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Hans Schoutens

Non-standard tight closure for affine $\mathbb{C}$-algebras

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Abstract. In this paper, non-standard tight closure is proposed as an alternative for classical tight closure on finitely generated algebras over $\mathbb{C}$. It has the advantage that it admits a functional definition, similar to the characteristic p definition of tight closure, where instead of the characteristic p Frobenius, its ultraproduct, the non-standard Frobenius, is used. This new closure operation $\operatorname{cl}(\cdot)$ has the same properties as classical tight closure, to wit, (1) if A is regular, then $\mathfrak{a} = \operatorname{cl}(\mathfrak{a})$; (2) if $A \subset B$ is an integral extension of domains, then $\operatorname{cl}(\mathfrak{a}B) \cap A \subset \operatorname{cl}(\mathfrak{a})$; (3) if A is local and $(x_1, \dots, x_n)$ is a system of parameters, then $((x_1, \dots, x_i) : x_{i+1}) \subset \operatorname{cl}(x_1, \dots, x_i)$ (*Colon-Capturing*); (4) if $\mathfrak{a}$ is generated by m elements, then $\operatorname{cl}(\mathfrak{a})$ contains the integral closure of $\mathfrak{a}^m$ and is contained in the integral closure of $\mathfrak{a}$ (*Briançon-Skoda*).

Key words. Tight closure–non-standard Frobenius–ultraproducts

1. Introduction

For A a ring of characteristic $p > 0$, HOCHSTER and HUNEKE have defined a closure operation on ideals of A, called *tight closure*. To be more precise, an element $z \in A$ belongs to the tight closure $\mathfrak{a}^*$ of an ideal $\mathfrak{a} = (f_1, \dots, f_s)A$, if there exists some $c \in A$ not contained in any minimal prime ideal of A, such that for all sufficiently high powers q of p, we have that

$$cz^q \in (f_1^q, \dots, f_s^q)A.$$

This closure operation admits all four properties listed in the abstract above; see for instance [7] or the excellent survey article [23]. The same authors have also extended the notion of tight closure to many rings containing a field of characteristic zero, including all finitely generated algebras over a field, and the same four properties still hold. However, even the very definition in characteristic zero is more complicated, involving some reduction process to characteristic p (see for instance [7, Appendix 1] or [4]). Moreover, to prove these properties in characteristic zero,

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Department of Mathematics
Ohio State University
Columbus, OH 43210 (USA)
e-mail: schoutens@math.ohio-state.edu

one needs some strong form of Artin Approximation. More precisely, the Artin-Rotthaus Theorem [2] is used to descend properties from complete local rings to arbitrary excellent rings.

In this paper, I propose an alternative closure operation (and several variants) for finitely generated $\mathbb{C}$-algebras, using the non-standard Frobenius, that is to say, the ultraproduct of the characteristic p Frobenii. To emphasize this, I termed the closure operation *non-standard tight closure*. The main goal of the present paper is to establish the four fundamental properties above for this new closure operation and its variants. Originally, I had also included in this paper a proof for a fifth fundamental property, to wit, *persistence* of non-standard tight closure (that is to say, the fact that for every homomorphism $A \to B$, the non-standard tight closure of an ideal $\mathfrak{a}$ of A is contained in the non-standard tight closure of its extension $\mathfrak{a}B$ in B). However, since the argument requires a substantial amount of non-Noetherian or non-standard commutative algebra, I decided to publish these results in a separate paper. In particular, I have made an attempt to prove the four properties using as little as possible, relying chiefly on some basic facts from tight closure theory in positive characteristic, and often, in fact, mimicking the elementary proofs these properties admit in positive characteristic.[1] After establishing the four main properties, I list in the last section a few applications. Further properties and applications can be found in [14, 18–22]. There is also some work in progress ([15, 16]) on extending the present ideas to mixed characteristic.

In the first part of this paper, §2–§4, some material from [12, 13] is reviewed and then used to describe exactly how the transfer from positive to zero characteristic is carried out using non-standard methods. Upon request of the referee of an earlier version of this paper, I will do this in considerable detail. Many of the other papers listed above also make extensive use of this method and therefore I have included several results which are not actually needed in the present paper, but only in the later papers. From §5 on, the theory of non-standard tight closure is then developed.

In the remainder of this introduction, I will briefly explain the main ideas and definitions. Let C be a finitely generated $\mathbb{C}$-algebra. Let $\mathfrak{a}$ be an ideal of C and $z \in C$. I will now explain what it means for z to belong to the *non-standard tight closure* of $\mathfrak{a}$. We can write C as a quotient of some polynomial ring $\mathbb{C}[X_1, \dots, X_n]$ by an ideal I. Rather than descending these data to a finitely generated $\mathbb{Z}$-algebra and then taking reductions modulo p as in the Hochster-Huneke tight closure case, we will invoke a different method of transfer. Namely, we observe that $\mathbb{C}$ can be viewed as a 'limit' of the fields $\mathbb{F}_p^{\text{alg}}$ (the algebraic closure of the p-element field). To make this 'limit' idea precise, we need some non-standard formalism, to wit, the notion of ultraproduct. I will not explain this at this point, but refer the reader to the next section for more details. Suffices to say here that there is a process that attaches, albeit not uniquely, to any complex number c a sequence c_p with each $c_p \in \mathbb{F}_p^{\text{alg}}$. Choose some generators $f_1, \dots, f_m$ of I and apply this to every coefficient of each f_i. We obtain polynomials f_{ip} defined over $\mathbb{F}_p^{\text{alg}}$. Let C_p be

[1] At present, there is one exception: to prove that non-standard tight closure is contained in integral closure, I need Néron p-desingularization.

the quotient of $\mathbb{F}_p^{\text{alg}}[X_1, \dots, X_n]$ modulo the ideal generated by the f_{ip}. These characteristic p rings C_p will play a role similar to the reductions modulo p in classical characteristic zero tight closure. Namely, on each C_p we have an action of Frobenius. As we could take the 'limit' (in the sense of ultraproducts) of the fields $\mathbb{F}_p^{\text{alg}}$, so can we similarly take the limit of the C_p. This ring, denoted C_∞, will be called the *non-standard hull* of C, and as such, it is uniquely defined up to $\mathbb{C}$-algebra isomorphism (whereas the C_p were not). It contains C as a subring and $C \subset C_\infty$ is faithfully flat. Moreover, the action of the Frobenius on each C_p (being entirely algebraic) extends to an action of an endomorphism $\mathbf{F}_\infty$ on C_∞, the 'limit' or *non-standard Frobenius* on C_∞. Nonetheless, we have to face the following serious obstructions: the non-standard hull is far from Noetherian and the action of $\mathbf{F}_\infty$ on it does not preserve the subring C. Notwithstanding, we say, in accordance with the characteristic p definition, that z lies in the non-standard tight closure of $\mathfrak{a}$, if there exists $c \in C$, not contained in any minimal prime ideal of C, such that for all m, we have

$$c\,\mathbf{F}_\infty^m(z) \in \mathbf{F}_\infty^m(\mathfrak{a})C_\infty \tag{1.1}$$

where $\mathbf{F}_\infty^m(\mathfrak{a})C_\infty$ simply means the ideal in C_∞ generated by all $\mathbf{F}_\infty^m(g)$ for $g \in \mathfrak{a}$.

A variant of this definition is obtained as follows. Apply the 'downwards process' to the data given by z and $\mathfrak{a} = (g_1, \dots, g_s)C$ and thus obtain elements $z_p, g_{ip} \in C_p$ whose coefficients have limits corresponding to the coefficients of z and g_i respectively. Then require that z_p lies in the (classical) tight closure of $(g_{1p}, \dots, g_{sp})C_p$. I will call this *generic tight closure*. It contains equational characteristic zero tight closure and is contained in non-standard tight closure; see Theorems 8.5 and 10.4.

The main advantage of non-standard tight closure is its functional definition in terms of the initial data. This is reflected by the ease with which the four main properties can be derived. That in practice non-standard tight closure is impossible to be calculated explicitly (as it depends on the knowledge of an ultrafilter) should not be held against it, for its cousin, classical tight closure, itself is hard to compute, even in positive characteristic. The final and weakest variant, *non-standard closure*, is obtained by only requiring (1.1) for $m = 1$. Surprising as it might be, the more since its characteristic p counterpart would be utterly useless, this still gives a closure operation for which the four fundamental properties hold. For an application of this weaker version, see [14].

2. Ultraproducts

In this section, W always denotes an infinite set.

2.1. Non-principal Ultrafilters

With a *non-principal ultrafilter* $\mathcal{U}$ on W, we mean a collection of infinite subsets of W closed under intersection, with the property that for any subset F of W, either F or its complement $-F$ belongs to $\mathcal{U}$. In particular, the empty-set does not belong to $\mathcal{U}$ and if $D \in \mathcal{U}$ and E is an arbitrary set containing D, then also $E \in \mathcal{U}$, for otherwise $-E \in \mathcal{U}$, whence $\emptyset = D \cap -E \in \mathcal{U}$, contradiction. Since every set in $\mathcal{U}$ must be infinite, it follows that any cofinite set belongs to $\mathcal{U}$. The existence of non-principal ultrafilters is equivalent with the Axiom of Choice, and we make this set-theoretic assumption henceforth. It follows that for any infinite subset of W, we can find a non-principal ultrafilter containing this set.

In the remainder of this section, we also fix a non-principal ultrafilter on W, and omit reference to this fixed ultrafilter from our notation. Non-principal ultrafilters play the role of a decision procedure on the collection of subsets of W by declaring some subsets 'large' (those belonging to $\mathcal{U}$) and declaring the remaining ones 'small'. More precisely, let o_w be elements indexed by $w \in W$, and let $\mathcal{P}$ be a property. We will use the expressions *almost all* o_w *satisfy property* $\mathcal{P}$ or o_w *satisfies property* $\mathcal{P}$ *for almost all* w as an abbreviation of the statement that there exists a set D in the ultrafilter $\mathcal{U}$, such that property $\mathcal{P}$ holds for the element o_w, whenever $w \in D$. Note that this is also equivalent with the statement that the set of *all* $w \in W$ for which o_w has property $\mathcal{P}$, lies in the ultrafilter.

2.2. *Ultraproducts*

Let O_w be sets, for $w \in W$. We define an equivalence relation on the Cartesian product $\prod O_w$, by calling two sequences $(a_w|w \in W)$ and $(b_w|w \in W)$ equivalent, if a_w and b_w are equal for almost all w. In other words, if the set of indices $w \in W$ for which $a_w = b_w$ belongs to the ultrafilter. We will denote the equivalence class of a sequence $(a_w|w \in W)$ by

$$\operatorname*{ulim}_{w\to\infty} a_w \quad \text{or} \quad a_\infty.$$

The set of all equivalence classes on $\prod O_w$ is called the *ultraproduct* of the O_w and is denoted

$$\operatorname*{ulim}_{w\to\infty} O_w \quad \text{or} \quad O_\infty.$$

Note that the element-wise and set-wise notations are reconciled by the fact that

$$\operatorname*{ulim}_{w\to\infty}\{o_w\} = \{\operatorname*{ulim}_{w\to\infty} o_w\}.$$

2.3. *Ultraproducts of Sets*

For the following properties, the easy proofs of which are omitted, let O_w be sets with ultraproduct O_∞.

2.3.1. *If Q_w is a subset of O_w for each w, then $\operatorname{ulim}_{w\to\infty} Q_w$ is a subset of $\operatorname{ulim}_{w\to\infty} O_w$.*

In fact, $\operatorname{ulim}_{w\to\infty} Q_w$ consists of all elements of the form $\operatorname{ulim}_{w\to\infty} o_w$, with almost all o_w in Q_w.

2.3.2. *If each O_w is the graph of a function $f_w\colon A_w \to B_w$, then O_∞ is the graph of a function $A_\infty \to B_\infty$, where A_∞ and B_∞ are the respective ultraproducts of A_w and B_w. We will denote this function by*

$$\operatorname*{ulim}_{w\to\infty} f_w \quad \textit{or} \quad f_\infty.$$

Moreover, we have an equality

$$\operatorname*{ulim}_{w\to\infty}(f_w(a_w)) = (\operatorname*{ulim}_{w\to\infty} f_w)(\operatorname*{ulim}_{w\to\infty} a_w),$$

for $a_w \in A_w$.

2.3.3. *If each O_w comes with an operation $*_w\colon O_w \times O_w \to O_w$, then*

$$*_\infty := \operatorname*{ulim}_{w\to\infty} *_w$$

is an operation on O_∞. If all (or, almost all) O_w are groups with multiplication $*_w$ and unit element 1_w, then O_∞ is a group with multiplication $*_\infty$ and unit element $1_\infty := \operatorname{ulim}_{w\to\infty} 1_w$. If almost all O_w are Abelian groups, then so is O_∞.

2.3.4. *If each O_w is a (commutative) ring with addition $+_w$ and multiplication $\cdot_w$, then O_∞ is a (commutative) ring with addition $+_\infty$ and multiplication $\cdot_\infty$.*

In fact, in that case, O_∞ is just the quotient of the product $\prod O_w$ modulo the ideal consisting of all sequences $(o_w|w \in W)$ for which almost all o_w are zero. From now on, we will drop subscripts on the operations and just write $+$ and $\cdot$ for the ring operations on the O_w and on O_∞.

2.3.5. *If almost all O_w are fields, then so is O_∞.*

Just to give an example of how to work with ultraproducts, let me give the proof: if $a \in O_\infty$ is non-zero, then by the previous description of the ring structure on O_∞, almost all a_w will be non-zero, for some (any) choice of $a_w \in O_w$ for which $\operatorname{ulim}_{w\to\infty} a_w = a$. Therefore, letting b_w be the inverse of a_w whenever this makes sense, and zero otherwise, one verifies that $\operatorname{ulim}_{w\to\infty} b_w$ is the inverse of a.

2.3.6. *If C_w are rings and O_w is an ideal in C_w, then O_∞ is an ideal in $C_\infty := \operatorname{ulim}_{w\to\infty} C_w$. In fact, O_∞ is equal to the subset of all elements of the form $\operatorname{ulim}_{w\to\infty} o_w$ with almost all $o_w \in O_w$. Moreover, the ultraproduct of the C_w/O_w is isomorphic to C_∞/O_∞.*

2.3.7. *If $f_w\colon A_w \to B_w$ are ring homomorphisms, then the ultraproduct f_∞ is again a ring homomorphism. In particular, if σ_w is an endomorphism on A_w, then the ultraproduct σ_∞ is a ring endomorphism on $A_\infty := \operatorname{ulim}_{w\to\infty} A_w$.*

These examples are just instances of the general principle that 'algebraic structure' carries over to the ultraproduct. The precise formulation of this principle is called Łos' Theorem and requires some terminology from model-theory. Since we will only apply the ultraproduct construction to rings, we do not need to introduce the formal language of model-theory and we can get by with the following ad hoc definition.

2.4. Formulae

With a *quantifier free formula without parameters* in the free variables $X = (X_1, \dots, X_n)$, we will mean in this paper an expression of the form

$$\varphi(X) := \bigvee_{j=1}^{m} f_{1j} = 0 \wedge \ldots \wedge f_{sj} = 0 \wedge g_{1j} \neq 0 \wedge \ldots \wedge g_{tj} \neq 0, \qquad (2.1)$$

where each f_{ij} and g_{ij} is a polynomial with integer coefficients in the variables X, and where $\wedge$ and $\vee$ are the logical connectives *and* and *or*. If instead we allow the f_{ij} and g_{ij} to have coefficients in a ring C, then we call $\varphi(X)$ a *quantifier free formula with parameters in C*. We allow all possible degenerate cases as well: there might be no variables at all (so that the formula simply declares that certain elements in $\mathbb{Z}$ or in C are zero and others are non-zero) or there might be no equations or no negations or perhaps no conditions at all. Put succinctly, a quantifier free formula is a Boolean combination of polynomial equations using the connectives $\wedge$, $\vee$ and $\neg$ (negation), with the understanding that we use distributivity and De Morgan's Laws to rewrite this Boolean expression in the (disjunctive normal) form (2.1).

With a *formula without parameters* in the free variables X, we mean an expression of the form

$$\varphi(X) := (\mathrm{Q}_1\, Y_1) \cdots (\mathrm{Q}_p\, Y_p)\, \psi(X, Y),$$

where $\psi(X,Y)$ is a quantifier free formula without parameters in the free variables X and $Y = (Y_1, \dots, Y_p)$ and where Q_i is either the universal quantifier $\forall$ or the existential quantifier $\exists$. If instead $\psi(X,Y)$ has parameters from C, then we call $\varphi(X)$ a *formula with parameters in C*. A formula with no free variables is called a *sentence*.

2.5. Satisfaction

Let $\varphi(X)$ be a formula in the free variables $X = (X_1, \dots, X_n)$ with parameters from C (this includes the case that there are no parameters by taking $C = \mathbb{Z}$ and the case that there are no free variables by taking $n = 0$). Let A be a C-algebra and let $\mathbf{a} = (a_1, \dots, a_n)$ be a tuple with entries from A. We will give meaning to the expression $\mathbf{a}$ *satisfies the formula* $\varphi(X)$ *in* A (sometimes abbreviated to $\varphi(\mathbf{a})$ *holds in* A or *is true in* A) by induction on the number of quantifiers. Suppose first that $\varphi(X)$ is quantifier free, given by the Boolean expression (2.1). Then $\varphi(\mathbf{a})$ holds in A, if for some j_0, we have that all $f_{ij_0}(\mathbf{a}) = 0$ and all $g_{ij_0}(\mathbf{a}) \neq 0$. For the general case, suppose $\varphi(X)$ is of the form $(\exists Y)\,\psi(X,Y)$ (respectively, $(\forall Y)\,\psi(X,Y)$), where the satisfaction relation is already defined for the formula $\psi(X,Y)$. Then $\varphi(\mathbf{a})$ holds in A, if there is some $b \in A$ such that $\psi(\mathbf{a}, b)$ holds in A (respectively, if $\psi(\mathbf{a}, b)$ holds in A, for all $b \in A$). The subset of A^n consisting of all tuples satisfying $\varphi(X)$ will be called the *subset defined by* φ, and any subset that arises in such way will be called a *definable subset* of A^n.

Note that if $n = 0$, then there is no mention of tuples in A. In other words, a sentence is either true or false in A. By convention, we set A^0 equal to the singleton $\{\emptyset\}$ (that is to say, A^0 consists of the empty tuple $\emptyset$). If φ is a sentence, then the set defined by it is either $\{\emptyset\}$ or $\emptyset$, according to whether φ is true or false in A.

There is a connection between definable sets and Zariski-constructible sets, where the relationship is the most transparent over algebraically closed fields, as we will explain below. In general, we can make the following observations.

2.6. Constructible Sets

Let C be a ring. Let $\varphi(X)$ a quantifier free formula with parameters from C, given as in (2.1). Let $\Sigma_{\varphi(X)}$ denote the C-constructible subset of $\mathbb{A}^n_C$ consisting of all prime ideals $\mathfrak{p}$ of $\operatorname{Spec} C[X]$ which for some j_0 contain all f_{ij_0} and do not contain any g_{ij_0}. In particular, if $n = 0$, so that $\mathbb{A}^0_C$ is by definition $\operatorname{Spec} C$, then the C-constructible subset Σ_φ associated to φ is a subset of $\operatorname{Spec} C$.

Let A be a C-algebra and assume moreover that A is a domain (we will never use constructible sets associated to formulae if A is not a domain). For an n-tuple $\mathbf{a}$ over A, we have that $\varphi(\mathbf{a})$ holds in A if, and only if, the prime ideal $\mathfrak{p}_{\mathbf{a}}$ of $A[X]$ generated by the $X_i - a_i$ lies in the constructible set $\Sigma_{\varphi(X)}$ (strictly speaking, we should say that it lies in the base change $\Sigma_{\varphi(X)} \times_{\operatorname{Spec} C} \operatorname{Spec} A$, but for notational clarity, we will omit any reference to base changes). We will express the latter fact by saying that $\mathfrak{p}_{\mathbf{a}}$ is an A-rational point of $\Sigma_{\varphi(X)}$. If φ is a sentence, then Σ_φ is a constructible subset of $\operatorname{Spec} C$ and hence its base change to $\operatorname{Spec} A$ is a constructible subset of $\operatorname{Spec} A$. Since A is a domain, $\operatorname{Spec} A$ has a unique A-rational point (corresponding to the zero-ideal) and hence φ holds in A if, and only if, this point belongs to Σ_φ.

Conversely, if Σ is a C-constructible subset of $\mathbb{A}^n_C$, then we can associate to it a quantifier free formula $\varphi_\Sigma(X)$ with parameters from C as follows. However, here there is some ambiguity, as a constructible set is more intrinsically defined than a formula. Suppose first that Σ is the Zariski closed subset $\mathrm{V}(I)$, where I is an ideal in $C[X]$. Choose a system of generators, so that $I = (f_1, \dots, f_s)C[X]$ and set $\varphi_\Sigma(X)$ equal to the quantifier free formula $f_1(X) = \dots = f_s(X) = 0$. Let A be a C-algebra without zero-divisors. It follows that an n-tuple $\mathbf{a}$ is an A-rational point of Σ if, and only if, $\mathbf{a}$ satisfies the formula φ_Σ. Therefore, if we make a different choice of generators $I = (f_1', \dots, f_s')C[X]$, although we get a different formula φ', it defines in any C-algebra A without zero-divisors the same definable set, to wit, the collection of A-rational points of Σ. To associate a formula to an arbitrary constructible set, we do this recursively by letting $\varphi_\Sigma \wedge \varphi_\Psi$, $\varphi_\Sigma \vee \varphi_\Psi$ and $\neg\varphi_\Sigma$ correspond to the constructible sets $\Sigma \cap \Psi$, $\Sigma \cup \Psi$ and $-\Sigma$ respectively.

We say that two formulae $\varphi(X)$ and $\psi(X)$ in the same free variables $X = (X_1, \dots, X_n)$ are *equivalent over a ring* A, if they hold on exactly the same tuples from A (that is to say, if they define the same subsets in A^n). In particular, if φ and ψ are sentences, then they are equivalent in A if they are both true or both false in A. If $\varphi(X)$ and $\psi(X)$ are equivalent for all rings A in a certain class $\mathcal{K}$, then we say that $\varphi(X)$ and $\psi(X)$ are *equivalent modulo the class* $\mathcal{K}$. In particular, if Σ is a constructible set in $\mathbb{A}^n_C$, then any two formulae associated to it are equivalent modulo the class of all C-algebras without zero-divisors. In this sense, there is a one-one correspondence between constructible subsets of $\mathbb{A}^n_C$ and quantifier free formulae with parameters from C modulo the above equivalence relation.

2.7. *Quantifier Elimination*

For certain rings (or classes of rings), every formula is equivalent to a quantifier free formula; this phenomenon is known under the name *Quantifier Elimination*. We will only encounter it for the following class.

Theorem 2.1 (Quantifier Elimination for algebraically closed fields). *If $\mathcal{K}$ is the class of all algebraically closed fields, then any formula without parameters is equivalent modulo $\mathcal{K}$ to a quantifier free formula without parameters.*

More generally, if F is a field and $\mathcal{K}(F)$ the class of all algebraically closed fields containing F, then any formula with parameters from F is equivalent modulo $\mathcal{K}(F)$ to a quantifier free formula with parameters from F.

Proof (Sketch of proof). These statements can be seen as translations in model-theoretic terms of Chevalley's Theorem which says that the projection of a constructible set is again constructible. I will only explain this for the first assertion. As already observed, a quantifier free formula $\varphi(X)$ (without parameters) corresponds to a constructible set $\Sigma_{\varphi(X)}$ in $\mathbb{A}^n_{\mathbb{Z}}$ and the tuples in K^n satisfying $\varphi(X)$ are precisely the K-rational points $\Sigma_{\varphi(X)}(K)$ of $\Sigma_{\varphi(X)}$. The key observation is now the following. Let $\psi(X, Y)$ be a quantifier free formula and put $\xi(X) := (\exists Y)\, \psi(X, Y)$, where $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_m)$. Let Ψ be the subset of K^{n+m} defined by $\psi(X, Y)$ and let Ξ be the subset of K^n defined by $\xi(X)$. Therefore, if we identify K^{n+m} with the collection of K-rational points of $\mathbb{A}^{n+m}_K$, then

$$\Psi = \Sigma_{\psi(X,Y)}(K).$$

Moreover, if $p\colon \mathbb{A}^{n+m}_K \to \mathbb{A}^n_K$ is the projection on the first n coordinates then $p(\Psi) = \Xi$. By Chevalley's Theorem, $p(\Sigma_{\psi(X,Y)})$ (as a subset in $\mathbb{A}^n_{\mathbb{Z}}$) is again constructible, say of the form $\Sigma_{\chi(X)}$, for some quantifier free formula $\chi(X)$. Hence

$\Xi = \Sigma_{\chi(X)}(K)$, showing that $\xi(X)$ is equivalent modulo K to $\chi(X)$. Since $\chi(X)$ does not depend on K, we have in fact an equivalence of formulae modulo the class $\mathcal{K}$. To get rid of an arbitrary chain of quantifiers, we use induction on the number of quantifiers, noting that the complement of a set defined by $(\forall Y)\,\psi(X,Y)$ is the set defined by $(\exists Y)\,\neg\psi(X,Y)$, where $\neg(\cdot)$ denotes negation.

For some alternative proofs, see [5, Corollary A.5.2] or [9, Theorem 1.6]. □

Therefore, when dealing with algebraically closed fields, we may forget altogether about formulae and use constructible sets instead. However, we will not always be able to work just in algebraically closed fields and so we need to formulate a general transfer principle for ultraproducts. Recall that a sentence is a formula without free variables.

Theorem 2.2 (Łos' Theorem). *Let C be a ring and let A_w be C-algebras. If φ is a sentence with parameters from C, then φ holds in almost all A_w if, and only if, φ holds in the ultraproduct A_∞.*

More generally, let $\varphi(X_1,\dots,X_n)$ be a formula with parameters from C and let $\mathbf{a}_w$ be an n-tuple in A_w with ultraproduct $\mathbf{a}_\infty$. Then $\varphi(\mathbf{a}_w)$ holds in almost all A_w if, and only if, $\varphi(\mathbf{a}_\infty)$ holds in A_∞.

The proof is tedious but not hard; one simply has to unwind the definition of formula (see [5, Theorem 9.5.1] for a more general treatment). Note that A_∞ is naturally a C-algebra, so that it makes sense to assert that φ is true or false in A_∞. Applying Łos' Theorem to a quantifier free formula, we get the following equational version.

Theorem 2.3 (Equational Łos' Theorem). *Suppose each A_w is a C-algebra and let A_∞ denote their ultraproduct. Let X be an n-tuple of variables, let $f \in C[X]$ and let $\mathbf{a}_w$ be an n-tuple in A_w. Then $f(\mathbf{a}_\infty) = 0$ in A_∞ if, and only if, $f(\mathbf{a}_w) = 0$ in A_w for almost all w.*

Moreover, instead of a single equation $f = 0$, we may take in the above statement any system of equations and negations of equations over C.

Let us list some applications of Łos' Theorem that are relevant for the present paper.

2.8. Ultraproducts of Fields

Let K_w be a collection of fields and K_∞ their ultraproduct, which is again a field by Assertion 2.3.5 (or by an application of Łos' Theorem). Note that the converse also holds: if an ultraproduct of rings is a field, then almost all of these rings are fields themselves.

2.8.1. If for each prime number p, only finitely many K_w have characteristic p, then K_∞ has characteristic zero.

Indeed, for every prime number p, the equation $pX - 1 = 0$ has a solution in all but finitely many of the K_w and hence it has a solution in K_∞, by Theorem 2.3.

2.8.2. *If almost all K_w are algebraically closed fields, then so is K_∞, and conversely.*

Indeed, for each $n \geq 2$, consider the sentence

$$\sigma_n := (\forall Y_0, \dots, Y_n)\,(\exists X)\, Y_n = 0 \;\vee\; Y_n X^n + \dots + Y_1 X + Y_0 = 0.$$

This sentence is true in any algebraically closed field, whence in almost all K_w, and therefore, by Łos' Theorem, in K_∞. However, a field in which every σ_n holds is algebraically closed. The converse is proven in the same way.

We have the following important corollary which can be thought of as a model theoretic Lefschetz Principle (here $\mathbb{F}_p^{\mathrm{alg}}$ denotes the algebraic closure of the p-element field).

Theorem 2.4 (Lefschetz Principle). *Let W be the set of prime numbers, endowed with some non-principal ultrafilter. The ultraproduct of the fields $\mathbb{F}_p^{alg}$ is isomorphic with the field $\mathbb{C}$ of complex numbers, that is to say, we have an isomorphism*

$$\operatorname*{ulim}_{p\to\infty} \mathbb{F}_p^{alg} \cong \mathbb{C}.$$

Proof. Let $\mathbb{F}_\infty$ denote the ultraproduct of the fields $\mathbb{F}_p^{\mathrm{alg}}$. By Assertion 2.8.2, the field $\mathbb{F}_\infty$ is algebraically closed, and by Assertion 2.8.1, its characteristic is zero. Using elementary set theory, one calculates that the cardinality of $\mathbb{F}_\infty$ is equal to that of the continuum. The theorem now follows since any two algebraically closed fields of the same uncountable cardinality are (non-canonically) isomorphic by Steinitz's Theorem (see [5]). □

Remark 2.5. We can extend the above result as follows: every uncountable algebraically closed field K of characteristic zero is the ultraproduct of algebraically closed fields of prime characteristic. More precisely, any ultraproduct of algebraically closed fields of different prime characteristic, but of the same uncountable cardinality as K, is isomorphic to K (in particular, we may do this for any choice of non-principal ultrafilter). So, if we would want to, we may view $\mathbb{C}$ also as an ultraproduct of algebraically closed fields of prime characteristic of cardinality the continuum, instead of the (countable) algebraically closed fields $\mathbb{F}_p^{\mathrm{alg}}$.

In the sequel, we will fix once and for all one such isomorphism between $\mathbb{C}$ and $\mathbb{F}_\infty$. Although some of the constructions will depend on the choice of this isomorphism (as well as on the choice of the non-principal ultrafilter), we will always obtain isomorphic objects (in the appropriate category), regardless of the particular choices made.

2.9. Ultraproducts of Rings

Let A_w be a collection of rings and let A_∞ be their ultraproduct.

2.9.1. *If each A_w is local with maximal ideal $\mathfrak{m}_w$ and residue field $k_w := A_w/\mathfrak{m}_w$, then A_∞ is local with maximal ideal $\mathfrak{m}_\infty := \operatorname{ulim}_{w\to\infty} \mathfrak{m}_w$ and residue field $k_\infty := \operatorname{ulim}_{w\to\infty} k_w$.*

Indeed, a ring is local if, and only if, the sum of any two non-units is again a non-unit. This statement is clearly expressible by means of a sentence, so that by Łos' Theorem, A_∞ is local. The remaining assertions now follow easily from Assertion 2.3.6. In fact, the same argument shows that the converse is also true: if A_∞ is local, then so are almost all A_w.

2.9.2. *If A_w are local rings of embedding dimension e, then so is A_∞.*

Indeed, by assumption almost all $\mathfrak{m}_w$ are generated by e elements x_{iw}. It follows that $\mathfrak{m}_\infty$ is generated by the e ultraproducts $x_{i\infty}$.

2.9.3. *Almost all A_w are domains (respectively, reduced) if, and only if, A_∞ is a domain (respectively, reduced).*

Indeed, being a domain is captured by the formula $(\forall X, Y)\, X = 0 \vee Y = 0 \vee XY \neq 0$ and being reduced by the formula $(\forall X)\, X = 0 \vee X^2 \neq 0$. In particular, using Assertion 2.3.6, we see that an ultraproduct of ideals is a prime (respectively, radical, maximal) ideal if, and only if, almost all ideals are prime (respectively, reduced, maximal).

2.9.4. *If almost all A_w are Artinian local rings of length l, then so is A_∞.*

This follows from [11]. If we do not restrict the length of the Artinian rings A_w, then A_∞ will no longer be Artinian (nor even Noetherian). In fact, apart from the above example, Noetherianity is never preserved in ultraproducts (for it cannot be expressed in general by means of formulae). This also shows that in general the ultraproduct of primary ideals need not be primary. Nonetheless, we still have the following unidirectional version: if A_∞ is Artinian, then so are almost all A_w. This yields:

2.9.5. *If I_w are ideals in the local rings $(A_w, \mathfrak{m}_w)$, such that in $(A_\infty, \mathfrak{m}_\infty)$, their ultraproduct I_∞ is $\mathfrak{m}_\infty$-primary, then almost all I_w are $\mathfrak{m}_w$-primary.*

2.9.6. *Suppose A_w and B_w are rings with respective ultraproducts A_∞ and B_∞. Then $A_w \cong B_w$ for almost all w if, and only if, $A_\infty \cong B_\infty$. Moreover, if S_w are rings such that A_w and B_w are S_w-algebras, then almost all A_w and B_w are isomorphic as S_w-algebras if, and only if, A_∞ and B_∞ are isomorphic as S_∞-algebras, where S_∞ is the ultraproduct of the S_w.*

This follows from Assertion 2.3.7 and Łos' Theorem, since we can express by means of formulae that a homomorphism is injective and surjective.

2.9.7. *For elements $x_{1w}, \dots, x_{dw}$ in A_w with ultraproduct $x_{i\infty}$, the sequence $(x_{1\infty}, \dots, x_{d\infty})$ is A_∞-regular if, and only if, the sequence $(x_{1w}, \dots, x_{dw})$ is A_w-regular for almost all w.*

Let me illustrate the argument by taking $d = 2$. We want to write down a formula in two free variables X and Y expressing that X is not a zero-divisor and Y is not a zero-divisor modulo X. The following formula $\varphi(X, Y)$ does exactly this

$$(\forall Z_1, Z_2)\,(\exists Z_3)\,(Z_1 = 0 \vee Z_1X \neq 0) \wedge (Z_1 = Z_3X \vee Z_1Y \neq Z_2X).$$

By Łos' Theorem, $\varphi(x_{1\infty}, x_{2\infty})$ holds in A_∞ if, and only if, $\varphi(x_{1w}, x_{2w})$ holds in almost all A_w.

I conclude this section, with discussing an example of an ultraproduct of rings that will play a crucial role in the sequel. For each w, let K_w be a field and put $A_w := K_w[X]$, where X is a finite fixed tuple of variables. Let A_∞ denote the ultraproduct $\operatorname{ulim}_{w\to\infty} A_w$. It follows from Assertion 2.3.1 that A_∞ contains the ultraproduct $K_\infty := \operatorname{ulim}_{w\to\infty} K_w$. Since each A_w is a $\mathbb{Z}[X]$-algebra, so is A_∞ by Łos' Theorem. Moreover, by that same theorem, the image of the variables X_i in A_∞ are algebraically independent over K_∞. Therefore, we have a canonical embedding of $K_\infty[X]$ into A_∞. The main property of this embedding was discovered by VAN DEN DRIES in [24].

Theorem 2.6 (VAN DEN DRIES). *If all K_w are fields with ultraproduct K_∞, then the canonical embedding*

$$K_\infty[X] \to \operatorname*{ulim}_{w\to\infty} K_w[X]$$

is faithfully flat, for every finite tuple of variables X.

Proof. See [10, Theorem 1.7]; for an alternative proof, see [17, A.2]. □

Set $A := K_\infty[X]$ and $A_\infty := \operatorname{ulim}_{w\to\infty} K_w[X]$. We have the following immediate corollary.

Corollary 2.7. *For every ideal I of A, we have that $IA_\infty \cap A = I$.*

Another useful property of this embedding is that an ideal I is prime in A if, and only if, IA_∞ is prime in A_∞; see Corollary 4.2 below.

3. Approximations and Non-standard Hulls

As observed in Assertion 2.9.4, in most cases, an ultraproduct is not Noetherian. In particular, no positive dimensional finitely generated $\mathbb{C}$-algebra arises as an ultraproduct. Nonetheless, these are the algebras on which we want to define a tight closure operation. The idea therefore is to embed such an algebra in an ultraproduct of rings of prime characteristic, where tight closure is already defined. Of course, this embedding should be canonical and there should be enough transfer between the algebra and the ultraproduct.

3.1. Notations

To treat a more general situation, we fix some field K of characteristic zero which arises as an ultraproduct of fields of prime characteristic. By Remark 2.5, this includes the case of an uncountable algebraically closed field of characteristic zero, and in particular, the case that $K = \mathbb{C}$. The underlying index set W will always be the set of prime numbers and we will no longer write it. Instead, we express this notationally by using for index the letter p, which always stands for an arbitrary prime number (there will never be need to specialize the particular prime number). Moreover, we fix some (unnamed) non-principal ultrafilter on the set of prime numbers. For each p, fix a field K_p of characteristic p, so that the ultraproduct $\operatorname{ulim}_{p\to\infty} K_p$ is isomorphic to K. Fix also an isomorphism between K and $\operatorname{ulim}_{p\to\infty} K_p$. Finally, fix a tuple of variables X, set $A_p := K_p[X]$ and let A_∞ be their ultraproduct. By the discussion at the end of the previous section, we may view $K[X]$ as a subring of A_∞ (after identification via the fixed isomorphism between K and $\operatorname{ulim}_{p\to\infty} K_p$), and we will denote this subring by A.

3.2. Approximations and Codes

By the isomorphism we just fixed, every element $c \in K$ can be written as an ultraproduct $\operatorname{ulim}_{p\to\infty} c_p$, where $c_p \in K_p$. If c'_p is a second choice of elements in K_p for which $c = \operatorname{ulim}_{p\to\infty} c'_p$, then almost all $c_p = c'_p$. We refer to any such choice of c_p as an *approximation* of c. If $\mathbf{c}$ is a tuple $(c_1, \dots, c_n)$, then taking an approximation c_{ip} for every c_i, yields a tuple $\mathbf{c}_p := (c_{1p}, \dots, c_{np})$, called an *approximation* of $\mathbf{c}$, and any two choices of approximations will be the same for almost all p. We want to extend the notion of approximation to other algebraic objects as well. At the same time, we will define the notion of family and fiber, which will be used in carrying out the transfer in the next section.

3.2.1. Polynomials. Let $f \in K[X]$. Write $f = \sum_\nu a_\nu X^\nu$ with $a_\nu \in K$. Choose for each a_ν an approximation $a_{\nu p}$ and put

$$f_p := \sum_\nu a_{\nu p} X^\nu.$$

If we make different choices of approximations for the a_ν, the resulting polynomials will still be equal for almost all p (since only finitely many coefficients are involved). We will call any such choice of polynomials f_p an *approximation* of f. It follows that the element $\operatorname{ulim}_{p\to\infty} f_p$ in A_∞ lies in the subring A and is equal to f.

More generally, let $\mathfrak{f} \in \mathbb{Z}[U, X]$, where U is another tuple of (parametric) variables. With a *fiber* of $\mathfrak{f}$ over a field F, we mean a polynomial $\mathfrak{f}(\mathbf{u}, X)$ in $F[X]$, obtained by substituting for U some tuple $\mathbf{u}$ over F. For instance, if $\mathfrak{f} = \sum U_\nu X^\nu$ and $\mathbf{a}$ is the tuple over K of the coefficients (a_ν) of f (as above), then $f = \mathfrak{f}(\mathbf{a}, X)$. For this reason, we call $\mathfrak{f}$ a *family of polynomials* and we call $\mathbf{a}$ a *code of f with respect to the family* $\mathfrak{f}$. It follows that if $\mathbf{a}_p$ is an approximation of $\mathbf{a}$, then $\mathfrak{f}(\mathbf{a}_p, X)$ is an approximation of f.

3.2.2. Ideals. Let I be an ideal in A, say of the form $(f_1, \dots, f_s)A$. Choose for each f_i an approximation f_{ip} and let I_p be the ideal in A_p generated by these f_{ip}. Again, one checks that if we make a different choice of approximations of the f_i, then for almost all p, the ideal obtained in this way is equal to I_p. In fact, by Łos' Theorem more is true: if the g_i are a different set of generators of I and if g_{ip} is an approximation for each g_i, then the ideals I_p and $(g_{1p}, \dots, g_{tp})A_p$ are equal for almost all p. We call any choice of I_p an *approximation* of I.

This time, the ultraproduct $I_\infty := \operatorname{ulim}_{p\to\infty} I_p$ of the I_p is no longer the original ideal I. Instead, we have that $I_\infty = IA_\infty$. In particular, we can retrieve I from the approximation I_p, since $I = I_\infty \cap A$ by Corollary 2.7.

To rephrase this in terms of families and fibers, let $\mathfrak{I}$ be an ideal in $\mathbb{Z}[U, X]$, where U is another tuple of (parametric) variables; we call any such an ideal a *family of ideals*. With a *fiber* of the family $\mathfrak{I}$ over a field F, we mean the image $\mathfrak{I}(\mathbf{u})$ of $\mathfrak{I}$ under the homomorphism $\mathbb{Z}[U, X] \to F[X]$ given by the substitution $U = \mathbf{u}$. In other words, if $\mathfrak{I}$ is generated by polynomials $\mathfrak{f}_i$, then $\mathfrak{I}(\mathbf{u})$ is generated by the fibers $\mathfrak{f}_i(\mathbf{u}, X)$. As for polynomials, we can find an approximation of an ideal I in A as follows. Choose some ideal $\mathfrak{I}$ in some polynomial ring $\mathbb{Z}[U, X]$ and choose a tuple $\mathbf{a}$ over K, such that $I = \mathfrak{I}(\mathbf{a})$. It is clear that we can always find such data. We call any such choice of tuple $\mathbf{a}$ a *code of I with respect to the family* $\mathfrak{I}$. If we take an approximation $\mathbf{a}_p$ of the code $\mathbf{a}$, then $\mathfrak{I}(\mathbf{a}_p)$ is an approximation of I.

3.2.3. Algebras. Let B be a finitely generated K-algebra of the form A/I, for some ideal I. Let I_p be an approximation of I and put $B_p := A_p/I_p$. I claim that taking a different presentation $B \cong K[Y]/J$ and a choice of approximation J_p yields algebras $K_p[Y]/J_p$ which are isomorphic as K_p-algebras to the B_p, for almost all p. It is therefore justified to call any choice of B_p an *approximation* of B. To prove the claim, let the assignment $X_i \mapsto P_i(Y)$, with $P_i \in K[Y]$, induce a K-algebra isomorphism between $K[X]/I$ and $K[Y]/J$. Let P_{ip} be an approximation of P_i. It follows that for almost all p, the K_p-algebra homomorphism $\phi_p \colon K_p[X] \to K_p[Y]$ given by $X_i \mapsto P_{ip}$ maps the ideal I_p into the ideal J_p. Moreover, by applying the same reasoning to an inverse of the given isomorphism, we see that ϕ_p induces an isomorphism between $K_p[X]/I_p$ and $K_p[Y]/J_p$, for almost all p (see also §3.2.4 below).

The ultraproduct B_∞ of the B_p is in general bigger than B, and will be called the *non-standard hull* of B. See §3.4 below for more details.

A finitely generated $\mathbb{Z}$-algebra $\mathfrak{A}$ of the form $\mathbb{Z}[U,X]/\mathfrak{I}$ with U some variables and $\mathfrak{I}$ an ideal in $\mathbb{Z}[U,X]$ (that is to say, a family of ideals), is called a *family of algebras*. A *fiber* of this family $\mathfrak{A}$ over a field F, is the algebra

$$\mathfrak{A}(\mathbf{u}) := F[X]/\mathfrak{I}(\mathbf{u})$$

for some fiber $\mathfrak{I}(\mathbf{u})$ over F of $\mathfrak{I}$ (that is to say, for some tuple $\mathbf{u}$ over F). If B is a finitely generated K-algebra of the form A/I and if $\mathbf{a}$ is a code for I with respect to some family $\mathfrak{I}$, then we call $\mathbf{a}$ also a *code* for B with respect to the family $\mathfrak{A} := \mathbb{Z}[U,X]/\mathfrak{I}$. If $\mathbf{a}_p$ is an approximation of $\mathbf{a}$, then $\mathfrak{A}(\mathbf{a}_p)$ is an approximation of $B = \mathfrak{A}(\mathbf{a})$.

3.2.4. Homomorphisms. Let B and C be finitely generated K-algebras of the form $B = A/I$ and $C = A/J$, for some ideals I and J, and let $\phi\colon B \to C$ be a K-algebra homomorphism given by the rule $X_i \mapsto F_i$, for some $F_i \in A$. Let I_p, J_p and F_{ip} be approximations of I, J and F_i respectively. It is an easy exercise to show, using Łos' Theorem, that with $B_p := A_p/I_p$ and $C_p := A_p/J_p$ (so that they are approximations of B and C respectively), the K_p-algebra endomorphism of A_p given by $X_i \mapsto F_{ip}$ induces a homomorphism $\phi_p\colon B_p \to C_p$, for almost all p. We call ϕ_p an *approximation* of ϕ. Its ultraproduct ϕ_∞ then gives a homomorphism $B_\infty \to C_\infty$ giving rise to a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\ \phi\ } & C \\ \downarrow & & \downarrow \\ B_\infty & \xrightarrow[\ \phi_\infty\ ]{} & C_\infty. \end{array} \tag{3.1}$$

We could similarly define the notions of family, fiber and code for homomorphisms; we leave the details to the reader.

3.3. *Summary*

In all of the above cases, we have the following underlying principle at work. Let O be an algebraic object defined over K (such as a polynomial, an ideal in $K[X]$, a finitely generated K-algebra or a K-algebra homomorphism). We can find a family $\mathfrak{O}$ (defined over $\mathbb{Z}[U,X]$) and a tuple $\mathbf{u}$ over K, such that the fiber $\mathfrak{O}(\mathbf{u})$ is precisely the original object O. Moreover, to obtain an approximation O_p of O, we then simply have to take an approximation $\mathbf{u}_p$ of $\mathbf{u}$ and take for O_p the fiber $\mathfrak{O}(\mathbf{u}_p)$. Put succinctly, an object is encoded by a tuple over K, that is to say, a code; an approximation is then encoded by an approximation of the code. Moreover, all the families discussed so far are made up from families of ideals.

3.4. Non-standard Hulls

Let B be a finitely generated K-algebra and let B_p be an approximation of B. We call the ultraproduct $B_\infty := \operatorname{ulim}_{p\to\infty} B_p$ the *non-standard hull* of B.

Since any two approximations of B are isomorphic as K_p-algebras by the argument in §3.2.3, it follows from Assertion 2.9.6 that B_∞ is uniquely defined up to K-algebra isomorphism. As we are only interested in objects up to isomorphism, it does not matter which approximation we take to calculate a non-standard hull.

Corollary 3.1. *The canonical homomorphism $B \to B_\infty$ is faithfully flat.*

Proof. Suppose B is of the form A/I, for some ideal I of A. We showed that $IA_\infty = I_\infty$ and from Assertion 2.3.6, after taking some approximation, we get that B_∞ is equal to A_∞/I_∞. Therefore, the homomorphism $B \to B_\infty$ is just the base change of the embedding $A \to A_\infty$ and hence is faithfully flat by Theorem 2.6. □

The next result explains somehow the terminology *non-standard*.

Proposition 3.2. *For each p, take some f_p in A_p. Then the f_p are an approximation of an element $f \in A$ if, and only if, there is a uniform bound N on the degree of each f_p.*

The proof is left as an exercise to the reader. As a corollary, one might say that A consist of all elements of *finite degree* in its non-standard hull A_∞, or, formulated differently, that A consists of the *standard elements* of A_∞.

4. Transfer

In this section, we keep the notations introduced in §3.1. In particular, K is the ultraproduct of characteristic p fields K_p, X is a fixed tuple of variables, $A := K[X]$ and $A_p := K_p[X]$, and A_∞ is the ultraproduct of the A_p (whence the non-standard hull of A).

Recall that for a finitely generated K-algebra $B = A/I$, we called the characteristic p rings $B_p := A_p/I_p$ an approximation of B, where I_p was obtained from I by replacing each coefficient $c \in K$ in each member of a generating set of I, by some $c_p \in K_p$ with the property that $\operatorname{ulim}_{p\to\infty} c_p = c$ (after identification of K with $\operatorname{ulim}_{p\to\infty} K_p$). Although neither I_p nor B_p is uniquely defined, any different choice of this data will be isomorphic for almost all p.

The objective of this section is to show how many algebraic properties pass from B to any of its approximations and vice versa. We will also deal with localizations of finitely generated K-algebras. Therefore, let us call a ring B a *K-affine algebra* if it is either a finitely generated K-algebra or a localization at a prime ideal of a finitely generated K-algebra (we express the latter fact by calling B a *local K-affine algebra*).

4.1. Definability in families

Let $\mathcal{P}$ be a property of t polynomials in a polynomial ring over a field in the variables X. We say that $\mathcal{P}$ is *definable in families*[2], if for each choice of t families of polynomials $\mathfrak{f}_1, \dots, \mathfrak{f}_t$ (in some parametric variables U; see §3.2), we can find a formula without parameters $\varphi(U)$, such that for each field F, a tuple $\mathbf{u}$ over F satisfies $\varphi(U)$ if, and only if, the t fibers $\mathfrak{f}_i(\mathbf{u})$ have property $\mathcal{P}$.

If we would allow only algebraically closed fields F in the above definition, then we could replace the formula φ by a $\mathbb{Z}$-constructible subset Σ by Theorem 2.1. This would suffice for treating the case of main interest, $K = \mathbb{C}$. However, in the current generality, where K is allowed to be non-algebraically closed, we need the formalism of formulae.

Since properties of ideals can be translated into properties of polynomials by considering a generating set of the ideal, we may include in the above definition properties about s ideals and t polynomials in a polynomial ring over a field. More precisely, if we are given s families of ideals $\mathfrak{I}_1, \dots, \mathfrak{I}_s$ and t families of polynomials $\mathfrak{f}_1, \dots, \mathfrak{f}_t$, then a property of the s fibers $\mathfrak{I}_i(\mathbf{u})$ and the t fibers $\mathfrak{f}_i(\mathbf{u})$, can be viewed as a property of the $sm + t$ fibers $\mathfrak{f}_i(\mathbf{u})$, where I_i is generated by the m polynomials $\mathfrak{f}_{t+(i-1)m+1}, \dots, \mathfrak{f}_{t+im}$. Since this allows for less cumbersome notation, we will in the sequel consider properties of polynomials and ideals. A similar convention can be made for families of algebras and elements and ideals in these algebras. We will leave it up to the reader to translate everything back to properties of polynomials (see also §4.4 below).

4.2. Properties which are definable in families

In the following, each given property (involving ideals $I, J, J_1, \dots$ and/or elements $f, g, g_1, \dots$), is definable in families.

4.2.1. The ideal membership property *expressing that "$f \in I$".*

This is definable in families for the following reason. There exists for each pair of natural numbers (d, n), a bound d' such that if $f_0, \dots, f_s$ are polynomials in n variables of degree at most d over a field and if f_0 lies in the ideal generated by the remaining f_i, then there exist polynomials g_i of degree at most d', such that $f_0 = g_1 f_1 + \dots + g_s f_s$ (see for instance [10], where this is shown to follow from Theorem 2.6). Now, let $\mathfrak{f}_0$ be a family of polynomials and let $\mathfrak{I}$ be a family of ideals, say generated by $\mathfrak{f}_1, \dots, \mathfrak{f}_s$. Let n be the number of X-variables and d be the maximum of the X-degree of all $\mathfrak{f}_i$. For a tuple $\mathbf{u}$ over a field F, we have that $\mathfrak{f}_0(\mathbf{u}, X)$ belongs to the fiber $\mathfrak{I}(\mathbf{u})$, if there exist polynomials g_i over F of degree at most d', such that

$$\mathfrak{f}_0(\mathbf{u}, X) = g_1 \mathfrak{f}_1(\mathbf{u}, X) + \dots + g_s \mathfrak{f}_s(\mathbf{u}, X). \tag{4.1}$$

[2] Note that in [12,13], to be definable in families was called there to be *asymptotically definable*.

Let $\mathbf{c}_i$ be the tuple of all coefficients of g_i (so that it is the code of g_i with respect to some appropriate choice of family of ideals of degree d'). The existence of the g_i satisfying (4.1) is then equivalent with the existence of tuples $\mathbf{c}_i$ over F satisfying the equations obtained by expanding the right hand side of (4.1) and comparing coefficients with the left hand side. Note that the latter set of equations involve the $\mathbf{u}$. Therefore, $\mathfrak{f}_0(\mathbf{u}, X)$ belongs to $\mathfrak{I}(\mathbf{u})$ if, and only if, this system has a solution in F for the $\mathbf{c}_i$. This latter statement, as a statement about the tuple $\mathbf{u}$, can be expressed by a formula (with only equations and existential quantifiers), showing that ideal membership is definable in families.

4.2.2. *The* ideal containment property *expressing that "$I \subset J$". Also, the properties expressing that one ideal is the sum, the product or the intersection of two (or some other fixed number of) ideals and the property that "$I = (J_1 : J_2)$".*

4.2.3. *The* primality *property expressing that "I is prime".*

As for the previous cases and for most cases that we will encounter, this property is definable in families due to the existence of certain bounds. More precisely, for each pair (d, n), we can find a bound d'', such that if I is an ideal generated by polynomials in n variables over a field of degree at most d and if for all polynomials f and g of degree at most d'', we have that $fg \in I$ implies that either f or g belongs to I, then in fact I is prime (see [10]). From this and the fact that the ideal membership problem is definable in families, it follows that primality is definable in families.

4.2.4. *The property expressing that "J is a prime ideal containing I".*

This is immediate from Assertions 4.2.2 and 4.2.3.

4.2.5. *The properties expressing that an ideal is radical, primary or maximal. Also, the properties expressing that one ideal is the radical of another, or that one ideal is an associated prime ideal of another, or that one ideal has a primary decomposition given by some other fixed number of ideals.*

Again this follows from the corresponding existence of bounds, proven in [10]. Note that there is also a bound on the possible number of associated primes of an ideal in terms of the degrees of its generators.

4.2.6. *For any fixed l, the property expressing that "J is a prime ideal containing I such that A_J/IA_J has length l".*

Immediate from Assertion 4.2.4 and the bounds proven in [12, Theorem 2.4].

4.2.7. For any fixed l, the property expressing that "J is a prime ideal with $I \subset J_1 \subset J$ such that $J_1(A_J/IA_J)$ has height (respectively, depth) l". Also, the property expressing that "$(x_1, \dots, x_d)$ is (part of) a system of parameters of (A_J/IA_J)".

Follows from the bounds proven in [12, Proposition 5.1].

4.2.8. The property expressing that "J is a prime ideal containing I such that the sequence $(g_1, \dots, g_d)$ is A_J/IA_J-regular".

Follows from the bound proven in [12, Corollary 5.2].

4.2.9. The property expressing that "the K-algebra endomorphism φ on A given by $X_i \mapsto g_i$ maps the ideal I inside the ideal J". Also the property expressing that the "induced homomorphism $A/I \to A/J$ is injective."

If $I = (f_1, \dots, f_s)A$, then the statement $\varphi(I) \subset J$ amounts to the ideal membership in J of the compositions $f_i(g_1, \dots, g_n)$ and, therefore, is definable in families by Assertion 4.2.1. It follows from [12, Theorem 2.7] that the property expressing that "the kernel of the induced homomorphism $\phi\colon A/I \to A/J$ is equal to an ideal $J_1(A/I)$" is definable in families. From this the definability in families of the injectivity of ϕ is clear.

Shortly, we will discuss properties of affine algebras and their ideals, but let us first indicate the main application of definability in families to the transfer problem. We keep the notations and definitions from §3.1.

Proposition 4.1. *Let $\mathcal{P}$ be a property about s ideals and t polynomials in a polynomial ring over a field. Assume that $\mathcal{P}$ is definable in families. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ and $g_1, \dots, g_t$ be ideals and polynomials in A. Choose approximations $\mathfrak{a}_{ip}$ and g_{jp} of $\mathfrak{a}_i$ and g_j respectively. The ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ and the elements $g_1, \dots, g_t$ satisfy property $\mathcal{P}$ if, and only if, the ideals $\mathfrak{a}_{1p}, \dots, \mathfrak{a}_{sp}$ and the polynomials $g_{1p}, \dots, g_{tp}$ satisfy property $\mathcal{P}$ for almost all p.*

Proof. Choose families of ideals $\mathfrak{I}_i$ and families of polynomials $\mathfrak{g}_i$ and choose a tuple $\mathbf{a}$ over K such that $\mathfrak{I}_i(\mathbf{a}) = \mathfrak{a}_i$ and $\mathfrak{g}_j(\mathbf{a}, X) = g_j(X)$ (this is always possible by the observations made in §3.3). Let $\mathbf{a}_p$ be an approximation of $\mathbf{a}$. As explained in §3.2, we can use $\mathfrak{I}_i(\mathbf{a}_p)$ and $\mathfrak{g}_j(\mathbf{a}_p, X)$ as approximations of $\mathfrak{a}_i$ and $g_j(X)$ respectively, so that we may assume that $\mathfrak{a}_{ip} = \mathfrak{I}_i(\mathbf{a}_p)$ and $g_{jp} = \mathfrak{g}_j(\mathbf{a}_p, X)$. Let $\varphi(U)$ be the formula witnessing that $\mathcal{P}$ is definable in families for the families $\mathfrak{I}_i$ and $\mathfrak{g}_j$.

Assume that the ideals $\mathfrak{a}_{ip}$ and the polynomials g_{jp} satisfy property $\mathcal{P}$ for almost all p. Therefore, $\varphi(\mathbf{a}_p)$ is true in K_p for almost all p. By Łos' Theorem, $\varphi(\mathbf{a})$ holds in K. This in turn means that the ideals $\mathfrak{a}_i$ and the polynomials g_j have property $\mathcal{P}$. The converse holds by reversing the arguments. □

Although we will not need this, the converse of Proposition 4.1 also holds: if a property $\mathcal{P}$ as above has the property that it holds on ideals $\mathfrak{a}_i$ and polynomials g_i if, and only if, it holds on almost all approximations $\mathfrak{a}_{ip}$ and g_{ip}, then $\mathcal{P}$ is definable in families.

Applying Proposition 4.1 to Assertions 4.2.2 and 4.2.3, we get the following corollary.

Corollary 4.2. *Let I and J be ideals in A with approximations I_p and J_p. Let I_∞ and J_∞ be their respective ultraproducts, viewed as ideals in A_∞. Then the following are true.*

1. *Almost all $I_p \subset J_p$ if, and only if, $I \subset J$ if, and only if, $I_\infty \subset J_\infty$.*
2. *Almost all I_p are prime if, and only if, I is prime if, and only if, I_∞ is.*

We will shortly obtain more transfer results of this kind, but first we extend our definition of approximation to incorporate local affine algebras.

4.3. Approximations and non-standard hulls–local case

Let R be a local K-affine algebra given as A_J/IA_J with $I \subset J$ and J prime. By Corollary 4.2, almost all J_p are prime ideals containing I_p. In particular, for almost all p, the ring

$$R_p := (A_p)_{J_p}/I_p(A_p)_{J_p}$$

is well-defined (in the remaining case, we can put $R_p := 0$). It follows from Corollary 4.2 that

$$\operatorname*{ulim}_{p\to\infty} R_p = (A_\infty)_{J_\infty}/I_\infty(A_\infty)_{J_\infty}, \tag{4.2}$$

We call the ultraproduct of the R_p the *non-standard hull* of R and denote it R_∞. This is well-defined up to K-algebra isomorphism. Indeed, if we take a different presentation of R as a localization of a finitely generated K-algebra, then the resulting local K_p-algebras will almost all be isomorphic to R_p, since both choices have the same ultraproduct equal to the right hand side of (4.2). Any such choice of R_p will therefore be called an *approximation* of R.

Corollary 4.3. *For R a $\mathbb{C}$-affine algebra R, the canonical homomorphism $R \to R_\infty$ is faithfully flat.*

Proof. Follows immediately from Theorem 2.6 (or from Corollary 3.1) by base change. □

Terminology. Let R be a local K-affine algebra with approximation R_p. Suppose $R = A_J/IA_J$ and let I_p and J_p be approximations of I and J respectively. For x an element of R, choose $f, g \in A$ with $g \notin J$, such that x is equal to the image in R of the fraction f/g. Let f_p and g_p be approximations of f and g respectively. It follows that almost all $g_p \notin J_p$. The collection of elements $x_p := f_p/g_p \in R_p$ is called an *approximation* of x. Similarly, for $\mathfrak{a}$ an ideal in R, choose an ideal J_1 in A such that $I \subset J_1 \subset J$ and $\mathfrak{a} = J_1R$, and let J_{1p} be an approximation of J_1. The collection of ideals $\mathfrak{a}_p := J_{1p}R_p$ is called an *approximation* of $\mathfrak{a}$. A similar convention is in place if R is just of the form A/I. In other words, we extend the notion of an approximation of an element or an ideal to arbitrary affine algebras.

4.4. Transfer for affine algebras

Let $\mathcal{P}$ be a property of s ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ and t elements $u_1, \dots, u_t$ in an arbitrary affine algebra R. Associated to $\mathcal{P}$, we define a property $\mathcal{P}'$ on ideals and polynomials in a polynomial ring A as follows. Consider first the case that R is finitely generated, that is to say, of the form A/I with I an ideal in A. Let $\mathcal{P}'$ be the property of the $s+1$ ideals $I, I_1, \dots, I_s$ and the t elements $f_1, \dots, f_t$ expressing that $I \subset I_i$, for all i, and that $\mathcal{P}$ holds on the ideals I_iR and the elements f_i viewed as ideals and elements in $R = A/I$. If R is local, say of the form A_J/IA_J with $I \subset J$ and J prime, then we let $\mathcal{P}'$ be the property on $s+2$ ideals $I, J, J_1, \dots, J_t$ and $2t$ elements $f_1, g_1, \dots, f_t, g_t$ expressing that $I \subset J$ with J a prime ideal, that $I \subset J_i \subset J$ and $g_i \notin J$, for all i, and that property $\mathcal{P}$ holds for the ideals $\mathfrak{a}_i$ and the elements u_i in R given by $\mathfrak{a}_i := J_iR$ and $u_i := f_i/g_i$.

We will say that a property $\mathcal{P}$ of s ideals and t elements in an affine algebra is *definable in families*, if the corresponding property $\mathcal{P}'$ is definable in families. In particular, it follows that any of the properties listed in §4.2 when extended to affine algebras, remain definable in families. The following list of theorems are all proved using Proposition 4.1 and the fact that the pertinent properties are definable in families. For instance, from Assertions 4.2.2–4.2.5, we get the following generalization of Corollary 4.2 for arbitrary K-affine algebras.

Theorem 4.4. *Let R be a K-affine algebra with approximation R_p and non-standard hull R_∞. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{b}_1, \dots$ be ideals in R with approximations $\mathfrak{a}_p, \mathfrak{b}_p, \mathfrak{b}_{1p}, \dots$ respectively. Let $\mathfrak{a}_\infty, \mathfrak{b}_\infty, \mathfrak{b}_{1\infty}, \dots$ be their respective ultraproducts, viewed as ideals in R_∞. Then the following are true.*

1. *Almost all $\mathfrak{a}_p$ are equal to $\mathfrak{b}_{1p} \cap \mathfrak{b}_{2p}$ (respectively, to $\mathfrak{b}_{1p}\mathfrak{b}_{2p}$, $\mathfrak{b}_{1p} + \mathfrak{b}_{2p}$, $(\mathfrak{b}_{1p} : \mathfrak{b}_{2p})$) if, and only if, $\mathfrak{a}$ is equal to $\mathfrak{b}_1 \cap \mathfrak{b}_2$ (respectively, to $\mathfrak{b}_1\mathfrak{b}_2$, $\mathfrak{b}_1 + \mathfrak{b}_2$, $(\mathfrak{b}_1 : \mathfrak{b}_2)$) if, and only if, $\mathfrak{a}_\infty$ is equal to $\mathfrak{b}_{1\infty} \cap \mathfrak{b}_{2\infty}$ (respectively, to $\mathfrak{b}_{1\infty}\mathfrak{b}_{2\infty}$, $\mathfrak{b}_{1\infty} + \mathfrak{b}_{2\infty}$, $(\mathfrak{b}_{1\infty} : \mathfrak{b}_{2\infty})$).*
2. *Almost all $\mathfrak{a}_p$ are prime (respectively, radical, maximal or primary) if, and only if, $\mathfrak{a}$ is prime (respectively, radical, maximal or primary) if, and only if, $\mathfrak{a}_\infty$ is prime (respectively, radical, maximal or primary).*
3. *For almost all p, the ideals $\mathfrak{b}_{1p}, \dots, \mathfrak{b}_{sp}$ are the minimal prime ideals (respectively, the associated prime ideals, the primary ideals in an irredundant primary decomposition) of $\mathfrak{a}_p$ if, and only if, $\mathfrak{b}_1, \dots, \mathfrak{b}_s$ are the minimal prime ideals (respectively, the associated prime ideals, the primary ideals in an irredundant primary decomposition) of $\mathfrak{a}$.*

Applying Łos' Theorem and Proposition 4.1 to Assertions 4.2.6– 4.2.8, we obtain:

Theorem 4.5. *Let R be a K-affine algebra with approximation R_p and non-standard hull R_∞. Let x_i be elements in R with approximation x_{ip} and let $\mathfrak{a}$ be an ideal in R with approximation $\mathfrak{a}_p$. Then the following are true (where for the last property, we assume R to be local).*

1. *For some fixed l, almost all $\mathfrak{a}_p$ have height (respectively, depth) l if, and only if, $\mathfrak{a}$ has height (respectively, depth) l.*

2. *For some fixed l, almost all R_p have dimension (respectively, depth, length) l if, and only if, R has dimension (respectively, depth, length) l.*
3. *For almost all p, the sequence $(x_{1p}, \dots, x_{dp})$ is R_p-regular if, and only if, $(x_1, \dots, x_d)$ is R-regular if, and only if, $(x_1, \dots, x_d)$ is R_∞-regular.*
4. *For almost all p, the sequence $(x_{1p}, \dots, x_{dp})$ is (part of) a system of parameters of R_p if, and only if, $(x_1, \dots, x_d)$ is (part of) a system of parameters of R.*

Using the results from [12, Theorem 5.2], we get by a similar argument on the existence of bounds together with Proposition 4.1 the following theorem.

Theorem 4.6. *Let R be a local K-affine algebra with approximation R_p. If $\mathcal{P}$ is any of the following properties of local rings*

1. *regular;*
2. *complete intersection;*
3. *Gorenstein;*
4. *Cohen-Macaulay;*
5. *normal;*

then R has property $\mathcal{P}$ if, and only if, almost all R_p have.

Sometimes, transfer only goes in one direction, namely from zero to positive characteristic–the right direction as far as we are concerned–as the next result shows.

Theorem 4.7. *Let $R \to S$ be a homomorphism of K-affine algebras with approximation $R_p \to S_p$. If $R \to S$ is finite (respectively, injective, surjective, bijective), then so are almost all $R_p \to S_p$. Moreover, $R_\infty \to S_\infty$ then also has this property.*

Proof. If $u_1, \dots, u_s$ generate S as an R-module, then $u_{1p}, \dots, u_{sp}$ generate S_p as an R_p-module, for almost all p, by Łos' Theorem, where u_{ip} is an approximation of u_i. Taking ultraproducts then shows that $u_1, \dots, u_s$ generate S_∞ as an R_∞-module, so that we proved the assertion for the property of $R \to S$ to be finite, and in particular, to be surjective ($s = 1$). For injectivity, use Assertion 4.2.9. □

Note the converse of the above result might fail, since if $u_{1p}, \dots, u_{sp}$ are generators for S_p as an R_p-module, their ultraproduct might not lie in S when their degree is not bounded.

5. Non-standard Tight Closure

In this section, we change notation slightly: A will always denote a $\mathbb{C}$-affine algebra (not just the polynomial ring), with approximation A_p and non-standard hull A_∞. All homomorphisms between $\mathbb{C}$-affine algebras are tacitly assumed to be $\mathbb{C}$-algebra homomorphisms which are essentially of finite type. Nonetheless, in all what follows, we could have worked in the same generality as in §3.1, that is to say, replacing $\mathbb{C}$ by any field of characteristic zero which is an ultraproduct of fields of positive characteristic.

5.1. Non-standard Frobenius

On each A_p we have an action of the Frobenius endomorphism $\mathbf{F}_p$ given by $a \mapsto a^p$. This defines an endomorphism $\mathbf{F}_\infty$ on A_∞ given by the rule

$$\mathbf{F}_\infty(a_\infty) := \operatorname*{ulim}_{p\to\infty} \mathbf{F}_p(a_p) (= \operatorname*{ulim}_{p\to\infty} (a_p)^p)$$

where the a_p are chosen in A_p such that their ultraproduct is a_∞ (note that we reserved the term *approximation* only for elements in A). We call $\mathbf{F}_\infty$ the *non-standard Frobenius*.

In particular, $\mathbf{F}_\infty$ is an automorphism of $\mathbb{C}$ (take $A = \mathbb{C}$). Its properties on $\mathbb{C}$ are extensively studied by HRUSHOVSKI, MACINTYRE and others; see for instance [6] and [8], where it is shown that these are generic automorphisms, satisfying some twisted form of the Lang-Weil Estimates. However, we will not need any of these results here.

Although $\mathbf{F}_\infty$ is no longer algebraic, some vestige remains: for any ideal I of A_∞, we have that $\mathbf{F}_\infty(I) \subset I$, and in fact, $\mathbf{F}_\infty(I)$ is contained in the intersection of all powers I^n. In particular, $\mathbf{F}_\infty$ commutes with any homomorphism which is essentially of finite type. This justifies the omission of the ring A_∞ in the notation of $\mathbf{F}_\infty$. Of course, in general, $\mathbf{F}_\infty(A)$ is no longer contained in A. In fact, since A can be thought of the elements of A_∞ of finite degree, it is clear that $A \cap \mathbf{F}_\infty(A) = \mathbb{C}$, whenever A is reduced.

We have finally come to the key definition of this article. Note the following notation that will be in effect henceforth.

Notation. For an ideal I in A or in A_∞, we let $\mathbf{F}_\infty(I)A_\infty$ denote the ideal in A_∞ generated by all $\mathbf{F}_\infty(f)$ with $f \in I$.

Definition 5.1 (Non-standard tight closure). *Let $\mathfrak{a}$ be an ideal of A and $z \in A$. We say that z lies in the* non-standard closure *of $\mathfrak{a}$, if there exists some $c \in A$, not contained in any minimal prime of A, such that*

$$c\,\mathbf{F}_\infty(z) \in \mathbf{F}_\infty(\mathfrak{a})A_\infty. \tag{5.1}$$

We write $\mathrm{cl}_{ns}(\mathfrak{a})$ for the non-standard closure of $\mathfrak{a}$.

Similarly, we say that z lies in the non-standard tight closure $\mathrm{cl}_{nst}(\mathfrak{a})$ *of $\mathfrak{a}$, if there exists some $c \in A$, not contained in any minimal prime of A, such that*

$$c\,\mathbf{F}^m_\infty(z) \in \mathbf{F}^m_\infty(\mathfrak{a})A_\infty \tag{5.2}$$

for all $m \in \mathbb{N}$.

There are also stable versions of these two notions, defined as follows. We say that z lies in the *stable non-standard closure* of $\mathfrak{a}$, if instead of (5.1), we have for some $m \geq 1$ that $\mathbf{F}^{m-1}_\infty(c)\,\mathbf{F}^m_\infty(z) \in \mathbf{F}^m_\infty(\mathfrak{a})A_\infty$, and that z lies in the *stable non-standard tight closure* of $\mathfrak{a}$, if (5.2) holds only for sufficiently large m. We denote these closures respectively by $\mathrm{cl}^{\mathrm{stab}}_{ns}(\mathfrak{a})$ and $\mathrm{cl}^{\mathrm{stab}}_{nst}(\mathfrak{a})$.

We say that an ideal is *non-standard closed* (respectively, *stably non-standard closed*, *non-standard tightly closed* or *stably non-standard tightly closed*) if $\mathfrak{a}$ is equal to its own respective closure. We leave it to the reader to verify that the closure of an ideal is again an ideal, containing the original one, for any of the four variants. Moreover, these ideals are then closed with respect to that closure.

The following inclusions are easily verified

$$\begin{aligned} \mathrm{cl}_{ns}(\mathfrak{a}) &\subset \mathrm{cl}_{ns}^{\mathrm{stab}}(\mathfrak{a}) \\ \mathrm{cl}_{nst}(\mathfrak{a}) &\subset \mathrm{cl}_{nst}^{\mathrm{stab}}(\mathfrak{a}) \\ \mathrm{cl}_{nst}(\mathfrak{a}) &\subset \mathrm{cl}_{ns}(\mathfrak{a}) \\ \mathrm{cl}_{nst}^{\mathrm{stab}}(\mathfrak{a}) &\subset \mathrm{cl}_{ns}^{\mathrm{stab}}(\mathfrak{a}), \end{aligned}$$

where in order to prove the last inclusion, we use that $\mathbf{F}_\infty(c)$ is a multiple of c.

Translated in terms of approximations the definitions become the following (I leave the stable versions to the reader). For $\mathfrak{a} = (f_1, \dots, f_s)A$, we have that $z \in \mathrm{cl}_{ns}(\mathfrak{a})$ if, and only if, there exist $h_{i,p} \in A_p$, such that

$$c_p(z_p)^p = h_{1,p}(f_{1p})^p + \cdots + h_{s,p}(f_{sp})^p \tag{5.3}$$

for almost all p, where c_p, z_p and f_{ip} are approximations of c, z and f_i respectively. Note that the $h_{i,p}$ will in general not have bounded degree whence their ultraproducts are no longer in A.

We have that $z \in \mathrm{cl}_{nst}(\mathfrak{a})$ if, and only if, for each $m \in \mathbb{N}$, there exists a set of rational primes D_m in the ultrafilter $\mathcal{U}$ and there exist $h_{i,p,m} \in A_p$, such that

$$c_p(z_p)^{p^m} = h_{1,p,m}(f_{1p})^{p^m} + \cdots + h_{s,p,m}(f_{sp})^{p^m} \tag{5.4}$$

for all $p \in D_m$. However, in general the intersection of all D_m will no longer belong to $\mathcal{U}$, so that it might very well be that for no prime p, the element z_p lies in the characteristic p tight closure of $\mathfrak{a}_p$. This calls for one further notion.

Definition 5.2 (Generic Tight Closure). *We say that $z \in A$ lies in the* generic tight closure *of $\mathfrak{a}$, if z_p lies in the tight closure $(\mathfrak{a}_p)^*$ of $\mathfrak{a}_p$, for almost all p.*

In other words, we may choose all D_m to be equal in the above discussion. Put differently, the generic tight closure of $\mathfrak{a}$ is the contraction to A of the ultraproduct of the $(\mathfrak{a}_p)^*$. We will show in Theorem 8.5 below, that the generic tight closure of an ideal is contained in its non-standard tight closure.

In the remainder of this paper, we prove the main properties of these closure operations. Since the proofs for the stable versions require only minor modifications, I have not included them here. Notwithstanding, see [22, Theorem 3.4], where stable non-standard closure (called there *non-standard difference closure*) is used, since it has some additional good properties. At present, it is not clear whether the stable versions coincide with their non-stable counterparts, or, for that matter, whether all these closure operations coincide. Therefore, it is probably better practice to always work with the stable versions.

6. Regular Rings

One of the most important predecessors of tight closure theory is Kunz's Theorem that the Frobenius $\mathbf{F}_p$ is a flat endomorphism on a regular ring R of characteristic p. A word of caution: what is meant here is that the inclusion $\mathbf{F}_p(R) \subset R$ is flat, where $\mathbf{F}_p(R)$ denotes the subring of all p-th powers of elements in R. Frobenius also induces an isomorphism $R \cong \mathbf{F}_p(R)$, which, of course, is therefore flat, but this is not special to regular rings: it holds for all reduced rings R of characteristic p. In characteristic zero, a minor obstacle arises in that for a positive dimensional regular $\mathbb{C}$-affine algebra A, the non-standard Frobenius $\mathbf{F}_\infty$ is (probably) no longer a flat endomorphism on the non-standard hull A_∞. However, its restriction to A is.

Proposition 6.1. *Let A be a $\mathbb{C}$-affine algebra. If A is regular, then the extension $\mathbf{F}_\infty(A) \subset A_\infty$ is faithfully flat.*

Proof. It is clear that any proper ideal of $\mathbf{F}_\infty(A)$, extends to a proper ideal in A_∞. So we only need to show that the inclusion $\mathbf{F}_\infty(A) \subset A_\infty$ is flat. I will provide a proof by direct verification. Another proof is given in [19, Corollary 3.8] by showing that A_∞ viewed as an A-module via the non-standard Frobenius $\mathbf{F}_\infty$ is a balanced big Cohen-Macaulay module, from which it follows that it is flat over A. This then amounts to the flatness of A_∞ over $\mathbf{F}_\infty(A)$.

To show that the inclusion $\mathbf{F}_\infty(A) \subset A_\infty$ is flat, we need to show that any solution $\mathbf{x}_\infty$ over A_∞ of the linear homogeneous equation

$$\mathbf{F}_\infty(a_1)X_1 + \cdots + \mathbf{F}_\infty(a_n)X_n = 0 \tag{6.1}$$

with $a_i \in A$, can be written as a linear combination

$$\mathbf{x}_\infty = b_{1\infty}\mathbf{F}_\infty(\mathbf{z}_1) + \cdots + b_{M\infty}\mathbf{F}_\infty(\mathbf{z}_M)$$

with $b_{i\infty} \in A_\infty$ and each $\mathbf{z}_j$ a solution in A of the linear equation

$$a_1X_1 + \cdots + a_nX_n = 0. \tag{6.2}$$

Choose some approximation a_{ip} of each a_i and choose tuples $\mathbf{x}_p$ in A_p, such that $\operatorname{ulim}_{p\to\infty} \mathbf{x}_p = \mathbf{x}_\infty$. It follows from (6.1) and Łos' Theorem that $\mathbf{x}_p$ is a solution in A_p of the linear equation

$$\mathbf{F}_p(a_{1p})X_1 + \cdots + \mathbf{F}_p(a_{np})X_n = 0 \tag{6.3}$$

for almost all p. Let $K_p \subset (A_p)^n$ be the A_p-submodule consisting of all solutions of

$$a_{1p}X_1 + \cdots + a_{np}X_n = 0.$$

By Theorem 4.6 almost all A_p are regular, so that by Kunz's Theorem, the inclusion $\mathbf{F}_p(A_p) \subset A_p$ is flat. Therefore, (6.3) implies that $\mathbf{x}_p$ lies in the submodule of $(A_p)^n$ generated by $\mathbf{F}_p(K_p)$.

Let N be the maximum of the degrees of the a_j. Hence almost all a_{jp} have degree at most N as well. It follows from [10] or [12], that there is an $N' \in \mathbb{N}$ only depending on N, such that each K_p is generated by at most N' elements, each of degree at most N'. Let $\mathbf{z}_{ip}$, for $i = 1, \dots, N'$ be these generators. Put $\mathbf{z}_{i\infty} := \operatorname{ulim}_{p\to\infty} \mathbf{z}_{ip}$. Using Proposition 3.2, we see that $\mathbf{z}_{i\infty}$ lies already in A. Moreover, by Łos' Theorem, each $\mathbf{z}_{i\infty}$ is a solution of the linear equation (6.2). Since $\mathbf{x}_p$ is a linear combination of the $\mathbf{F}_p(\mathbf{z}_{ip})$, it follows that $\mathbf{x}_\infty$ is a linear combination of the $\mathbf{F}_\infty(\mathbf{z}_{i\infty})$, as required. □

Theorem 6.2. *Let A be a $\mathbb{C}$-affine algebra. If A is regular, then any ideal is non-standard closed.*

Proof. Let $\mathfrak{a} = (f_1, \dots, f_s)A$ be an ideal in A and let z be an element in its non-standard closure. Therefore, we can find a $c \in A$ not contained in any minimal prime of A, such that

$$c\,\mathbf{F}_\infty(z) = h_1\mathbf{F}_\infty(f_1) + \dots + h_s\mathbf{F}_\infty(f_s),$$

for some $h_i \in A_\infty$. In other words, c belongs to $(\mathbf{F}_\infty(\mathfrak{a})A_\infty :_{A_\infty} \mathbf{F}_\infty(z))$. By Proposition 6.1, the extension $\mathbf{F}_\infty(A) \subset A_\infty$ is flat, which implies that

$$(\mathbf{F}_\infty(\mathfrak{a})A_\infty :_{A_\infty} \mathbf{F}_\infty(z)) = \mathbf{F}_\infty(\mathfrak{a} :_A z)A_\infty.$$

If $z \notin \mathfrak{a}$, then $J := (\mathfrak{a} :_A z)$ is a proper ideal of A. In particular, we get that $c \in \mathbf{F}_\infty(J)A_\infty$. Since $\mathbf{F}_\infty(J)A_\infty \cap A \subset \mathbb{C}$ and J is a proper ideal, we must have that $\mathbf{F}_\infty(J)A_\infty \cap A = 0$, implying that $c = 0$, contradiction. □

Remark 6.3. The same holds true for non-standard tight closure, since it is contained in the non-standard closure, and an easy adaptation of the proof then gives the result for the stable versions as well.

7. Contractions under Finite Extensions

Theorem 7.1. *Let $A \subset B$ a finite (or, more generally, an integral) extension of $\mathbb{C}$-affine domains. Let $\mathfrak{a}$ be an ideal in A. Then $\operatorname{cl}_{ns}(\mathfrak{a}B) \cap A \subset \operatorname{cl}_{ns}(\mathfrak{a})$.*

Proof. It suffices to prove the theorem for finite extensions (since any relation in B already holds in a finite A-subalgebra of B). Let $z \in \operatorname{cl}_{ns}(\mathfrak{a}B) \cap A$, so that for some non-zero $b \in B$, we have that

$$b\,\mathbf{F}_\infty(z) \in \mathbf{F}_\infty(\mathfrak{a}B)B_\infty. \tag{7.1}$$

By a well-known argument (reason with the field of fractions), there exists an A-module homomorphism $\phi\colon B \to A$, sending b to some non-zero element a of A. Let $\phi_\infty\colon B_\infty \to A_\infty$ be its extension to the non-standard hulls B_∞ and A_∞ of B and A respectively (see Theorem 4.7). Applying ϕ_∞ to (7.1), we obtain

$$a\,\mathbf{F}_\infty(z) \in \mathbf{F}_\infty(\mathfrak{a})A_\infty,$$

showing that z lies in the non-standard closure of $\mathfrak{a}$. □

Remark 7.2. Alternatively, we can make the observation that b in (7.1) can be taken in A (namely, replace b by the constant term of an integral equation of b over A of minimal degree). Therefore, $b\,\mathbf{F}_\infty(z)$ lies in $\mathbf{F}_\infty(\mathfrak{a})B_\infty \cap A_\infty$. Since $A_\infty \subset B_\infty$ is finite by Theorem 4.7 and since any finite extension of integral domains in characteristic zero is split (by taking traces), we get that $\mathbf{F}_\infty(\mathfrak{a})B_\infty \cap A_\infty = \mathbf{F}_\infty(\mathfrak{a})A_\infty$, showing that z lies in the non-standard closure of $\mathfrak{a}$.

By either of these arguments we an also show that the Theorem is true with non-standard closure replaced by generic tight closure or non-standard tight closure.

8. Colon Capturing

Theorem 8.1 (Colon Capturing). *Let A be a local $\mathbb{C}$-affine algebra with system of parameters $(x_1,\dots,x_n)$. For each i, the colon ideal $((x_1,\dots,x_i)A : x_{i+1})$ is contained in the non-standard tight closure of $(x_1,\dots,x_i)A$.*

Proof. I will only give the argument in case A is equidimensional–for the general case, see Remark 8.3 below. Write $A = B/I$ with B some localization of a polynomial ring over $\mathbb{C}$. Consider the x_i already as elements of B. Suppose I has height e. By an easy prime avoidance argument, we can find $f_1,\dots,f_e \in I$, such that

$$\operatorname{ht}(f_1,\dots,f_e,x_1,\dots,x_i)B = e+i$$

for each i. In particular, $J := (f_1,\dots,f_e)B$ has height e. Since B is regular, J has no embedded associated primes. Since $J \subset I$ both have height e and since all minimal primes of I have height e by equidimensionality, we see that every minimal prime of I is also a minimal prime of J. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_m$ be the minimal primes of J, so that $\mathfrak{p}_1,\dots,\mathfrak{p}_l$ are the minimal primes of I, for some $l \le m$. Let J_k be the primary component of J belonging to the minimal prime $\mathfrak{p}_k$, for $k = 1,\dots,l$, and let $\tilde{J}$ be the intersection of the remaining primary components, so that

$$J = J_1 \cap \dots \cap J_l \cap \tilde{J}.$$

Let I_k be the $\mathfrak{p}_k$-primary component of I and $\tilde{I}$ be the intersection of the embedded primary components (this is not uniquely defined, but this does not matter here), so that

$$I = I_1 \cap \dots \cap I_l \cap \tilde{I}.$$

There is some N, such that $\mathfrak{p}_k^N \subset J_k$, for all k. Choose $c \in \tilde{J}$, but not in any $\mathfrak{p}_k$, for $k = 1,\dots,l$. It follows that

$$cI^N \subset \tilde{J} \cap \bigcap_{k=1}^{l} \mathfrak{p}_k^N \subset J_1 \cap \dots \cap J_l \cap \tilde{J} = J. \tag{8.1}$$

Moreover, the image of c in A is not contained in any minimal prime ideal of A.

Let $z \in B$ so that its image in A lies in $((x_1, \dots, x_i)A :_A x_{i+1})$. Therefore, we can find $b_i \in B$, such that $zx_{i+1} + b_1x_1 + \cdots + b_ix_i \in I$. Take approximations of everything in sight, so that for almost all p, we have a relation

$$z_p x_{i+1\,p} + b_{1p}x_{1p} + \cdots + b_{ip}x_{ip} \in I_p$$

for almost all p. Let q denote any power p^m bigger than N, then by (8.1) we have

$$c_p\left[(z_p)^q(x_{i+1\,p})^q + (b_{1p})^q(x_{1p})^q + \cdots + (b_{ip})^q(x_{ip})^q\right] \in J_p. \tag{8.2}$$

By Theorem 4.5, almost all $J_p + (x_{1p}, \dots, x_{dp})B_p$ have height $d+e$. Since almost all B_p are regular by Theorem 4.6, it follows that

$$(f_{1p}, \dots, f_{ep}, (x_{1p})^q, \dots, (x_{dp})^q)$$

is a regular sequence. Therefore, from (8.2), we get that

$$c_p(z_p)^q \in (f_{1p}, \dots, f_{ep}, (x_{1p})^q, \dots, (x_{ip})^q)B_p.$$

Taking reduction modulo I_p and then ultraproducts, we get that

$$c\,\mathbf{F}^m_\infty(z) \in (\mathbf{F}^m_\infty(x_1), \dots, \mathbf{F}^m_\infty(x_i))A_\infty,$$

for all $m \in \mathbb{N}$. Since we already observed that c is not contained in any minimal prime of A, we get that z lies in the non-standard tight closure of $(x_1, \dots, x_i)A$, as required. □

Remark 8.2. Colon Capturing then also holds for non-standard closure, as this contains non-standard tight closure.

One can avoid taking approximations in the above proof, using the fact that since $(f_1, \dots, f_e, x_1, \dots, x_d)$ is B-regular, $(f_1, \dots, f_e, \mathbf{F}^m_\infty(x_1), \dots, \mathbf{F}^m_\infty(x_d))$ is B_∞-regular, and then carry out the above proof directly in B_∞. To prove the latter fact, use an argument similar to the one proving Assertion 2.9.7; see [19, Theorem 5.1] for details.

Remark 8.3. Colon Capturing also holds for generic tight closure. Indeed, $z \in ((x_1, \dots, x_i)A : x_{i+1})$ leads to a similar relation for approximations. Since almost all $(x_{1p}, \dots, x_{ip})$ are a system of parameters of A_p by Theorem 4.5, tight closure Colon Capturing ([7, Theorem 3.1]) yields that z_p lies in the tight closure of $(x_{1p}, \dots, x_{ip})A_p$, that is to say, z lies in the generic tight closure of $(x_1, \dots, x_i)A$. Using this, we can give an alternative proof of Theorem 8.1, by showing that generic tight closure is contained in non-standard tight closure. This argument does not require us to assume equidimensionality. However, it relies on a non-trivial result from tight closure theory: the existence of test elements; moreover, we need the following uniform version.

Proposition 8.4. *For each $\mathbb{C}$-affine algebra A, we can find an element $c \in A$ with the property that c_p is a test element for almost all A_p, where c_p and A_p are approximations of c and A respectively.*

Proof. We use [3, Theorem 7.1 and 7.2] to generate test elements. First assume that A is a domain with field of fractions L. Take a Noether Normalization $\mathbb{C}[X] \hookrightarrow A$ (that is to say, a finite injective homomorphism). Since $\mathbb{C}[X] \hookrightarrow A$ is generically smooth, L is a finite separable extension of $\mathbb{C}(X)$. Choose $u_1, \dots, u_N \in A$, such that they form a basis of L over $\mathbb{C}(X)$. Put c equal to the determinant of the matrix with entries the traces of all possible products $u_i u_j$ (where the trace is taken with aid of the Galois group of L over $\mathbb{C}(X)$). I claim that almost all approximations c_p are test elements. Indeed, let $\mathbb{F}_p^{\text{alg}}[X] \to A_p$ and u_{ip} be approximations of $\mathbb{C}[X] \to A$ and of the u_i respectively. By Theorem 4.7, almost all $\mathbb{F}_p^{\text{alg}}[X] \to A_p$ are Noether Normalizations, and for p sufficiently large, they are generically smooth. By Łos' Theorem, almost all c_p are given as the determinant of the matrix with entries the traces of all possible products $u_{ip}u_{jp}$ and hence are test elements by [3, Theorem 7.1 and 7.2].

For A arbitrary, a simple argument explained in [7, Exercise 2.10] shows that if we can find for each $A/\mathfrak{p}$, with $\mathfrak{p}$ a minimal prime of A, an element c with the desired properties, then we can also find such an element in A, taking into account that by Theorem 4.4, the minimal primes of A_p are just the approximations of the minimal primes of A, for almost all p. Details are left to the reader. □

Theorem 8.5. *Generic tight closure is contained in non-standard tight closure.*

Proof. Let A be a $\mathbb{C}$-affine algebra. Let $z \in A$ and let $\mathfrak{a}$ be an ideal of A. Let A_p be an approximation of A and choose approximations z_p and $\mathfrak{a}_p$ in A_p of z and $\mathfrak{a}$ respectively. We have to show that if z_p belongs to the tight closure of almost all $\mathfrak{a}_p$, then z lies in the non-standard tight closure of $\mathfrak{a}$. By Proposition 8.4, there is a $c \in A$ whose approximation c_p is a test element in almost all A_p. Therefore, for almost all p, we have that

$$c_p\,\mathbf{F}_p^m(z_p) \in \mathbf{F}_p^m(\mathfrak{a}_p)A_p$$

for all $m \geq 1$. Taking ultraproducts, we get that

$$c\,\mathbf{F}_\infty^m(z) \in \mathbf{F}_\infty^m(\mathfrak{a})A_\infty,$$

for all $m \geq 1$, showing that z lies in the non-standard tight closure of $\mathfrak{a}$. □

As an immediate consequence we derive from Theorem 6.2 the following:

Corollary 8.6. *Any ideal in a regular $\mathbb{C}$-affine algebra is equal to its generic tight closure.*

9. Integral Closure and the Briançon-Skoda Theorem

The next result is easy to proof for classical tight closure, but seems to require something like Néron p-desingularization in the non-standard case.

Theorem 9.1. *In any $\mathbb{C}$-affine algebra A, the non-standard closure of an arbitrary ideal is contained in its integral closure.*

Proof. Let z be in the non-standard closure of an ideal $\mathfrak{a}$. This means that we can find $c \in A$, not contained in any minimal prime of A, such that

$$c\,\mathbf{F}_\infty(z) \in \mathbf{F}_\infty(\mathfrak{a})A_\infty, \tag{9.1}$$

with A_∞ the non-standard hull of A. In order to show that z lies in the integral closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$, it suffices to show that for each discrete valuation ring V and any homomorphism $\phi\colon A \to V$ with kernel a minimal prime of A, we have that $\phi(z) \in \mathfrak{a}V$. Fix one such homomorphism ϕ and let $\mathfrak{p}$ be its kernel. After taking completion and using Cohen's Structure Theorem, we may assume that $V = K[[t]]$ for some extension field K of $\mathbb{C}$ and t a single variable. The image of ϕ in V lives already inside $k[[t]]$ for some algebraically closed subfield k of cardinality the continuum. Since $k \cong \mathbb{C}$, we may take $V = \mathbb{C}[[t]]$ from the start. By (9.1) and the fact that $c \notin \mathfrak{p}$, the image of z in $A/\mathfrak{p}$ will still lie in the non-standard closure of $\mathfrak{a}(A/\mathfrak{p})$. So we may as well assume that ϕ is injective, whence that A is a subalgebra of V. Next, we may replace A by the A-subalgebra of V generated by t and then localize A so that it contains $V_0 := \mathbb{C}[t]_{(t)}$. By Néron p-desingularization (see for instance [1, §4]), there exists a smooth V_0-algebra B and there exist V_0-algebra homomorphisms $s\colon A \to B$ and $\psi\colon B \to V$, such that ψs is equal to the inclusion ϕ. In particular, B is a regular $\mathbb{C}$-affine domain. Applying s to (9.1) and observing that $s(c) \neq 0$, lest $\psi s(c) = c$ would vanish, we get that $s(z)$ lies in the non-standard closure of $\mathfrak{a}B$. By Theorem 6.2, $s(z)$ lies in $\mathfrak{a}B$ since B is regular. Applying ψ yields that $z = \psi(s(z))$ lies in $\mathfrak{a}V$, as required. □

Theorem 9.2 (Briançon-Skoda). *Let A be a $\mathbb{C}$-affine algebra and let $\mathfrak{a}$ be an ideal of A. If $\mathfrak{a}$ can be generated by m elements, then the integral closure $\overline{\mathfrak{a}^m}$ of $\mathfrak{a}^m$ lies inside the non-standard tight closure of $\mathfrak{a}$.*

Proof. Let $z \in \overline{\mathfrak{a}^m}$. This means that we can find $g_j \in \mathfrak{a}^{jm}$ and an integral equation

$$z^d + g_1 z^{d-1} + \cdots + g_d = 0.$$

Let A_p be an approximation of A and choose approximations z_p, g_{jp} and $\mathfrak{a}_p$ in A_p of z, g_j and $\mathfrak{a}$ respectively. For almost all p, we have an equation

$$(z_p)^d + g_{1p}(z_p)^{d-1} + \cdots + g_{dp} = 0 \tag{9.2}$$

in A_p and, moreover, $g_{jp} \in (\mathfrak{a}_p)^{im}$. In other words, for those p, we have that z_p belongs to the integral closure of $(\mathfrak{a}_p)^m$. By the tight closure Briançon-Skoda Theorem ([7, Theorem 5.7]), we have that z_p belongs to the tight closure of $\mathfrak{a}_p$, for almost all p. In other words, z belongs to the generic tight closure of $\mathfrak{a}$, whence to its non-standard tight closure, by Theorem 8.5.

In fact, if $\mathfrak{a}$ has positive height, we can altogether avoid the use of Theorem 8.5 (which relies on a non-trivial result about test elements in characteristic p), by repeating the (elementary) argument in [7, Theorem 5.7] for each z_p. Firstly, from (9.2), we get that

$$(\mathfrak{a}_p)^{m(d-1)}(z_p)^N \subset (\mathfrak{a}_p)^{Nm}$$

for all N. If $N = q = p^l$ is a power of p, then $(\mathfrak{a}_p)^{qm} \subset \mathbf{F}_q(\mathfrak{a}_p)A_p$, since $\mathfrak{a}_p$ is generated by m elements. Therefore, $(\mathfrak{a}_p)^{m(d-1)}\mathbf{F}_q(z_p) \subset \mathbf{F}_q(\mathfrak{a}_p)A_p$, for almost all p. Taking ultraproducts, we get that

$$\mathfrak{a}^{m(d-1)}\mathbf{F}^l_\infty(z) \subset \mathbf{F}^l_\infty(\mathfrak{a})A_\infty,$$

for all $l \geq 1$, showing that $z \in \mathrm{cl}_{nst}(\mathfrak{a})$. □

10. Applications

The first application is an easy proof of the following Theorem of HOCHSTER and ROBERTS.

Theorem 10.1. *Let A be a regular $\mathbb{C}$-affine algebra. Let G be a reductive linear algebraic group (so that G is the complexification of a compact real Lie group). If G acts $\mathbb{C}$-rationally on A by $\mathbb{C}$-algebra automorphisms, then the fixed ring A^G is Cohen-Macaulay.*

Proof. Since the problem is local, we may localize and assume from the start that A is a local. Let $(x_1, \dots, x_d)$ be a system of parameters in A^G. We need to show that $(I_j : x_j) = I_j$ in A^G, for every $j = 1, \dots, d$, where $I_j = (x_1, \dots, x_{j-1})A$. Let $z \in A^G$, so that $zx_j \in I_j$. By Colon-Capturing (Theorem 8.1), we get that $z \in \mathrm{cl}_{ns}(I_j)$. Since $A^G \subset A$, it follows that z lies in the non-standard closure of I_jA. Since A is regular, it follows from Theorem 6.2 that $z \in I_jA$.

From Lie theory or a general argument about linearly reductive groups, it follows that there exists a so-called *Reynolds operator* $\rho_G\colon A \to A^G$, that is to say, a homomorphism of A^G-modules. Therefore, applying ρ_G to $z \in I_jA$ gives $z \in I_j$, as required. □

The above proof shows the following more general result, since all we needed of the embedding $A^G \subset A$ is its cyclic purity (recall that in general a homomorphism $R \to A$ is called *cyclically pure* if $\mathfrak{a}A \cap R = \mathfrak{a}$, for every ideal $\mathfrak{a}$ in R).

Corollary 10.2. *Let $R \to A$ be a cyclically pure extension of $\mathbb{C}$-affine algebras. If A is regular, then R is Cohen-Macaulay.*

In [19, Theorem B], this result is generalized to hold for any cyclically pure homomorphism of Noetherian local rings containing a field, by extending the definition of non-standard tight closure to an arbitrary complete Noetherian algebra over a field. Using deeper results from tight closure theory and singularity theory, but staying within the category of $\mathbb{C}$-affine algebras, further generalizations can be found in [18] and [22].

The classical Briançon-Skoda Theorem for affine algebras is a formal consequence of the properties of any of the three closure operations defined in this paper. In particular, by combining Theorem 6.2 and Theorem 9.2, we get the following result.

Theorem 10.3. *Let A be a $\mathbb{C}$-affine algebra and let $\mathfrak{a}$ be an ideal of A. If A is regular and $\mathfrak{a}$ can be generated by m elements, then $\overline{\mathfrak{a}^m} \subset \mathfrak{a}$.*

In particular, if f is a polynomial in n variables over $\mathbb{C}$, without constant term, then there exists a polynomial s with $s(0) \neq 0$, such that sf^n lies in the Jacobian ideal *of f, that is to say, the ideal generated by the partial derivatives of f.*

See [7, Chapter 5] or [21] for an argument why the last assertion follows from the first. In [21], using some non-standard methods as well, this is extended to power series rings over $\mathbb{C}$, thus recovering the original Briançon-Skoda Theorem. Conceivably, one could as well give an argument for this latter result via non-standard tight closure, using its extension to power series rings discussed in [19, §6].

Classical tight closure is contained in non-standard tight closure

Recall the definition made by HOCHSTER and HUNEKE of the *(equational) tight closure* $\mathfrak{a}^*$ of an ideal $\mathfrak{a}$ in a $\mathbb{C}$-affine algebra A (see for instance [7, Appendix 1, Definition 3.1]). An element $z \in A$ lies in $\mathfrak{a}^*$, if there exists some finitely generated $\mathbb{Z}$-subalgebra D of A containing z, such that, with $\mathfrak{d} := \mathfrak{a} \cap D$, the image of z modulo p lies in the characteristic p tight closure of $\mathfrak{d}(D/pD)$, for all but finitely many p. Set $D_p := D/pD$ and let z_p and $\mathfrak{d}_p$ be the respective image of z and $\mathfrak{d}$ in D_p. Observe that there is a canonical embedding of $\mathbb{Z}$ into $\operatorname{ulim}_{p\to\infty} \mathbb{F}_p$. To be more precise, for $s \in \mathbb{Z}$, if s_p denotes the image of s in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, then $\operatorname{ulim}_{p\to\infty} s_p$ is equal to s viewed as an element in $\mathbb{C}$ via *any* isomorphism given by the Lefschetz Principle (Theorem 2.4). We now may choose approximations A_p and $\mathfrak{a}_p$ of A and $\mathfrak{a}$ respectively, such that $D_p \subset A_p$ and $\mathfrak{d}_p = \mathfrak{a}_p \cap D_p$. It follows from our above observation that z_p, viewed as an element of A_p, is an approximation of z (see [18, Corollary 4.9] for a more precise result). By assumption, z_p lies in the tight closure of $\mathfrak{d}_p$, whence in the tight closure of $\mathfrak{a}_p$ in A_p. By definition, this means that z lies in the generic tight closure of $\mathfrak{a}$, whence also in its non-standard (tight) closure by Theorem 8.5. In conclusion, we showed the following.

Theorem 10.4. *For $\mathbb{C}$-affine algebras, equational tight closure is contained in any of the three closure operations 'generic tight closure', 'non-standard tight closure' and 'non-standard closure'.*

Some Questions

1. At present there is no counterexample for any of the closure operations to be different. It is quite possible that they all coincide with equational tight closure (as it is possible that all classical tight closure operations in characteristic zero coincide). Nor is it clear whether a different choice of non-principal ultrafilter, or a different choice of an isomorphism in Theorem 2.4, yields a different closure operation. There is some evidence that the choice of ultrafilter should not matter, at least not in the definition of non-standard tight closure.

2. There is not yet a satisfactory theory of non-standard tight closure in the non-affine setup. The best results so far are obtained in [19], where a closure operation, called *tight difference closure*, is defined for complete algebras over an uncountable algebraically closed field. It will come as no surprise that there one needs Artin Approximation (although it is plausible that something weaker than the Artin-Rotthaus Theorem suffices; see [19, Theorem 6.12]). Neither is there yet any concrete notion of tight closure in mixed characteristic, although I have some hope to develop some asymptotical version, using the Ax-Kochen-Ershov Theorem. For some positive results in this direction see [15,16].
3. Although Proposition 8.4 seems to indicate that test elements exist in the non-standard world, it is not yet clear whether this is actually so. In other words, given a $\mathbb{C}$-affine algebra A, can we find an element c not contained in any minimal prime of A, such that $c\mathbf{F}^m_\infty(z)$ lies always in $\mathbf{F}^m_\infty(\mathfrak{a})A_\infty$, for every m and every z in the non-standard tight closure of $\mathfrak{a}$? Notwithstanding our ignorance on this aspect, it is possible to show that non-standard tight closure persists (although it is not yet clear whether the same is true for non-standard closure). This will be the content of a future paper.
4. Our new closure operations prompt for a study of the analogues of F-regularity and F-rationality. Some work in this direction has been done in the papers [18–20,22], but the general theory still needs development.

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