. Corollary 6.4 yields that $\Omega/\mathfrak{p}\Omega$ is flat as an $R/\mathfrak{p}$-module. Therefore, by definition of flatificator, $\mathfrak{f} \subseteq \mathfrak{p}$. I claim that the intersection of all one-dimensional prime ideals is equal to the nilradical $\mathfrak{n}$. Assuming the claim, it follows that $\mathfrak{f} \subseteq \mathfrak{n}$, showing that $\Omega/\mathfrak{n}\Omega$ is flat over $R/\mathfrak{n}$.

To prove the claim, we need to show that the intersection of all one-dimensional prime ideals lies in any minimal prime ideal $\mathfrak{g}$ of R. Therefore, we may replace R by $R/\mathfrak{g}$ and assume, without loss of generality, that R is a domain. Suppose x is a non-zero element lying in every one-dimensional prime ideal of R. By prime avoidance (see for instance [10, Theorem 14.1]), we can choose a sequence $x_1 = x, x_2, \dots, x_d$ in R such that $(x_{i+1}, \dots, x_d)R$ has height $d - i$ for all i, where $d = \dim R$. Let $\mathfrak{p}$ be a minimal prime ideal of $(x_2, \dots, x_d)R$. Since $\mathfrak{p}$ has height $d - 1$, it is one-dimensional. By assumption, $x \in \mathfrak{p}$, contradicting that $(x_1, \dots, x_d)R$ has height d. □

7. Further remarks

We start with a couple of examples showing that some of the results obtained in this paper are sharp.

7.1. **Example** (Theorem 6.5 is false in higher dimensions)**.** Let R be a regular local ring and let E be the injective hull of the residue field k. It is well-known that $\operatorname{Tor}_n^R(E, k) = 0$ for all $n \neq d$ and that $\operatorname{Tor}_d^R(E, k) \cong k$, where d is the dimension of R. Clearly, E is not separated, so that the various separatedness conditions made in this paper cannot be omitted. Moreover, if $d \geq 3$ and we take an exact sequence

$$0 \to \Lambda \to \Phi \to E \to 0$$

with Φ free, then Λ is separated, has finite projective dimension (since R is regular) and $\operatorname{Tor}_1^R(\Lambda, k) = 0$, but Λ is not flat (since $\operatorname{Tor}_{d-1}^R(\Lambda, k) \neq 0$).

7.2. **Example** (Lazard)**.** In [9], Lazard uses the following example to disprove that every module of flat dimension one is the inductive limit of modules of projective dimension one. Let k be a field, put $A := k[x, y, z]$ and let R be the localization of $A/(x^2, xy, xz)A$ at the maximal ideal $\mathfrak{m} := (x, y, z)A$. One checks that R has dimension two and depth zero, so that R is not Cohen-Macaulay. Moreover, $\operatorname{Sing} R = \{\mathfrak{m}\}$, so that R has an isolated singularity. Let $\mathfrak{g} := xR$ and $\mathfrak{p} := (x, y)R$. Consider the exact sequence

$$0 \to R_{\mathfrak{p}} \to R_{\mathfrak{g}} \to \Pi \to 0$$

Since $R_{\mathfrak{g}}$ is flat and $R_{\mathfrak{p}} \otimes k = 0$, tensoring with k yields $\operatorname{Tor}_1^R(\Pi, k) = 0$. Tensoring the above sequence on the other hand with R/yR yields $\operatorname{Tor}_1^R(\Pi, R/yR) \cong R_{\mathfrak{p}}/yR_{\mathfrak{p}}$ and the latter is isomorphic to $k(\mathfrak{p})$, showing that Π is not flat. Note that $R_{\mathfrak{p}} \cong k(z)[y]_{yk(z)[y]}$ is a discrete valuation ring, although $\mathfrak{p}$ has depth zero. However, since $R_{\mathfrak{p}}$ and $R_{\mathfrak{g}}$ are flat, Π has flat dimension one. So, all $\operatorname{Tor}_n^R(\Pi, k) = 0$ for $n \geq 1$, and Π has finite projective dimension. Therefore, one cannot hope that the vanishing of $\operatorname{Tor}_1^R(\Omega, k)$ implies flatness, that is to say, Theorem 5.4 cannot be improved, even not if we require that all Betti numbers vanish. Neither can we drop the separatedness condition in Corollary 5.13.

7.3. **Example** (Bartijn-Strooker)**.** The following example of a separated non-balanced big Cohen-Macaulay module of finite projective dimension is taken from [2, Exemple 3.11 a)]. Let $R := k[[x, y, z]]$ with k an arbitrary field and let Φ be a free R-module of infinite rank. Let $\Omega := \Phi + (x, y)\widehat{\Phi}$. Clearly Ω is separated, as $\widehat{\Phi}$ is. I claim that Ω is a non-flat (non-balanced) big Cohen-Macaulay R-module for which all Betti numbers vanish. In order to prove this, it suffices to show that (x, y) is not Ω-regular but (z, x, y) is. Indeed, it then follows that Ω is is a non-flat big Cohen-Macaulay module. Moreover, since $\operatorname{Tor}_n^R(\Omega, k) \cong \operatorname{Tor}_n^{R/(z,x,y)R}(\Omega/(z, x, y)\Omega, k) = 0$, all Betti numbers vanish.

We start with showing that y is not $(\Omega/x\Omega)$-regular. To this end, take any $\theta \in \widehat{\Phi} - \Omega$. It follows that $x\theta$ and $y\theta$ both lie in Ω. Let $\omega := x\theta$, then $\omega \notin x\Omega$ and $y\omega = x(y\theta) \in x\Omega$, showing that y is a zero divisor modulo $x\Omega$. Next, I claim that $\Omega/z\Omega$ is isomorphic to the completion of $\Phi/z\Phi$. Since by Proposition 4.3, the latter is a flat R/zR-module, (z, x, y) is Ω-regular. To prove the isomorphism, we need to show that $\Omega \cap z\widehat{\Phi} = z\Omega$ and $\widehat{\Phi} = \Omega + z\widehat{\Phi}$. The last equality is clear since Φ is dense in $\widehat{\Phi}$ and the former is easily verified using Lemma 4.1.

Note that the completion of Ω is $\widehat{\Phi}$, which is clearly flat. The reader can verify that Ω has flat dimension one. Let Λ be a first syzygy of Ω, that is to say, Λ is the kernel of an epimorphism from a free module onto Ω. It follows that Λ is flat and separated. However, it cannot be universally separated, for otherwise Theorem 5.2 would imply that Ω is flat. In particular, Proposition 4.6 does not hold with separated instead of complete. Also note that Ω is an $\mathfrak{m}$-pure submodule of the (complete) flat and separated R-module $\widehat{\Phi}$. This shows that if a module is separated, has finite projective dimension and is $\mathfrak{m}$-pure in a complete flat module, then it is not necessarily flat.

Residual homological dimension. Let us call the *residual homological dimension* of a module Ω over a Noetherian local ring R the largest $n \geq 0$ for which $\operatorname{Tor}_n^R(\Omega, k) \neq 0$, where k is the residue field of R (if no largest n exist, we set the residual homological dimension equal to ∞ and if all $\operatorname{Tor}_n^R(\Omega, k) = 0$, we set it equal to -1). The following identity is proven in [2].

7.4. **Theorem** ([2, Théorème 4.1])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let Ω be an R-module. If Ω has finite projective dimension and is non-degenerated, then the residual* *homological dimension of Ω is equal to the difference between the depth of R and the depth of Ω.*

According to the modern practice, the *depth* of an R-module Ω is the least n for which $\operatorname{Ext}^n_R(k,\Omega) \neq 0$ (in [2] this was called *E-profondeur* and depth referred there to the length of a maximal Ω-regular sequence). The depth may be infinite, but it is finite for non-degenerated modules. In particular, separated modules have finite depth.

7.5. *Remark.* As an immediate corollary of this theorem, we get $\operatorname{Tor}^R_n(\Omega,k) = 0$ for any non-degenerated R-module Ω of finite projective dimension and any n strictly bigger than the depth of R, since the depth of Ω cannot exceed the depth of R. Hence we showed the following improvement of Remark 5.8 (note that the degenerated case holds trivially):

7.6. **Theorem.** *Let $(R,\mathfrak{m})$ be a Noetherian local ring with residue field k and let q be the depth of R. Let Ω be an arbitrary R-module. If Ω has finite projective dimension and all $\operatorname{Tor}^R_n(\Omega,k)$ vanish for $n = 1,\dots,q$, then $\widehat{\Omega}$ is flat.* □

This yields immediately the following flatness criterion for Noetherian local rings of depth zero.

7.7. **Corollary.** *Let $(R,\mathfrak{m})$ be a depth zero Noetherian local ring and let Ω be an arbitrary R-module. If Ω has finite projective dimension, then its completion is flat. In particular, for a complete R-module, flatness is the same as having finite projective dimension.* □

Another corollary is the following flatness criterion, which is a generalization of [13, Theorem IV.1] for complete modules.

7.8. **Corollary.** *Let $(R,\mathfrak{m})$ be a Noetherian local ring with residue field k and let Ω be an R-module of finite projective dimension. If there exists some maximal R-regular sequence which is also Ω-regular, or more generally, if $\operatorname{Ext}^n_R(k,\Omega) = 0$ for all $n = 0,\dots,q-1$, where q is the depth of R, then $\widehat{\Omega}$ is flat. If Ω is moreover universally separated, then Ω itself is flat.*

Proof. Since the degenerated case holds trivially, we may assume Ω is non-degenerated, and therefore has finite depth. If Ω admits a regular sequence of length q, then $\operatorname{Ext}^n_R(k,\Omega)$ vanishes, for each $n = 0,\dots,q-1$ (see for instance [7]). Therefore, the first assertion is a special case of the second, so that we may assume that all the *Ext* modules vanish in the specified range. By definition of depth, Ω has depth at least q. It follows that the residual homological dimension of Ω is zero by Theorem 7.4. Therefore, Theorem 5.6 gives the flatness of $\widehat{\Omega}$ and Corollary 5.13 gives the flatness of Ω under the additional assumption that Ω is universally separated. □

Finitude of projective dimension. Of course, if R is regular, then the condition on a module to have finite projective dimension is automatically satisfied. In the paper [13], various homological criteria are given to ensure the finitude of the projective dimension of an R-module Ω when R is not regular. The most general criterion states that if $\operatorname{Tor}^R_n(\Omega,R/\mathfrak{p}) = 0$ for all $\mathfrak{p}$ in the singular locus of R, then Ω has finite projective dimension, where in the Cohen-Macaulay case, we only need to check for a single $n \geq d =: \dim(R)$ and in the non-Cohen-Macaulay case for all n in an interval in $\mathbb{Z}_{>0}$ of length $d+1$.

This criterion takes a particularly easy form if R has an *isolated singularity*, that is to say, if all proper localizations of R are regular. Indeed, in that case, it suffices that all $\operatorname{Tor}^R_n(\Omega,k)$ vanish in the specified region. For instance, Theorem 6.5 for an isolated two-dimensional singularity takes the following form: *if Ω is separated and* $\operatorname{Tor}^R_1(\Omega,k) =$

$\mathrm{Tor}_2^R(\Omega, k) = 0$, *and in the non-Cohen-Macaulay case, also* $\mathrm{Tor}_3^R(\Omega, k) = 0$, *then* Ω *is flat.* Similarly, Theorem 6.6 can be reformulated as follows: *if* R *is a* d*-dimensional local Cohen-Macaulay ring with at most an isolated singularity and* Ω *is an universally separated* R*-module for which* $\mathrm{Tor}_1^R(\Omega, k) = \mathrm{Tor}_d^R(\Omega, k) = 0$*, then* Ω *is flat.*

The Main Theorem for isolated singularities becomes:

7.9. **Theorem.** *Let* $(R, \mathfrak{m})$ *be a* d*-dimensional Noetherian local ring with an isolated singularity. Let* k *be its residue field and let* Ω *be an* R*-module. If* $\mathrm{Tor}_n^R(\Omega, k) = 0$ *for all* $n = 1, \dots, d+1$*, then* $\widehat{\Omega}$ *is flat, and if* Ω *is moreover universally separated, then* Ω *itself is flat. In fact, if* R *is Cohen-Macaulay, we only need to check vanishing for* $n = 1, \dots, d$*.*

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