

LOG-TERMINAL SINGULARITIES AND VANISHING THEOREMS VIA NON-STANDARD TIGHT CLOSURE

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Abstract

Generalizing work of Smith and Hara, we give a new characterization of log-terminal singularities for finitely generated algebras over \mathbb{C} , in terms of purity properties of ultraproducts of characteristic p Frobenii.

As a first application we obtain a Boutot-type theorem for log-terminal singularities: given a pure morphism $Y \rightarrow X$ between affine \mathbb{Q} -Gorenstein varieties of finite type over \mathbb{C} , if Y has at most a log-terminal singularities, then so does X . The second application is the Vanishing for Maps of Tor for log-terminal singularities: if $A \subseteq R$ is a Noether Normalization of a finitely generated \mathbb{C} -algebra R and S is an R -algebra of finite type with log-terminal singularities, then the natural morphism $\mathrm{Tor}_i^A(M, R) \rightarrow \mathrm{Tor}_i^A(M, S)$ is zero, for every A -module M and every $i \geq 1$. The final application is Kawamata-Viehweg Vanishing for a connected projective variety X of finite type over \mathbb{C} whose affine cone has a log-terminal vertex (for some choice of polarization). As a corollary, we obtain a proof of the following conjecture of Smith: if G is the complexification of a real Lie group acting algebraically on a projective smooth Fano variety X , then for any numerically effective line bundle \mathcal{L} on any GIT quotient $Y := X//G$, each cohomology module $H^i(Y, \mathcal{L})$ vanishes for $i > 0$, and, if \mathcal{L} is moreover big, then $H^i(Y, \mathcal{L}^{-1})$ vanishes for $i < \dim Y$.

1. Introduction

The work of Smith, Hara et al., has led to a characterization of log-terminal singularities (equivalence (1) \Leftrightarrow (1') below) in terms of purity properties of the Frobenius on a general reduction modulo p . Although this characterization has proven to be very useful, one of its main drawbacks is the fact that it is not known to be inherited by quotients of group actions. Our first result is a similar characterization without this defect.

Theorem 1. *Let R be a local \mathbb{Q} -Gorenstein domain essentially of finite type over a field of characteristic zero. Then the following are equivalent:*

- (1) *R has log-terminal singularities.*

Received 14.11.2003. Partially supported by a grant from the National Science Foundation and by visiting positions at Paris VII and at the Ecole Normale Supérieure.

- (1') R is F -regular type.
- (2) R is ultra- F -regular.

The implication (1') \Rightarrow (1) is proven in [39, Corollary 4.16] or [9], using Smith's work on rational singularities in [38]; the converse implication is proven by Hara in [8, Theorem 5.2]. We will give a proof in §3.8 for the implications (1') \Rightarrow (2) \Rightarrow (1). The notion of 'ultra- F -regularity' should be viewed as a non-standard version of the notion of 'strong F -regularity'. More precisely, let R be a local domain essentially of finite type over \mathbb{C} (see Remark 1.2 below for arbitrary fields). In [32], we associated to R a canonically defined extension R_∞ , called the *non-standard hull* of R , which is realized as the ultraproduct of certain local rings in characteristic p , called *approximations* of R (see §2.1 below for exact definitions). One should view an approximation of R as a more canonical way of reducing R modulo p (see §2.19), and a non-standard hull of R , as a convenient way of storing all these reductions into a single algebraic object (of characteristic zero). With an *ultra-Frobenius* on R , we mean the ring homomorphism into the non-standard hull R_∞ given by the rule $x \mapsto x^\pi$, where π is a non-standard integer obtained as the ultraproduct of various powers of prime numbers (see §3.2 for precise definitions). We call R *ultra- F -regular*, if for each non-zero c in R , we can find an ultra-Frobenius $x \mapsto x^\pi$ such that the morphism $R \rightarrow R_\infty: x \mapsto cx^\pi$ is pure. One should compare this with the Hochster-Huneke notion of *strong F -regularity* of a domain R of prime characteristic p : for each non-zero c in R , there is a power q of p , such that the morphism $R \rightarrow R: x \mapsto cx^q$ is split (which under these conditions is equivalent with it being pure). If R is moreover \mathbb{Q} -Gorenstein then strong F -regularity is equivalent by [24] with *weakly F -regularity*, that is to say, with the property that every ideal is tightly closed.

Application 1: Quotients of Log-terminal Singularities. The first application (see §3.12 for the proof) is the result that log-terminal singularities are preserved under quotients of reductive groups, provided the quotient is \mathbb{Q} -Gorenstein. Although this seems to be a result that ought to have a proof using Kodaira Vanishing (as is the case for the corresponding statement for rational singularities by [2]), I do not know of any argument other than the one provided here.

Theorem 2. *Let $R \rightarrow S$ be a local homomorphism of \mathbb{Q} -Gorenstein local domains essentially of finite type over a field of characteristic zero. If $R \rightarrow S$ is cyclically pure (that is to say, if $\mathfrak{a} = \mathfrak{a}S \cap R$ for all ideals $\mathfrak{a} \subseteq R$) and if S has log-terminal singularities, then so has R .*

In particular, let G be a reductive group acting algebraically on an affine \mathbb{Q} -Gorenstein variety X . If X has at most log-terminal singularities, then so has the quotient space X/G , provided it is \mathbb{Q} -Gorenstein.

The present proof is entirely elementary in case X is assumed to be smooth in the last assertion (see Remark 1.1). If G is moreover finite, then the theorem was already proven in [19] using canonical covers and Boutot's result [2]. In this case,

the condition that the quotient is \mathbb{Q} -Gorenstein is automatically satisfied, but this is not so in the general case. For some similar descent properties using non-standard methods, see also [34].

Application 2: Vanishing of Maps of Tor. The next result (see Theorem 4.2 for the proof) was previously only known for S regular, or more generally, for S weakly CM^n -regular ([13, Theorems 4.1 and 4.12]).

Theorem 3. *Let $R \rightarrow S$ be a homomorphism of \mathbb{C} -affine algebras such that S is a domain with at most log-terminal singularities (or, more generally, a pure subring of such a ring). Let A be a regular subring of R over which R is module finite. Then for every A -module M and every $i \geq 1$, the natural morphism $\text{Tor}_i^A(M, R) \rightarrow \text{Tor}_i^A(M, S)$ is zero.*

Application 3: Vanishing Theorems. Purity of Frobenius was used effectively in [16] to prove the Cohen-Macaulayness of rings of invariants. Exploiting this further, Mehta and Ramanathan deduced Vanishing Theorems for Schubert varieties from purity properties of Frobenius in [25]. The approach in this paper is a non-standard analogue of these ideas, especially those from [40]. Let X be a connected, normal projective variety of characteristic zero. Recall that $\text{Spec } S$ is called an *affine cone* of X , if S is some finitely generated graded algebra such that $X = \text{Proj } S$ (for each choice of ample invertible sheaf on X , one obtains such a graded ring S ; see §5 below). The *vertex* of the affine cone is by definition the closed point on $\text{Spec } S$ determined by the irrelevant maximal ideal of S (generated by all homogeneous elements of positive degree). We call X *globally ultra-F-regular*, if some affine cone of X has an ultra-F-regular vertex. In particular, in view of Theorem 1, if the vertex of an affine cone is a log-terminal singularity, then X is globally ultra-F-regular. Since the anti-canonical cone of a smooth Fano variety (or more generally, of a Fano variety with rational singularities) has this property (see Theorem 7.1 below), every smooth Fano variety is globally ultra-F-regular. By Theorem 2, the same also holds for any GIT (Geometric Invariant Theory) quotient of a smooth Fano variety. In Corollaries 6.6 and 6.7 we will show the following vanishing of cohomology for globally ultra-F-regular varieties.

Theorem 4. *Let X be a globally ultra-F-regular projective variety and let \mathcal{L} be a numerically effective line bundle on X (this includes the case $\mathcal{L} := \mathcal{O}_X$). For each $i > 0$, the cohomology module $H^i(X, \mathcal{L})$ vanishes. Moreover, if \mathcal{L} is also big, then $H^i(X, \mathcal{L}^{-1})$ vanishes for each $i < \dim X$.*

Together with our previous remarks, we get the following result, which was originally conjectured by Smith in [40].

Theorem 5. *Let G be a reductive group acting algebraically on a projective Fano variety X with rational singularities and let $Y := X//G$ be a GIT quotient of X (with respect to some linearization of the action of G). If \mathcal{L} is a numerically effective*

line bundle on Y , then each cohomology module $H^i(Y, \mathcal{L})$ vanishes for $i > 0$, and, if \mathcal{L} is moreover big, then $H^i(Y, \mathcal{L}^{-1})$ vanishes for $i < \dim Y$.

Theorem 1 also begs the question what (weakly) F-regular type and ultra-F-regularity would amount to if we drop the \mathbb{Q} -Gorenstein condition. Without this assumption, one should actually use the notion of strongly F-regular type (which is only conjecturally equivalent with weakly F-regular type), but even then it is no longer clear that this is equivalent with ultra-F-regularity (one direction holds by Theorem 3.5 below). In [9, §4.6], the authors propose Nakayama's notion of *admissible singularities* ([27]) as a candidate for an equivalent condition to strongly F-regular type. They point out that an affine cone of a smooth Fano variety has in general only admissible singularities (although its anti-canonical cone has log-terminal singularities). The fact that any such cone is ultra-F-regular (see Remark 6.3) corroborates hence their claim.

1.1. Remark on Kodaira Vanishing. Hara's proof of implication (1) \Rightarrow (1') in Theorem 1 relies heavily on Kodaira Vanishing (in fact, on Akizuki-Kodaira-Nakano Vanishing). Therefore, it is of interest to see which of the results in this paper do not make use of Kodaira Vanishing. If we let S be regular in Theorem 2, then we do not need the implication (1) \Rightarrow (1') and hence no Vanishing Theorem is used (see Remark 3.13 below). Similarly, our proof of Theorem 4 uses only elementary results from cohomology theory and hence does not rely on Kodaira Vanishing. Nonetheless, in order to prove that smooth Fano varieties are globally ultra-F-regular, and hence to obtain Theorem 5, we do need Hara's result and hence Kodaira Vanishing.

1.2. Remark on the base field. To make the exposition more transparent, I have only dealt in the text with the case that the base field is \mathbb{C} . However, the results extend to arbitrary base fields of characteristic zero by the following two observations. First, any algebraically closed field of characteristic zero and of cardinality 2^λ (for some infinite cardinal λ) is the ultraproduct of (algebraically closed) fields of positive characteristic by the Lefschetz Principle (see for instance [32, Remark 2.5]) and this is the only property we used of \mathbb{C} (cf. (1) below). Second, since all properties admit faithfully flat descent, we can always make a base change to an algebraically closed field of sufficiently large cardinality whence in particular to an algebraically closed field realized as the ultraproduct of fields of positive characteristic.

Acknowledgement. I am grateful to Karen Smith for drawing my attention to the question of Vanishing Theorems for GIT quotients of Fano varieties. Without her continuous help and encouragement, this paper would not have been possible.

2. Transfer and Approximations

In this section, I will briefly discuss an alternative construction of the usual 'reduction modulo p ' construction from algebraic geometry. The advantage is that we can work just with schemes of finite type over fields, that is to say, there is no need to work with relative versions. The one drawback is that we need the base field to be algebraically closed of cardinality 2^λ for some infinite cardinal λ (under the assumption of the Generalized Continuum Hypothesis, each uncountable cardinal is of the form 2^λ). However, as explained in Remark 1.2, this is not too much of a constraint. Moreover, in order to simplify the exposition, I will in the sequel only discuss the case that the base field is \mathbb{C} .

For generalities on ultraproducts, including Łos' Theorem, see [32, §2]. Recall that an ultraproduct of rings C_p is a certain homomorphic image of the direct product of the C_p . This ultraproduct will be denoted by $\text{ulim}_{p \rightarrow \infty} C_p$, or simply by C_∞ , and similarly, the image of a sequence $(a_p \mid p)$ in C_∞ will be denoted by $\text{ulim}_{p \rightarrow \infty} a_p$, or simply by a_∞ . Any choice of sequence of elements a_p whose ultraproduct is equal to a_∞ will be called an *approximation* of a_∞ (note that we are using the term more loosely than in [32], where we reserved the notion of approximation only for *standard* a_∞). The key ingredient for transfer between zero and positive characteristic is the following fundamental isomorphism

$$(1) \quad \mathbb{C} \cong \text{ulim}_{p \rightarrow \infty} \mathbb{F}_p^{\text{alg}},$$

where $\mathbb{F}_p^{\text{alg}}$ denotes the algebraic closure of the p -element field. I will refer to (1) as the *Lefschetz Principle* for algebraically closed fields; see [32, Theorem 2.4] or [33, Fact 4.2] for proofs.

2.1. Affine Algebras. Let me briefly recall from [32] the construction of an approximation of a finitely generated (for short, an *affine*) \mathbb{C} -algebra C . For a fixed tuple of variables ξ , let A_∞ be the ultraproduct of the $A_p := \mathbb{F}_p^{\text{alg}}[\xi]$. We call A_∞ the *non-standard hull* of $A := \mathbb{C}[\xi]$ and A_p an *approximation* of A . By [41], the canonical homomorphism $A \rightarrow A_\infty$ is faithfully flat (see also [28, Theorem 1.7] or [35, A.2]). For an arbitrary affine \mathbb{C} -algebra C , say of the form A/I , we let

$$(2) \quad C_\infty := A_\infty / I A_\infty = C \otimes_A A_\infty$$

and call it the *non-standard hull* of C . One shows that C_∞ is the ultraproduct of affine $\mathbb{F}_p^{\text{alg}}$ -algebras C_p . Any such choice of C_p is called an *approximation* of C . There are two ways to construct these: either one observes that $I A_\infty$ is the ultraproduct of ideals I_p in A_p (we call I_p an *approximation* of I ; see [32, §3]) and takes C_p to be A_p / I_p , or alternatively, one takes a model of C , looks at its reductions modulo p and takes a suitably chosen base change over $\mathbb{F}_p^{\text{alg}}$ (see Proposition 2.18 and §2.19 below). By [32, 3.4], the non-standard hull C_∞ is independent (up to an

isomorphism of \mathbb{C} -algebras) of the presentation of C as a homomorphic image of a polynomial ring. Consequently, if C'_p are obtained from C by the above process starting from a different presentation of C , then $C_p \cong C'_p$ (as $\mathbb{F}_p^{\text{alg}}$ -algebras), for almost all p . The non-standard hull does depend though on the choice of ultrafilter and on the choice of the isomorphism (1).

We can similarly define the non-standard hull for a local ring R essentially of finite type over \mathbb{C} (a *local affine* \mathbb{C} -algebra, for short). Suppose R is of the form $C_{\mathfrak{p}}$ with C an affine \mathbb{C} -algebra and \mathfrak{p} a prime ideal of C . It follows from [32, Corollary 4.2] that $\mathfrak{p}C_{\infty}$ is prime. We define the *non-standard hull* of R to be the localization

$$R_{\infty} := (C_{\infty})_{\mathfrak{p}R_{\infty}}.$$

This is again independent from the choice of presentation $R := C_{\mathfrak{p}}$. As explained above, $\mathfrak{p}C_{\infty}$ is an ultraproduct of ideals \mathfrak{p}_p in an approximation C_p of C . By Łos' Theorem, almost all \mathfrak{p}_p are prime. We call the localization $R_p := (C_p)_{\mathfrak{p}_p}$ (for those p for which it makes sense), an *approximation* of R . It follows that the ultraproduct of the R_p is R_{∞} .

In the forthcoming [1], we will drop the finite type condition and show the existence of approximations and non-standard hulls for arbitrary Noetherian local rings containing \mathbb{Q} .

2.2. Homomorphisms. Let $\varphi: C \rightarrow D$ be a (local) homomorphism of finite type between (local) affine \mathbb{C} -algebras. This corresponds to a presentation of D as $C[\xi]/I$ (or a localization of the latter), for some finite tuple of variables ξ . Let C_p and D_p be approximations of C and D respectively, where we use the presentation $D := C[\xi]/I$ to construct the D_p . This shows that almost every D_p is a C_p -algebra. The corresponding ring homomorphism $\varphi_p: C_p \rightarrow D_p$ is called an *approximation* of φ . The ultraproduct of the φ_p is a homomorphism $\varphi_{\infty}: C_{\infty} \rightarrow D_{\infty}$, called the *non-standard hull* of φ , where C_{∞} and D_{∞} are the non-standard hulls of C and D respectively. We have a commutative diagram

$$(3) \quad \begin{array}{ccc} C & \xrightarrow{\varphi} & D \\ \downarrow & & \downarrow \\ C_{\infty} & \xrightarrow{\varphi_{\infty}} & D_{\infty} \end{array}$$

Note that if we choose a polynomial ring A of which both C and D are homomorphic images, then $C_{\infty} \cong C \otimes_A A_{\infty}$ and $D_{\infty} \cong D \otimes_A A_{\infty}$ by (2), and φ_{∞} is just the base change of φ over A_{∞} .

2.3. Affine Schemes. If X is an affine scheme of finite type over \mathbb{C} , say of the form $\text{Spec } C$ with C an affine \mathbb{C} -algebra, then we call $X_p := \text{Spec } C_p$ an *approximation* of X , for any choice of approximation C_p of C . One has to be careful however: it is not true that the ultraproduct $\text{ulim}_{p \rightarrow \infty} X_p$ of the X_p is equal to $\text{Spec } C_\infty$. In fact, $\text{ulim}_{p \rightarrow \infty} X_p$ is the subset of $\text{Spec } C_\infty$ consisting of all prime ideals of the form $(I_\infty : a_\infty)$ for I_∞ a finitely generated ideal in C_∞ (in fact, we may take I_∞ to be generated by at most $\dim X$ elements) and a_∞ an element of C_∞ (and hence in general is no longer a scheme). Instead, we call $X_\infty := \text{Spec}(C_\infty)$ the *non-standard hull* of X . We have a faithfully flat canonical morphism $X_\infty \rightarrow X$. Since X_∞ is no longer a Noetherian scheme, it is more prudent to reason on its approximations X_p instead, and that is the course we will take in this paper.

2.4. Affine Morphisms. Let $f: Y \rightarrow X$ be a morphism of finite type between the affine schemes $Y := \text{Spec } D$ and $X := \text{Spec } C$ of finite type over \mathbb{C} . This induces a \mathbb{C} -algebra homomorphism $\varphi: C \rightarrow D$. Let $\varphi_p: C_p \rightarrow D_p$ be an approximation of φ (as in §2.2) and let $f_p: Y_p \rightarrow X_p$ be the corresponding morphism between the approximations $Y_p := \text{Spec } D_p$ and $X_p := \text{Spec } C_p$. We call f_p an *approximation* of f . It follows from the corresponding transfer for affine algebras (see [32, §4]) that if f is an (open, closed, locally closed) immersion (respectively, injective, surjective, an isomorphism, flat, faithfully flat), then so are almost all f_p . We leave the details to the reader.

2.5. Modules. Let \mathcal{F} be a coherent \mathcal{O}_X -module. Any such module is of the form \widetilde{M} with M a finitely generated C -module (see [10, II.5] for the notation). Write M as the cokernel of a matrix Γ over C , that is to say, given by an exact sequence

$$C^a \xrightarrow{\Gamma} C^b \rightarrow M \rightarrow 0.$$

Let Γ_p be an approximation of Γ (that is to say, the Γ_p are $(a \times b)$ -matrices over C_p with ultraproduct equal to Γ) and let M_p be the cokernel of Γ_p . We call M_p an *approximation* of M and we call their ultraproduct M_∞ the *non-standard hull* of M . Again one shows that M_∞ does not depend on the choice of matrix Γ ; in fact, we have an isomorphism

$$(4) \quad M_\infty \cong M \otimes_C C_\infty.$$

The \mathcal{O}_{X_p} -module $\mathcal{F}_p := \widetilde{M_p}$ associated to M_p is called an *approximation* of \mathcal{F} .

2.6. Schemes. Let X be a scheme of finite type over \mathbb{C} . Let U_i be a finite covering of X by affine open subsets. For each i , let $U_{i,p}$ be an approximation of U_i . I claim that for almost all p , the $U_{i,p}$ glue together into a scheme X_p of finite type over $\mathbb{F}_p^{\text{alg}}$, and, for any other choice of open affine covering $\{U'_i\}$ of X , if the resulting glued schemes are denoted X'_p , then $X_p \cong X'_p$, for almost all p . This justifies calling the X_p an *approximation* of X . The proof of the claim is not hard, but is a little tedious, in that we have to check that the whole construction of glueing schemes

is constructive and hence passes by Łos' Theorem through ultraproducts. Here is a rough sketch: for each pair $i < j$, we have an isomorphism

$$\varphi_{ij}: \mathcal{O}_{U_i}|_{U_i \cap U_j} \cong \mathcal{O}_{U_j}|_{U_i \cap U_j}.$$

Taking approximations φ_{ijp} of these maps as described in §2.4, we get from Łos' Theorem that φ_{ijp} defines an isomorphism

$$\mathcal{O}_{U_{ip}}|_{U_{ip} \cap U_{jp}} \cong \mathcal{O}_{U_{jp}}|_{U_{ip} \cap U_{jp}}$$

for almost all p . Hence the U_{ip} glue together to get a scheme X_p . If we start from a different open affine covering $\{U'_i\}$, then to see that the resulting schemes X'_p agree for almost all p , reason on a common refinement of these two coverings.

Similarly, if C_i is the affine coordinate ring of U_i , then the $\text{Spec}(C_{i\infty})$ glue together and the resulting scheme X_∞ will be called the *non-standard hull* of X . In particular, the canonical morphism $X_\infty \rightarrow X$ is faithfully flat (since it is so locally).

2.7. Morphisms. Let $f: Y \rightarrow X$ be a morphism of finite type between schemes of finite type over \mathbb{C} . Let X_p and Y_p be approximations of X and Y respectively. Choose finite affine open coverings \mathfrak{U} and \mathfrak{V} of respectively X and Y , such that \mathfrak{V} is a refinement of $f^{-1}(\mathfrak{U})$. In other words, for each $V \in \mathfrak{V}$, we can find $U \in \mathfrak{U}$, such that $f(V) \subseteq U$. Let us write $f|_V$ for the restriction $V \rightarrow U$ induced by f . Choose approximations $\mathfrak{U}_p, \mathfrak{V}_p$ and $(f|_V)_p$ of $\mathfrak{U}, \mathfrak{V}$ and the affine morphisms $f|_V$ respectively (use §2.4 for the latter). It follows that for any two opens $V, V' \in \mathfrak{V}$, the morphisms $(f|_V)_p$ and $(f|_{V'})_p$ agree on the intersection $V_p \cap V'_p$, for almost all p , and therefore determine a morphism $f_p: Y_p \rightarrow X_p$, which we will call an *approximation* of f . As for affine morphisms, most algebraic properties descend to the approximations in the sense that f has a certain property (such as being a closed immersion or flat) if, and only if, almost all f_p have.

2.8. Coherent Sheaves. Let \mathcal{F} be a coherent \mathcal{O}_X -module. For each i , let \mathcal{G}_{ip} be an approximation of the coherent \mathcal{O}_{U_i} -module $\mathcal{F}|_{U_i}$ as explained in §2.5 and §2.6 and with the notations therein. Again one easily checks that these \mathcal{G}_{ip} glue together to give rise to a coherent \mathcal{O}_{X_p} -module \mathcal{F}_p , which we therefore call an *approximation* of \mathcal{F} , and, moreover, the construction does not depend on the choice of open affine covering.

If \mathcal{F} is a coherent sheaf of ideals on X with approximation \mathcal{F}_p , then almost all \mathcal{F}_p are sheaves of ideals, and the closed subscheme they determine on X_p is an approximation of the closed subscheme determined by \mathcal{F} . More generally, many local properties (such as being invertible, locally free) hold for the sheaf \mathcal{F} if, and only if, they hold for almost all of its approximations \mathcal{F}_p , since they can be checked locally and hence reduce to a similar transfer property for affine algebras discussed at large in [32].

2.9. Graded Rings and Modules. Recall that a ring S is called (\mathbb{Z}) -graded if it can be written as a direct sum $\bigoplus_{j \in \mathbb{Z}} [S]_j$, where each $[S]_j$ is an additive subgroup of S (called the j -th homogeneous piece of S), with the property that $[S]_i \cdot [S]_j \subseteq [S]_{i+j}$, for all $i, j \in \mathbb{Z}$. In particular, each $[S]_i$ is an $[S]_0$ -module. If all $[S]_j$ are zero for $j < 0$, we call S positively graded. An S -module M is called graded if it admits a decomposition $\bigoplus_{j \in \mathbb{Z}} [M]_j$, where each $[M]_j$ is an additive subgroup of M (called the j -th homogeneous piece of M), with the property that $[S]_i \cdot [M]_j \subseteq [M]_{i+j}$, for all $i, j \in \mathbb{Z}$. We will write $M(m)$ for the m -th twist of M , that is to say, for the graded S -module for which $[M(m)]_j := [M]_{m+j}$.

Let S be a graded affine \mathbb{C} -algebra. Let S_p be an approximation of S and S_∞ its non-standard hull. Our goal is to show that almost all S_p are graded. Let x_i be homogeneous algebra generators of S over \mathbb{C} , say of degree d_i . Put $A := \mathbb{C}[\xi]$ and let $\varphi: A \rightarrow S$ be given by $\xi_i \mapsto x_i$. We make A into a graded ring by giving ξ_i weight d_i , that is to say, $[A]_j$ is the vector space over \mathbb{C} generated by all monomials $\xi_1^{e_1} \cdots \xi_n^{e_n}$ such that $d_1 e_1 + \cdots + d_n e_n = j$. Hence the kernel I of $A \rightarrow S$ is generated by homogeneous polynomials in this new grading. Give each A_p the same grading as A (using the weights d_i) and let I_p be an approximation of I . It follows from Łos' Theorem that almost all I_p are generated by homogeneous elements. Since $S_p \cong A_p/I_p$ for almost all p , we proved that almost all approximations are graded. Moreover, if S is positively graded, then so are almost all S_p .

However, the non-standard hull S_∞ is no longer a graded ring. Nonetheless, for each non-standard integer j (that is to say, any element $j := \text{ulim}_{p \rightarrow \infty} j_p$ in the ultrapower \mathbb{Z}_∞ of \mathbb{Z}), we can define the j -th homogeneous piece $[S_\infty]_j$ of S_∞ as the ultraproduct of the $[S_p]_{j_p}$. It follows that each $[S_\infty]_j$ is a direct summand of S_∞ (and in fact, $S_\infty^{\text{gr}} := \bigoplus_{j \in \mathbb{Z}_\infty} [S_\infty]_j$ is a (proper) direct summand of S_∞), and $[S_\infty]_i \cdot [S_\infty]_j \subseteq [S_\infty]_{i+j}$, for all $i, j \in \mathbb{Z}_\infty$ (so that S_∞^{gr} is a \mathbb{Z}_∞ -graded ring). If j is a standard integer (that is to say, $j \in \mathbb{Z}$, whence $j_p = j$ for almost all p), the embedding $S \subseteq S_\infty$ induces an embedding

$$(5) \quad [S]_j \subseteq [S_\infty]_j.$$

Note that this is not necessarily an isomorphism. For instance, if $S := \mathbb{C}[\xi, \zeta, 1/\xi]$ with ξ and ζ having weight 1 (and $1/\xi$ weight -1), then $[S]_0 \cong \mathbb{C}[\zeta/\xi]$ whereas $[S_\infty]_0$ contains for instance the ultraproduct of the ζ^p/ξ^p .

Let M be a finitely generated graded S -module. Let M_p be an approximation of M and M_∞ its non-standard hull. By the same argument as above, almost all M_p are graded S_p -modules. We define similarly the j -th homogeneous piece $[M_\infty]_j$ of M_∞ as the ultraproduct of the $[M_p]_{j_p}$. It follows that $[S_\infty]_i \cdot [M_\infty]_j \subseteq [M_\infty]_{i+j}$, for each $i, j \in \mathbb{Z}_\infty$, and $[M]_j \subseteq [M_\infty]_j$ for each standard j .

If $M \rightarrow N$ is a degree preserving morphism of finitely generated graded S -modules (so that $[M]_j$ maps inside $[N]_j$, for all j), then the same is true for almost

all approximations $M_p \rightarrow N_p$. Hence the base change $M_\infty \rightarrow N_\infty$ sends $[M_\infty]_j$ inside $[N_\infty]_j$, for each $j \in \mathbb{Z}_\infty$.

2.10. Projective Schemes. Suppose that $X := \text{Proj } S$ is a projective scheme, with S an affine, positively graded \mathbb{C} -algebra. Let S_p be an approximation of S . Then S_p is an affine, positively graded $\mathbb{F}_p^{\text{alg}}$ -algebra by §2.9, and $X_p \cong \text{Proj } S_p$, for almost all p . Indeed, this is clear for $S := \mathbb{C}[\xi_0, \dots, \xi_n]$ (so that $X = \mathbb{P}_\mathbb{C}^n$), and the general case follows from this since any projective scheme of finite type over \mathbb{C} is a closed subscheme of some $\mathbb{P}_\mathbb{C}^n$.

2.11. Polarizations. Let X be a projective variety and \mathcal{P} an ample line bundle on X (we will call \mathcal{P} a *polarization* and study this situation in more detail in §5). Let X_p and \mathcal{P}_p be approximations of X and \mathcal{P} respectively. I claim that almost all \mathcal{P}_p are ample line bundles. That almost all are invertible is clear from the discussion in §2.8. Suppose first that \mathcal{P} is very ample. Hence there is an embedding $f: X \rightarrow \mathbb{P}_\mathbb{C}^N$ such that $\mathcal{P} \cong f^*\mathcal{O}(1)$. From the discussion in §2.7 and §2.9, the approximation $f_p: X_p \rightarrow \mathbb{P}_{\mathbb{F}_p^{\text{alg}}}^N$ is an embedding and $\mathcal{P}_p \cong f_p^*\mathcal{O}(1)$ for almost all p , showing that almost all \mathcal{P}_p are very ample. If \mathcal{P} is just ample, then \mathcal{P}^m is very ample for some $m > 0$ by [10, II. Theorem 7.6]. Hence by our previous argument, almost all \mathcal{P}_p^m are very ample. By another application of [10, II. Theorem 7.6], almost all \mathcal{P}_p are ample.

Presumably the converse also holds, but this requires a finer study of the dependence of the exponent m on the ample sheaf: it should only depend on the degree complexity of the sheaf (that is to say, on the maximum of the degrees of the polynomials needed in describing the sheaf).

2.12. Complexes. Let \mathcal{C}^\bullet be an arbitrary bounded complex in which each term \mathcal{C}^m is a finitely generated module over an affine \mathbb{C} -algebra. Using §2.2 and §2.3, we can choose an approximation for each term and each homomorphism in this complex. Let $(\mathcal{C}^\bullet)_p$ denote the corresponding object. By Łos' Theorem, almost all $(\mathcal{C}^\bullet)_p$ are complexes. This justifies calling $(\mathcal{C}^\bullet)_p$ an *approximation* of \mathcal{C}^\bullet . Let A be a polynomial ring over \mathbb{C} such that each \mathcal{C}^m is an A -module. It follows from (4) that we have an isomorphism of complexes

$$(6) \quad \mathcal{C}^\bullet \otimes_A A_\infty \cong \text{ulim}_{p \rightarrow \infty} (\mathcal{C}^\bullet)_p.$$

Since taking cohomology consists of taking kernels, images and quotients, each of which commutes with ultraproducts, taking cohomology also commutes with ultraproducts. Applying this to (6), we get for each i , an isomorphism

$$(7) \quad H^i(\mathcal{C}^\bullet) \otimes_A A_\infty \cong H^i(\mathcal{C}^\bullet \otimes_A A_\infty) \cong \text{ulim}_{p \rightarrow \infty} H^i((\mathcal{C}^\bullet)_p)$$

where we used that $A \rightarrow A_\infty$ is faithfully flat for the first isomorphism.

Our next goal is to show that an approximation of the cohomology of a coherent \mathcal{O}_X -module \mathcal{F} is obtained by taking the cohomology of its approximations. In order

to prove this, we use Čech cohomology to calculate sheaf cohomology (this will be studied further in §5.3.1 below).

2.13. Čech Cohomology. Recall that the Čech complex $\mathcal{C}^\bullet(\mathfrak{U}; \mathcal{F})$ of \mathcal{F} associated to an open affine covering $\mathfrak{U} := \{U_1, \dots, U_s\}$ of X is by definition the complex in which the m -th term for $m \geq 1$ is

$$\mathcal{C}^m(\mathfrak{U}; \mathcal{F}) := \bigoplus_{\mathbf{i}} H^0(U_{\mathbf{i}}, \mathcal{F})$$

where \mathbf{i} runs over all m -tuples of indices $1 \leq i_1 < i_2 < \dots < i_m \leq s$ and where $U_{\mathbf{i}} := U_{i_1} \cap U_{i_2} \cap \dots \cap U_{i_s}$ (see [10, III.4] for more details). Note that $\mathcal{C}^\bullet(\mathfrak{U}; \mathcal{F})$ is a bounded complex of affine \mathbb{C} -algebras.

Lemma 2.14. *Let X be a scheme of finite type over \mathbb{C} , let \mathfrak{U} be a finite affine open covering of X and let \mathcal{F} be a coherent \mathcal{O}_X -module. If X_p , \mathfrak{U}_p and \mathcal{F}_p are approximations of X , \mathfrak{U} and \mathcal{F} respectively, then the complexes $\mathcal{C}^\bullet(\mathfrak{U}_p; \mathcal{F}_p)$ are an approximation of the complex $\mathcal{C}^\bullet(\mathfrak{U}; \mathcal{F})$.*

Proof. Since an approximation of \mathfrak{U} is obtained by choosing an approximation for each affine open in it, we get from Łos' Theorem that \mathfrak{U}_p is an open covering of X_p for almost all p . Moreover, if U is an affine open with approximation U_p , then $H^0(U_p, \mathcal{F}_p)$ is an approximation of $H^0(U, \mathcal{F})$. The assertion readily follows from these observations. \square

If X is separated and of finite type over \mathbb{C} and if \mathcal{F} is a coherent \mathcal{O}_X -module, then the cohomology modules $H^i(X, \mathcal{F})$ can be calculated as the cohomology of the Čech complex $\mathcal{C}^\bullet(\mathfrak{U}; \mathcal{F})$, for any choice of finite open affine covering \mathfrak{U} ([10, Theorem 4.5]). More precisely,

$$(8) \quad H^i(X, \mathcal{F}) \cong H^{i+1}(\mathcal{C}^\bullet(\mathfrak{U}; \mathcal{F}))$$

(some authors start numbering the Čech complex from zero, so that there is no shift in the superscripts needed). If \mathfrak{U} consists of affine opens $\text{Spec } C_i$, we can choose a polynomial ring A over \mathbb{C} so that every C_i is a homomorphic image of A . It follows that each module in $\mathcal{C}^\bullet(\mathfrak{U}; \mathcal{F})$ is a finitely generated A -module, and hence so is each $H^i(X, \mathcal{F})$.

If X is moreover projective, then each $H^i(X, \mathcal{F})$ is a finite dimensional vector space over \mathbb{C} and its dimension will be denoted by $h^i(X, \mathcal{F})$.

Theorem 2.15. *Let X be a separated scheme of finite type over \mathbb{C} and let \mathcal{F} be a coherent \mathcal{O}_X -module. Let X_p and \mathcal{F}_p be approximations of X and \mathcal{F} respectively. For an appropriate choice of a polynomial ring A over \mathbb{C} and for each i , we have an isomorphism*

$$(9) \quad H^i(X, \mathcal{F}) \otimes_A A_\infty \cong \varinjlim_{p \rightarrow \infty} H^i(X_p, \mathcal{F}_p).$$

In particular, if X is moreover projective, then $h^i(X, \mathcal{F})$ is equal to $h^i(X_p, \mathcal{F}_p)$ for almost all p .

Proof. By Lemma 2.14, the $\mathcal{C}^\bullet(\mathcal{U}_p; \mathcal{F}_p)$ are an approximation of $\mathcal{C}^\bullet(\mathcal{U}; \mathcal{F})$. The first assertion now follows from (7) and (8). The last assertion follows from the first, by taking lengths of both sides and using [29, Proposition 1.5]. \square

Therefore, if $H^i(X, \mathcal{F})$ vanishes for some i , then so do almost all $H^i(X_p, \mathcal{F}_p)$. More precisely, for a fixed choice of approximation, let $\Sigma_{i, \mathcal{F}}$ be the collection of prime numbers p for which $H^i(X_p, \mathcal{F}_p)$ vanishes. By the above result, $\Sigma_{i, \mathcal{F}}$ belongs to the ultrafilter if, and only if, $H^i(X, \mathcal{F}) = 0$. However, if we have an infinite collection of coherent sheaves \mathcal{F}_n with zero i -th cohomology, the intersection of all $\Sigma_{i, \mathcal{F}_n}$ will in general no longer belong to the ultrafilter, and therefore can very well be empty. The next result shows that by imposing some further algebraic relations among the \mathcal{F}_n , the intersection remains in the ultrafilter.

Corollary 2.16. *Let X be a projective scheme of finite type over \mathbb{C} . Let \mathcal{L} be an invertible \mathcal{O}_X -module and let \mathcal{E} be a locally free \mathcal{O}_X -module. Let X_p, \mathcal{L}_p and \mathcal{E}_p be approximations of X, \mathcal{L} and \mathcal{E} respectively. If for some i and some n_0 , we have that $H^i(X, \mathcal{E} \otimes \mathcal{L}^n) = 0$ for all $n \geq n_0$, then for almost all p , we have, for all $n \geq n_0$, that $H^i(X_p, \mathcal{E}_p \otimes \mathcal{L}_p^n) = 0$.*

Proof. Let \mathcal{A} denote the symmetric algebra $\bigoplus_{n \geq 0} \mathcal{L}^n$ of \mathcal{L} and let $\mathcal{F} := \mathcal{A} \otimes \mathcal{E} \otimes \mathcal{L}^{n_0}$. Note that $\mathcal{F} = \bigoplus_{n \geq n_0} \mathcal{E} \otimes \mathcal{L}^n$, so that our assumption becomes $H^i(X, \mathcal{F}) = 0$. We cannot apply Theorem 2.15 directly, as \mathcal{F} is not a coherent \mathcal{O}_X -module. Let $Y := \mathbf{Spec} \mathcal{A}$ be the scheme over X associated to \mathcal{A} (see [10, II. Ex. 5.17]). Since \mathcal{A} is a finitely generated sheaf of \mathcal{O}_X -algebras, the morphism $f: Y \rightarrow X$ is of finite type. Moreover, f is affine (that is to say, $f^{-1}(U) \cong \mathbf{Spec} \mathcal{A}(U)$ for every affine open U of X) and $\mathcal{A} \cong f_* \mathcal{O}_Y$. Let $\mathcal{G} := f^*(\mathcal{E} \otimes \mathcal{L}^{n_0})$, so that \mathcal{G} is a coherent \mathcal{O}_Y -module. We have isomorphisms

$$f_* \mathcal{G} \cong f_* \mathcal{O}_Y \otimes \mathcal{E} \otimes \mathcal{L}^{n_0} \cong \mathcal{A} \otimes \mathcal{E} \otimes \mathcal{L}^{n_0} = \mathcal{F},$$

where the first isomorphism follows from the projection formula (see [10, II. Ex. 5.1]). Therefore,

$$(10) \quad H^i(Y, \mathcal{G}) \cong H^i(X, f_* \mathcal{G}) = H^i(X, \mathcal{F}) = 0$$

where the first isomorphism holds by [10, III. Ex. 4.1].

Let $f_p: Y_p \rightarrow X_p$ be an approximation of f (as described in §2.7) and let \mathcal{F}_p and \mathcal{G}_p be approximations of \mathcal{F} and \mathcal{G} respectively. By Łos' Theorem, we have isomorphisms

$$(11) \quad \mathcal{G}_p \cong f_p^*(\mathcal{E}_p \otimes \mathcal{L}_p^{n_0})$$

$$(12) \quad (f_p)_*\mathcal{G}_p \cong \mathcal{F}_p$$

$$(13) \quad (f_p)_*\mathcal{O}_{Y_p} \cong \bigoplus_{n \geq 0} \mathcal{L}_p^n$$

$$(14) \quad \mathcal{F}_p \cong \bigoplus_{n \geq n_0} \mathcal{E}_p \otimes \mathcal{L}_p^n$$

for almost all p . Applying Theorem 2.15 to (10), we get that almost all $H^i(Y_p, \mathcal{G}_p)$ vanish. Hence, by the analogue of (10) in characteristic p , almost all $H^i(X_p, \mathcal{F}_p)$ vanish. In view of (14), this proves the assertion. \square

Let us conclude this section with showing that this non-standard formalism just introduced is closely related to the usual reduction modulo p (as for instance used in the definition of tight closure in characteristic zero in [14]).

2.17. Models. Let K be a field and R a K -affine algebra. With a *model of R* relative to K (called *descent data* in [14]) we mean a pair (Z, R_Z) consisting of a subring Z of K which is finitely generated over \mathbb{Z} and a Z -algebra R_Z essentially of finite type, such that $R \cong R_Z \otimes_Z K$. Oftentimes, we will think of R_Z as being the model. Clearly, the collection of models R_Z of R forms a direct system whose union is R . We say that R is *F-rational type* (respectively, *weakly F-regular type*, or *strongly F-regular type*), if there exists a model (Z, R_Z) , such that $R_Z/\mathfrak{p}R_Z$ is F-rational (respectively, weakly F-regular or strongly F-regular) for all maximal ideals \mathfrak{p} of Z (note that $R_Z/\mathfrak{p}R_Z$ has positive characteristic). See [14] or [17, App. 1] for more details.

The following was proved in [34, Corollary 5.9] for local rings; the general case is proven by the same argument.

Proposition 2.18. *Let R be a \mathbb{C} -affine domain. For each finite subset of R , we can find a model (Z, R_Z) of R containing this subset, and, for almost all p , a maximal ideal \mathfrak{p}_p of Z and a separable extension $Z/\mathfrak{p}_p \subseteq \mathbb{F}_p^{\text{alg}}$, such that the collection of base changes $R_Z \otimes_Z \mathbb{F}_p^{\text{alg}}$ gives an approximation of R . Moreover, for any $r \in R_Z$, the collection of images of r under the various homomorphisms $R_Z \rightarrow R_Z \otimes_Z \mathbb{F}_p^{\text{alg}}$ gives an approximation of r .*

2.19. Approximations as Universal Reductions. Suppose $(R_{Z'}, Z')$ is another model of R satisfying the assertion of the previous proposition (so that we have homomorphisms $Z' \rightarrow \mathbb{F}_p^{\text{alg}}$). Since any two approximations agree almost everywhere as mentioned in §2.1, we get that $R_Z \otimes_Z \mathbb{F}_p^{\text{alg}} \cong R_{Z'} \otimes_{Z'} \mathbb{F}_p^{\text{alg}}$ for almost all p .

3. Log-terminal Singularities

In [9], the authors show that a \mathbb{Q} -Gorenstein \mathbb{C} -affine algebra has log-terminal singularities if it is weakly F-regular type (note that weakly F-regular type and strongly F-regular type are equivalent under the \mathbb{Q} -Gorenstein condition by [24]), whereas the converse is proved in [8]. In this section we will define a third condition in terms of ultraproducts of Frobenii and prove its equivalence with the other ones. The advantage of the latter property is that it is easily seen to descend under pure homomorphisms (see Proposition 3.11). We recall some terminology first.

3.1. \mathbb{Q} -Gorenstein Singularities. A normal scheme X is called *\mathbb{Q} -Gorenstein* if some positive multiple of its canonical divisor K_X is Cartier; the least such positive multiple is called the *index* of X . If $f: \bar{X} \rightarrow X$ is a resolution of singularities of X and E_i are the irreducible components of the exceptional locus, then the canonical divisor $K_{\bar{X}}$ is numerically equivalent to $f^*(K_X) + \sum a_i E_i$ (as \mathbb{Q} -divisors), for some unique $a_i \in \mathbb{Q}$ (a_i is called the *discrepancy* of X along E_i ; see [20, Definition 2.22]). If all $a_i > -1$, we call X *log-terminal* (in case we only have a weak inequality, we call X *log-canonical*).

3.2. Ultra-Frobenii. Any ring R of characteristic p is endowed with the *Frobenius* endomorphism $\varphi_p: x \mapsto x^p$, and its powers $\varphi_q := \varphi_p^e$, where $q := p^e$. We can therefore view R as a module over itself via the homomorphism φ_q , and to emphasize this, we will use the notation $\varphi_{q*}R$ (a notation borrowed from algebraic geometry; other authors use notations such as R^F , R^{φ_q} or eR). Similarly, for an arbitrary R -module M , we will write $\varphi_{q*}M$ for the R -module structure on M via φ_q (that is to say, $x \cdot m = x^q m$). It follows that $\varphi_{q*}M \cong M \otimes_R \varphi_{q*}R$.

For each prime number p , choose a positive integer e_p and let π be the non-standard integer given as the ultraproduct of the powers p^{e_p} . To each such π , we associate an *ultra-Frobenius* in the following way. For each \mathbb{C} -affine domain R with non-standard hull R_∞ , consider the homomorphism

$$R \rightarrow R_\infty: x \mapsto x^\pi := \text{ulim}_{p \rightarrow \infty} (x_p)^{p^{e_p}}$$

where x_p is an approximation of x (one easily checks that this does not depend on the choice of approximation). We will denote this ultra-Frobenius by φ_π , or simply φ ; whenever we want to emphasize the ring R on which it operates, we write $\varphi_{\pi;R}$ or simply φ_R . This assignment is functorial, in the sense that for any homomorphism $f: R \rightarrow S$ of finite type, we have a commutative diagram

$$(15) \quad \begin{array}{ccc} R & \xrightarrow{f} & S \\ \varphi_R \downarrow & & \downarrow \varphi_S \\ R_\infty & \xrightarrow{f_\infty} & S_\infty \end{array}$$

Note that φ is the restriction to R of the ultraproduct of the $\varphi_{p^{e_p}}$. In particular, if all $e_p = 1$, then the corresponding ultra-Frobenius φ_∞ was called the *non-standard Frobenius* in [32].

Each ultra-Frobenius induces an R -module structure on R_∞ , which we will denote by $\varphi_* R_\infty$ (so that $x \cdot r_\infty = \varphi(x)r_\infty$). It follows that $\varphi_* R_\infty$ is the ultraproduct of the $(\varphi_{p^{e_p}})_* R_p$, with R_p an approximation of R . If M is a finitely generated R -module with non-standard hull M_∞ , then the R -module structure on $M_\infty \cong M \otimes_R R_\infty$ via the action of φ on the second factor, will be denoted $\varphi_* M$. It follows that $\varphi_* M$ is isomorphic to $M \otimes_R \varphi_* R_\infty$ and hence isomorphic to the ultraproduct of the $(\varphi_{p^{e_p}})_* M_p$.

Definition 3.3. We say that a \mathbb{C} -affine domain R is *ultra-F-regular*, if for each non-zero $c \in R$, we can find an ultra-Frobenius φ such that the R -module morphism

$$c\varphi_R: R \rightarrow \varphi_* R_\infty: x \mapsto c\varphi(x)$$

is pure.

For M an R -module, we will write $c\varphi_M: M \rightarrow M \otimes_R \varphi_* R_\infty$ for the base change of $c\varphi_R$. Since purity is preserved under localization, one easily verifies that the localization of an ultra-F-regular ring is again ultra-F-regular.

Remark 3.4. If R is normal, so that purity and cyclical purity are the same by [11, Theorem 2.6], then purity of $c\varphi_R$ is equivalent to the condition that for every $y \in R$ and every ideal I in R , if $c\varphi(y) \in \varphi(I)R_\infty$, then $y \in I$. From this and the fact that any ultra-Frobenius on a regular local ring is flat (same proof as for [32, Proposition 6.1]), one easily checks that a regular (local) \mathbb{C} -affine domain is ultra-F-regular.

In [34], we called a local \mathbb{C} -affine domain with approximation R_p *weakly generically F-regular* (respectively, *generically F-rational*), if each ideal I in R (respectively, some ideal I generated by a system of parameters), is equal to its generic tight closure. Recall from [32] that an element $x \in R$ lies in the *generic tight closure* of an ideal I , if x_p lies in the tight closure of I_p , for almost p , where I_p and x_p are approximations of I and x respectively. We proved in [34, Theorem C] that being generically F-rational is equivalent with having rational singularities. Let us call R *generically F-regular* if every localization of R is weakly generically F-regular.

Theorem 3.5. *Let R be a \mathbb{C} -affine domain.*

- (1) *If R is strongly F-regular type, then it is ultra-F-regular.*

(2) *If R is ultra-F-regular, then it is generically F-regular.*

Proof. To prove (1), let c be a non-zero element of R . Let (Z, R_Z) be a model of R containing c . By Proposition 2.18, there exists for almost all p , a maximal ideal \mathfrak{p}_p of R and a separable extension $Z/\mathfrak{p}_p \subseteq \mathbb{F}_p^{\text{alg}}$, such that $R_p := R_Z \otimes_Z \mathbb{F}_p^{\text{alg}}$ is an approximation of R and such that for any element $r \in R_Z$, its image in R_p under the base change $\gamma_p: R_Z \rightarrow R_p$ is an approximation of r . In particular, c is the ultraproduct of the $\gamma_p(c)$. By definition, we may choose the model in such way that almost all $S_p := R_Z/\mathfrak{p}_p R_Z$ are strongly F-regular. In particular, we can find powers $q := p^{e_p}$, such that the morphism

$$S_p \rightarrow \varphi_{q_*}(S_p): x \mapsto \gamma_p(c)x^q$$

is pure. By base change, the R_p -module morphism

$$R_p \rightarrow \varphi_{q_*}(S_p) \otimes R_p: x \mapsto \gamma_p(c)x^q \otimes 1$$

is also pure. Since $Z/\mathfrak{p}_p \subseteq \mathbb{F}_p^{\text{alg}}$ is separable, we get that $\varphi_{q_*}(S_p) \otimes R_p \cong \varphi_{q_*}R_p$, showing that the R_p -module morphism

$$(16) \quad R_p \rightarrow \varphi_{q_*}R_p: x \mapsto \gamma_p(c)x^q$$

is pure. Let φ be the ultra-Frobenius given as the (restriction to R of the) ultraproduct of the φ_q and let g_∞ be the ultraproduct of the morphisms given in (16). It follows that $g_\infty(x_\infty) = c\varphi(x_\infty)$, so that the restriction of g_∞ to R is precisely $c\varphi_R$. Moreover, from the purity of (16), it follows, using Łos' Theorem, that every finitely generated ideal J of R_∞ is equal to the contraction of its extension under g_∞ . Since $R \rightarrow R_\infty$ is faithfully flat, whence cyclically pure, it follows that the restriction of g_∞ to R , that is to say, $c\varphi_R$, is cyclically pure. Since R is in particular normal, $c\varphi_R$ is pure by [11, Theorem 2.6], showing that R is ultra-F-regular.

Assume next that R is ultra-F-regular. Without loss of generality, we may assume that R is moreover local. Let I be an ideal in R and let x be an element in the generic tight closure of I . We need to show that $x \in I$. Let x_p and I_p be approximations of x and I respectively. By [32, Proposition 8.3], we can choose $c \in R$ such that almost every c_p is a test element for R_p , where R_p and c_p are approximations of R and c respectively. Let φ be an ultra-Frobenius such that the R -module morphism $c\varphi_R$ is pure. In particular this implies for every $y \in R$ that

$$(17) \quad \text{if } c\varphi(y) \in \varphi(I)R_\infty, \text{ then } y \in I.$$

Suppose φ is the ultraproduct of the $\varphi_p^{e_p}$. Therefore, (17) translated in terms of an approximation y_p of an element $y \in R$, becomes the statement

$$(18) \quad \text{if } c_p \varphi_p^{e_p}(y_p) \in \varphi_p^{e_p}(I_p)R_p, \text{ then } y_p \in I_p,$$

for almost all p .

By assumption, almost every x_p lies in the tight closure of I_p . Since c_p is a test element, this means that

$$c_p \varphi_p^N(x_p) \in \varphi_p^N(I_p)R_p,$$

for all N . With $N := e_p$, we get from (18) that $x_p \in I_p$. Taking ultraproducts shows that x lies in IR_∞ , whence in I , by the faithful flatness of $R \rightarrow R_\infty$. \square

Remark 3.6. Note that if R is ultra-F-regular with approximation R_p , then it is not necessary the case that almost all R_p are strongly F-regular. Namely, the set $\Sigma_{y,I}$ of prime numbers p for which (18) holds, depends a priori on y and I , and therefore, their intersection over all possible y and I might very well be empty.

The prime characteristic analogue of the next result was first observed in [42]; we follow the argument given in [39, Theorem 4.15].

Proposition 3.7. *Let $R \subseteq S$ be a finite extension of local \mathbb{C} -affine domains, étale in codimension one. Let c be a non-zero element of R and φ an ultra-Frobenius. If $c\varphi_R: R \rightarrow \varphi_*R_\infty$ is pure, then so is $c\varphi_S: S \rightarrow \varphi_*S_\infty$.*

In particular, if R is ultra-F-regular, then so is S .

Proof. Let $R \subseteq S$ be an arbitrary finite extension of d -dimensional local \mathbb{C} -affine domains and fix a non-zero element c and an ultra-Frobenius φ . Let \mathfrak{n} be the maximal ideal of S and ω_S its canonical module. I claim that if $R \subseteq S$ is étale, then $S \otimes_R \varphi_*R_\infty \cong \varphi_*S_\infty$. Assuming the claim, let $R \subseteq S$ now only be étale in codimension one. It follows from the claim that the supports of the kernel and the cokernel of the base change $S \otimes_R \varphi_*R_\infty \rightarrow \varphi_*S_\infty$ have codimension at least two. Hence the same is true for the base change

$$\omega_S \otimes_S S \otimes_R \varphi_*R_\infty \rightarrow \omega_S \otimes_S \varphi_*S_\infty.$$

Applying the top local cohomology functor $H_{\mathfrak{n}}^d$, we get, in view of Grothendieck Vanishing and the long exact sequence of local cohomology, an isomorphism

$$(19) \quad H_{\mathfrak{n}}^d(\omega_S \otimes_R \varphi_*R_\infty) \cong H_{\mathfrak{n}}^d(\omega_S \otimes_S \varphi_*S_\infty).$$

By Grothendieck duality, $H_{\mathfrak{n}}^d(\omega_S)$ is the injective hull E of the residue field of S . Taking the base change of $c\varphi_R$ and $c\varphi_S$ over ω_S , and then taking the top local cohomology, yields the following commutative diagram

$$\begin{array}{ccccc} E = H_{\mathfrak{n}}^d(\omega_S) & \longrightarrow & E \otimes_R \varphi_*R_\infty & \longrightarrow & H_{\mathfrak{n}}^d(\omega_S \otimes_R \varphi_*R_\infty) \\ & & \downarrow & & \downarrow \cong \\ E = H_{\mathfrak{n}}^d(\omega_S) & \longrightarrow & E \otimes_S \varphi_*S_\infty & \longrightarrow & H_{\mathfrak{n}}^d(\omega_S \otimes_S \varphi_*S_\infty) \end{array}$$

where the last vertical arrow in this diagram is the isomorphism (19). Since by assumption, $c\varphi_R: R \rightarrow \varphi_*R_\infty$ is pure, so is the base change $\omega_S \rightarrow \omega_S \otimes_R \varphi_*R_\infty$. Since purity is preserved after taking cohomology, the top composite arrow is injective, and hence so is the lower composite arrow. In particular, its first factor

$E \rightarrow E \otimes_S \varphi_* S_\infty$ is injective. Note that this morphism is still given as multiplication by c , and hence is equal to the base change $c\varphi_E$ of $c\varphi_S$. By [13, Lemma 2.1(e)], the injectivity of $c\varphi_E = c\varphi_S \otimes E$ then implies that $c\varphi_S$ is pure, as we set out to prove.

To prove the claim, observe that if $R \rightarrow S$ is étale with approximation $R_p \rightarrow S_p$, then almost all of these are étale. Indeed, by [26, Corollary 3.16], we can write S as $R[\xi]/I$, with $\xi := (\xi_1, \dots, \xi_n)$ and $I := (f_1, \dots, f_n)R[\xi]$, such that the Jacobian $J(f_1, \dots, f_n)$ is a unit in R , and by Łos' Theorem, this property is preserved for almost all approximations. In general, if $C \rightarrow D$ is an étale extension of characteristic p domains, then we have an isomorphism $\varphi_{q*} C \otimes_C D \cong \varphi_{q*} D$ (see for instance [12, p. 50] or the proof of [39, Theorem 4.15]). Applied to the current situation, we get that $S_p \otimes_{R_p} \varphi_{q*} R_p \cong \varphi_{q*} S_p$, for q any power of p ([12, p. 50]). Therefore, after taking ultraproducts, we obtain the required isomorphism $S \otimes_R \varphi_* R_\infty \cong \varphi_* S_\infty$ (note that $S_\infty \cong R_\infty \otimes_R S$ since $R \rightarrow S$ is finite).

To prove the last assertion, we have to show that we can find for each non-zero $c \in S$ an ultra-Frobenius φ such that $c\varphi_S$ is pure. However, if we can do this for some non-zero multiple of c , then we can also do this for c , and hence, since S is finite over R , we may assume without loss of generality that $c \in R$. Since R is ultra-F-regular, we can find therefore an ultra-Frobenius φ such that $c\varphi_R$ is pure, and hence by the first assertion, so is then $c\varphi_S$, proving that S is ultra-F-regular. \square

3.8. Proof of Theorem 1. The equivalence of (1) and (1') is proven by Hara in [8, Theorem 5.2]. Theorem 3.5 proves (1') \Rightarrow (2). Hence remains to prove (2) \Rightarrow (1).

To this end, assume R is ultra-F-regular. Recall the construction of the canonical cover of R due to Kawamata. Let r be the index of R , that is to say, the least r such that $\mathcal{O}_X(rK_X) \cong \mathcal{O}_X$, where $X := \text{Spec } R$ and K_X the canonical divisor of X . This isomorphism induces an R -algebra structure on

$$\tilde{R} := H^0(X, \mathcal{O}_X \oplus \mathcal{O}_X(K_X) \oplus \dots \oplus \mathcal{O}_X((r-1)K_X)),$$

which is called the *canonical cover* of R ; see [19]. Since $R \rightarrow \tilde{R}$ is étale in codimension one (see for instance [39, 4.12]), we get from Proposition 3.7 that \tilde{R} is ultra-F-regular. Hence \tilde{R} is weakly generically F-regular, by Theorem 3.5. In particular, \tilde{R} is generically F-rational, whence has rational singularities, by [37, Theorem 6.2]. By [19, Theorem 1.7], this in turn implies that R has log-terminal singularities. \square

Remark 3.9. Note that without relying on Hara's result (which uses Kodaira Vanishing), we proved the implications (1') \Rightarrow (2) \Rightarrow (1), recovering the result of Smith in [38, 39].

Remark 3.10. There are at least eight more conditions which are expected to be equivalent with the ones in Theorem 1 for a local \mathbb{Q} -Gorenstein \mathbb{C} -affine domain R , namely

- (3) R is weakly generically F-regular;

- (3') R is generically F-regular;
- (4) R is weakly F-regular (that is to say, every ideal is equal to its tight closure in the sense of [14]);
- (4') R is F-regular (that is to say, every localization of R is weakly F-regular);
- (5) $R \rightarrow \mathcal{B}(R)$ is cyclically pure (we will say that R is \mathcal{B} -regular; see §4 below for the definition of $\mathcal{B}(R)$);
- (5') $S \rightarrow \mathcal{B}(S)$ is cyclically pure for every localization S of R ;
- (6) R is weakly difference regular (that is to say, every ideal is equal to its non-standard tight closure in the sense of [32]; see [36, §3.10]);
- (6') R is difference regular (that is to say, every localization of R is weakly difference regular);

The implications (6) \Rightarrow (3) \Rightarrow (4) and (3) \Rightarrow (5) follow respectively from the facts that generic tight closure is contained in non-standard tight closure [32, Theorem 10.4], that classical tight closure is contained in generic tight closure [32, Theorem 8.4], and that \mathcal{B} -closure is contained in generic tight closure [34, Corollary 4.5]. We have similar implications among the accented conditions, and, of course, the accented conditions trivially imply their weak counterparts. By [34, Theorem 5.2] conditions (5) and (5') are equivalent. Finally, Theorem 3.5 proves that (2) \Rightarrow (3').

Conjecturally, weakly F-regular is the same as weakly F-regular type, so that therefore all (weak) conditions (1)–(4) would be equivalent for local \mathbb{Q} -Gorenstein \mathbb{C} -affine domains. If we conjecture moreover that \mathcal{B} -closure is the same as generic tight closure (as plus closure is expected to be the same as tight closure), (1)–(5) would be equivalent. Without these assumptions, it is not hard to show that if R is \mathcal{B} -regular, then any ultra-Frobenius is pure. The fact that we allow in the definition of ultra-F-regularity any ultra-Frobenius, and not just powers of the non-standard Frobenius, causes an obstruction in proving that (2) \Rightarrow (6).

The importance of this new characterization of log-terminal singularities in Theorem 1 is the fact that unlike the first two properties, ultra-F-regularity is easily proved to descend under (cyclically) pure homomorphisms.

Proposition 3.11. *Let $R \rightarrow S$ be a cyclically pure homomorphism between \mathbb{C} -affine algebras. If S is ultra-F-regular, then so is R .*

Proof. Since S is in particular normal, so is R (see for instance [37, Theorem 4.7]). Therefore, the embedding $R \rightarrow S$ is pure, by [11, Theorem 2.6]. Let c be a non-zero element of R . By assumption, we can find an ultra-Frobenius φ such that the S -module morphism

$$c\varphi_S: S \rightarrow \varphi_*S_\infty: x \mapsto c\varphi(x)$$

is pure, and whence so is its composition with $R \rightarrow S$. However, this composite morphism factors as $c\varphi_R$ followed by the inclusion $\varphi_*R_\infty \subseteq \varphi_*S_\infty$. Therefore, the first factor, $c\varphi_R$, is already pure, showing that R is ultra-F-regular. \square

Remark 3.12. Proposition 3.11 together with Theorem 1 and Remark 1.2 therefore yield the first assertion of Theorem 2. The last assertion is then a direct consequence of this (after localization), since the hypotheses imply that the inclusion $A^G \subseteq A$ is cyclically pure (in fact even split), where A is the affine coordinate ring of X and A^G the subring of G -invariant elements (so that $X/G = \text{Spec } A^G$).

Remark 3.13. Under the stronger assumption that S is regular in Theorem 2 rather than just having log-terminal singularities, we can still conclude that R has log-terminal singularities, without having to rely on the deep result by Hara. Namely, since S is regular, it is ultra-F-regular (Remark 3.4), whence so is R by Proposition 3.11, and therefore R has log-terminal singularities by (2) \Rightarrow (1) in Theorem 1.

3.14. Log-canonical singularities. The following is yet unclear. *If the non-standard Frobenius $\varphi_\infty: R \rightarrow R_\infty$ is pure, for R a \mathbb{Q} -Gorenstein local \mathbb{C} -affine domain, does R have log-canonical singularities? Is the converse also true? What if we only require that some ultra-Frobenius is pure?* Note that F-pure type implies log-canonical singularities by [43, Corollary 4.4], and this former condition is supposedly the analogue of φ_∞ being pure. If the question and its converse are both answered in the affirmative, we also have a positive solution to the following question: *if $R \rightarrow S$ is a cyclically pure homomorphism of \mathbb{Q} -Gorenstein local \mathbb{C} -affine domains and if S has log-canonical singularities, does then so have R ?* See also Remark 6.8 below for some related issues.

4. Vanishing of Maps of Tor

We start with providing a proof of Theorem 3 from the introduction. To this end, we need to review some results from [34] on the canonical construction of big Cohen-Macaulay algebras. For R a local \mathbb{C} -affine domain, let $\mathcal{B}(R)$ be the ultraproduct of the absolute integral closures $(R_p)^+$, where R_p is some approximation of R . We showed in [34, Theorem A] that $\mathcal{B}(R)$ is a (balanced) big Cohen-Macaulay algebra of R . It follows that if R is regular, then $R \rightarrow \mathcal{B}(R)$ is faithfully flat ([34, Corollary 2.5]).

This construction is weakly functorial in the sense that given any local homomorphism $R \rightarrow S$ of local \mathbb{C} -affine domains, we can find a (not necessarily unique) homomorphism $\mathcal{B}(R) \rightarrow \mathcal{B}(S)$ making the following diagram commute

$$(20) \quad \begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathcal{B}(R) & \longrightarrow & \mathcal{B}(S). \end{array}$$

If $R \rightarrow S$ is finite, then $\mathcal{B}(R) = \mathcal{B}(S)$ (see [34, Theorem 2.4]). As already observed in Remark 3.10, we have the following purity result.

Proposition 4.1. *If a local \mathbb{C} -affine domain R is ultra- F -regular, then $R \rightarrow \mathcal{B}(R)$ is cyclically pure (that is to say, R is \mathcal{B} -regular).*

Theorem 3 is a special case of the next result in view of Theorem 1 and Proposition 3.11. It generalizes [13, Theorem 4.12] (I will only deal with the Tor functor here; the more general form of loc. cit., can be proved by the same arguments).

Theorem 4.2 (Vanishing of maps of Tor). *Let $R \rightarrow S$ be a homomorphism of \mathbb{C} -affine algebras such that S is an ultra- F -regular domain. Let A be a regular subring of R over which R is module finite. Then for every A -module M and every $i \geq 1$, the natural morphism $\mathrm{Tor}_i^A(M, R) \rightarrow \mathrm{Tor}_i^A(M, S)$ is zero.*

Proof. If the map is non-zero, then it remains so after a suitable localization of S , so that we may assume that S is local. We then may localize A and R at the respective contractions of the maximal ideal of S , and assume that A and R are already local. Let \mathfrak{p} be a minimal prime of R contained in the kernel of the homomorphism $R \rightarrow S$. The composition $R \rightarrow R/\mathfrak{p} \rightarrow S$ induces a factorization $\mathrm{Tor}_i^A(M, R) \rightarrow \mathrm{Tor}_i^A(M, R/\mathfrak{p}) \rightarrow \mathrm{Tor}_i^A(M, S)$. Thus, in order to prove the statement, it suffices to show that the second homomorphism is zero, so that we may assume that R is a domain.

We have a commutative diagram

$$(21) \quad \begin{array}{ccccc} A & \longrightarrow & R & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B}(A) & \xlongequal{\quad} & \mathcal{B}(R) & \longrightarrow & \mathcal{B}(S). \end{array}$$

Let ϕ be the composite morphism

$$(22) \quad \mathrm{Tor}_i^A(M, R) \rightarrow \mathrm{Tor}_i^A(M, \mathcal{B}(R)) \rightarrow \mathrm{Tor}_i^A(M, \mathcal{B}(S)).$$

By the preceding discussion and our assumptions, $\mathcal{B}(A) = \mathcal{B}(R)$ and $\mathcal{B}(A)$ is flat over A . Therefore, the middle module in (22) is zero, whence so is ϕ . Using the commutativity of (21), we see that ϕ also factors as

$$\mathrm{Tor}_i^A(M, R) \rightarrow \mathrm{Tor}_i^A(M, S) \rightarrow \mathrm{Tor}_i^A(M, \mathcal{B}(S)).$$

By Proposition 4.1, the embedding $S \rightarrow \mathcal{B}(S)$ is cyclically pure, whence pure, by [11, Theorem 2.6] and the fact that S is normal. Therefore, $\mathrm{Tor}_i^A(M, S) \rightarrow \mathrm{Tor}_i^A(M, \mathcal{B}(S))$ is injective (see [13, Lemma 2.1(h)]). It follows from $\phi = 0$ that then also $\mathrm{Tor}_i^A(M, R) \rightarrow \mathrm{Tor}_i^A(M, S)$ must be zero, as required. \square

The next two results follow already from Hara's characterization of log-terminal singularities (equivalence (1) \iff (1') in Theorem 1) together with the pertinent facts on zero characteristic tight closure. However, we can give more direct proofs using our methods. The first of these is a Briançon-Skoda type result. For regular rings, it was first proved in [22]; a tight closure proof was given in [12].

Theorem 4.3 (Briançon-Skoda). *Let R be a local domain essentially of finite type over a field of characteristic zero and assume R has at most log-terminal singularities (or, more generally, is ultra- F -regular). If I is an ideal in R generated by at most n elements, then the integral closure of I^{n+k} is contained in I^{k+1} , for every $k \geq 0$.*

Proof. The proof is an immediate consequence of Proposition 4.1 applied to [34, Theorem B]. For the reader's convenience, we repeat the argument. Let R be ultra- F -regular and let I an ideal generated by at most n elements. Let z be an element in the integral closure of I^{n+k} , for some $k \in \mathbb{N}$. Take approximations R_p, I_p and z_p of R, I and z respectively. Since z satisfies an integral equation

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

with $a_i \in I^{(n+k)i}$, we have for almost all p an equation

$$(z_p)^n + a_{1p}(z_p)^{n-1} + \cdots + a_{np} = 0$$

with $a_{ip} \in (I_p)^{(n+k)i}$ an approximation of a_i . In other words, z_p lies in the integral closure of $(I_p)^{n+k}$, for almost all p . By [13, Theorem 7.1], almost every z_p lies in $(I_p)^{k+1} R_p^+ \cap R_p$. Taking ultraproducts, we get that $z \in I^{k+1} \mathcal{B}(R) \cap R$. By Proposition 4.1, we get that $z \in I^{k+1}$ as required. \square

Recall that the *symbolic power* $I^{(n)}$ of an ideal I in a ring R is by definition the collection of all $a \in R$ for which there exists an R/I -regular element $s \in R$ such that $sa \in I^n$. We always have an inclusion $I^n \subseteq I^{(n)}$. If $I := \mathfrak{p}$ is prime, then $\mathfrak{p}^{(n)}$ is just the \mathfrak{p} -primary component of \mathfrak{p}^n . The following generalizes the main results of [4] and [15] to log-terminal singularities.

Theorem 4.4. *Let R be a log-terminal (or, more generally, ultra- F -regular) \mathbb{C} -affine domain. Let \mathfrak{a} be an ideal in R and let h be the largest height of an associated prime of \mathfrak{a} (or more generally, the largest analytic spread of $\mathfrak{a}R_{\mathfrak{p}}$, for \mathfrak{p} an associated prime of R). If \mathfrak{a} has finite projective dimension, then $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$, for all n .*

Proof. The same argument that deduces [31, Theorem 3.4] from its positive characteristic counterpart [15, Theorem 1.1(c)], can be used to obtain the zero characteristic counterpart of [15, Theorem 1.1(b)], to wit, the fact that $\mathfrak{a}^{(hn)}$ lies in the

generic tight closure of \mathfrak{a}^n (use [30, Proposition 6.3] in conjunction with the techniques from [32, §4]). By Theorem 1 and Theorem 3.5, each ideal is equal to its generic tight closure, proving the assertion. \square

5. Polarizations

In this section, X denotes a projective scheme of finite type over some algebraically closed field K . Given an ample invertible \mathcal{O}_X -module \mathcal{P} , we will call the pair (X, \mathcal{P}) a *polarized scheme* and we call \mathcal{P} a *polarization* of X .

Fix a polarized scheme (X, \mathcal{P}) . For each \mathcal{O}_X -module \mathcal{F} , we define its *polarization* to be the sheaf

$$\mathcal{F}^{\text{pol}} := \bigoplus_{n \in \mathbb{Z}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^n.$$

In particular, for each \mathcal{F} , we have an isomorphism

$$\mathcal{F}^{\text{pol}} \cong \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\text{pol}}.$$

Definition 5.1. The *section ring* S of (X, \mathcal{P}) is the ring of global sections of $\mathcal{O}_X^{\text{pol}}$, that is to say,

$$(23) \quad S := \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{P}^n).$$

Note that S is a finitely generated graded algebra over $H^0(X, \mathcal{O}_X) = K$ by letting $[S]_n := H^0(X, \mathcal{P}^n)$ (ampleness is used to guarantee that S is finitely generated). In fact, S is positively graded, since \mathcal{P}^n has no global sections for $n < 0$.

The polarization can be completely recovered from the section ring S by the rules

$$X \cong \text{Proj } S \quad \text{and} \quad \mathcal{P} \cong \widetilde{S(1)}.$$

In fact, $\mathcal{P}^n = \widetilde{S(n)}$, for any $n \in \mathbb{Z}$. Global properties of X can now be studied via local properties of S (or more accurately, of $S_{\mathfrak{m}}$, where \mathfrak{m} is the irrelevant maximal ideal generated by all homogeneous elements of positive degree).

Definition 5.2. The *section module* of an \mathcal{O}_X -module \mathcal{F} (with respect to the polarization \mathcal{P}) is the module of global sections $H^0(X, \mathcal{F}^{\text{pol}})$ of \mathcal{F}^{pol} and is denoted $M_{\mathcal{P}}(\mathcal{F})$, or just $M(\mathcal{F})$, if the polarization is clear.

In particular, the section module $M(\mathcal{O}_X)$ of \mathcal{O}_X is just S itself. Let $F := M(\mathcal{F})$. We make F into a \mathbb{Z} -graded S -module by $[F]_n := H^0(X, \mathcal{F} \otimes \mathcal{P}^n)$, for $n \in \mathbb{Z}$. Indeed, for each $m, n \in \mathbb{Z}$, we have $[S]_m \cdot [F]_n \subseteq [F]_{m+n}$ because we have canonical isomorphisms $\mathcal{P}^m \otimes (\mathcal{F} \otimes \mathcal{P}^n) \cong \mathcal{F} \otimes \mathcal{P}^{m+n}$. If \mathcal{F} is coherent, then there is some n_0 such that $\mathcal{F} \otimes \mathcal{P}^n$ is generated by its global sections for all $n \geq n_0$, since \mathcal{P} is

ample. Therefore, if \mathcal{F} is coherent, F is finitely generated as an S -module. For each $n \in \mathbb{Z}$, we have an isomorphism

$$(24) \quad M(\widetilde{\mathcal{F}})(n) \cong \mathcal{F} \otimes \mathcal{P}^n.$$

Indeed, it suffices to prove that both sheaves have the same sections on each open $D_+(x)$ ($:=$ the set of all homogeneous prime ideals not containing the homogeneous element x), and this is straightforward. Note that we can in particular recover \mathcal{F} from its section module $M(\mathcal{F})$ since $\mathcal{F} \cong \widetilde{M(\mathcal{F})}$.

5.3. Čech Cohomology and Polarizations. Let (X, \mathcal{P}) be a polarized scheme with section ring $S := M(\mathcal{O}_X)$. Let \mathfrak{m} be the maximal irrelevant ideal of S and let $\mathbf{x} := (x_1, \dots, x_s)$ be a homogeneous system of parameters of S (so that $\mathbf{x}S$ is in particular \mathfrak{m} -primary). For each tuple \mathbf{i} of indices given by $1 \leq i_1 < i_2 < \dots < i_m \leq s$, set

$$\mathbf{x}_{\mathbf{i}} := x_{i_1} x_{i_2} \cdots x_{i_m} \quad \text{and} \quad U_{\mathbf{i}} := D_+(\mathbf{x}_{\mathbf{i}})$$

Let $\mathcal{U}_{\mathbf{x}}$ be the affine open covering of X given by the $U_{\mathbf{i}} := D_+(x_i)$.

5.3.1. Čech complex of a sheaf. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. We generalize the discussion in §2.13 to polarizations as follows. The *Čech complex* of the polarization of \mathcal{F} with respect to the covering $\mathcal{U}_{\mathbf{x}}$ is the complex $\mathcal{C}^\bullet(\mathcal{U}_{\mathbf{x}}; \mathcal{F}^{\text{pol}})$ given as

$$0 \rightarrow \mathcal{C}^1 := \bigoplus_i H^0(U_i, \mathcal{F}^{\text{pol}}) \rightarrow \dots \rightarrow \mathcal{C}^m := \bigoplus_{\mathbf{i}} H^0(U_{\mathbf{i}}, \mathcal{F}^{\text{pol}}) \rightarrow \dots$$

where in \mathcal{C}^m the index \mathbf{i} runs over all m -tuples of indices $1 \leq i_1 < i_2 < \dots < i_m \leq s$ and where the morphisms are, up to sign, given by restriction (see [10, Chapt. III. §4] for more details). Using (24), we see that $H^0(U_{\mathbf{i}}, \mathcal{F} \otimes \mathcal{P}^n)$ is isomorphic to $[M(\mathcal{F})_{\mathbf{x}_{\mathbf{i}}}]_n$. Therefore, we have a (degree preserving) isomorphism

$$(25) \quad H^0(U_{\mathbf{i}}, \mathcal{F}^{\text{pol}}) \cong M(\mathcal{F})_{\mathbf{x}_{\mathbf{i}}}.$$

5.3.2. Čech complex of a module. More generally, we associate to an arbitrary S -module F a *Čech complex* $\mathcal{C}^\bullet(\mathbf{x}; F)$ given as

$$(26) \quad 0 \rightarrow \mathcal{C}^1 := \bigoplus_i F_{x_i} \rightarrow \dots \rightarrow \mathcal{C}^m := \bigoplus_{\mathbf{i}} F_{\mathbf{x}_{\mathbf{i}}} \rightarrow \dots$$

(with notation as above) where the morphisms are, up to sign, the natural inclusions (in fact, this construction can also be made in the non-graded case, with \mathbf{x} an arbitrary tuple of elements in S ; see [3, p. 129] for more details). For an \mathcal{O}_X -module \mathcal{F} , we get using (25), an isomorphism of complexes

$$(27) \quad \mathcal{C}^\bullet(\mathcal{U}_{\mathbf{x}}; \mathcal{F}^{\text{pol}}) \cong \mathcal{C}^\bullet(\mathbf{x}; M(\mathcal{F})).$$

In particular, the cohomology of either complex can be used to compute the sheaf cohomology of \mathcal{F} .

5.3.3. Local cohomology. On the other hand, for an arbitrary S -module F , the complex $\mathcal{C}^\bullet(\mathbf{x}; F)$ can also be used to calculate local cohomology. Recall that $H_m^0(F)$ is equal to the \mathfrak{m} -torsion $\Gamma_m(F)$ of F , that is to say, equal to the (homogeneous) submodule of all elements of F which are annihilated by some power of \mathfrak{m} ; the derived functors of $\Gamma_m(\cdot)$ are then the local cohomology modules $H_m^i(\cdot)$. By [3, Theorem 3.5.6], the local cohomology of an S -module F can be computed as the cohomology of the augmented complex $0 \rightarrow F \rightarrow \mathcal{C}^\bullet(\mathbf{x}; F)$ (that is, we inserted an additional term $\mathcal{C}^0 := F$ in (26)). Hence for $i > 1$, we have an isomorphism

$$(28) \quad H^i(\mathcal{C}^\bullet(\mathbf{x}; F)) \cong H_m^i(F),$$

whereas for $i = 1$, we have a short exact sequence

$$(29) \quad 0 \rightarrow H_m^0(F) \rightarrow F \rightarrow H^1(\mathcal{C}^\bullet(\mathbf{x}; F)) \rightarrow H_m^1(F) \rightarrow 0.$$

Since all local cohomology modules are Artinian, F and $H^1(\mathcal{C}^\bullet(\mathbf{x}; F))$ have the same localizations at the various x_i . If F is a finitely generated graded S -module, then $H^1(\mathcal{C}^\bullet(\mathbf{x}; F)) = M(\tilde{F})$, where \tilde{F} is the \mathcal{O}_X -module associated to F (use (33) below). In conclusion, we have an equality of complexes

$$(30) \quad \mathcal{C}^\bullet(\mathbf{x}; F) = \mathcal{C}^\bullet(\mathbf{x}; M(\tilde{F}))$$

and (29) becomes the exact sequence (see also [5, Theorem A4.1])

$$(31) \quad 0 \rightarrow H_m^0(F) \rightarrow F \rightarrow M(\tilde{F}) \rightarrow H_m^1(F) \rightarrow 0.$$

5.3.4. Comparison of cohomology. Let us summarize some of these observations. By (8) and (27), we have for each $i \geq 0$, isomorphisms of graded S -modules

$$(32) \quad H^i(X, \mathcal{F}^{\text{pol}}) \cong H^{i+1}(\mathcal{C}^\bullet(\mathfrak{U}_x; \mathcal{F}^{\text{pol}})) \cong H^{i+1}(\mathcal{C}^\bullet(\mathbf{x}; M(\mathcal{F}))).$$

Moreover, for $i \geq 1$, these modules are also isomorphic to $H_m^{i+1}(M(\mathcal{F}))$ by (28). In particular, with $i = 0$, isomorphism (32) becomes

$$(33) \quad M(\mathcal{F}) = H^0(X, \mathcal{F}^{\text{pol}}) \cong H^1(\mathcal{C}^\bullet(\mathfrak{U}_x; \mathcal{F}^{\text{pol}})).$$

Using that the isomorphisms in (32) preserve degree, we have for each $i \geq 0$ and each $n \in \mathbb{Z}$, isomorphisms

$$(34) \quad H^i(X, \mathcal{F} \otimes \mathcal{P}^n) \cong H^{i+1}([\mathcal{C}^\bullet(\mathbf{x}; M(\mathcal{F}))]_n) \cong [H_m^{i+1}(M(\mathcal{F}))]_n$$

(where the final isomorphism only holds for $i \geq 1$).

Lemma 5.4. *Let (X, \mathcal{P}) be a polarized scheme with section ring S and let \mathcal{F} and \mathcal{G} be two coherent \mathcal{O}_X -modules. If \mathcal{P} is very ample, then there is a short exact sequence (of degree preserving morphisms)*

$$(35) \quad 0 \rightarrow H_m^0(M(\mathcal{F}) \otimes_S M(\mathcal{G})) \rightarrow M(\mathcal{F}) \otimes_S M(\mathcal{G}) \rightarrow M(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \rightarrow H_m^1(M(\mathcal{F}) \otimes_S M(\mathcal{G})) \rightarrow 0.$$

Proof. Let $F := M(\mathcal{F})$ and $G := M(\mathcal{G})$ be the respective section modules of \mathcal{F} and \mathcal{G} . By (31), it suffices to show that the \mathcal{O}_X -module $\widetilde{F \otimes_S G}$ associated to $F \otimes_S G$ is isomorphic with $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. Since \mathcal{P} is very ample, S is generated by its linear forms and it suffices to check that both sheaves agree over each open $D_+(x)$ with x homogeneous of degree one. To this end, we get, using (24), isomorphisms

$$\begin{aligned} (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(D_+(x)) &\cong \mathcal{F}(D_+(x)) \otimes_{\mathcal{O}_X(D_+(x))} \mathcal{G}(D_+(x)) \\ &\cong [F_x]_0 \otimes_{[S_x]_0} [G_x]_0 \\ &\cong [(F \otimes_S G)_x]_0 \\ &\cong (\widetilde{F \otimes_S G})(D_+(x)) \end{aligned}$$

where the penultimate isomorphism follows from [7, II. Proposition 2.5.13] (to apply this, it is necessary that x has degree one). \square

The main application of Frobenius is through the following easy fact.

Lemma 5.5. *If (X, \mathcal{P}) is a polarized scheme of characteristic $p > 0$ and \mathcal{F} an invertible sheaf on X , then for any power q of p , we have*

$$(36) \quad (\varphi_{q*}(\widetilde{M(\mathcal{F})})) \cong \mathcal{F}^q.$$

Proof. Let $S := M(\mathcal{O}_X)$ be the section ring of the given polarization. Since (36) can be checked locally, one easily reduces to the case that \mathcal{F} is the ideal-sheaf associated to a principal ideal aS , with a a non-zero divisor in S . Moreover, the zeroth and first local cohomology of aS vanish, so that $M(\mathcal{F}) = aS$ by (31). Therefore, $\varphi_{q*}(M(\mathcal{F})) = \varphi_{q*}(aS) \cong a^q S$, from which (36) follows by taking associated sheaves. \square

We also want to remind the reader of the following observation made in [40].

Proposition 5.6. *The section ring of a polarized scheme (X, \mathcal{P}) is Cohen-Macaulay if, and only if, $H^i(X, \mathcal{P}^n) = 0$ for all n and all $0 < i < \dim X$.*

Under the additional assumption that X is Cohen-Macaulay, some section ring of X is Cohen-Macaulay if, and only if, $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < \dim X$.

Proof. Let S be the section ring of (X, \mathcal{P}) and let \mathfrak{m} be its maximal irrelevant ideal. As explained in [18, Proposition 2.1], the local cohomology groups $H_{\mathfrak{m}}^0(S)$ and $H_{\mathfrak{m}}^1(S)$ always vanish. By a theorem of Grothendieck ([3, Theorem 3.5.7]), S is Cohen-Macaulay if, and only if, $H_{\mathfrak{m}}^{i+1}(S) = 0$, for all $i < \dim X$. By (32) in §5.3.4, this in turn is equivalent with $H^i(X, \mathcal{P}^n) = 0$, for all $0 < i < \dim X$ and all n , proving the first assertion.

Suppose X is moreover Cohen-Macaulay. Since \mathcal{P} is invertible, $H^i(X, \mathcal{P}^n) = 0$ for all $0 < i < \dim X$ and $n \ll 0$ by Serre duality ([10, III. Theorem 7.6]). The same is true for $n \gg 0$, since \mathcal{P} is ample ([10, III. Proposition 5.3]). Therefore polarizing X with respect to a sufficiently large power \mathcal{P}^s instead of \mathcal{P} , we may even assume

that $H^i(X, \mathcal{P}^n) = 0$ for all $n \neq 0$ and $0 < i < \dim X$. The second assertion then follows from this and the first assertion (applied to the new polarization). \square

6. Vanishing Theorems

In [40], Smith introduces the notion of a globally F -regular type variety and shows that it admits several vanishing theorems. She moreover conjectures that any GIT quotient by a reductive group of a smooth Fano variety satisfies these vanishing theorems. We will establish this conjecture using similar arguments as in loc. cit., but substituting ultra- F -regularity for F -regularity.

Let X be a connected projective scheme of finite type over \mathbb{C} (a *projective variety*, for short).

Definition 6.1. We say that X is *globally ultra- F -regular*, if some section ring S of X is ultra- F -regular at its vertex, that is to say, $S_{\mathfrak{m}}$ is ultra- F -regular where \mathfrak{m} is the maximal irrelevant ideal of S .

Remark 6.2. In [40], Smith calls X *globally F -regular* if some section ring S is strongly F -regular type (note that since S is positively graded, this is equivalent by [23] with S being weakly F -regular type). By Theorem 3.5, this implies that S is ultra- F -regular and hence that X is globally ultra- F -regular. In particular, if $S_{\mathfrak{m}}$ is (\mathbb{Q}) -Gorenstein, where \mathfrak{m} is the maximal irrelevant ideal, then globally F -regular type and globally ultra- F -regular are equivalent in view of Theorem 1.

So we could deduce the desired vanishing theorems from Theorem 1 and the work of Smith in [40], if we are willing to use Hara's characterization of F -regular type. However, using a non-standard version of her arguments, we can as easily derive the vanishing theorems directly, without any appeal to Hara's work (and hence without using Kodaira Vanishing).

Remark 6.3. As in [40], one can prove directly that if X is globally ultra- F -regular, then every section ring is locally ultra- F -regular at its vertex. Alternatively, this follows from [40, Theorem 3.10] (even without localizing at the irrelevant maximal ideal), if we use Theorem 1 as in the previous remark.

In that respect, note that if the section ring S with respect to the polarization \mathcal{P} is ultra- F -regular at its vertex, then so is any Veronese subring $S^{(r)} := \bigoplus_n [S]_{rn}$ by Proposition 3.11, as it is a pure subring. In particular, any section ring corresponding to a positive power of \mathcal{P} is ultra- F -regular at its vertex. In particular, we may always assume, without relying on the results from [40], that a globally ultra- F -regular variety admits a *very ample* polarization whose section ring is ultra- F -regular at its vertex.

As already mentioned, the main advantage of using ultra-F-regularity instead of F-regular type is the fact that it descends under pure homomorphism (Proposition 3.11). In particular, we get the following descent property for quotient singularities.

Theorem 6.4. *Let X be a connected projective variety over \mathbb{C} . Let G be a reductive group acting algebraically on X and let $X//G$ be an arbitrary GIT quotient of X . If X is globally ultra-F-regular, then so is $X//G$.*

Proof. Any GIT quotient of X is obtained by taking some polarization \mathcal{P} of X , extending the G -action to \mathcal{P} , taking the section ring S with the induced G -action and letting $X//G := \text{Proj } S^G$, where S^G is the ring of invariants of S . In particular, S^G is a section ring of $X//G$. Since S is ultra-F-regular by Remark 6.3, so is S^G by Proposition 3.11 as $S^G \subseteq S$ is pure (even split). \square

We now proceed with the main technical result in this section, which will be used to derive the Kawamata-Viehweg vanishing stated in Theorem 4.

Theorem 6.5. *Let X be a globally ultra-F-regular connected projective variety over \mathbb{C} and let \mathcal{F} be an invertible \mathcal{O}_X -module. If for some $i > 0$ and some effective Cartier divisor D , all $H^i(X, \mathcal{F}^n(D))$ vanish for $n \gg 0$, then $H^i(X, \mathcal{F})$ vanishes.*

Proof. Choose a polarization \mathcal{P} of X with section ring S , so that $S_{\mathfrak{m}}$ is ultra-F-regular, where \mathfrak{m} is the maximal irrelevant ideal of S . By Remark 6.3, we may assume without loss of generality that \mathcal{P} is very ample. Let I be the section module of $\mathcal{T} := \mathcal{O}_X(D)$. Let \mathbf{x} be a homogeneous system of parameters of S and let $\mathcal{U}_{\mathbf{x}}$ be the open affine covering given by the $D_+(x_i)$ ($= \text{Proj}([S_{x_i}]_0)$). Since D is Cartier, I is a fractional ideal, that is to say, a finitely generated rank-one S -submodule of the field of fractions K of S . Clearing denominators in the inclusion $I \subseteq K$, we can find an S -module morphism $g: I \rightarrow S$. Since D is effective, I admits a canonical section $s \in [I]_0 = H^0(X, \mathcal{T})$ (see for instance [6, B.4.5]). In particular, the morphism $S \rightarrow I: 1 \mapsto s$ is degree preserving. Put $c := g(s)$. By ultra-F-regularity, there is an ultra-Frobenius φ such that $c\varphi_{S_{\mathfrak{m}}}$ is pure. The composition $S \rightarrow I \xrightarrow{g} S: 1 \mapsto s \mapsto c$ is equal to multiplication with c on S (where we disregard the grading). Tensoring this composite homomorphism with φ_*S_{∞} gives

$$\varphi_*S_{\infty} \rightarrow I \otimes \varphi_*S_{\infty} \rightarrow \varphi_*S_{\infty}: 1 \mapsto s \otimes 1 \mapsto c$$

which composed with the inclusion $S \hookrightarrow \varphi_*S_{\infty}$ therefore gives the morphism $c\varphi_S$. By assumption, the base change $c\varphi_{S_{\mathfrak{m}}}$ is pure. Since

$$S_{\mathfrak{m}} \rightarrow S_{\mathfrak{m}} \otimes I \otimes \varphi_*S_{\infty}$$

is a factor of the pure morphism $c\varphi_{S_{\mathfrak{m}}}$, it is also pure. Let $F := M(\mathcal{F})$ be the section module of \mathcal{F} . Tensoring with F yields a pure $S_{\mathfrak{m}}$ -module morphism

$$F_{\mathfrak{m}} \rightarrow (F \otimes I \otimes \varphi_*S_{\infty})_{\mathfrak{m}}.$$

Using the isomorphism $\varphi_*F \cong F \otimes \varphi_*S_{\infty}$ (see §3.2), we can identify $F \otimes I \otimes \varphi_*S_{\infty}$ with $I \otimes \varphi_*F$. Taking Čech complexes with respect to the tuple \mathbf{x} yields a pure

homomorphism of Čech complexes

$$\mathcal{C}^\bullet(\mathbf{x}; F_{\mathfrak{m}}) \rightarrow \mathcal{C}^\bullet(\mathbf{x}; (I \otimes \varphi_* F)_{\mathfrak{m}}).$$

It is well-known that purity is preserved after taking cohomology, so that we have a pure morphism

$$H^{i+1}(\mathcal{C}^\bullet(\mathbf{x}; F_{\mathfrak{m}})) \hookrightarrow H^{i+1}(\mathcal{C}^\bullet(\mathbf{x}; (I \otimes \varphi_* F)_{\mathfrak{m}})).$$

Since at a maximal ideal, the cohomology of a localized Čech complex is the same as the cohomology of the non-localized Čech complex (see for instance [3, Remark 3.6.18]), we get an injective morphism

$$(37) \quad H^{i+1}(\mathcal{C}^\bullet(\mathbf{x}; F)) \hookrightarrow H^{i+1}(\mathcal{C}^\bullet(\mathbf{x}; I \otimes \varphi_* F))$$

By a similar argument as in §2.9, each module in $\mathcal{C}^\bullet(\mathbf{x}; I \otimes \varphi_* F)$, although not graded, has a graded piece in each (standard or non-standard) degree. This property is inherited by the cohomology groups and (37) preserves degrees. Hence in degree zero, we get an injective morphism

$$(38) \quad [H^{i+1}(\mathcal{C}^\bullet(\mathbf{x}; F))]_0 \hookrightarrow [H^{i+1}(\mathcal{C}^\bullet(\mathbf{x}; I \otimes \varphi_* F))]_0.$$

I claim that the right hand side of (38) is zero, whence by injectivity, so is the left hand side. Since the latter is just $H^i(X, \mathcal{F})$ by (34), the theorem follows from the claim.

To prove the claim, let $(X_p, \mathcal{P}_p), S_p, \mathbf{x}_p, \mathcal{F}_p$ and \mathcal{I}_p be approximations of $(X, \mathcal{P}), S, \mathbf{x}, \mathcal{F}$ and \mathcal{I} respectively, and suppose the ultra-Frobenius φ is given as the ultraproduct of the Frobenii φ_q (for $q := p^{e_p}$ some power of p). By §2.11, almost all (X_p, \mathcal{P}_p) are polarized. Using Theorem 2.15, the section modules $F_p := M(\mathcal{F}_p)$ and $I_p := M(\mathcal{I}_p)$ are approximations of respectively F and I . In particular, the ultraproduct of the $\varphi_{q_*} F_p \cong F_p \otimes \varphi_{q_*} S_p$ is equal to $\varphi_* F$ and we have an isomorphism of Čech complexes

$$\mathcal{C}^\bullet(\mathbf{x}; I \otimes \varphi_* F) \cong \text{ulim}_{p \rightarrow \infty} \mathcal{C}^\bullet(\mathbf{x}_p; I_p \otimes \varphi_{q_*} F_p).$$

Since cohomology commutes with ultraproducts, we get an isomorphism

$$H^{i+1}(\mathcal{C}^\bullet(\mathbf{x}; I \otimes \varphi_* F)) \cong \text{ulim}_{p \rightarrow \infty} H^{i+1}(\mathcal{C}^\bullet(\mathbf{x}_p; I_p \otimes \varphi_{q_*} F_p)).$$

Therefore, the claim follows if we can show that almost all

$$(39) \quad [H^{i+1}(\mathcal{C}^\bullet(\mathbf{x}_p; I_p \otimes \varphi_{q_*} F_p))]_0 = 0.$$

Let $\mathfrak{U}_{\mathbf{x}_p}$ be the affine covering of X_p given by the $D_+(x_{i_p})$. By Lemma 5.5, we have isomorphisms of \mathcal{O}_{X_p} -modules

$$(\widetilde{\varphi_{q_*} F_p}) \cong \mathcal{F}_p^q.$$

Applying Lemma 5.4 twice shows that $M(\mathcal{I}_p \otimes \mathcal{F}_p^q)$ and $I_p \otimes \varphi_{q*} F_p$ are isomorphic up to m -torsion. In particular, this yields an isomorphism of Čech complexes

$$\mathcal{C}^\bullet(\mathbf{x}_p; I_p \otimes \varphi_{q*} F_p) \cong \mathcal{C}^\bullet(\mathfrak{U}_{\mathbf{x}_p}; (\mathcal{I}_p \otimes \mathcal{F}_p^q)^{\text{pol}}).$$

Taking cohomology, we get

$$H^{i+1}(\mathcal{C}^\bullet(\mathbf{x}_p; I_p \otimes \varphi_{q*} F_p)) \cong H^{i+1}(\mathcal{C}^\bullet(\mathfrak{U}_{\mathbf{x}_p}; (\mathcal{I}_p \otimes \mathcal{F}_p^q)^{\text{pol}})).$$

By (34) in §5.3.4, the zero-th homogeneous part of the right hand side is isomorphic to $H^i(X_p, \mathcal{I}_p \otimes \mathcal{F}_p^q)$, and this is zero for large enough p by Corollary 2.16 and our assumption. Thus, we showed (39) and hence completed the proof. \square

As in [40], we immediately obtain the following corollaries; we include their proofs for the reader's convenience. Together with Remark 1.2, they prove Theorem 4 from the introduction.

Corollary 6.6. *Let X be a globally ultra- F -regular connected projective variety over \mathbb{C} and let \mathcal{L} be an invertible \mathcal{O}_X -module. If \mathcal{L} is numerically effective (NEF), then $H^i(X, \mathcal{L})$ vanishes, for all $i > 0$.*

Proof. Suppose first that \mathcal{L} is ample. By Serre Vanishing, $H^i(X, \mathcal{L}^n) = 0$ for $n \gg 0$ and $i > 0$. Hence $H^i(X, \mathcal{L}) = 0$ by Theorem 6.5. Suppose now that \mathcal{L} is merely NEF. This means that we can find an ample effective Cartier divisor D such that $\mathcal{L}^n(D)$ is ample, for all $n \geq 0$. Since we already proved the ample case, $H^i(X, \mathcal{L}^n(D)) = 0$, for all $n \geq 0$ and $i > 0$. Therefore, $H^i(X, \mathcal{L}) = 0$ by another application of Theorem 6.5. \square

Corollary 6.7 (Kawamata-Viehweg Vanishing). *Let X be a connected projective variety over \mathbb{C} and let \mathcal{L} be an invertible \mathcal{O}_X -module. If X is globally ultra- F -regular and if \mathcal{L} is big and NEF, then $H^i(X, \mathcal{L}^{-1}) = 0$, for all $i < \dim X$.*

Proof. Fix some $i < \dim X$. Because \mathcal{L} is big and NEF, we can find an effective Cartier divisor D such that $\mathcal{L}^m(-D)$ is ample for all $m \gg 0$, by [20, Proposition 2.61]. Let S be a section ring of X which is ultra- F -regular. Since S is therefore Cohen-Macaulay, so is X . Given an ample invertible sheaf \mathcal{P} , Serre duality yields $H^i(X, \mathcal{P}^n) = 0$, for all n of sufficiently large absolute value (see the argument in the proof of Proposition 5.6). Applied to $\mathcal{P} := \mathcal{L}^m(-D)$, this gives

$$H^i(X, (\mathcal{L}^{-m}(D))^n) = H^i(X, (\mathcal{L}^m(-D))^{-n}) = 0$$

for all sufficiently large m and n . Hence, for fixed m , Theorem 6.5 yields the vanishing of $H^i(X, \mathcal{L}^{-m}(D))$. Since this holds for all large m , another application of Theorem 6.5 then gives $H^i(X, \mathcal{L}^{-1}) = 0$. \square

Remark 6.8. Call a \mathbb{C} -affine domain R *ultra- F -pure*, if $R \rightarrow \varphi_* R_\infty$ is pure for some ultra-Frobenius φ . Call a connected projective variety X over \mathbb{C} *globally ultra- F -pure*, if some section ring of X is ultra- F -pure. Inspecting the proof of Theorem 6.5, we get the following weaker version ($D = 0$): *if X is globally ultra- F -pure*

and \mathcal{L} invertible with $H^i(X, \mathcal{L}^n) = 0$ for all $n \gg 0$, then $H^i(X, \mathcal{L}) = 0$. In particular, the argument in the proof of Corollary 6.6 shows that on a globally ultra-F-pure variety, an ample invertible sheaf has no higher cohomology.

In fact, we can even prove Kodaira Vanishing for this class of varieties: *if X is globally ultra-F-pure and Cohen-Macaulay, then $H^i(X, \mathcal{L}^{-1}) = 0$ for all $i < \dim X$ and all ample invertible sheaves \mathcal{L} on X .* Indeed, by Serre duality ([10, III. Corollary 7.7]), the dual of $H^i(X, \mathcal{L}^{-n})$ is $H^{d-i}(X, \omega_X \otimes \mathcal{L}^n)$ where d is the dimension of X and ω_X the dualizing sheaf on X . Because \mathcal{L} is ample, the latter cohomology group vanishes for large n , and hence so does the first. Applying the weaker version of the vanishing theorem to this, we get that $H^i(X, \mathcal{L}^{-1})$ vanishes.

Because of the analogy with the notion of *Frobenius split* (see [40, Proposition 3.1]) and the fact that a Schubert variety has this property ([25, Theorem 2]), it is reasonable to expect that a Schubert variety is globally ultra-F-pure. The referee has pointed out to me that the analogue of this in positive characteristic has recently been proven in the preprint [21]. In particular, if this result on Schubert varieties also holds in characteristic zero, then we get Kodaira Vanishing for any GIT quotient of a Schubert variety, since ultra-F-purity descends under pure homomorphisms (by the same argument as for Proposition 3.11).

7. Fano Varieties

Let X be a connected, normal projective variety over \mathbb{C} . The canonical (or, *dualizing*) sheaf ω_X of X is the unique reflexive sheaf which agrees with the sheaf $\wedge^d \Omega_{X/\mathbb{C}}$ on the smooth locus of X . We call X *Fano*, if its anti-canonical sheaf ω_X^{-1} is ample (we do not require X to be smooth).

Theorem 7.1. *A Fano variety with rational singularities is globally ultra-F-regular.*

Proof. Let X be a Fano variety with rational singularities. Let S be the anti-canonical section ring of X , that is to say, the section ring with respect to the polarization given by the ample sheaf ω_X^{-1} . It is well-known (see for instance [40, Proposition 6.2]), that S is Gorenstein and has again rational singularities. Since rational Gorenstein singularities are log-terminal, we obtain from Theorem 1 that S_m is ultra-F-regular, showing that X is globally ultra-F-regular. \square

Remark 7.2. In proving that a Fano variety with rational singularities is globally ultra-F-regular, we have used Kodaira Vanishing twice: via Hara's result in Theorem 1 and via [40, Proposition 6.2]. Combining Theorems 4 and 6.4 with the previous theorem yields Theorem 5 from the introduction.

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