IDEALS IN LOCAL RINGS OF FINITE EMBEDDING DIMENSION

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ABSTRACT. Ideal theory for local rings of finite embedding dimension.

1. FINITE EMBEDDING DIMENSION

By a *local* ring $(R, \mathfrak{m})$, we mean a (not necessarily Noetherian) commutative ring R with a unique maximal ideal $\mathfrak{m}$. The minimal number of generators of $\mathfrak{m}$ is called the *embedding dimension* of R. We studied the local algebra of local rings of finite embedding dimension in [3]. In this paper, we study their ideal theory.

A local ring $(R, \mathfrak{m})$ comes with a canonically defined topology given by the maximal ideal (called the $\mathfrak{m}$-*adic* topology). All topological terminology therefore refers to this topology. In particular, R is called *separated* if it is Hausdorff in its $\mathfrak{m}$-adic topology. This is equivalent with $\mathfrak{I}_R = 0$, where $\mathfrak{I}_R$ is the *ideal of infinitesimals*, defined as the intersection of all powers $\mathfrak{m}^n$. We call the residue ring $R/\mathfrak{I}_R$ therefore the *separated quotient* of R. Similarly, an ideal I is called *closed*, if it is so in the $\mathfrak{m}$-adic topology, or, equivalently, if the corresponding residue ring R/I is separated. The *completion* $\widehat{R}$ of R is its $\mathfrak{m}$-adic completion, given as the inverse limit of all $R/\mathfrak{m}^n$. If R has finite embedding dimension, the completion $\widehat{R}$ is again a local ring with maximal ideal $\mathfrak{m}\widehat{R}$, which is moreover complete, by [3, ?]. In fact, this implies:

Theorem A. *The completion of local ring of finite embedding dimension is Noetherian.* □

1.1. **Corollary.** *Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension and let I be an arbitrary ideal in R. The completion of R/I is $\widehat{R}/I\widehat{R}$. In particular, the closure of I is $I\widehat{R} \cap R$.*

Proof. Let $\bar{R} := R/I$ and let $S := \widehat{R}/I\widehat{R} = \widehat{R} \otimes_R \bar{R}$. The isomorphism $R/\mathfrak{m}^n \cong \widehat{R}/\mathfrak{m}^n\widehat{R}$ induces by base change an isomorphism $\bar{R}/\mathfrak{m}^n\bar{R} \cong S/\mathfrak{m}^n S$. Hence $\bar{R}$ and S have the same completion. However, since $\widehat{R}$ is complete, so is S, showing that it is the completion of $\bar{R}$.

Applied with I an $\mathfrak{m}$-primary ideal, we get an isomorphism $R/I \cong \widehat{R}/I\widehat{R}$ showing that $I\widehat{R} \cap R = I$, that is to say, that I is *contracted from* $\widehat{R}$. Since this property is preseved under arbitrary intersections, every closed ideal I is contracted from $\widehat{R}$, as it is the intersection of the $\mathfrak{m}$-primary ideals $I + \mathfrak{m}^n$. Conversely, if $I\widehat{R} \cap R = I$, then R/I embeds in $\widehat{R}/I\widehat{R}$, and by the first assertion, this is its completion. In particular, R/I is separated, that is to say, I is closed. □

In particular, $\mathsf{Cl}(R)$ satisfies the ascending chain condition. Indeed, if $I_1 \subseteq I_2 \subseteq \dots$ is an increasing chain of closed ideals in R, then, since $\widehat{R}$ is Noetherian, their extension to

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$\widehat{R}$ must become stationary, say $I_n\widehat{R} = I_{n+k}\widehat{R}$, for all k, and hence contracting back to R gives $I_n = I_{n+k}$, for all k.

1.2. **Corollary.** *If $(R, \mathfrak{m})$ is a local ring of finite embedding dimension, then the image of the map $\mathsf{Spec}(\widehat{R}) \to \mathsf{Spec}(R)$ is $\mathsf{ClSpec}(R)$.*

Proof. By Corollary 1.1, the image of the map consists of closed prime ideals. To prove the converse, let $\mathfrak{p}$ be an arbitrary closed prime ideal of R. By Corollary 1.1, we have $\mathfrak{p} = \mathfrak{p}\widehat{R} \cap R$. Let $\mathfrak{N}$ be maximal in $\widehat{R}$ with the property that $\mathfrak{p} = \mathfrak{N} \cap R$. I claim that $\mathfrak{N}$ is a prime ideal, showing that $\mathfrak{p}$ lies in the image of $\mathsf{Spec}(\widehat{R}) \to \mathsf{Spec}(R)$. To prove the claim, suppose $fg \in \mathfrak{N}$ but $f, g \notin \mathfrak{N}$. By maximality, there exist $a, b \in R \setminus \mathfrak{p}$ such that $a \in \mathfrak{N} + f\widehat{R}$ and $b \in \mathfrak{N} + g\widehat{R}$. Hence $ab \in \mathfrak{N} + fg\widehat{R} = \mathfrak{N}$ and since $ab \in R$, we get $ab \in \mathfrak{N} \cap R = \mathfrak{p}$, contradicting that $\mathfrak{p}$ is prime. □

1.3. **Lemma.** *If the completion of a local ring $(R, \mathfrak{m})$ of finite embedding dimension is Artinian, then so is R.*

Proof. By assumption, $\mathfrak{m}^n\widehat{R} = 0$, for some n. Since $R/\mathfrak{m}^{n+1} \cong \widehat{R}/\mathfrak{m}^{n+1}\widehat{R} = \widehat{R}$, we get $\mathfrak{m}^n/\mathfrak{m}^{n+1} = 0$. Since $\mathfrak{m}$ is finitely generated, we may apply Nakayama's Lemma and conclude that $\mathfrak{m}^n = 0$, which implies that R is Artinian. □

Ultra-Noetherian rings. In [3], the most important class of local rings of finite embedding dimension was the class of all *ultra-Noetherian* local rings, defined as any ultraproduct of Noetherian local rings of bounded embedding dimension. More precisely, let m be some positive integer, let W be an infinite index set, and for each $w \in W$, let $(R_w, \mathfrak{m}_w)$ be a Noetherian local ring of embedding dimension at most m. Choose a non-principal ultrafilter on W and let $\mathtt{u}R$ be the ultraproduct of the R_w. It follows that $\mathtt{u}R$ is a local ring whose maximal ideal is given by the ultraproduct of the $\mathfrak{m}_w$, and hence is generated by at most m elements. The R_w are called *components* of $\mathtt{u}R$ (inspite of the fact that they are not uniquely determined by $\mathtt{u}R$). Moreover, if W is countable or if the ultrafilter is countably incomplete, then by [3, ? and ?], we have:

Theorem B. *Let $\mathtt{u}R$ be an ultra-Noetherian local ring with components R_w. The separated quotient $\mathtt{c}R := \mathtt{u}R/\mathfrak{I}_{\mathtt{u}R}$, called the* cataproduct *of the R_w, is equal to completion of $\mathtt{u}R$ and hence in particular is Noetherian. The closure of an ideal $I \subseteq \mathtt{u}R$ is equal to $I + \mathfrak{I}_R$.*

Moreover, if $\mathtt{u}R$ is the ultrapower of a Noetherian local ring R, that is to say, if all R_w are equal to R, then there is a canonical embedding $R \to \mathtt{c}R$, which is moreover faithfully flat. □

2. Primary decomposition for closed ideals

Let $I \subseteq A$ be an ideal and $\mathfrak{p}$ a prime ideal containing I. We say that $\mathfrak{p}$ *belongs* to I, if there is some x such that $\mathfrak{p}$ is the radical of $(I : x)$. In case $\mathfrak{p}$ can actually be made equal to $(I : x)$, we call $\mathfrak{p}$ an *associated* prime of I. We say that I has a *primary decomposition*, if it can be written as a finite intersection of primary ideals, say $I = \mathfrak{g}_1 \cap \dots \cap \mathfrak{g}_s$. We call such a decomposition *minimal*, if any two $\mathfrak{g}_i$ have distinct radical and are incomparable (with respect to inclusion).

2.1. **Theorem** (Primary decomposition)**.** *Let R be a local ring of finite embedding dimension and let $I \subseteq R$ be a closed ideal.*

(2.1.1) *I has a primary decomposition*

$$I = \mathfrak{g}_1 \cap \cdots \cap \mathfrak{g}_s; \tag{1}$$

(2.1.2) *we may choose (1) so that it is minimal and all $\mathfrak{g}_i$ are closed;*
(2.1.3) *if (1) is minimal and all $\mathfrak{g}_i$ are closed, then the associated primes of I are precisely the radicals of the $\mathfrak{g}_i$ and they are all closed;*
(2.1.4) *the radical of I is also closed and some power of the radical lies inside I;*
(2.1.5) *$\mathsf{V}(I)$ has finitely many irreducible components.*

Proof. Since $\widehat{R}$ is Noetherian by Theorem A, we have a primary decomposition

$$I\widehat{R} = \mathfrak{G}_1 \cap \cdots \cap \mathfrak{G}_s$$

with $\mathfrak{G}_i$ primary ideals in $\widehat{R}$. Let $\mathfrak{P}_i$ be the radical of $\mathfrak{G}_i$. Let $\mathfrak{g}_i$ and $\mathfrak{p}_i$ be the respective contraction of $\mathfrak{G}_i$ and $\mathfrak{P}_i$ to R. One easily checks that $\mathfrak{g}_i$ is primary with radical equal to $\mathfrak{p}_i$. Moreover, by Corollary 1.1, both $\mathfrak{g}_i$ and $\mathfrak{p}_i$ are closed. By taking the intersection of all $\mathfrak{g}_i$ with the same radical, we may arrange for all $\mathfrak{p}_i$ to be distinct. After further omitting redundant terms, we can assume therefore that (1) is a minimal decomposition. The radical $J := \operatorname{rad}(I)$ is the intersection of all $\mathfrak{p}_i$ (and in fact, of the minimal primes only) and therefore is again closed. Choose m large enough such that $\mathfrak{P}_i^m \subseteq \mathfrak{G}_i$ for all i. Hence $J^m \subseteq \mathfrak{p}_i^m \subseteq \mathfrak{G}_i \cap R = \mathfrak{g}_i$ and it follows therefore from (1) that $J^m \subseteq I$. Assertion (2.1.5) is then clear since $\mathsf{V}(I) = \mathsf{V}(J) = \mathsf{V}(\mathfrak{p}_1) \cup \cdots \cup \mathsf{V}(\mathfrak{p}_s)$.

By [1, Theorem 4.5], the prime ideals belonging to I are precisely the $\mathfrak{p}_i$. So remains to show that each $\mathfrak{p}_i$ is in fact an associated prime. By passing to R/I, we may assume that I is the zero ideal and R is separated. Let (1) be a minimal decomposition of the zero ideal and let $\mathfrak{a} := \mathfrak{g}_2 \cap \cdots \cap \mathfrak{g}_s$. By minimality, $\mathfrak{a}$ is not contained in $\mathfrak{g}_1$, so that any non-zero $x \in \mathfrak{a}$ does not belong to $\mathfrak{g}_1$. If $xy = 0$, then $xy \in \mathfrak{g}_1$ and hence $y \in \mathfrak{p}_1$ since $x \notin \mathfrak{g}_1$. This shows that $\operatorname{Ann}(x) \subseteq \mathfrak{p}_1$. Since $\mathfrak{p}_1^m \subseteq \mathfrak{g}_1$, we have $\mathfrak{p}_1^m \mathfrak{a} = 0$. Choose k minimal such that $\mathfrak{p}_1^k \mathfrak{a} = 0$ and let x be a non-zero element in $\mathfrak{p}_1^{k-1}\mathfrak{a}$. Hence $x\mathfrak{p}_1 = 0$ and $\mathfrak{p}_1 \subseteq \operatorname{Ann}(x)$. By our previous argument, the converse inclusion also holds, showing that $\mathfrak{p}_1$ is an associated prime. □

By the general theory of minimal primary decompositions (see for instance [1, Theorem 4.10]), the $\mathfrak{g}_i$ in (1) belonging to the minimal prime ideals of I are uniquely determined as $\mathfrak{g}_i = IR_{\mathfrak{p}_i} \cap R$.

2.2. **Corollary.** *If R is a separated local ring of finite embedding dimension, then its depth is positive if and only if it contains an R-regular element.*

Proof. Note that in general, a local ring $(R, \mathfrak{m})$ can have positive depth without admitting an R-regular element (the converse always holds). Assume R has positive depth, so that $\mathfrak{m}$ is not an associated prime. If R is moreover separated, then it has only finitely associated primes by Theorem 2.1. Hence by prime avoidance, we can find $x \in \mathfrak{m}$ outside these finitely many ideals, which is therefore an R-regular element. □

3. Noetherian ideals

Let A be a ring and $I \subseteq A$ an ideal. If `blue` is a property of rings, then by abuse of terminology, we often will say that the ideal I is *`blue`* when we mean to say that its residue ring A/I has that property (inspite of the potential confusion this can cause by viewing the ideal as a module having this property; this latter perspective is never intended). The collection of all ideals of A will be denoted $\mathsf{Gr}(A)$ (the *Grassmanian* of A), viewed as a set ordered by inclusion. Note that we have three operations on $\mathsf{Gr}(A)$, to wit, sum, product and intersection. In fact, sum and intersection turn $\mathsf{Gr}(A)$ into a lattice with smallest element the zero ideal and greatest element the unit ideal. Let $\mathsf{Spec}(A)$ denote the collection of all prime ideals of A and let $\mathsf{Spec}_d(A)$ be the subcollection of all d-dimensional prime ideals of A, that is to say, all prime ideals $\mathfrak{p}$ such that $A/\mathfrak{p}$ has (Krull) dimension d. Let $\mathsf{Fg}(A)$ denote the collection of all finitely generated ideals of A; let $\mathsf{Noe}(A)$ be the collection of all *Noetherian* ideals of A, that is to say, all ideals $I \subseteq A$ such that A/I is Noetherian; and let $\mathsf{FgNoe}(A)$ be the intersection of $\mathsf{Fg}(A)$ and $\mathsf{Noe}(A)$, the collection of all finitely generated Noetherian ideals. If $(R, \mathfrak{m})$ is a local ring, then we also define $\mathsf{Cl}(R)$, the collection of all *closed* ideals. By Krull's Intersection Theorem (see for instance [2, Theorem 8.10]), we have $\mathsf{Noe}(R) \subseteq \mathsf{Cl}(R)$. By Theorem B, this is an equality for ultra-Noetherian local rings, but no so in general.

Clearly, $\mathsf{Fg}(A)$, $\mathsf{Noe}(A)$ and $\mathsf{FgNoe}(A)$ are all closed under sums. As far as the two other operations are concerned: $\mathsf{Fg}(A)$ is closed under products, but not necessarily under intersections, whereas conversely, $\mathsf{Noe}(A)$ is closed under intersections (Corollary 3.2 below) but not necessarily under products. In particular, $\mathsf{Noe}(A)$ is a sub-lattice of $\mathsf{Gr}(A)$ (with the same greatest element, the unit ideal, but possibly without a smallest element). Combining both conditions yields an ever better behaved lattice:

3.1. **Proposition.** *For any ring A, the collection of ideals $\mathsf{FgNoe}(A)$ is closed under (finite) products, sums and intersections. If $I \subseteq J$ is an inclusion of ideals in A with $I \in \mathsf{FgNoe}(A)$, then also $J \in \mathsf{FgNoe}(A)$. If an ideal I belongs to $\mathsf{FgNoe}(A)$, then so does any ideal with the same radical as I.*

Proof. We start with proving the second assertion: let $I \subseteq J$ and $I \in \mathsf{FgNoe}(A)$. Hence A/I is Noetherian, showing J is finitely generated modulo I. It follows that J itself is finitely generated, and clearly A/J is again Noetherian, proving the second assertion.

Next, we show that $\mathsf{FgNoe}(A)$ is closed under products. To this end, let $I, J \in \mathsf{FgNoe}(A)$. Clearly IJ is finitely generated, so remains to show that A/IJ is Noetherian. Let $I := (a_1, \dots, a_m)A$ and $J := (b_1, \dots, b_n)A$. We will induct on the sum $m + n$, starting at the first non-trivial case $m + n = 2$, that is to say, when $m = n = 1$, so that $I := aA$ and $J := bA$. Let $\mathfrak{a}$ be an arbitrary ideal containing ab. We need to show that $\mathfrak{a}$ is finitely generated. By assumption, both $\mathfrak{a} + aA$ and $J' := (\mathfrak{a} : a)$ are finitely generated (since they contain respectively a and b). Choose a finitely generated ideal $I' \subseteq \mathfrak{a}$ such that $\mathfrak{a} + aA = I' + aA$. It is now easy to see that $\mathfrak{a} = I' + aJ'$, whence finitely generated.

For the general case, that is to say, $m+n > 2$, we may assume by symmetry that $n > 1$. Put $B := A/b_nA$. Clearly $IB, JB \in \mathsf{FgNoe}(B)$ and our induction hypothesis applies, yielding that $IJB \in \mathsf{FgNoe}(B)$. This means that $IJ + b_nA$ lies in $\mathsf{FgNoe}(A)$. Now put $C := A/IJ$. It follows that $(IJ + b_nA)C = b_nC$ and IC both belong to $\mathsf{FgNoe}(C)$, whence so does their product by induction. Hence $b_nI + IJ = IJ \in \mathsf{FgNoe}(A)$, as we wanted to show.

Since $IJ \subseteq I \cap J$, it follows that $\mathsf{FgNoe}(A)$ is closed under intersections as well. To prove the final assertion, let $I \in \mathsf{FgNoe}(A)$ and let J be such that $\operatorname{rad} I = \operatorname{rad} J$. Since then also $\operatorname{rad} J$ belongs to $\mathsf{FgNoe}(A)$, it is finitely generated, and hence $(\operatorname{rad} J)^n \subseteq J$ for some n. Hence $I^n \subseteq J$ and since $I^n \in \mathsf{FgNoe}(A)$ by the result on products, we also get $J \in \mathsf{FgNoe}(A)$. □

3.2. **Corollary.** *Let A be an arbitrary ring. The collection $\mathsf{Noe}(A)$ is closed under finite intersections.*

Proof. Let $\mathfrak{a}, \mathfrak{b} \in \mathsf{Noe}(A)$. By assumption, $A/\mathfrak{a}$ is Noetherian and hence we can find a finitely generated ideal $J \subseteq \mathfrak{b}$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{a} + J$. I claim that $(\mathfrak{a} \cap \mathfrak{b}) + J = \mathfrak{b}$. Indeed, one direction is obvious, so let $b \in \mathfrak{b}$. Hence we can find $a \in \mathfrak{a}$ and $j \in J$ such that $b = a + j$. Our claim is now clear since $b - j = a$ lies in $\mathfrak{a} \cap \mathfrak{b}$. Put $B := A/(\mathfrak{a} \cap \mathfrak{b})$. We showed that $\mathfrak{b}B \in \mathsf{FgNoe}(B)$, and by the same argument, also $\mathfrak{a}B \in \mathsf{FgNoe}(B)$. Hence by Proposition 3.1, the intersection $\mathfrak{a}B \cap \mathfrak{b}B$ lies in $\mathsf{FgNoe}(B)$. However, since this intersection is just the zero ideal, we showed that B is Noetherian. □

3.3. **Theorem.** *A ring A is Noetherian if and only if every Noetherian prime ideal is finitely generated.*

More precisely, if A is not Noetherian and $\mathfrak{p}$ is maximal among all the ideals not in $\mathsf{FgNoe}(A)$, then $\mathfrak{p}$ is a non-finitely generated, Noetherian prime ideal.

Proof. Let us prove the second assertion first. Assume A is not Noetherian. It is easy to see that any ascending chain of non-finitely generated ideals has a union which is neither finitely generated. Hence, by Zorn's Lemma, there exist a maximal non-finitely generated ideal, say $\mathfrak{p}$. Any ideal strictly containing $\mathfrak{p}$ is therefore finitely generated. In particular, $A/\mathfrak{p}$ is Noetherian. In fact, if $\mathfrak{p} \subsetneq I$ then $I \in \mathsf{FgNoe}(A)$. Hence remains to show that $\mathfrak{p}$ is a prime ideal. Suppose $a, b \notin \mathfrak{p}$. By what we just proved, $\mathfrak{p} + aA$ and $\mathfrak{p} + bA$ both belong to $\mathsf{FgNoe}(A)$, whence so does their product by Proposition 3.1. Since $\mathfrak{p} + abA$ contains this product, $\mathfrak{p} + abA$ also belongs to $\mathsf{FgNoe}(A)$, again by Proposition 3.1, showing that $ab \notin \mathfrak{p}$.

To prove the first assertion, one direction is clear, so assume every Noetherian prime ideal is finitely generated. However, if A were not Noetherian, then the above prime ideal $\mathfrak{p}$ violates our assumption. □

3.4. **Corollary** (Cohen)**.** *A ring is Noetherian if and only if all its prime ideals are finitely generated.* □

In fact we get the following sharper result in the local case (note that a Noetherian ideal is closed by Krull's Intersection theorem).

3.5. **Corollary.** *A local ring $(R, \mathfrak{m})$ is Noetherian if and only if all its closed prime ideals are finitely generated.* □

From Proposition 3.1 we also get immediately:

3.6. **Corollary.** *If a local ring $(R, \mathfrak{m})$ has finite embedding dimension, then each $\mathfrak{m}$-primary ideal is finitely generated.*

4. Geometric dimension

When dealing with rings of finite embedding dimension, the following invariant has proven to be a good substitute for Krull dimension ([3]): the *geometric dimension* of a local ring $(R, \mathfrak{m})$ of finite embedding dimension is defined recursively as follows. We say that R has geometric dimension zero, and we write $\operatorname{geodim}(R) = 0$, if and only if R is Artinian. In general, we say that $\operatorname{geodim}(R) \leq d$, if there exists $x \in R$ such that $\operatorname{geodim}(R/xR) \leq d - 1$. Finally, we say that R has geometric dimension equal to d if $\operatorname{geodim}(R) \leq d$ but not $\operatorname{geodim}(R) \leq d - 1$, and we simply write $\operatorname{geodim}(R) := d$. It follows that $\operatorname{geodim}(R) \leq \operatorname{embdim}(R)$. We recall the following characterization [3, Theorem ? and Theorem ?] of geometric dimension.

Theorem C. *Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension. The following numbers are all equal.*

- *the geometric dimension* d *of* R*;*
- *the least possible number of elements* d' *generating an* $\mathfrak{m}$*-primary ideal;*
- *the dimension* $\widehat{\mathrm{d}}$ *of the completion* $\widehat{R}$ *of* R*;*
- *the geometric dimension* d_{sep} *of the separated quotient* R_{sep}*;*
- *the combinatorial dimension* $\bar{\mathrm{d}}$ *of* $\mathsf{ClSpec}(R)$*.*

□

From this and Corollary 1.1, we get for each ideal $I \subseteq R$ an equality

$$\operatorname{geodim}(I) = \dim(I\widehat{R}). \tag{2}$$

Generic tuples. A tuple $\overline{x}$ in R is called *generic*, if it generates an $\mathfrak{m}$-primary ideal and has length equal to the geometric dimension of R. We call an element x *generic*, if it is the first (or for that matter, any) entry of a generic tuple. By [3, Proposition ?], a tuple in R is generic if and only if its image in $\widehat{R}$ is a system of parameters. An element $x \in R$ is generic if and only if $\operatorname{geodim}(R/xR) = \operatorname{geodim}(R) - 1$.

Threshold primes. Let d be the geometric dimension of R and let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the d-dimensional prime ideals of $\widehat{R}$. Note that $\widehat{R}$ itself has dimension d by Theorem C, so that all its d-dimensional primes are minimal (but there may be other minimal prime ideals, of lower dimension). We call the $\mathfrak{p}_i := \mathfrak{q}_i \cap R$ the *threshold* primes of R. By Corollary 1.2, the threshold primes are closed and contain no proper closed primes. In fact, by (2), a prime ideal $\mathfrak{p}$ of R is a threshold prime if and only if it is closed and has the same geometric dimension as R. Moreover, an element $x \in R$ is generic if and only if it is not contained in any threshold prime of R.

5. Finite Krull dimension

Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension. The dimension of R is zero if and only if R is Artinian if and only if its geometric dimension is zero. This is a trivial instance in which dimension and geometric dimension coincide. In this section, we will investigate the relationship between geometric dimension and dimension in case both are finite.

5.1. **Theorem.** *Let R be a local ring of geometric dimension one. If R is separated, then it has dimension one and all radical ideals are Noetherian.*

Proof. Since R is separated, Theorem 2.1 yields that R has finitely many minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$, all of which are closed. Since $\mathsf{ClSpec}(R)$ has dimension one by Theorem C, the only closed prime ideals are therefore the $\mathfrak{p}_i$ together with $\mathfrak{m}$. Hence, the only closed prime ideals of $R/\mathfrak{p}_i$ are the zero ideal and the maximal ideal. Since both are finitely generated, $R/\mathfrak{p}_i$ is Noetherian by Corollary 3.5. Since the nilradical $\operatorname{nil}(R)$ is equal to the intersection $\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s$, it is a Noetherian ideal by Corollary 3.2. In particular, $R_{\text{red}} := R/\operatorname{nil}(R)$ is a one-dimensional Noetherian local ring, and $\mathfrak{p}_1, \dots, \mathfrak{p}_s, \mathfrak{m}$ are therefore its only prime ideals. But these are then also the only prime ideals of R itself, proving that R is one-dimensional and all its prime ideals are Noetherian. By Corollary 3.2, every radical ideal, being an intersection of some of these prime ideals, is also Noetherian. □

5.2. **Corollary.** *A separated local ring of geometric dimension one is Noetherian if and only if* $\operatorname{nil}(R)$ *is finitely generated.*

Proof. By Theorem 5.1, the reduction $R_{\text{red}} := R/\mathfrak{n}$ is Noetherian, where $\mathfrak{n} := \text{nil}(R)$. Hence, if $\mathfrak{n}$ is finitely generated, it belongs to $\mathsf{FgNoe}(R)$. Since $\mathfrak{n}$ is nilpotent (as it is so in $\widehat{R}$), Proposition 3.1 yields that $(0) \in \mathsf{FgNoe}(R)$, that is to say, R is Noetherian. □

Non-separated local rings of geometric dimension one have in general infinitely many prime ideals and are therefore far from being Noetherian (even if they are reduced). We investigate now other conditions implying that Krull dimension and geometric dimension agree. A one-dimensional local ring of finite geometric dimension must have geometric dimension equal to one as well, for its geometric dimension is at most one by Theorem C, and it cannot be zero, lest R be Artinian. In higher dimensions, equality fails in general, and so we make the following definition: the *geometric defect* of a local ring R is the difference $\dim R - \text{geodim}(R)$ whenever both are finite; otherwise we put its geometric defect equal to ∞. By the theorem, the geometric defect is always non-negative or infinite. We say that R is *geometric*, if its geometric defect is zero. Theorem C yields that R is geometric if and only if it has the same (Krull) dimension as its completion. In particular, Noetherian ideals are geometric.

5.3. **Proposition.** *Let R be a local ring of finite embedding dimension.*

(5.3.1) *Any two ideals with the same radical have the same dimension and the same geometric dimension. In particular, R is geometric if and only if R_{red} is.*

(5.3.2) *If $I \subseteq J$ are two ideals with the same closure and if I is geometric, then so is J.*

(5.3.3) *A radical, geometric ideal is closed.*

(5.3.4) *If I is geometric, then it has only finitely many minimal primes, all of which are closed; the maximal dimensional ones are the threshold primes and they are geometric. In particular, $\mathsf{V}(I)$ has finitely many irreducible components.*

Proof. In order to prove (5.3.1), it suffices to show that an ideal I and its radical $J := \text{rad}(I)$ have the same dimension and the same geometric dimension. Since $\mathsf{V}_R(I) = \mathsf{V}_R(J)$, both ideals have the same dimension. Moreover, one easily verifies that also $\mathsf{V}_{\widehat{R}}(I\widehat{R}) = \mathsf{V}_{\widehat{R}}(J\widehat{R})$, so that also $I\widehat{R}$ and $J\widehat{R}$ have the same dimension. Hence I and J have the same geometric dimension by (2).

To prove (5.3.2), let J be any ideal between I and its closure. It follows from Theorem C that I and J have the same geometric dimension. Hence the inequalities

$$\text{geodim}(J) \leq \dim(J) \leq \dim(I) = \text{geodim}(I)$$

must be equalities, showing that J is geometric.

To prove (5.3.3), let I be a radical and geometric ideal. Since I is an intersection of prime ideals and since being closed is preserved under intersections, we may moreover assume that I is a prime ideal. However, if I is not closed, then its closure must have strictly smaller dimension, violating (5.3.2).

To prove (5.3.4), let I be a geometric ideal, say, of dimension d, and let $\bar{I}$ and $\text{rad}(I)$ be respectively its closure and radical. It follows from (5.3.1) that $\text{rad}(I)$ is geometric, whence from (5.3.3) that it is closed. In particular, $\bar{I} \subseteq \text{rad}(I)$. By Theorem 2.1, the minimal primes of $\bar{I}$ are all closed and there are only finitely many of them. Hence the first and last assertion of (5.3.4) follows since $\mathsf{V}(I) = \mathsf{V}(\text{rad}(I)) = \mathsf{V}(\bar{I})$. Let $\mathfrak{p}$ be a threshold prime of R/I. By (2), the geometric dimension of $\mathfrak{p}$ is d, and since its dimension is at most d, $\mathfrak{p}$ is geometric. □

Before we give a general criterium for being geometric, we study some low dimensional cases first. Since $\mathsf{Spec}(A)$ is in general not well-ordered, there might very well be no prime ideals of finite dimension other than the maximal ideals (which always have dimension zero). For rings of finite embedding dimension we do have:

5.4. **Lemma.** *If $(R, \mathfrak{m})$ is a local ring of finite embedding dimension, then $\mathsf{Spec}_1(R)$ is non-empty.*

Proof. Let $(\mathfrak{p}_i)_i$ be a chain in $\mathsf{Spec}(R) \setminus \{\mathfrak{m}\}$ and let $\mathfrak{p}$ be their union $\mathfrak{p}$. If $\mathfrak{p} = \mathfrak{m}$ then $\mathfrak{m} \subseteq \mathfrak{p}_i$ for some i, since $\mathfrak{m}$ is finitely generated, contradiction. Hence the conditions of Zorn's lemma are met, and $\mathsf{Spec}(R) \setminus \{\mathfrak{m}\}$ has a maximal element, say $\mathfrak{p}$. It follows that $\mathfrak{p}$ has dimension one. □

5.5. **Theorem.** *If $(R, \mathfrak{m})$ is a one-dimensional local ring of finite embedding dimension, then it is geometric, $\mathsf{Spec}(R)$ is finite, every radical ideal is Noetherian and every infinitesimal is nilpotent.*

In particular, any one-dimensional radical ideal in a local ring of finite embedding dimension is Noetherian.

Proof. We already argued that the geometric dimension of R has to be one too. Let $\mathfrak{p}$ be a prime ideal other than $\mathfrak{m}$. The only two prime ideals of $R/\mathfrak{p}$ are finitely generated, showing that $R/\mathfrak{p}$ is Noetherian by Corollary 3.4. Hence $\mathfrak{p}$ is Noetherian whence closed, and therefore of the form $\mathfrak{P} \cap R$ for some prime ideal $\mathfrak{P}$ of $\widehat{R}$ by Corollary 1.2. It follows that $\mathfrak{P}$ must have dimension one as well and hence $\mathfrak{p}$ is a threshold prime. In particular, $\mathsf{Spec}(R)$ is finite. Since $\operatorname{nil} R$ is the intersection of these finitely many prime ideals, it also belongs to $\mathsf{Noe}(R)$ by Corollary 3.2. In particular, $\mathfrak{I}_R \subseteq \operatorname{nil}(R)$. □

5.6. **Theorem** (Dichotomy for two-dimensional primes)**.** *Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension and let $\mathfrak{p} \subseteq R$ be a two-dimensional prime ideal. Exactly one of the the following two cases holds:*

(5.6.1) *$\mathfrak{p}$ is closed and geometric, and $\mathsf{V}(\mathfrak{p})$ is infinite;*

(5.6.2) *$\mathfrak{p}$ is not closed and has geometric defect one, and $\mathsf{V}(\mathfrak{p})$ is finite.*

Proof. Let $S := R/\mathfrak{p}$. By assumption, S has dimension two and hence its geometric dimension d is at most two by Theorem C. Since S is not Artinian, $d = 1$ or $d = 2$. Suppose S admits a prime ideal $\mathfrak{q}$ other than (0), $\mathfrak{m}S$ and its threshold primes. For dimension reasons, $\mathfrak{q}$ cannot be contained in any threshold prime. Hence, by prime avoidance, we can find an element $x \in \mathfrak{q}$ outside each threshold prime. Hence x is generic in S and S/xS has geometric dimension $d - 1$. On the other hand, S/xS being a homomorphic image of a two-dimensional domain, has dimension at most one, and the chain $\mathfrak{q} \subsetneq \mathfrak{m}S$ then shows that its dimension is exactly one. By Theorem 5.5, its geometric dimension is then also one, showing that $d = 2$.

Therefore, if S has geometric dimension one, then $\mathsf{V}_R(\mathfrak{p})$ (which we identify with $\mathsf{Spec}(S)$) consists of the finitely many threshold primes of S together with $\mathfrak{p}$ and $\mathfrak{m}$, and hence in particular is finite. To prove the converse, assume $\mathsf{V}_R(\mathfrak{p})$ is finite. By prime avoidance, we may choose $x \in \mathfrak{m}$ outside any other prime ideal of S. It follows that S/xS has dimension zero whence also geometric dimension zero, so that S has geometric dimension one, as we wanted to show. Moreover, since S has only two possibilities for its geometric dimension, we also showed that S is geometric if and only if $\mathsf{V}_R(\mathfrak{p})$ is infinite. To complete the proof, it remains to show that the latter two conditions are also equivalent with $\mathfrak{p}$ being closed.

Let $\mathfrak{q} \subseteq S$ be a non-zero prime ideal of S. It follows that $\mathfrak{q}$ has dimension at most one, whence is closed by Theorem 5.5. Hence $\mathfrak{I}_S$ is contained in the intersection $\mathfrak{a}$ of all non-zero prime ideals of S. If $\mathfrak{a}$ is non-zero, then since S is a domain, $\mathfrak{a}$ must have dimension one, and therefore is Noetherian by Theorem 5.5. Therefore, $\mathsf{V}_S(\mathfrak{a})$ is finite, and hence so is $\mathsf{V}_R(\mathfrak{p}) = \mathsf{V}_S(\mathfrak{a}) \cup \{\mathfrak{p}\}$, contradicting our assumption. Hence $\mathfrak{a}$ is the zero ideal, whence so is $\mathfrak{I}_S$.

Conversely, assume S is separated. If $\mathfrak{q}$ is a non-zero prime of S, then its dimension is at most one, whence is closed by Theorem 5.5. Therefore, every prime ideal of S is closed, that is to say, $\mathsf{ClSpec}(S) = \mathsf{Spec}(S)$, and since the latter has dimension two by assumption, so has the former. Hence S also has geometric dimension two by Theorem C. □

5.7. **Corollary.** *A two-dimensional local ring of finite embedding dimension is geometric if and only if $\mathsf{Spec}_1(R)$ is infinite if and only if there exists a one-dimensional prime which is not a threshold prime.*

Proof. Let R be a two-dimensional local ring of finite geometric dimension. Suppose first that itis geometric and let $\mathfrak{q}$ be a threshold prime. Since $R/\mathfrak{q}$ has geometric dimension two too, $R/\mathfrak{q}$ has dimension two by Theorem C, whence is geometric. Theorem 5.6 then shows that $\mathsf{Spec}_1(R/\mathfrak{q})$ is infinite, whence afortiori, so is $\mathsf{Spec}_1(R)$ and hence there are one-dimensional non-threshold primes. Conversely, assume $\mathfrak{p}$ is a one-dimensional prime which is not a threshold prime. Let $\mathfrak{P}$ be a minimal prime of $\mathfrak{p}\widehat{R}$. Since $R/\mathfrak{p}$ has dimension one, $\mathfrak{p} = \mathfrak{P} \cap R$. If R has geometric dimension one, then $\widehat{R}$ has dimension one by Theorem C and hence $\mathfrak{p}$ would be a threshold prime, contradiction. □

5.8. **Corollary.** *A three-dimensional separated local domain of finite embedding dimension has geometric defect at most one. It is geometric if and only if some two-dimensional prime ideal is closed.*

Proof. By assumption and Theorem 5.5, the only prime ideals that can be non-closed must have dimension two. If there exists a closed two-dimensional prime, then $\mathsf{ClSpec}(R)$ has combinatorial dimension three, whence R has geometric dimension three by Theorem C. If there is no closed two-dimensional prime, then $\mathsf{ClSpec}(R)$ has combinatorial dimension two, so R has geometric dimension two, whence geometric defect one. □

The previous results already hint at some combinatorial properties of the prime spectrum distinguishing geometric local rings. The precise phenomenon is coined by the following definition. We say that the spectrum of R has a *bottleneck*, if for some $i > 0$, there exists an $(i+1)$-dimensional prime ideal $\mathfrak{p}$ which is contained only in finitely many i-dimensional prime ideals (in other words, $\mathsf{Spec}_i(R/\mathfrak{p})$ is finite).

5.9. **Theorem.** *Let R be a local ring of finite dimension and finite embedding dimension. The following are equivalent:*

(5.9.1) *R is geometric;*
(5.9.2) *the spectrum of R has no bottlenecks;*
(5.9.3) *every ideal of R is geometric;*
(5.9.4) *every prime ideal of R is geometric;*
(5.9.5) *every prime ideal of R is closed;*
(5.9.6) *the natural map $\mathsf{Spec}(\widehat{R}) \to \mathsf{Spec}(R)$ is surjective.*

Proof. Let us first prove by induction on the geometric dimension d of R, that if R is not geometric, then $\mathsf{Spec}(R)$ has a bottleneck. If $\mathsf{Spec}_d(R)$ is finite, then clearly any $d+1$-dimensional prime gives rise to a bottleneck in $\mathsf{Spec}(R)$ (and by assumption, such primes exist, since R has dimension at least $d+1$). Hence we may assume that $\mathsf{Spec}_d(R)$ is infinite, so that there exists in particular a d-dimensional prime ideal $\mathfrak{p}$ which is not a threshold prime. Hence $\mathfrak{p}$ contains a generic element x and R/xR has geometric dimension $d-1$. By construction, however, R/xR has dimension at least d, so that R/xR is not geometric. By our induction hypothesis, the spectrum of R/xR contains a bottleneck, and it is easy to see that this implies that also $\mathsf{Spec}(R)$ contains a bottleneck. In fact, if $\mathsf{Spec}(R)$ has no bottleneck, then neither does $\mathsf{Spec}(R/I)$, for every ideal I in R. This observation, together with what we just proved, establishes the implication (5.9.2) $\Rightarrow$ (5.9.3).

Implication (5.9.3) $\Rightarrow$ (5.9.4) holds trivially. To show (5.9.4) $\Rightarrow$ (5.9.2), suppose (5.9.4) holds but $\mathsf{Spec}(R)$ has a bottleneck, that is to say, an $(i+1)$-dimensional prime ideal $\mathfrak{p}$ which lies in only finitely many i-dimensional prime ideals, for some $i > 0$. Let $x \in \mathfrak{m}$ be outside each of these i-dimensional prime ideals and put $S := R/\mathfrak{p}$. It follows that S/xS has dimension at most $i-1$. However, since S has dimension $i+1$ and is geometric, its geometric dimension is also $i+1$. Hence S/xS must have geometric dimension at least i, whence dimension at least i by Theorem C, contradiction.

The equivalence of (5.9.5) and (5.9.6) follows from Corollary 1.2. If (5.9.5) holds, so that $\mathsf{ClSpec}(R) = \mathsf{Spec}(R)$, then R is geometric by Theorem C. Since (5.9.5) passes to any homomorphic image of R, we get (5.9.3). As for (5.9.4) $\Rightarrow$ (5.9.5), this follows from (5.3.3) in Proposition 5.3.

Remains to show the equivalence of (5.9.1) with the rest. Since it is trivially implied by (5.9.3), we only need to prove the converse and so we assume R is geometric. We will prove by induction on the geometric dimension d of R that all its prime ideals are closed. Let $\mathfrak{p}$ be an arbitrary prime ideal of R. By (5.3.4), we are done if $\mathfrak{p}$ is a minimal prime, so assume it is not. In particular, we can find some $x \in \mathfrak{p}$ outside all minimal primes of R. Since the latter include all the threshold primes, x is generic and hence R/xR has geometric dimension $d-1$. Moreover, since by construction $\mathsf{V}(xR) = \mathsf{Spec}(R/xR)$ contains no minimal, whence no d-dimensional prime of R, the dimension of R/xR is at most $d-1$. Hence R/xR must have dimension $d-1$, and therefore is geometric. By induction, all prime ideals of R/xR are closed, whence, in particular, so is $\mathfrak{p}$. □

We have the following version of Krull's Principal Ideal Theorem.

5.10. **Corollary.** *Let A be a ring of finite Krull dimension and assume the spectrum of each localization of A is without bottlenecks. If $\mathfrak{p}$ is a minimal prime of $(a_1,\dots,a_n)A$, for some $a_i \in A$, and $\mathfrak{p}$ is finitely generated, then $\mathfrak{p}$ has height at most n.*

Proof. Let $R := A_{\mathfrak{p}}$. By assumption, R has finite embedding dimension, whence finite geometric dimension, say d. By Theorem 5.9, the dimension of R is also d, and hence $\mathfrak{p}$ has height d. On the other hand, since $(a_1,\dots,a_n)R$ is $\mathfrak{p}R_{\mathfrak{p}}$-primary, Theorem C implies that $d \leq n$. □

6. Coherence

Recall that a ring A is called *coherent* if and only if the kernel of any linear map between finite free A-modules is finitely generated.

6.1. **Lemma.** *If A is coherent ring and $\mathrm{u}A$ is an ultrapower of A, then the diagonal embedding $A \to \mathrm{u}A$ is flat.*

Proof. By the equational characterization of flatness, we need to show that if $\bar{b}$ is a tuple with entries in $\mathrm{u}A$ which is a solution to a linear homogeneous equation $L = 0$ with coefficients in A, then $\bar{b}$ is a linear combination of solutions of $L = 0$ with entries in A. Let V be the kernel of the linear map $A^n \to A$ induced by L. By coherence, V is finitely generated, say, by tuples $\bar{a}_1,\dots,\bar{a}_m$. Choose tuples $\bar{b}_w$ in A^n such that their ultraproduct is

$\bar{b}$. By Łos' Theorem, almost each $\bar{b}_w$ is a solution of $L = 0$ and hence there exist $s_{iw} \in A$ such that

$$\bar{b}_w = s_{1w}\bar{a}_1 + \cdots + s_{mw}\bar{a}_m.$$

If we let s_i be the ultraproduct of the s_{iw}, then $\bar{b} = s_1\bar{a}_1 + \cdots + s_m\bar{a}_m$ by Łos' Theorem, as required. □

6.2. **Lemma.** *Let $A \to B$ be a flat map of rings and let I be an ideal of B. Then the following are equivalent.*

(6.2.1) *The map $A \to B/I$ is flat.*

(6.2.2) *For every finitely generated ideal $\mathfrak{a}$ of A, we have $\mathfrak{a}B \cap I = \mathfrak{a}I$.*

Proof. From the exact sequence

$$0 \to I \to B \to B/I \to 0$$

we get, after tensoring with $A/\mathfrak{a}$, an exact sequence

$$0 = \operatorname{Tor}_1^A(B, A/\mathfrak{a}) \to \operatorname{Tor}_1^A(B/I, A/\mathfrak{a}) \to I/\mathfrak{a}I \to B/\mathfrak{a}B$$

The kernel of the last arrow is equal to $(\mathfrak{a}B \cap I)/\mathfrak{a}I$, showing that

$$\mathfrak{a}B \cap I = \mathfrak{a}I \quad \text{if and only if} \quad \operatorname{Tor}_1^A(B/I, A/\mathfrak{a}) = 0.$$

Hence the direction (6.2.1) ⇒ (6.2.2) is clear, and the converse follows from a well-known Tor-criterion for flatness (see for instance [2, Theorem 7.8]). □

6.3. **Theorem.** *Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension. If R is coherent, then the following are equivalent:*

(6.3.1) *R is Noetherian;*

(6.3.2) *every finitely generated ideal is closed;*

(6.3.3) *for every finitely generated ideal $\mathfrak{a}$ and every n, there exists some $c := c(\mathfrak{a}; n)$ such that $\mathfrak{a} \cap \mathfrak{m}^c \subseteq \mathfrak{a}\mathfrak{m}^n$;*

(6.3.3') *for every finitely generated ideal $\mathfrak{a}$, the $\mathfrak{m}$-adic topology on $\mathfrak{a}$ coincides with its induced topology as a subspace $\mathfrak{a} \subseteq R$.*

Proof. The equivalence of (6.3.3) and (6.3.3') is immediate from the definitions. The implication (6.3.1) ⇒ (6.3.2) is trivial and (6.3.1) ⇒ (6.3.3) follows from the Artin-Rees Lemma, since we may choose $c(\mathfrak{a}; n)$ equal to $c + n$ for some c only depending on $\mathfrak{a}$. We will simulatenously prove both converses. Let $\mathrm{u}R$ and $\mathrm{c}R$ be respectively the ultrapower and catapower of R. Since $\mathrm{u}R$ has finite embedding dimension, $\mathrm{c}R = \mathrm{u}R/\mathfrak{I}_{\mathrm{u}R}$ is Noetherian by Theorem B. By Lemma 6.1, the natural map $R \to \mathrm{u}R$ is faithfully flat. We want to show that $R \to \mathrm{c}R$ is faithfully flat, for if this holds, then R is Noetherian, since $\mathrm{c}R$ is. In view of Lemma 6.2, it suffices to show that $\mathfrak{a}(\mathrm{u}R) \cap \mathfrak{I}_{\mathrm{u}R} = \mathfrak{a}\mathfrak{I}_{\mathrm{u}R}$, for every finitely generated ideal $\mathfrak{a} := (f_1, \ldots, f_s)R$. To this end, let $a \in \mathfrak{a}(\mathrm{u}R) \cap \mathfrak{I}_{\mathrm{u}R}$. Choose $a_w \in R$ so that their ultraproduct equals a. By Łos' Theorem, $a_w \in \mathfrak{a}$ for almost all w. Let $n(w)$ be maximal such that $a_w \in \mathfrak{a}\mathfrak{m}^{n(w)}$. Choose $a_{iw} \in \mathfrak{m}^{n(w)}$ so that

$$a_w = a_{1w}f_1 + \cdots + a_{sw}f_s.$$

Let a_i be the ultraproduct of the a_{iw}. Since $a = a_1f_1 + \cdots + a_sf_s$, we are done once we showed that each a_i belongs to $\mathfrak{I}_{\mathrm{u}R}$. By way of contradiction, suppose the latter does not hold, so that after renumbering, we may assume that $a_1 \notin \mathfrak{m}^N(\mathrm{u}R)$, for some N. I claim that $a_w \in \mathfrak{a}\mathfrak{m}^N$, for almost all w. By maximality of $n(w)$, this then implies that $N \leq n(w)$ and hence $a_{1w} \in \mathfrak{m}^N$. By Łos' Theorem, $a_1 \in \mathfrak{m}^N(\mathrm{u}R)$, yielding the desired contradiction.

So remains to prove that almost all a_w belong to $\mathfrak{a}\mathfrak{m}^N$. In case (6.3.3) holds, we can choose $c := c(\mathfrak{a}; N)$ such that $\mathfrak{a} \cap \mathfrak{m}^c \subseteq \mathfrak{a}\mathfrak{m}^N$. Since $a \in \mathfrak{I}_{\mathrm{u}R} \subseteq \mathfrak{m}^c(\mathrm{u}R)$, almost all a_w belong to $\mathfrak{m}^c$ whence to $\mathfrak{a}\mathfrak{m}^N$, so that we are done in this case. Hence assume condition (6.3.2) holds. Let $\mathrm{u}S$ and $\mathrm{c}S$ be respectively the ultrapower and catapower of $S := \widehat{R}$. Since S is Noetherian by Theorem A, the natural map $S \to \mathrm{c}S$ is faithfully flat by Theorem B. By Lemma 6.1, so is the map $S \to \mathrm{u}S$. Hence by Lemma 6.2, we have $\mathfrak{a}(\mathrm{u}S) \cap \mathfrak{I}_{\mathrm{u}S} = \mathfrak{a}\mathfrak{I}_{\mathrm{u}S}$. In particular, $a \in \mathfrak{a}\mathfrak{I}_{\mathrm{u}S}$. By Łos' Theorem, almost each a_w lies in $\mathfrak{a}\mathfrak{m}^N S$. Since $\mathfrak{a}\mathfrak{m}^N$ is finitely generated, it is closed, and hence $\mathfrak{a}\mathfrak{m}^N S \cap R = \mathfrak{a}\mathfrak{m}^N$ by Corollary 1.1. Therefore, almost each a_w lies in $\mathfrak{a}\mathfrak{m}^N$, as we wanted to show. □

In fact, without assuming R to be coherent, (6.3.3) ⇒ (6.3.2) always holds. Indeed, let $\mathfrak{a}$ be a finitely generated ideal and b an element in its closure. By assumption, there is some c such that

$$(\mathfrak{a} + Rb) \cap \mathfrak{m}^c \subseteq \mathfrak{m}(\mathfrak{a} + Rb). \tag{3}$$

Write $b = a + m$ with $a \in \mathfrak{a}$ and $m \in \mathfrak{m}^c$. Hence $m = b - a$ belongs to $\mathfrak{m}(\mathfrak{a} + Rb)$ by (3) and therefore, can be written as $m = a' + bm'$ with $a' \in \mathfrak{a}$ and $m' \in \mathfrak{m}$. Hence $b(1 - m') = a + a' \in \mathfrak{a}$ and since $1 - m'$ is a unit, we get $b \in \mathfrak{a}$, showing that $\mathfrak{a}$ is closed.

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