FLATTENING AND SUBANALYTIC SETS IN RIGID ANALYTIC GEOMETRY

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Abstract. Let K be an algebraically closed field endowed with a complete non-archimedean norm with valuation ring R. Let $f\colon Y \to X$ be a map of K-affinoid varieties. In this paper we study the analytic structure of the image $f(Y) \subset X$; such an image is a typical example of a subanalytic set. We show that the subanalytic sets are precisely the **D**-semianalytic sets, where **D** is the truncated division function first introduced by Denef and van den Dries. This result is most conveniently stated as a Quantifier Elimination result for the valuation ring R in an analytic expansion of the language of valued fields.

To prove this we establish a Flattening Theorem for affinoid varieties in the style of Hironaka, which allows a reduction to the study of subanalytic sets arising from flat maps, i.e., we show that a map of affinoid varieties can be rendered flat by using only finitely many local blowing ups. The case of a flat map is then dealt with by a small extension of a result of Raynaud and Gruson showing that the image of a flat map of affinoid varieties is open in the Grothendieck topology.

Using Embedded Resolution of Singularities, we derive in the zero characteristic case a Uniformization Theorem for subanalytic sets: a subanalytic set can be rendered semianalytic using only finitely many local blowing ups with smooth centres. As a corollary we obtain that any subanalytic set in the plane R^2 is semianalytic.

0. Introduction

Subanalytic sets arise naturally in real analytic geometry as images of proper analytic maps. The structure of such an image can be quite complicated: it is not necessarily definable by means of inequalities between analytic functions, i.e., it is not in general *semianalytic*. Therefore, a compact subset of a real analytic manifold is called *subanalytic*, if it is (at least locally) the projection of a relatively compact semianalytic set. It is then a non-trivial fact that the complement of a subanalytic set is again subanalytic. Nevertheless, subanalytic sets share the same tameness properties as for instance semialgebraic or analytic sets: local finiteness of the number of connected components (which are subanalytic again); the distance between two disjoint closed subanalytic sets is strictly positive; various Łojasiewicz inequalities hold. Real subanalytic sets were first introduced by Łojasiewicz and subsequently studied by Gabrielov [**Gab**] and Hironaka [**Hi 2-4**] by complex

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analytic methods (flattening, Voûte Etoilée) and by geometric techniques (Resolution of Singularities). A new approach appeared in a paper [**DvdD**] by DENEF and VAN DEN DRIES, where a model-theoretic point of view was taken, resulting in a much more concise form and having the enormous advantage to be applicable to the p-adic case as well. In this seminal paper, all of the old results in the real case were reproven together with their p-adic analogs.

Motivated by problems of elliptic curves, TATE constructed a theory of rigid analytic geometry over complete non-archimedean algebraically closed fields. This theory was further developed by KIEHL, GRAUERT et al., largely in analogy with complex analysis. A little later RAYNAUD gave an alternative treatment through formal schemes and more recently still, BERKOVICH approached the subject from the viewpoint of spectral theory.

From the point of view of model theory, non-archimedean fields provide a very fruitful study. After the work of AX and KOCHEN, the theory of $\mathbb{Q}_p$ (the p-adics) has been exhaustively studied by MACINTYRE and for algebracially closed valued fields long before by ABRAHAM ROBINSON. With the recent massive application of model theory to real analytic geometry through the work of VAN DEN DRIES, WILKIE et al., the time seemed right to add analytic structure to the complete algebraically closed valued fields.

Such a study was initiated by LIPSHITZ, later on joined by ROBINSON, who developed in [**Lip**] and [**LR 1-2**] a theory, allowing more general functions than rigid analytic ones in the description of semi- and subanalytic sets, thus obtaining a theory of *weak rigid subanalytic sets*. This yielded some important results on rigid subanalytic sets as well (see **(2.4-2.6)** for a further discussion). At the same time the second author obtained a different theory in [**Sch 1-3**] , were he used a restricted class of analytic functions, yielding the theory of *strong rigid subanalytic sets*. Both of these theories were based on the model-theoretic approach introduced by [**DvdD**] , but the method seemed to resist treatment for the general case of a rigid subanalytic set.

It was the insight of DENEF that the methods of HIRONAKA might be used in the rigid case as well. The key observation is a result due to RAYNAUD and GRUSON describing the image of a flat map between affinoid varieties; this serves as a replacement for the Fibre Cutting Lemma of flat maps in HIRONAKA's work. To make the reduction to the flat case, one needs a good theory of rigid analytic flatificators (to be used as centres of local blowing ups) and the construction of the Voûte Etoilée (a compact Hausdorff space encoding finite sequences of local blowing ups). The former is carried out by the second author in [**Sch 7**] and the latter by the first author in [**Gar**] . However, in order to make the construction of the Voûte Etoilée, it seems necessary to add extra points to the rigid analytic variety, following BERKOVICH. The present paper will put all these results together to obtain the sought for theory of rigid subanalytic sets. Our main theorem states that any rigid subanalytic set can be described by inequalities among functions which are obtained by composition and division of analytic functions (see Section 2 for details). Using the theorem of the complement [**LR 2**] , it is enough to show this for images of rigid analytic maps. Therefore, we need to *flatten* an arbitrary analytic map by means of local blowing ups. Blowing ups are the cause for having to introduce division in the description of a subanalytic set.

The first Section contains this flattening procedure. A local form in the Berkovich category is derived first from which then a global rigid analytic flattening theorem is deduced. In the proof of the former result, we briefly recall some concepts and results from [**Gar**] and [**Sch 7**] . The next Section then contains our main result, preceded by a discussion of the link between blowing ups and the truncated division operator **D**. In Section 3 we show how using our main result together with Embedded Resolution of Singularities, one derives a uniformization theorem for rigid subanalytic sets. This is then used to show that subanalytic sets of the plane are in fact semianalytic. The treatment here is analogous to the one in [**DvdD**] or [**Sch 2-3**] . In an appendix we have gathered some material on the forementioned result of Raynaud and Gruson. Most of this is well-known, but we needed a small extension of the result, which required an adaptation in the proof as it appeared in [**Meh**] . For this reason, it has been added here.

Remark 1. Rigid analytic geometry works over any complete non-archimedean ordered field, but it is most convenient to take the field to be algebraically closed as well, so that all points are rational. Since it seems easier to make such an extra assumption, we will do so in this paper, so that the unit disk can be identified with the maximal spectrum of $K\langle S\rangle$. However, all the results of the first section remain true for arbitrary complete non-archimedean fields. To conveniently formulate Quantifier Elimination one does need the base field to be algebraically closed. The authors do not know whether the result of [**DvdD**] over the p-adics can be deduced from the present theory.

Remark 2. The authors have restricted their attention only to the rigid analytic case, but a treatment of Berkovich subanalytic sets seems now to be accessible, using the same methods.[1] Such a theory would be desirable since then topological properties of subanalytic sets can be studied, such as the finiteness of connected components or other homotopic invariants, triangularization, ...

Remark 3. As far as the characteristic of the base field is concerned, no assumption is needed, except in Section 3 where an application of Embedded Resolution of Singularities is used. If one would have a version of Embedded Resolution of Singularities in positive characteristic, or, at least of its corollary mentioned in the proof of **(3.1)**, the assumption on the characteristic could be removed.

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1. Rigid Analytic Flattening

1.1. Blowing Ups. Let X be a rigid analytic variety. We will be concerned in this section with local blowing up maps and their compositions. For the definition and elementary properties of rigid analytic blowing up maps, we refer to [**Sch 5**] . Suffice it to say here that they are characterised by the familiar property whereby a coherent sheaf of ideals is made invertible. Any blowing up map is proper and

[1] To this end, a uniform version of the model-completeness result of Lipshitz and Robinson is required.

an isomorphism away from the centre. If its centre is nowhere dense, then it is also surjective. A local blowing up π of X is a composition of a blowing up map $\pi'\colon \tilde{X} \to U$ and an open immersion $U \hookrightarrow X$. We will always assume that U is affinoid.[2]

If Z is the centre of the blowing up π' (and hence in particular a closed analytic subvariety of U), then we call Z also the centre of π and we will say that π is the local blowing up of X with locally closed centre Z.

Let $f\colon Y \to X$ be a map of rigid analytic varieties and let $\pi\colon \tilde{X} \to U \hookrightarrow X$ be a local blowing up with centre $Z \subset U$. If $\theta\colon \tilde{Y} \to f^{-1}(u) \hookrightarrow Y$ denotes the local blowing up of Y with (locally closed) centre $f^{-1}(Z)$, then by universality of the blowing up, there exists a unique map $\tilde{f}\colon \tilde{Y} \to \tilde{X}$, making the following diagram commute

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\theta} & Y \\
{\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\
\tilde{X} & \xrightarrow[\pi]{} & X.
\end{array}
\tag{1}
$$

This unique map $\tilde{f}$ is called the *strict transform* of f under π and the above diagram will be referred to as the *diagram of the strict transform*.

In general, we will not be able to work with just a single local blowing up, but we will make use of maps which are finite compositions of local blowing up maps. Therefore, if $\pi\colon \tilde{X} \to X$ is the composite map $\psi_1 \circ \cdots \circ \psi_m$, with each $\psi_{i+1}\colon X_{i+1} \to U_i \hookrightarrow X_i$ a local blowing up map with centre Z_i, for $i < m$, (with $X = X_0$ and $\tilde{X} = X_m$), then we define recursively $f_i\colon Y_i \to X_i$ as the strict transform of f_{i-1} under ψ_i where $f_0 = f$ and $Y_0 = Y$. The last strict transform f_m is called the *(final) strict transform* of f under π and the other strict transforms f_i, for $i < m$, will be referred to as the *intermediate* strict transforms.

For us, the following three possible properties of a map π as above, will be crucial.

- (i) The centres Z_i are nowhere dense.
- (ii) The intermediate strict transforms are flat over their centre, i.e., the restriction $f_i^{-1}(Z_i) \to Z_i$ is flat, for $i < m$.
- (iii) The final strict transform $\tilde{f}$ of f under π is flat.

Our Flattening Theorem states that given a map of affinoid varieties $f\colon Y \to X$, we can find finitely many maps $\pi_1, \dots, \pi_s$ as above with these three properties (i)-(iii), such that, furthermore, the union of their images contains $\operatorname{Im} f$.

Hironaka's proof of the complex analytic Flattening Theorem, heavily exploits the fact that complex spaces are Hausdorff and locally compact. Rigid analytic spaces do not admit a genuine topology and hence the original proofs do not carry over to this situation. However, the work of Berkovich provides us with new analytic spaces, equivalent to rigid analytic spaces as far as their sheaf theory is concerned, but admitting a locally compact Hausdorff topology. We briefly recall their construction. Let A be an affinoid algebra and $X = \operatorname{Sp} A$ the corresponding affinoid variety. We fix once and for all a complete normed field Ω extending K and

[2] One can always make such an assumption by perhaps shrinking the admissible open, since whenever we use local blowing ups, we will only be interested in a local situation.

of cardinality big enough so that it contains any completion of any normed extension field of K which is topologically of finite type over K. An *analytic point* x of X is defined to be a continuous K-algebra morphism $x\colon A \to \Omega$. Let $U = \operatorname{Sp} C$ be an affinoid subdomain of X containing x, then U is called an *affinoid neighbourhood* of x, if the map $x\colon A \to \Omega$ factors through a map $C \to \Omega$. Two analytic points are said to be congruent if they admit the same system of neighbourhoods. The *affinoid Berkovich space* $\mathbb{X}$ associated to X is then the collection of all congruence classes of analytic points of X. Put the weakest topology on $\mathbb{X}$ making all maps $x \mapsto |x(f)|$ continuous, for $f \in A$. This turns the space $\mathbb{X}$ into a compact space, which is Hausdorff, if, moreover, if X is reduced. As a special case of an analytic point, take any $x \in X$ (i.e., some maximal ideal $\mathfrak{m}$ of A), then the composite map $A \twoheadrightarrow A/\mathfrak{m} = K \hookrightarrow \Omega$ yields an analytic point, called a *geometric point* of X. Any affinoid subdomain containing a geometric point is a neighbourhood of that point, and hence X viewed as the set of geometric points can be identified with a subspace of $\mathbb{X}$, and as such is everywhere dense in it.

Finally, one can put a K-analytic structure on $\mathbb{X}$, by defining a structure sheaf $\mathcal{O}_{\mathbb{X}}$ on it. The category of coherent sheaves on X is then equivalent with the category of coherent sheaves on $\mathbb{X}$. So far we have only introduced the local models of the category of Berkovich analytic spaces, but with some care certain global models and their morphisms can be defined. This leads to a satisfactory theory of analytic spaces which are Hausdorff, paracompact strictly K-analytic in the sense of [**Ber 2**] . We refer the reader for further details to [**Ber 1**] , [**Ber 2**] or [**SvdP**] .

1.2. Local Flattening of Berkovich Spaces. *Let $f\colon \mathbb{Y} \to \mathbb{X}$ be a map of Hausdorff, paracompact strictly K-analytic Berkovich spaces with $\mathbb{X}$ reduced. Pick $x \in \operatorname{Im}(f)$ and let $\mathbb{L}$ be a non-empty compact subset of $f^{-1}(x)$. There exists a finite collection E of maps $\pi\colon \mathbb{X}_\pi \to \mathbb{X}$, with each $\mathbb{X}_\pi$ affinoid, such that the following four properties hold, where we put $\mathbb{X}_0 = \mathbb{X}$, $\mathbb{Y}_0 = \mathbb{Y}$ and $f_0 = f$ and where $\pi \in E$ is arbitrary in the first three conditions.*

- (**i**) *The map π is a composition $\psi_0 \circ \cdots \circ \psi_m$ of finitely many local blowing up maps $\psi_i\colon \mathbb{X}_i \to \mathbb{X}_{i-1}$ with locally closed nowhere dense centre $\mathbb{Z}_i \subset \mathbb{X}_i$, for $i = 1, \dots, m$.*
- (**ii**) *Let f_i be defined inductively as the strict transform of f_{i-1} under the local blowing up ψ_i. Then $f_i^{-1}(\mathbb{Z}_i) \to \mathbb{Z}_i$ is flat, for $i = 1, \dots, m$.*
- (**iii**) *The final strict transform $f_m\colon \mathbb{Y}_m \to \mathbb{X}_m$ of f under the whole map π given by the strict transform diagram*

$$\begin{array}{ccc} \mathbb{Y}_m & \xrightarrow{\theta} & \mathbb{Y} \\ {\scriptstyle f_m}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{X}_m & \xrightarrow[\pi]{} & \mathbb{X}, \end{array}$$

  is flat at each point of $\mathbb{Y}_m$ lying above a point of $\mathbb{L}$.
- (**iv**) *The union of all the $\operatorname{Im}\pi$, for $\pi \in E$, is a neighbourhood of x.*

Proof. For the duration of this proof, *spaces* will be Hausdorff, paracompact strictly K-analytic Berkovich spaces and *maps* between them will be analytic maps in the Berkovich sense. Topological notions are taken with respect to the Berkovich topology. In particular, a *local blowing up* will be the composition of a blowing up followed by an open immersion of an affinoid Berkovich space. We use black board bold letters $\mathbb{X}, \mathbb{Y}, \dots$ to denote Berkovich spaces.

Step 1. Our first task is to define the Voûte Etoilée of an arbitrary space $\mathbb{X}$. The details of this process are in [**Gar**], but the method is wholly due to HIRONAKA who makes the construction for complex analytic spaces. Let $\mathcal{E}(\mathbb{X})$ denote the collection of all maps $\pi\colon \mathbb{X}' \to \mathbb{X}$ which are finitely many compositions of local blowing up maps. One can define a partial order relation on $\mathcal{E}(\mathbb{X})$ by calling $\psi\colon \mathbb{X}'' \to \mathbb{X}$ smaller than π, if ψ factors as πq, for some $q\colon \mathbb{X}'' \to \mathbb{X}'$. We denote this by $\psi \leq \pi$. Such a q is then necessarily unique and must belong to $\mathcal{E}(\mathbb{X}')$ ([**Gar**, Proposition 3.2]). If, moreover, the image $q(\mathbb{X}'')$ of q is relatively compact (i.e., its closure is compact), then we denote this by $\psi \ll \pi$. Any two maps $\pi_1, \pi_2 \in \mathcal{E}(\mathbb{X})$ admit a unique minimum or *meet* $\pi_3 \in \mathcal{E}(\mathbb{X})$ with respect to the order $\leq$ ([**Gar**, Lemma 3.3]), denoted by $\pi_1 \wedge \pi_2$. This meet π_3 is just the strict transform of π_2 under π_1 (or vice versa). The set $\mathcal{E}(\mathbb{X})$ then becomes a semi-lattice with smallest element the empty map $\emptyset\colon \emptyset \to \mathbb{X}$. A subset e of $\mathcal{E}(\mathbb{X})$ is called a *filter*, if it does not contain $\emptyset$, if its closed under meets, and, if for any $\psi \in e$ and $\pi \in \mathcal{E}(\mathbb{X})$, with $\psi \leq \pi$, we have that also $\pi \in e$. An *étoile* e on X is now defined as a maximal filter on the semi-lattice $\mathcal{E}(\mathbb{X})$ subject to the extra condition that for any $\pi \in e$ we can find $\psi \in e$, with $\psi \ll \pi$.

The collection of all étoiles on $\mathbb{X}$ is called the Voûte Etoilée of $\mathbb{X}$ and is denoted by $\mathcal{E}_{\mathbb{X}}$. This space is topologised by taking for opens the sets of the form $\mathcal{E}_\pi$ given as the collection of all étoiles on $\mathbb{X}$ containing $\pi\colon \mathbb{X}' \to \mathbb{X}$, for some $\pi \in \mathcal{E}(\mathbb{X})$. In fact, $\mathcal{E}_\pi$ is isomorphic with $\mathcal{E}_{\mathbb{X}'}$ via the map $J_\pi\colon \mathcal{E}_{\mathbb{X}'} \to \mathcal{E}_{\mathbb{X}}$, sending e' to the collection of all $\theta \in \mathcal{E}(\mathbb{X})$ for which there exists some $\psi \in e'$ such that $\pi \circ \psi \leq \theta$ ([**Gar**, Proposition 3.6]). The Voûte Etoilée is Hausdorff in this topology ([**Gar**, Theorem 3.11]). Moreover, for any étoile $e \in \mathcal{E}_{\mathbb{X}}$, the intersection of all $\operatorname{Im}\pi$, where π runs through the maps in e, is a singleton $\{x\}$ and any open immersion $1|_{\mathbb{U}}\colon \mathbb{U} \hookrightarrow \mathbb{X}$ with $x \in \mathbb{U}$, belongs to e ([**Gar**, Proposition 3.9]). We denote the thus defined map $e \mapsto x$ by $p_{\mathbb{X}}\colon \mathcal{E}_{\mathbb{X}} \to \mathbb{X}$. It is a continuous and surjective map (see remark in loc. cit.). It is a highly non-trivial result that this map is also proper in the sense that the inverse image of a compact is compact ([**Gar**, Theorem 3.13]).

Step 2. Next we will introduce the concept of a flatificator. Let $f\colon \mathbb{Y} \to \mathbb{X}$ be a map and let $x \in \mathbb{X}$. A *flatificator* of f at x is a locally closed subspace $\mathbb{Z}$ of $\mathbb{X}$ containing x, such that f is flat over it (i.e., the restriction $f^{-1}(\mathbb{Z}) \to \mathbb{Z}$ is flat), and such that, whenever $\mathbb{V}$ is a second locally closed subspace containing x over which f is also flat, at least locally around x, then $\mathbb{V}$ is a subspace of $\mathbb{Z}$ locally around x. In other words, a flatificator is a largest locally closed subspace over which f becomes flat in a neighbourhood of x. Such a flatificator is called *universal*, if it is stable under base change (i.e., if $g\colon \mathbb{X}' \to \mathbb{X}$ is arbitrary, then $g^{-1}(\mathbb{Z})$ is the flatificator of the base change $\mathbb{Y} \times_{\mathbb{X}} \mathbb{X}' \to \mathbb{X}'$ at x', for any x' in the fibre above x). In [**Sch 7**, Theorem A.2] it is shown that any map $f\colon \mathbb{Y} \to \mathbb{X}$ admits a universal flatificator $\mathbb{Z}$ in each point x of $\operatorname{Im} f$. If $\mathbb{X}$ is moreover reduced then we can detect flatness via the flatificator: blowing up the flatificator exhibits some non-trivial portion of non-flatness as torsion. More precisely, it is shown in ([**Sch 7**, Theorem A.6]) that whenever f is not flat in some point of the fibre $f^{-1}(x)$, then there exists a nowhere dense subspace $\mathbb{Z}_0$ of $\mathbb{Z}$, such that the local blowing up $\psi_1\colon \mathbb{X}_1 \to \mathbb{X}$ with centre $\mathbb{Z}_0$ *renders the fibre above x smaller*. With this we mean the following. Let

$$\begin{array}{ccc} \mathbb{Y}_1 & \xrightarrow{\zeta_1} & \mathbb{Y} \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{X}_1 & \xrightarrow[\psi_1]{} & \mathbb{X} \end{array}$$

denote the strict transform diagram induced by ψ_1. Then, for every $y \in \mathbb{X}_1$ lying above x, we have a non-trivial embedding of closed subspaces

$$f_1^{-1}(y) \times_K \Omega \subsetneq f^{-1}(x) \times_K \Omega, \tag{†}$$

where Ω is our universal domain. (Note the extension of scalars is necessary in order to compare these two fibres as subspaces of $\mathbb{Y} \times_K \Omega$.) We refer to this result as the *Fibre Lemma*.

Step 3. Fix an étoile e on $\mathbb{X}$ such that $p_{\mathbb{X}}(e) = x$. Put $x_0 = x$, $e_0 = e$ and put $f_0 \colon \mathbb{Y}_0 \to \mathbb{X}_0$ equal to the original map $f \colon \mathbb{Y} \to \mathbb{X}$. The Fibre Lemma will enable us to define, by induction, points $x_i \in \mathbb{X}_i$ and étoiles $e_i \in \mathcal{E}_{\mathbb{X}_i}$ with $p_{\mathbb{X}_i}(e_i) = x_i$, local blowing up maps $\psi_i \colon \mathbb{X}_i \to \mathbb{X}_{i-1}$ with nowhere dense centre $\mathbb{Z}_{i-1}$, maps $f_i \colon \mathbb{Y}_i \to \mathbb{X}_i$ and non-empty compact subsets $\mathbb{L}_i$ of the fibre $f_i^{-1}(x_i)$, at least as long as f_i is not flat in some point of $\mathbb{L}_i$. Each f_i will be the strict transform

$$\begin{array}{ccc} \mathbb{Y}_i & \xrightarrow{\zeta_i} & \mathbb{Y}_{i-1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i-1}} \\ \mathbb{X}_i & \xrightarrow[\psi_i]{} & \mathbb{X}_{i-1}. \end{array}$$

of the previous map f_{i-1} under the local blowing up ψ_i. Moreover, each f_i will be flat above the centre $\mathbb{Z}_i$ and have the property on the fibres (†) in the point x_{i-1}. Let us show how to define from the point $x_{i-1} \in \mathbb{X}_{i-1}$ a new point $x_i \in \mathbb{X}_i$ and a new étoile e_i on $\mathbb{X}_i$. Since $\mathbb{Z}_{i-1}$ is nowhere dense and contains x_{i-1}, we deduce that $\psi_i \in e_{i-1}$ (i.e., $e_{i-1} \in \mathcal{E}_{\psi_i}$) from [**Gar**, Corollary 3.10] . The isomorphism $J_{\psi_i} \colon \mathcal{E}_{\mathbb{X}_i} \to \mathcal{E}_{\psi_i}$ then yields a uniquely determined étoile e_i on $\mathbb{X}_i$ and this in turns uniquely determines the point $x_i = p_{\mathbb{X}_i}(e_i)$ of $\mathbb{X}_i$. By diagram chasing, one checks that this implies $\psi_i(x_i) = x_{i-1}$. Finally, we define a compact subset of $f_i^{-1}(x_i)$ by

$$\mathbb{L}_i = \mathbb{Y}_i \cap (\{x_i\} \times_{\mathbb{X}_{i-1}} \mathbb{L}_{i-1}).$$

Let $\mathcal{I}_i$ denote the coherent ideal of $\mathbb{Y} \times_K \Omega$ defining the fibre $f_i^{-1}(x_i) \times_K \Omega$, so that by (†) the chain

$$\mathcal{I}_0 \subsetneq \mathcal{I}_1 \subsetneq \mathcal{I}_2 \subsetneq \dots$$

is strictly increasing on the compact set $\mathbb{L} \times_K \Omega$. Therefore this chain must become stationary, say at level m, meaning that f_m is flat in each point of $\mathbb{L}_m$.

Flatness is an open condition in the source,[3] so that we can find an open $\mathbb{V}$ of $\mathbb{Y}_m$ containing $\mathbb{L}_m$, such that $f_m|_{\mathbb{V}} \colon \mathbb{V} \to \mathbb{X}_m$ is flat. Let $\mathbb{M} = \theta^{-1}(\mathbb{L}) \setminus \mathbb{V}$, where θ

[3]This is for instance proven in [**Sch 7**, Corollary A.3] as a consequence of the existence of a flatificator.

is the compositum $\zeta_m \circ \cdots \circ \zeta_1$. Note that $\mathbb{M}$ is compact, since θ is proper. We claim that after some further local blowing up (in fact, an open immersion will suffice), we may assume that $\mathbb{M}$ is empty, so that the new strict transform will be flat at each point lying above a point of $\mathbb{L}$. To this end, suppose $\mathbb{M}$ is non-empty and pick some $y \in \mathbb{M}$. Since $f_m(y) \neq x_m$, we can find a compact neighbourhood $\mathbb{K}_y$ of x_m in $\mathbb{X}_m$, such that $y \notin f_m^{-1}(\mathbb{K}_y)$. Hence

$$\emptyset = \bigcap_{y \in \mathbb{M}} (f_m^{-1}(\mathbb{K}_y) \cap \mathbb{M}). \tag{1}$$

The compactness of each $f_m^{-1}(\mathbb{K}_y) \cap \mathbb{M}$ means that already a finite number of them, say $f_m^{-1}(\mathbb{K}_{y_i}) \cap \mathbb{M}$, for $i < t$, have empty intersection. Let $\mathbb{K}$ be the intersection of these finitely many $\mathbb{K}_{y_i}$, which is then still a compact neighbourhood of x_m, with the property that

$$f_m^{-1}(\mathbb{K}) \cap \theta^{-1}(\mathbb{L}) \subset \mathbb{V}.$$

Let $\mathbb{X}_e$ be an open of $\mathbb{X}_m$ containing x_m and contained in $\mathbb{K}$. Then the restriction of f_m above $\mathbb{X}_e$ has now the property that it is flat in each point lying above a point of $\mathbb{L}$.

Step 4. Summarising, we found for each étoile e on $\mathbb{X}$ with $p_{\mathbb{X}}(e) = x$ a local blowing up map $\pi_e \colon \mathbb{X}_e \to \mathbb{X}$, such that the strict transform

$$\begin{array}{ccc} \mathbb{Y}_e & \xrightarrow{\theta_e} & \mathbb{Y} \\ {\scriptstyle f_e}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbb{X}_e & \xrightarrow[\pi_e]{} & \mathbb{X} \end{array}$$

has the property that it is flat in each point lying above a point of $\mathbb{L}$. Moreover, there is a canonically defined point x_e on $\mathbb{X}_e$ lying above x. Let $\mathbb{C}_e$ be a relatively compact open neighbourhood of x_e and set

$$\mathbb{D}_e = J_{\pi_e}(p_{\mathbb{X}_e}^{-1}(\mathbb{C}_e)).$$

Then, for each $e \in p_{\mathbb{X}}^{-1}(x)$, the set $\mathbb{D}_e$ is an open neighbourhood of e and the union of all the $\mathbb{D}_e$ is a neighbourhood of $p_{\mathbb{X}}^{-1}(x)$. Let $\{\mathbb{H}_\lambda\}_{\lambda \in \Lambda}$, be the collection of all non-empty relatively compact open neighbourhoods of x in $\mathbb{X}$. Pick arbitrary $\lambda_0, \lambda_1 \in \Lambda$ with $\overline{\mathbb{H}}_{\lambda_1} \subset \mathbb{H}_{\lambda_0}$, where, in general, $\overline{\mathbb{H}_\lambda}$ denotes the topological closure of $\mathbb{H}_\lambda$. Consider the the open covering of $p_{\mathbb{X}}^{-1}(\overline{\mathbb{H}}_{\lambda_1})$ given by the sets

$$\left\{ p_{\mathbb{X}}^{-1}(\mathbb{H}_{\lambda_0}) \cap \mathbb{D}_e \mid e \in p_{\mathbb{X}}^{-1}(x) \right\} \cup \left\{ p_{\mathbb{X}}^{-1}(\mathbb{H}_{\lambda_0} \setminus \overline{\mathbb{H}}_\lambda) \mid \lambda \in \Lambda \right\}. \tag{2}$$

This is indeed a covering, since by the Hausdorff property, we can find for each $y \neq x$ a $\lambda \in \Lambda$ for which $y \notin \overline{\mathbb{H}}_\lambda$. Since $p_{\mathbb{X}}$ is proper, we have that $p_{\mathbb{X}}^{-1}(\overline{\mathbb{H}}_{\lambda_1})$ is compact and hence there exists a finite subset $E \subset p_{\mathbb{X}}^{-1}(x)$ and a finite subset $\Gamma \subset \Lambda$, such that the collection of all sets of (2) with $e \in E$ and $\lambda \in \Gamma$ remains a covering of $p_{\mathbb{X}}^{-1}(\overline{\mathbb{H}}_{\lambda_1})$. Putting $\mathbb{H}$ equal to the intersection of the $\mathbb{H}_\lambda$, for $\lambda \in \Gamma$, this is still a neighbourhood of x and

$$p_{\mathbb{X}}^{-1}(\mathbb{H}) \subset \bigcup_{e \in E} \mathbb{D}_e.$$

Observing that $p_{\mathbb{X}}(\mathbb{D}_e) = \pi_e(\mathbb{C}_e)$ and using that $p_{\mathbb{X}}$ and $p_{\mathbb{X}_e}$ are surjective, we deduce that

$$\mathbb{H} \subset \bigcup_{e \in E} \pi_e(\mathbb{C}_e),$$

as required. ■

We will work in the more familiar category of rigid analytic varieties and consequently we must translate this flattening theorem into a version appropiate for the context. This also calls for a more global result.

1.3. Flattening Theorem. *Let $f\colon Y \to X$ be a map of affinoid varieties with X reduced. Then there exists a finite collection E of maps $\pi\colon X_\pi \to X$, with each X_π again affinoid such that the following properties hold.*

(i) *Each $\pi \in E$ is the composition $\psi_1 \circ \cdots \circ \psi_m$ of finitely many local blowing up maps ψ_i with locally closed nowhere dense centre Z_{i-1}, for $i = 1, \ldots, m$.*

(ii) *For each $\pi \in E$, let f_i be inductively defined as the strict transform of f_{i-1} under the local blowing up ψ_i. Then $f_i^{-1}(Z_i) \to Z_i$ is flat, for $i = 1, \ldots, m$. The diagram of strict transform is*

$$\begin{array}{ccc} Y_i & \xrightarrow{\zeta_i} & Y_{i-1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i-1}} \\ X_i & \xrightarrow[\psi_i]{} & X_{i-1}. \end{array}$$

(iii) *The strict transform $f_\pi\colon Y_\pi \to X_\pi$ of f under the whole map π (which is f_m according to our enumeration) is flat. The diagram of strict transform is*

$$\begin{array}{ccc} Y_\pi & \xrightarrow{\theta} & Y \\ {\scriptstyle f_\pi}\downarrow & & \downarrow{\scriptstyle f} \\ X_\pi & \xrightarrow[\pi]{} & X. \end{array} \tag{1}$$

(iv) *The union of all the* $\mathrm{Im}(\pi)$, *for $\pi \in E$, contains the image* $\mathrm{Im}\, f$.

Proof. Let $\mathbb{X}$ and $\mathbb{Y}$ be the corresponding Berkovich spaces of X and Y respectively and let us continue to write f for the corresponding map $\mathbb{Y} \to \mathbb{X}$. Fix an analytic point x of X (i.e., a point of $\mathbb{X}$), contained in the image of f. Let $\mathbb{L} = f^{-1}(x)$, which is closed in $\mathbb{Y}$ whence compact since $\mathbb{Y}$ is. By **(1.2)**, we can find a finite collection E_x of maps $\pi\colon \mathbb{X}_\pi \to \mathbb{X}$ with $\mathbb{X}_\pi$ affinoid, such that the conditions (**i**)-(**iv**) hold. For each $\pi \in E_x$, let

$$\begin{array}{ccc} \mathbb{Y}_\pi & \xrightarrow{\theta} & \mathbb{Y} \\ {\scriptstyle f_\pi}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{X}_\pi & \xrightarrow[\pi]{} & \mathbb{X} \end{array}$$

be the corresponding strict transform diagram.

By (**iii**) of **(1.2)** we have that the strict transform f_π is flat in each point of $\theta^{-1}(f^{-1}(x)) = f_\pi^{-1}(\pi^{-1}(x))$. Let us first show that we can modify the data in such way that f_π becomes flat everywhere. Since flatness is open in the source by [**Sch 7**, Theorem 3.8] , we can find an open neighbourhood $\mathbb{V}'$ of $f_\pi^{-1}(\pi^{-1}(x))$ in $\mathbb{Y}_\pi$ over which f_π is flat. Since $\mathbb{X}_\pi$ and $\mathbb{Y}_\pi$ are compact Hausdorff spaces, we can find an open neighbourhood $\mathbb{U}'$ of $\pi^{-1}(x)$, such that $f_\pi^{-1}(\mathbb{U}') \subset \mathbb{V}'$. Similarly, we can find an open neighbourhood $\mathbb{U}$ of x in $\mathbb{X}$, such that $\pi^{-1}(\mathbb{U}) \subset \mathbb{U}'$. The neighbourhood $\mathbb{U}$ can be taken inside the union of all the $\operatorname{Im}(\pi)$, for all $\pi \in E_x$. Set $\mathbb{U}_\pi = \pi^{-1}(\mathbb{U})$. Note that $\mathbb{U}_\pi \hookrightarrow \mathbb{X}_\pi$ is the strict transform of the open immersion $\mathbb{U} \hookrightarrow \mathbb{X}$ under π. Let ψ be the restriction of π to $\mathbb{U}_\pi$. The strict transform of f under ψ is the map

$$f_\pi^{-1}(\mathbb{U}_\pi) \to \mathbb{U}_\pi,$$

which by construction is flat, since

$$f_\pi^{-1}(\mathbb{U}_\pi) \subset f_\pi^{-1}(\mathbb{U}') \subset \mathbb{V}'.$$

This establishes our claim upon replacing π by ψ.

Hence we may assume that f_π is flat. Note also that in the above process, we have not violated condition (**iv**) of **(1.2)**, so that the $\pi(\mathbb{X}_\pi)$, for all $\pi \in E_x$, form a covering of an affinoid neighbourhood $\mathbb{W}_x$ of x in $\mathbb{X}$. We can translate all these diagrams to the rigid analytic setup and assume that the same diagrams hold with the spaces now rigid analytic varieties (see Remark 1 below), where we keep the same names for our spaces and maps, but just replace any blackboard letter, such as $\mathbb{X}, \dots$, by its corresponding roman equivalent $X, \dots$, denoting the corresponding rigid analytic variety. In particular, (i)-(iii) hold and we show how to obtain (iv).

Let us now vary the analytic point x over $\operatorname{Im} f$, so that the W_x cover all analytic points of $\operatorname{Im} f$. Since $\mathbb{Y}$ is compact in the Berkovich topology so is $f(\mathbb{Y})$. Therefore, by [**Ber 2**, Lemma 1.6.2] , already finitely many of the W_x cover all analytic points of $\operatorname{Im} f$. In particular, there is a finite collection S of analytic points, such that the union of all $\operatorname{Im}(\pi)$, for all $\pi \in E_x$ and all $x \in S$, cover $\operatorname{Im} f$, i.e., condition (iv) is now verified as well. ■

Remark 1. In this translation process from Berkovich data to rigid analytic data, one needs the following. Let us denote by $\mathbb{M}(M)$ the corresponding Berkovich space of a quasi-compact (i.e., quasi-separated with a finite admissible affinoid covering) rigid analytic variety M. Let $\mathbb{X} = \mathbb{M}(X)$ and suppose $\pi\colon \tilde{\mathbb{X}} \to \mathbb{U} \hookrightarrow \mathbb{X}$ is a local blowing up with centre $\mathbb{Z}$, where the latter is a closed subspace of the open $\mathbb{U}$. We can find a wide affinoid V of X, such that its closure $\mathbb{M}(V)$ in $\mathbb{X}$ is contained inside $\mathbb{U}$. Hence there exists a closed analytic subvariety Z of V, such that $\mathbb{M}(Z) = \mathbb{Z} \cap \mathbb{M}(V)$. Let $p\colon \tilde{X} \to V$ be the blowing up of V with this centre Z, then $\mathbb{M}(\tilde{X}) \subset \tilde{\mathbb{X}}$ (see [**Gar**, Lemma 2.2] for the details). So in our translation we will replace π by the (rigid analytic) local blowing up $\tilde{X} \to V \hookrightarrow X$. Moreover, if $\mathbb{W}$ is an open inside $\mathbb{U}$ such that its closure $\overline{\mathbb{W}}$ is still contained in $\mathbb{U}$, then we can take V such that $\mathbb{W} \subset \mathbb{M}(V)$ and hence

$$\pi^{-1}(\mathbb{W}) \subset \mathbb{M}(\tilde{X}) \subset \tilde{\mathbb{X}}.$$

Note that the local blowing up $\tilde{\mathbb{W}} \to \mathbb{W} \hookrightarrow \mathbb{X}$ of $\mathbb{X}$ with centre $\mathbb{Z} \cap \mathbb{W}$ coincides with the restriction $\pi^{-1}(\mathbb{W}) \to \mathbb{X}$, so that the rigid analytic local blowing up $\tilde{X} \to X$

is sandwiched by the Berkovich local blowing ups $\pi^{-1}(\mathbb{W}) \to \mathbb{X}$ and $\tilde{\mathbb{X}} \to \mathbb{X}$. The picture is

$$\begin{array}{ccccc} \tilde{\mathbb{W}} & \longrightarrow & \mathbb{W} & \longrightarrow & \mathbb{X} \\ \downarrow & & \downarrow & & \| \\ \mathbb{M}(\tilde{X}) & \longrightarrow & \mathbb{M}(V) & \longrightarrow & \mathbb{X} \\ \downarrow & & \downarrow & & \| \\ \tilde{\mathbb{X}} & \longrightarrow & \mathbb{U} & \longrightarrow & \mathbb{X}, \end{array}$$

where the composite vertical maps are open immersions and the outer composite horizontal maps are local blowing ups.

Moreover, in this way we can maintain in the rigid analytic version all covering properties which were already satisfied in the Berkovich version.

Remark 2. Note that we proved something stronger than condition (iv), namely the union of the images of all $\pi \in E$ covers not only all geometric points of $\operatorname{Im} f$, but also all analytic points.

2. Subanalytic Sets

2.1. Definition. We now introduce the notion of semianalytic and subanalytic sets in rigid analytic geometry. There are essentially two different ways of viewing these objects, one is geometrical in nature and the other is model-theoretic. We give both point of views and leave it to the reader to pick his favourite. In what follows, let $X = \operatorname{Sp} A$ be a reduced affinoid variety (i.e., A has no non-trivial nilpotent elements), although the assumption that X is reduced is not always necessary.

2.1.1. The Geometric Point of View.

A subset Σ of X is called *globally (rigid) semianalytic* in X, if Σ is the union of finitely many *basic* subsets, where the latter are of the form

$$(1)\quad \{\, x \in X \mid |p_i(x)| \le |q_i(x)|\,,\ \text{for } i < n \text{ and } |p_i(x)| < |q_i(x)|\,,\ \text{for } n \le i < m \,\}\,,$$

with the $p_i, q_i \in A$. The set Σ is just called *(rigid) semianalytic* in X, if there exists a finite admissible affinoid covering $\{X_j\}_{j<t}$ of X, such that $\Sigma \cap X_j$ is globally semianalytic in X_j, for each $j < t$.

The set Σ is called *(rigid) subanalytic* in X, if there exists a globally semianalytic subset Ω of $X \times R^N$, for some N, such that $\Sigma = \pi(\Omega)$, where $\pi\colon X \times R^N \to X$ is the projection on the first factor. Whereas the collection of all (globally) semianalytic subsets of X is easily seen to be a Boolean algebra, this is no longer obvious at all for the class of subanalytic sets. Recently, Lipshitz and Robinson gave a proof of this result in [**LR 2**, Corollary 1.6] . Below, we give a short review of their results, since we will make use of them in the proof of our Quantifier Elimination **(2.7)**.

The reader might wonder whether one should not introduce more local versions of subanalyticity, for instance, what about the projection of a semianalytic set which is not globally semianalytic? A moments reflection shows quite easily that we would not enlarge the class of sets at all. We extend the notion of semianalytic and subanalytic to an arbitrary quasi-compact rigid analytic variety X as follows: let $\Sigma \subset X$ then Σ is semianalytic (respectively, subanalytic) in X, if there exists a finite admissible affinoid covering $\{X_i\}_{i<s}$ of X, such that each $\Sigma \cap X_i$ is semianalytic (respectively, subanalytic) in X_i.

In order to give a neat description of a subanalytic set, it is convenient to introduce a special function $\mathbf{D}$, first introduced by Denef and van den Dries in their paper [**DvdD**] , in which they describe p-adic subanalytic sets. Put

$$\mathbf{D}\colon R^2 \to R\colon (a,b) \mapsto \begin{cases} a/b & \text{if } |a| \le |b| \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We define the *algebra* $A^{\mathbf{D}}$ *of* $\mathbf{D}$*-functions* on X, as the smallest K-algebra of K-valued functions on X containing A and closed under the following two operations.

(i) If $p, q \in A^{\mathbf{D}}$, then also $\mathbf{D}(p,q) \in A^{\mathbf{D}}$.
(ii) If $p \in A\langle T_1, \dots, T_N\rangle$ and $q_i \in A^{\mathbf{D}}$ with $|q_i| \le 1$, for $i = 1, \dots, N$, then also $p(q_1, \dots, q_N) \in A^{\mathbf{D}}$.

Here, the function $\mathbf{D}(p,q)$ is to be considered as a pointwise division, i.e., defined by $x \mapsto \mathbf{D}(p(x), q(x))$. Note also that if $p \in A^{\mathbf{D}}$ then p defines a bounded function on X and hence it makes sense to define $|p| = \sup_{x\in X} |p(x)|$. If we allow in the definition of (globally) semianalytic sets also $\mathbf{D}$-functions rather than just elements of A, we may now formulate the definition of *(globally)* $\mathbf{D}$*-semianalytic* sets: the functions appearing in (1) may be elements of $A^{\mathbf{D}}$. The class of globally $\mathbf{D}$-semianalytic sets coincides with the class of $\mathbf{D}$-semianalytic sets. Our main result now will be that a set is $\mathbf{D}$-semianalytic, if and only if, it is subanalytic.

2.1.2. The Model-Theoretic Point of View.

If one wants to initiate the model-theoretic study of the field K with its analytic structure, it is more convenient to consider the valuation ring R instead. This is because the unit ball, which is a domain of convergence for the ring of strictly convergent power series, may be identified with a Cartesian product of R. We propose the following language.

The *analytic language* $\mathcal{L}_{\text{an}}$ for R consists of two 2-ary relation symbols $\mathbf{P}_{\le}$ and $\mathbf{P}_{<}$ and an n-ary function symbol F_f, for every strictly convergent power series f in n-variables of norm at most one, i.e., for every $f \in R\langle X_1, \dots, X_n\rangle$, where $n = 0, 1, \dots$. The interpretation of R as an $\mathcal{L}_{\text{an}}$-structure is as follows. Each n-ary function symbol F_f is interpreted as the corresponding function $f\colon R^n \to R$, defined by the strictly convergent power series f (note that $|f| \le 1$, so that f is indeed R-valued). The relation symbol $\mathbf{P}_{\le}$ interprets the subset $\{\, (x,y) \in R^2 \mid |x| \le |y| \,\}$ of R^2, and likewise, $\mathbf{P}_{<}$ describes the subset $\{\, (x,y) \in R^2 \mid |x| < |y| \,\}$. Hence, the atomic formulae in this language (or rather, their interpretation in R) are of the following three types

$$f(x) = g(x), \tag{1}$$

$$|f(x)| \le |g(x)|\,, \tag{2}$$

$$|f(x)| < |g(x)|\,. \tag{3}$$

Note that the first type can be rewritten as $|f(x) - g(x)| \le 0$, so that we actually only have to deal with types (2) and (3). One can of course define $\mathbf{P}_{<}(x,y)$ as

$\neg \mathbf{P}_{\leq}(y,x)$, but the advantage of not doing so is that all formulae can now be made equivalent with positive ones, i.e., without using the negation symbol. One cannot expect R to have elimination of quantifiers in this language, as it has neither in the real or the p-adic case (basically the same counterexample, in essence due to Osgood, can be used in all three cases).

In an attempt to remedy this, we introduce an expansion $\mathcal{L}_{\rm an}^{\mathbf{D}}$ of $\mathcal{L}_{\rm an}$ with one new 2-ary function symbol $\mathbf{D}$, which we will interpret in our structure as the function $\mathbf{D}$ of above. If K were the p-adic field (and hence $R = \mathbb{Z}_p$), then by a theorem of Denef and van den Dries [**DvdD**] , R admits Elimination of Quantifiers in an expansion of this language where one needs to add extra predicates, one for each $n = 2, 3, \dots$, to express that an element is an n-th power; a similar expansion occurs in Macintyre's algebraic Quantifier Elimination for $\mathbb{Z}_p$. In the algebraically closed case these predicates are clearly obsolete. Hence the following is the natural rigid analytic analogue: the valuation ring R of K admits Elimination of Quantifiers in the language $\mathcal{L}_{\rm an}^{\mathbf{D}}$.

Let us see how this ties in with the above notion of subanalyticity. A subset of R^N which is definable in the language $\mathcal{L}_{\rm an}$ by a quantifier free formula, is precisely a globally semianalytic set whereas an existentially definable set is a projection of a globally semianalytic set, so consequently subanalytic. It is not too hard to see that the function $\mathbf{D}$ is existentially definable and whence also every $\mathbf{D}$-function on R^N, so that any $\mathbf{D}$-semianalytic subset of R^N is subanalytic. The claim that every subanalytic set is $\mathbf{D}$-semianalytic is then equivalent with the aforementioned Quantifier Elimination in the language $\mathcal{L}_{\rm an}^{\mathbf{D}}$.[4]

We remark that any affinoid variety X is quantifier-free definable in $\mathcal{L}_{\rm an}$ since there is a closed immersion $X \hookrightarrow R^N$ for some $N \in \mathbb{N}$. More generally any quasi-compact rigid analytic variety is also quantifier-free definable in $\mathcal{L}_{\rm an}$. Also note that semianalytic sets (respectively, subanalytic sets) in such a variety X correspond to quantifier-free definable (respectively, existentially definable) subsets of X.

We will be adopting from now on the geometric point of view. Certain theorems hold also in case K is not algebraically closed. However, for sake of simplicity we will maintain this assumption in what follows. In particular, one can and we will identify $\mathrm{Sp}(K\langle S_1,\dots,S_n\rangle)$ with R^n.

2.1.3. Example. If $f\colon Y \to X$ is a map of affinoid varieties, then the image $f(Y)$ is a typical subanalytic subset of X (not necessarily semianalytic!). Subanalyticity follows from projecting the graph of f (which is analytic, whence semianalytic) onto X. More generally, it follows that $f(\Sigma) \subset X$ is subanalytic whenever $\Sigma \subset Y$ is subanalytic. This example shows that even when one is merely interested in closed analytic subsets, one needs to study subanalytic sets as well. However, there are some particular kind of maps which have better understood image. For instance, Kiehl's Proper Mapping Theorem [**Ki**] (or [**BGR**, 9.6.3. Proposition 3]) states that the image of a proper map is closed analytic. However, this does not tell us anything on the image of a semianalytic set under a proper map. In fact, in [**Sch 1**] and [**Sch 2**] the second author shows that if $\Sigma \subset Y$ is semianalytic and $f\colon Y \to X$ is proper, then $f(\Sigma)$ is $\mathbf{D}$-semianalytic in X; he carries out a systematic study of the sets arising in this way–the *strongly subanalytic* sets. One might hope

[4]By an easy logic argument, it is enough to eliminate only existential quantifiers to obtain Quantifier Elimination.

though that certain proper maps, viz. blowing up maps, nevertheless behave better with respect to semianalyticity. It is the contents of **(2.3)** below that this is true provided one replaces semianalytic by **D**-semianalytic and then, unfortunately, this is only true generically, i.e., away from the centre. It is because of this (rather straightforward) result that **D**-functions are needed. Noteworthy here is that in case of the blowing up of the plane in a single (reduced) point, the image of a semianalytic set is nevertheless semianalytic again. This (much harder) result will be used implicitly in the proof of **(3.2)**.

A second class of affinoid maps with well-understood images are the flat maps: their images are finite unions of rational domains and hence in particular semianalytic. This highly non-trivial result is due to Raynaud and Gruson (a full account by Mehlmann appeared in [**Meh**]). Because of its crucial role in our argument and since we need a slight improvement of their original result in the form **(2.2)** below, we will provide most of the details in an Appendix.

2.2. Theorem (Raynaud-Gruson-Mehlmann). *Let $f\colon Y \to X$ be a flat map of affinoid varieties. Let Σ be a semianalytic subset of Y defined by finitely many inequalities of the form $|h(y)| < 1$ or $|h(y)| \geq 1$, where each h belongs to the affinoid algebra of Y and has supremum norm at most one. Then $f(\Sigma)$ is semianalytic in X.*

Proof. See the Appendix. ∎

2.3. Proposition. *Let $\pi\colon \tilde{X} \to X$ be a map of rigid analytic varieties and let Σ be a **D**-semianalytic subset of $\tilde{X}$. If π is a locally closed immersion, then $\pi(\Sigma)$ is **D**-semianalytic in X. If π is a local blowing up map with centre Z, then $\pi(\Sigma) \setminus Z$ is **D**-semianalytic in X.*

Proof. For closed immersions the statement is trivial. If $U = \operatorname{Sp} C \hookrightarrow X = \operatorname{Sp} A$ is a rational affinoid subdomain, then $C = A\langle f/g\rangle$, where $f = (f_1, \dots, f_n)$ with $f_i, g \in A$ having no common zero. Hence any function $h \in C$ defined on U is **D**-definable on X (just replace any occurrence of f_i/g by $\mathbf{D}(f_i, g)$). Now, any affinoid subdomain is a finite union of rational subdomains by [**BGR**, 7.3.5. Corollary 3] and hence we proved the proposition for any affinoid open immersion as well. From this the general locally closed immersion case follows easily.

This leaves us with the case of a blowing up. Without loss of generality, we may assume X to be affinoid. Let us briefly recall the construction of a blowing up map as described in [**Sch 5**] . Let $X = \operatorname{Sp} A$ and let Z be a closed analytic subvariety of X defined by the ideal $(g_1, \dots, g_n)$ of A. We can represent A as a quotient of some $K\langle S\rangle$, with $S = (S_1, \dots, S_m)$, so that X becomes a closed analytic subvariety of R^m. However, in order to construct the blowing up of X with centre Z, we need a different embedding, given by the surjective algebra morphism

$$K\langle S, T\rangle \twoheadrightarrow A\colon T_j \mapsto g_j,$$

for $j = 1, \dots, n$, extending the surjection $K\langle S\rangle \twoheadrightarrow A$ and where $T = (T_1, \dots, T_n)$. This gives us a closed immersion $i\colon X \hookrightarrow R^m \times R^n$ and after identifying X with its image $i(X)$, we see that $Z = X \cap (R^m \times 0)$. Now, the blowing up $\pi\colon \tilde{X} \to X$ is given by a strict transform diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi} & X \\ {\scriptstyle \tilde{\imath}}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ W & \xrightarrow[\gamma]{} & R^m \times R^n \end{array}$$

where γ denotes the blowing up of $R^m \times R^n$ with centre the linear space $R^m \times 0$. There is a standard finite admissible affinoid covering $\{W_1, \dots, W_n\}$ of W where each W_j has affinoid algebra

$$C_j = \frac{K\langle S, T, U\rangle}{(T_j U_1 - T_1, \dots, T_j U_n - T_n)},$$

so that $\gamma(s,t,u) = (s,t)$ for any point $(s,t,u) \in W_j$, where the latter is considered as a closed analytic subset of $R^m \times R^n \times R^n$ via the above representation of C_j. Moreover, $\tilde{X}$ is a closed analytic subvariety of $X \times_{(R^m \times R^n)} W$. Therefore, if we set $\tilde{X}_j = \tilde{\imath}^{-1}(W_j)$, then $\{\tilde{X}_1, \dots, \tilde{X}_n\}$ is a finite admissible affinoid covering of $\tilde{X}$ with the affinoid algebra $\tilde{A}_j$ of each $\tilde{X}_j$ some quotient of the affinoid algebra

$$\frac{A\langle \hat{U}_j\rangle}{(g_j U_1 - g_1, \dots, g_j U_n - g_n)} \tag{1}$$

of $W_j \times_{(R^m \times R^n)} X$, where $\hat{U}_j$ means all variables U_k save U_j.

With this notation, let us return to the proof of the proposition. We are given some **D**-semianalytic set Σ of $\tilde{X}$ and we seek to describe the image $\pi(\Sigma) \setminus Z$. Let us focus for the time being at one $\Sigma \cap \tilde{X}_j$, where $j \in \{1, \dots, n\}$. Since $\Sigma \cap \tilde{X}_j$ is **D**-semianalytic, we can find a quantifier free $\mathcal{L}_{\mathrm{an}}^{\mathbf{D}}$-formula $\varphi(\bar{\boldsymbol{s}}, \bar{\boldsymbol{u}})$, such that $(s,u) \in R^m \times R^n$ belongs to $\Sigma \cap \tilde{X}_j$, if and only if, $\varphi(s,u)$ holds. Hence, for $s \in R^m$, we have that $s \in \pi(\Sigma \cap \tilde{X}_j)$, if and only if,

$$(\exists \bar{\boldsymbol{u}})\varphi(s, \bar{\boldsymbol{u}}). \tag{2}$$

Note that by (1), if $\varphi(s,u)$ holds, then in particular $(s,u) \in \tilde{X}_j$ and hence $g_j(s)u_k = g_k(s)$, for all $k = 1, \dots, n$. Now, a point $s \in R^m$ does not belong to Z, precisely when one of the $g_k(s)$ does not vanish. Therefore, as j ranges through the set $\{1, \dots, n\}$ and using (2), it is not too hard to see that $s \in R^m$ belongs to $\pi(\Sigma) \setminus Z$, if and only if, $s \in X$ and

$$\bigvee_{j=1}^{n} \bigwedge_{k=1}^{n} |g_k(s)| \leq |g_j(s)| \wedge g_j(s) \neq 0 \wedge \varphi(s, \mathbf{D}(g_1(s), g_j(s)), \dots, \mathbf{D}(g_n(s), g_j(s))),$$

which is indeed a **D**-semianalytic description of $\pi(\Sigma) \setminus Z$. ■

Remark. The above result is unsatisfactory in so far as it does not tell us anything about $\pi(\Sigma)$ restricted to the centre Z of the blowing up. If we could prove that also $\pi(\Sigma) \cap Z$ were **D**-semianalytic, then the whole image $\pi(\Sigma)$ would be **D**-semianalytic, as we would very much like to show. But, above Z, the map π looks like a projection map, so that we can't say much more about $\pi(\Sigma\cap\pi^{-1}(Z)) = \pi(\Sigma)\cap Z$ except that it is a subanalytic set. If Z would be zero dimensional and whence finite, then clearly also $\pi(\Sigma)\cap Z$ is **D**-semianalytic. This suggests that we might be able to use the above result in order to prove Quantifier Elimination by an induction argument on the dimension of X, as soon as we can arrange that Z has strictly smaller dimension than X. This will be the case, if Z is nowhere dense; a condition we ensure will always be fulfilled.

Another point ought to be mentioned here: although a blowing up $\pi\colon \tilde{X} \to X$ is an isomorphism outside its centre Z, this does *not* automatically imply that one can deduce from the **D**-semianalyticity of $\Sigma \setminus \pi^{-1}(Z)$ the same property for its (isomorphic) image $\pi(\Sigma)\setminus Z$. What is going on here is that being (**D**-)semianalytic is not an intrinsic property of a set, but of its embedding in a larger space. In other words, being isomorphic as point sets is not enough and thus the above statement is not a void one.

Before we turn to the proof of our main theorem, let us give a brief review on the model-completeness result of Lipshitz and Robinson. Geometrically, this amounts to the fact that the complement of a subanalytic set is again subanalytic. This is by no means a straightforward result. In the real case it was shown by Gabrielov using quite involved arguments and it was only since the appearance of the paper [**DvdD**] of Denef and van den Dries that one has a conceptual proof through a much stronger result, namely, the class of subanalytic sets is equal to the class of **D**-semianalytic sets. Closure under complementation is now immediate. Using the result of [**LR 2**] we exploit their dimension theory to prove our main Quantifier Elimination Theorem.

2.4. Theorem (Lipshitz-Robinson). *The complement $X \setminus \Sigma$ and the closure $\overline{\Sigma}$ (in the canonical topology) of a subanalytic set Σ in X, where X is a reduced quasi-compact rigid analytic variety, is again subanalytic.*

2.5. Theorem (Lipshitz-Robinson). *Let X be a reduced quasi-compact rigid analytic variety and let Σ be a subanalytic set in X. Then there exists a finite partition of Σ by pairwise disjoint rigid analytic submanifolds X_i of X such that their underlying set is subanalytic in X.*

The proofs of both Theorems rely on a certain Quantifier Elimination result in some appropriate language and we refer the reader to the paper [**LR 2**, Corollary 1.2 and 1.3] by Lipshitz and Robinson. Let us just show how one can derive a good dimension theory for subanalytic sets from these results. First of all, there is the notion of the dimension of a quasi-compact rigid analytic variety. This is defined as the maximum of the (Krull) dimension of all its local rings (we give the empty space dimension $-\infty$). In case $X = \operatorname{Sp} A$ is affinoid, this is just the dimension of A. Next, we define the dimension of a subanalytic set Σ in X as the maximum of all $\dim Y$, where $Y \subset \Sigma$ is a submanifold of X. If Σ carries already the structure of a manifold, then clearly its subanalytic dimension equals its manifold dimension.

The relevant properties for this dimension function are now summarised by the following proposition.

2.6. Proposition. *Let X be a quasi-compact rigid analytic variety and let Σ and Σ' be (non-empty) subanalytic sets in X. Then the following holds.*

(i) *If $\Sigma \subset \Sigma'$, then the dimension of Σ is at most the dimension of Σ'.*
(ii) *The dimension of Σ is zero, if and only if, Σ is finite.*
(iii) *The dimension of Σ equals the dimension of its closure (in the canonical topology) $\overline{\Sigma}$.*
(iv) *The dimension of the* boundary $\overline{\Sigma} \setminus \Sigma$ *is strictly smaller than the dimension of Σ.*
(v) *If $f\colon X \to Y$ is a map of quasi-compact rigid analytic varieties, then the dimension of $f(\Sigma)$ is at most the dimension of Σ, with equality in case f is injective.*
(vi) *If Σ is semianalytic, then the dimension of Σ is equal to the (usual) dimension of its Zariski closure.*

Remark. Note that by **(2.4)** both the closure $\overline{\Sigma}$ and the boundary $\overline{\Sigma} \setminus \Sigma$ are indeed subanalytic.

Proof. The first two statements follow from the fact that the dimension of a subanalytic set is the maximum of the dimensions of each manifold in any finite subanalytic manifold partitioning (as in **(2.5)**). The other statements require more work. See [**Lip**] and also [**DvdD**, 3.15-3.26] for the p-adic analogues–the proofs just carry over to our present situation, once one has **(2.5)**. ■

2.7. Theorem (Quantifier Elimination). *Let X be a reduced affinoid variety, then the subanalytic subsets of X are precisely the* **D***-semianalytic subsets of X.*

Proof. We have already seen that **D**-semianalytic sets are subanalytic. To prove the converse, let Σ be a subanalytic set of X. We will induct on the dimension of Σ and then on the dimension of X. The zero-dimensional case follows immediately from (ii) in **(2.6)**. Hence fix $\dim \Sigma = k > 0$ and $\dim X = d > 0$.

Step 1. It suffices to take Σ closed in the canonical topology (i.e., the induced topology coming from the norm). Indeed, assume the theorem proven for all subanalytic sets which are closed in the canonical topology. Let $\overline{\Sigma}$ be the closure of Σ with respect to the canonical topology. By **(2.4)** and (iii) of **(2.6)**, also $\overline{\Sigma}$ is subanalytic and of dimension equal to the dimension of Σ. Hence by our assumption $\overline{\Sigma}$ is even **D**-semianalytic. Let Γ be the boundary $\overline{\Sigma} \setminus \Sigma$, which is again subanalytic by **(2.4)**. Moreover, by (iv) of **(2.6)**, Γ has strictly smaller dimension than Σ. Hence, by our induction hypothesis on the dimension of a subanalytic set, we have that also Γ is **D**-semianalytic. Therefore also $\Sigma = \overline{\Sigma} \setminus \Gamma$, as required.

Step 2. Hence we may assume that Σ is closed in the canonical topology. There exists a globally semianalytic subset $\Omega' \subset X \times R^N$, for some N, such that $\Sigma = f'(\Omega')$, where $f'\colon X \times R^N \to X$ is the projection on the first factor. The union of finitely many **D**-semianalytic sets is again such. Therefore, without loss of generality, we may even take Ω' to be a basic set, i.e., of the form

$$\left\{ (x,t) \in X \times R^N \mid \bigwedge_{i<m} |p_i(x,t)| \leq |q_i(x,t)| \ \wedge \bigwedge_{m \leq i < n} |p_i(x,t)| < |q_i(x,t)| \right\},$$

where the p_i and q_i are in $A\langle T\rangle$, with $X = \operatorname{Sp} A$ and $T = (T_1, \dots, T_N)$. Introduce n new variables Z_i and consider the following closed analytic subset Y of $X \times R^{N+n}$ given by the equations $p_i - Z_i q_i = 0$, for $i < n$. Let Ω be the basic subset of Y

given by $(x, t, z) \in Y$ belongs to Ω whenever $|z_i| < 1$, for $m \le i < n$. Let q be the product of all the q_i, for $m \le i < n$, and we obviously can assume that $q \neq 0$ lest Σ is non-empty. If $f\colon Y \to X$ denotes the composition of the closed immersion $Y \hookrightarrow X \times R^{N+n}$ followed by the projection $X \times R^{N+n} \to X$, then $f(\Omega \cap U) = \Sigma$, where U is the complement in Y of the zero-set of q. Using [**Sch 6**, Corollary 2.2] , we may, after perhaps modifying some of the equations defining Y, assume that the closure of U in the canonical topology equals the whole of Y and hence the closure (in the canonical topology) of $\Omega \cap U$ is Ω. Now $\Omega \cap U \subset f^{-1}(\Sigma)$ and so $\Omega = \overline{\Omega \cap U} \subset f^{-1}(\Sigma)$, since Σ is closed and f is continuous. Hence $f(\Omega) = \Sigma$.

Step 3. Before giving the details of the proof, let's pause to give a brief outline of how we will go about. According to our Flattening Theorem, we can find finitely many diagrams

$$(\dagger)_\pi \qquad \begin{array}{ccc} Y_\pi & \xrightarrow{\theta_\pi} & Y \\ {\scriptstyle f_\pi}\downarrow & & \downarrow{\scriptstyle f} \\ X_\pi & \xrightarrow[\pi]{} & X \end{array}$$

indexed by maps π, where each such π is a finite composition of local blowing up maps with the properties (i)-(iii) and such that $\operatorname{Im} f$ is contained in the union of all the $\operatorname{Im}(\pi)$. Now, in order to study $\Sigma = f(\Omega)$, we will chase Ω around these diagrams $(\dagger)_\pi$. There are only finitely many π to consider; it will suffice to do this for one such π since the analysis for the others is identical. First we take the preimage $\theta_\pi^{-1}(\Omega)$, which is again a semianalytic set defined by inequalities of the form $|h| < 1$ where the h are functions on Y_π of supremum norm at most one. Next we take the image of the latter set under f_π. Our extension of Raynaud's Theorem **(2.2)** guarantees that this image is semianalytic. Finally we push this set back to X via π and denote this set temporarily by Σ'. If we had the full version of **(2.3)**, i.e., a local blowing up map preserves **D**-semianalyticity, then this last set would be indeed **D**-semianalytic.

Of course, in chasing Ω around the diagram, we might have lost some points, i.e., it may well be the case that $\Sigma' \neq \Sigma$. But this could happen only for points coming from one of the centres of the local blowing ups that make up π (since outside its centre, a blowing up map is an isomorphism). Above each of these centres the strict transform is flat so we account for those missing points using **(2.2)** once more. Hence the only problem in the above reasoning lies in the application of **(2.3)**: it is not the whole image that we can account for by means of that proposition, but only for the part outside the centre. However, the latter has dimension strictly smaller and by an induction argument on the dimension, we could also deal with this part.

Step 4. Our second induction hypothesis says that any subanalytic set in an affinoid variety of dimension strictly smaller than d is **D**-semianalytic. Let us first draw the following strengthening of **(2.3)**:

(2.3)′ *Let $\pi\colon \tilde{W} \to W$ be any local blowing up of a quasi-compact rigid analytic variety W of dimension at most d whose centre Z is nowhere dense. If $\Gamma \subset \tilde{W}$ is* **D***-semianalytic, then $\pi(\Gamma) \subset W$ is also* **D***-semianalytic.*

The key point is that Z has dimension strictly smaller than the dimension d of W, which is also the dimension of $\tilde{W}$. Now

$$\pi(\Gamma) = (\pi(\Gamma) \setminus Z) \cup (Z \cap \pi(\Gamma)).$$

By **(2.3)** we know that $\pi(\Gamma) \setminus Z$ is **D**-semianalytic and by our induction hypothesis on the dimension we have that also $Z \cap \pi(\Gamma)$ is (take a finite affinoid covering to reduce to the affinoid case).

Step 5. Now, according to **(1.3)**, there exists a finite collection E of maps $\pi\colon X_\pi \to X$, such that each $\pi \in E$ induces a strict transform diagram $(\dagger)_\pi$ with properties (i)-(iv) of loc. cit. (The intermediate strict transform diagrams are given by $(\dagger)_i$ below). By (iv), if we could show that each $\mathrm{Im}(\pi) \cap \Sigma$ is **D**-semianalytic in X, then the same would hold for Σ, since there are only finitely many π. Therefore, let us concentrate on one such $\pi = \pi_1 \circ \ldots \circ \pi_m$ and adopt the notation from **(1.1)** for this map, so that in particular, (i)-(iii) of loc. cit. holds. Let each π_{i+1} be the blowing up of the admissible affinoid $U_i \subset X_i$ with nowhere dense centre $Z_i \subset U_i$. The diagram of strict transform is given by

$$(\dagger)_i \qquad \begin{array}{ccc} Y_{i+1} & \xrightarrow{\theta_{i+1}} & Y_i \\ {\scriptstyle f_{i+1}}\downarrow & & \downarrow{\scriptstyle f_i} \\ X_{i+1} & \xrightarrow[\pi_{i+1}]{} & X_i. \end{array}$$

Define inductively $\Omega_i \subset Y_i$ as $\theta_i^{-1}(\Omega_{i-1})$ starting from $\Omega_0 = \Omega$. Note that each Ω_i is a semianalytic set of Y_i defined by several inequalities of the type $|h| < 1$, where each $h \in \mathcal{O}(Y_i)$ is of supremum norm at most one. Define also inductively, but this time by downwards induction, the sets $W_{i-1} = \pi_i(W_i) \subset U_i \subset X_i$ where we start with $W_m = X_m = X_\pi$. In particular, we have that $W_0 = \mathrm{Im}(\pi)$. By **(2.3)′** each W_i is **D**-semianalytic in X_i. In order to describe Σ, we will furthermore make use of the sets Γ_i defined as $f_i(\Omega_i) \cap W_i$, for $i \leq m$. In particular, note that Γ_0 is nothing else than $f(\Omega) \cap W_0 = \Sigma \cap \mathrm{Im}(\pi)$, which we aim to show is **D**-semianalytic.

The next claim shows how two successive members in the chain of commutative diagrams $(\dagger)_i$ relate the Γ_i: for each $i < m$, we have an equality

$$(\ddagger)_i \qquad \Gamma_i = \pi_{i+1}(\Gamma_{i+1}) \cup (\Gamma_i \cap Z_i).$$

Assume we have established already $(\ddagger)_i$, for each $i < m$. We will prove, by downwards induction on $i \leq m$, that each Γ_i is **D**-semianalytic in X_i, so that in particular Γ_0 would be **D**-semianalytic in X, as required. First of all, since $f_\pi = f_m$ is assumed to be flat, we can apply **(2.2)** to Ω_m to conclude that $\Gamma_m = f_m(\Omega_m)$ is semianalytic whence **D**-semianalytic in X_m. Assume now that we have already proven that Γ_{i+1} is **D**-semianalytic in X_{i+1} and we want to obtain the same conclusion for Γ_i in X_i. Using $(\ddagger)_i$, it is enough to establish this for both sets in the right hand side of that equality. The first of these, $\pi_{i+1}(\Gamma_{i+1})$, is **D**-semianalytic since we have now the strong version **(2.3)′** of **(2.3)** at our disposal. As for the second set, $\Gamma_i \cap Z_i$, also this one is **D**-semianalytic, since f_i restricted to $f_i^{-1}(Z_i)$ is flat and since

$$\Gamma_i \cap Z_i = f_i(\Omega_i \cap f_i^{-1}(Z_i)) \cap W_i,$$

so that **(2.2)** applies. Note that we already established that W_i is **D**-semianalytic.

Therefore, it only remains to prove $(\ddagger)_i$. The inclusion $\supset$ is straightforward and we omit the details. To prove $\subset$, let $x_i \in \Gamma_i$. That means that there exists $y_i \in \Omega_i$ and $w_{i+1} \in W_{i+1}$ such that $f_i(y_i) = x_i = \pi_{i+1}(w_{i+1})$. If $x_i \in Z_i$, we are done. Hence assume that $x_i \notin Z_i$ so that $y_i \notin f_i^{-1}(Z_i)$. However, since $W_i \subset U_i$ we have that $y_i \in f_i^{-1}(U_i)$. Since θ_{i+1} is the blowing up of $f_i^{-1}(U_i)$ with centre $f_i^{-1}(Z_i)$ and whence an isomorphism outside this centre, we can even find $y_{i+1} \in Y_{i+1}$, such that $\theta_{i+1}(y_{i+1}) = y_i$. From $y_i \in \Omega_i$ it then follows that $y_{i+1} \in \Omega_{i+1}$. Put $x_{i+1} = f_{i+1}(y_{i+1})$. Commutativity of the strict transform diagram implies that $\pi_{i+1}(x_{i+1}) = x_i = \pi_{i+1}(w_{i+1})$. Since $x_i \notin Z_i$, the blowing up π_{i+1} is an isomorphism in that point, so that $w_{i+1} = x_{i+1}$ which therefore belongs to $f_{i+1}(\Omega_{i+1}) \cap W_{i+1} = \Gamma_{i+1}$, proving our claim, and hence also our main theorem. ∎

Remark. We can derive from the above proof also a weak uniformization as follows. Define Σ_i inductively as the inverse image of Σ_{i-1} under π_i, for $1 \leq i \leq m$, with $\Sigma_0 = \Sigma$. With notations as in the above proof, we can derive, for $i < m$, from $(\ddagger)_i$ the following identity

$$\Sigma_{i+1} \cap W_{i+1} = \Gamma_{i+1} \cup \big(\pi_{i+1}^{-1}(\Gamma_i \cap W_i) \cap W_{i+1}\big).$$

For $i = m-1$, this takes the simplified form $\Sigma_m = \Gamma_m \cup \pi_m^{-1}(\Gamma_{m-1} \cap Z_{m-1})$. Now, as already observed, Γ_m is semianalytic in $X_m = X_\pi$ and similarly $\Gamma_{m-1} \cap Z_{m-1}$ is semianalytic in X_{m-1} and whence also its preimage under π_m. In other words, we showed the following proposition.

2.8. Corollary. *Let X be a reduced affinoid variety and let Σ be a subanalytic set in X. There exists a finite collection of compositions of finitely many local blowing up maps $\pi_1, \dots, \pi_n$ with nowhere dense centre, such that the union of the $\mathrm{Im}(\pi_i)$ contains Σ, and such that each preimage $\pi_i^{-1}(\Sigma)$ has become semianalytic.*

Proof. This follows from the above discussion in the case where Σ is closed in the canonical topology. The reduction to this case uses an induction argument similar to the one in the proof of the theorem. ∎

Note also that to prove the corollary, we do not make use of **(2.3)** but only of **(2.2)**. For an improvement of **(2.8)**, at least in the zero characteristic case, see the Uniformization Theorem **(3.1)** below, where we will be able to take smooth centres for the blowing ups involved.

3. Uniformization

In [**Sch 2**, Theorem 4.4] it was proved that for any strongly subanalytic set Σ in an affinoid manifold X, there exists a finite covering family of compositions π of finitely many local blowing ups with smooth and nowhere dense centre, such that the preimage $\pi^{-1}(\Sigma)$ is semianalytic, provided the characteristic of K is zero. The restriction to zero characteristic is entirely due to the lack of an Embedded Resolution of Singularities in positive characteristic. A proof of this rigid analytic Embedded Resolution of Singularities for zero characteristic can be found in [**Sch 4**, Theorem 3.2.5] . In the present paper, we will extend the above Uniformization Theorem to the class of all subanalytic sets. The proof is entirely the same as for the strong subanalytic case, in that we only make use of the fact that a subanalytic set is **D**-semianalytic. For the convenience of the reader we give below an outline of the argument.

3.1. Uniformization Theorem. *Let X be an affinoid manifold (i.e., all its local rings are regular) and assume K has characteristic zero. Let Σ be a subanalytic subset of X. Then there exists a finite collection E of maps $\pi\colon X_\pi \to X$, with each X_π again affinoid, such that the following properties hold.*

(i) *Each $\pi \in E$ is the composition $\psi_1 \circ \cdots \circ \psi_m$ of finitely many local blowing up maps ψ_i with nowhere dense and smooth centre, for $i < m$.*
(ii) *The union of all the $\mathrm{Im}(\pi)$, for $\pi \in E$, equals X.*
(iii) *For each $\pi \in E$, we have that $\pi^{-1}(\Sigma)$ is semianalytic in X_π.*

Proof. Let $X = \mathrm{Sp}\, A$. As already mentioned, we will use Embedded Resolution of Singularities on X and more particularly the following corollary to it: given $p, q \in A$, then there exists a finite collection E' of maps, such that (i) and (ii) hold, for each $\pi\colon X_\pi \to X$ in E', and furthermore either $p \circ \pi$ divides $q \circ \pi$, or vice versa, $q \circ \pi$ divides $p \circ \pi$, in the affinoid algebra of X_π. See for instance [**Sch 2**, Lemma 4.2] for a proof.

From our Quantifier Elimination **(2.7)**, we know that Σ is $\mathbf{D}$-semianalytic. By a (not too difficult) argument, involving an induction on the number of times the function $\mathbf{D}$ appears in one of the describing functions of Σ (for details see [**Sch 2**, Theorem 4.4]), we can reduce to the case that there is only one such occurrence. In other words, we may assume that there exist a quantifier free formula $\psi(\bar{\boldsymbol{x}}, \boldsymbol{y})$ in the language $\mathcal{L}_{\mathrm{an}}$ and functions $p, q \in A$, such that $x \in \Sigma$, if and only if,

$$\psi(x, \mathbf{D}(p(x), q(x))) \quad \text{holds.} \tag{1}$$

After an appeal to the aforementioned corollary of Embedded Resolution of Singularities to p and q, and since we only seek to prove our result modulo finite collections of maps for which (i) and (ii) holds, we may already assume that either p divides q or q divides p. In the former case, there is some $h \in A$, such that $q = hp$ in A. Therefore, $\mathbf{D}(p(x), q(x)) = 0$, unless $q(x) \neq 0$ and $|h(x)| = 1$, in which case it is equal to $1/h(x)$. Let U_1 be the affinoid subdomain defined by $|h(x)| \leq 1/2$ and U_2 by $|h(x)| \geq 1/2$, so that $\{U_1, U_2\}$ is an admissible affinoid covering of X. Hence $x \in U_1$ belongs to Σ, if and only if, $\psi(x, 0)$ holds and $x \in U_2$ belongs to Σ, if and only if,

$$[|h(x)| \geq 1 \wedge q(x) \neq 0 \wedge \psi(x, 1/h(x))] \vee [(|h(x)| < 1 \vee q(x) = 0) \wedge \psi(x, 0)]$$

holds. Observe that $1/h$ belongs to the affinoid algebra of U_2, since h does not vanish on U_2. In other words, Σ is semianalytic on both sets and whence on the whole of X.

In the remaining case that q divides p, i.e. there is some $h \in A$, such that $qh = p$ in A, we have an even simpler description of Σ, namely $x \in \Sigma$, if and only if,

$$[p(x) \neq 0 \wedge \psi(x, h(x))] \vee [p(x) = 0 \wedge \psi(x, 0)]$$

holds, again showing that Σ is semianalytic. ■

3.2. Corollary. *Suppose K has characteristic zero and let $\Sigma \subset R^2$. If Σ is subanalytic, then in fact it is semianalytic.*

Proof. In [**Sch 3**, Theorem 3.2] this is proved for the subclass of strongly subanalytic sets. However, in its proof, nowhere we have made essential use of the strongness (=overconvergency), and hence the same proof applies verbatim (see also the final remark in the introduction of loc. cit.). ■

Remark. Using ABHYANKAR's Embedded Resolution of Singularities in positive characteristic for excellent local rings of dimension two, one can remove the assumption on the characteristic in the above Corollary.

APPENDIX: ELIMINATION ALONG FLAT MAPS

A.1. Definition. This section will be devoted to a proof of **(2.2)**. In it, we will need some properties of the *reduction functor* applied to an affinoid algebra. However, for our purposes, we do not need to introduce the whole machinery of reductions but can do with an ad hoc construction to be presented below. First of all, let us fix some notation. As before, R denotes the valuation ring of K, i.e., all $r \in K$ with $|r| \leq 1$, and $\wp$ will denote the maximal ideal of R, i.e., all $r \in K$, such that $|r| < 1$. Note that R is a *rank-one* valuation ring, i.e., the only prime ideals are (0) and $\wp$. The residue field $R/\wp$ will be denoted by $\bar{K}$. Notice that it is also an algebraically closed field.

We will call an R-algebra A° an *admissible* algebra, if A° is flat as an R-algebra and *topologically of finite type*, meaning of the form $R\langle S\rangle/I^\circ$, for some finitely generated ideal I° and some variables $S = (S_1, \dots, S_N)$. From a given admissible algebra A°, we can construct an affinoid algebra by tensoring over K, namely let $A = A^\circ \otimes_R K$. Flatness now guarantees that $A^\circ \subset A$. If we start with an affinoid algebra $A = K\langle S\rangle/I$ and define A° as $R\langle S\rangle/I^\circ$ with $I^\circ = I \cap R\langle S\rangle$ as above, then I° is finitely generated and A° is torsion-free whence flat over R, that is to say, A° is admissible. By tensoring over K we recover our original affinoid algebra, i.e., $A = A^\circ \otimes_R K$. However, A° depends on the particular choice of representing A as a quotient of some $K\langle S\rangle$.

For the sake of simplicity, let us assume that K is algebraically closed.[5] Let A° be an admissible R-algebra and let $A = A^\circ \otimes_R K$ be the corresponding affinoid algebra. With respect to the structure map $R \to A^\circ$, any prime ideal of A° lies either above (0) or above $\wp$. The former prime ideals are in one-one correspondence with the prime ideals of A. Hence, in particular, we will consider $\operatorname{Sp} A$ as a subset of $\operatorname{Spec}(A^\circ)$. Let us call a map $x^\circ\colon \operatorname{Spec} R \to \operatorname{Spec}(A^\circ)$ an *R-rational point*. Then to give a point $x \in \operatorname{Sp} A$ (i.e., a maximal ideal $\mathfrak{m}$ of A) is the same as to give an R-rational point x° (given by the ideal $\mathfrak{m} \cap A^\circ$). The *reduction* of x is the restriction of x° to the closed immersion $\operatorname{Spec} \bar{K} \hookrightarrow \operatorname{Spec} R$ and is denoted by $\bar{x}$. In other words, $\bar{x}$ is given by the maximal ideal $(\mathfrak{m} \cap A^\circ) + \wp A^\circ$. Let us denote the maximal spectrum of A° by $\operatorname{Max}(A^\circ)$. The *reduction map* $\xi\colon \operatorname{Sp} A \to \operatorname{Max}(A^\circ)$ is the map given by sending x to its reduction $\bar{x}$. A word of caution: the reduction map is not induced by any algebra morphism.

The reduction map ξ is functorial in the following sense. Let $\varphi^\circ\colon A^\circ \to B^\circ$ be an R-algebra morphism of admissible algebras and let $\varphi\colon A \to B$ be the morphism of affinoid algebras obtained by tensoring φ° with K. Then we have a commutative

[5] This assumption is not essential, although the proofs would require some modifications for the general case.

diagram

$$(\diamond)\qquad \begin{array}{ccc} \operatorname{Sp} B & \xrightarrow{f} & \operatorname{Sp} A \\ \xi\downarrow & & \downarrow\xi \\ \operatorname{Max}(B^\circ) & \xrightarrow[f^\circ]{} & \operatorname{Max}(A^\circ) \end{array}$$

where f and f° are the respective maps on the maximal spectra induced by φ and φ°.

It is well-known (see for instance [**Meh**]) that the reduction map is surjective (regardless whether K is algebraically closed or not). This is an immediate consequence of the Flat Lifting Lemma below.

A.2. Lemma. *Let $A^\circ \to B^\circ$ be a flat R-algebra morphism of admissible R-algebras and let $f\colon Y^\circ = \operatorname{Spec}(B^\circ) \to X^\circ = \operatorname{Spec}(A^\circ)$ denote the corresponding map of affine schemes. Let $x^\circ\colon \operatorname{Spec} R \to X^\circ$, be an R-rational point of X° and let $\bar{x}$ denote its reduction $\operatorname{Spec} \bar{K} \to X^\circ$. Suppose there exists a $\bar{K}$-rational point $\bar{y}\colon \operatorname{Spec} \bar{K} \to Y^\circ$, such that*

$$(1)\qquad \begin{array}{ccc} \operatorname{Spec} \bar{K} & \xrightarrow{\bar{y}} & Y^\circ \\ \Big\| & & \downarrow f \\ \operatorname{Spec} \bar{K} & \xrightarrow[\bar{x}]{} & X^\circ \end{array}$$

commutes. Then there exists an R-rational point y° of Y°, which has reduction $\bar{y}$, and is such that

$$(2)\qquad \begin{array}{ccc} \operatorname{Spec} R & \xrightarrow{y^\circ} & Y^\circ \\ \Big\| & & \downarrow f \\ \operatorname{Spec} R & \xrightarrow[x^\circ]{} & X^\circ \end{array}$$

commutes. We call y° a factorisation of x° lifting $\bar{y}$.

Proof. Let $\mathfrak{p}^\circ$ be the prime ideal of A° associated to x° (i.e., the image of the generic point under x°). Let $\bar{\mathfrak{p}} = \mathfrak{p}^\circ + \wp A^\circ$, so that it is the maximal ideal of A° associated to $\bar{x}$. Finally, let $\bar{\mathfrak{q}}$ be the maximal ideal of B° associated to $\bar{y}$, so that the commutativity of (1) translates into

$$\bar{\mathfrak{p}} = \bar{\mathfrak{q}} \cap A^\circ. \tag{3}$$

Since $A^\circ \to B^\circ$ is flat, the Going Down Theorem (see for instance [**Mats**, Theorem 9.5]) guarantees the existence of a prime ideal $\mathfrak{n}^\circ$ of B°, such that $\mathfrak{n}^\circ \subset \bar{\mathfrak{q}}$ and

$$\mathfrak{p}^\circ = \mathfrak{n}^\circ \cap A^\circ. \tag{4}$$

Let $\mathfrak{q}^\circ$ be an ideal of B°, maximal with respect to the following two conditions

$$\mathfrak{n}^\circ \subset \mathfrak{q}^\circ \subset \bar{\mathfrak{q}} \tag{5}$$

$$\mathfrak{q}^\circ \cap R = (0). \tag{6}$$

The reader easily verifies that such an ideal is necessarily a prime ideal. Moreover, by (3), (4) and (5), we must have inclusions

$$\mathfrak{p}^\circ \subset \mathfrak{q}^\circ \cap A^\circ \subset \bar{\mathfrak{p}}.$$

In view of (6), the latter of these must be strict. Since $A^\circ/\mathfrak{p}^\circ \cong R$, the only two prime ideals of A° containing $\mathfrak{p}^\circ$ are $\mathfrak{p}^\circ$ itself and $\bar{\mathfrak{p}}$. Therefore, we conclude that

$$\mathfrak{q}^\circ \cap A^\circ = \mathfrak{p}^\circ. \tag{7}$$

Let $B = B^\circ \otimes_R K$ be the associated affinoid algebra. We claim that $\mathfrak{q}^\circ B$ is a maximal ideal of B. Assuming the claim, we have an inclusion of R-algebras

$$R \hookrightarrow B^\circ/\mathfrak{q}^\circ \hookrightarrow B/\mathfrak{q}^\circ B \cong K.$$

Again the last inclusion must be strict and since R is a rank-one valuation ring, the first inclusion is in fact an isomorphism. In other words, if y° is the point of Y° corresponding to $\mathfrak{q}^\circ$, then it is an R-rational point. Moreover, $\mathfrak{q}^\circ + \wp B^\circ$ is then a maximal ideal, containing $\mathfrak{q}^\circ$ and contained in $\bar{\mathfrak{q}}$ by (5), and hence equal to the latter. This shows that y° is a lifting of $\bar{y}$, as required.

It remains to prove the claim. To this end, let $b \in B$ not belonging to $\mathfrak{q}^\circ B$. We can find $0 \neq \pi \in \wp$, such that πb belongs to B° and even to $\wp B^\circ$. In particular, it belongs to $\bar{\mathfrak{q}}$. By the maximality of $\mathfrak{q}^\circ$, we must have that

$$(\mathfrak{q}^\circ + \pi b B^\circ) \cap R \neq (0).$$

From this it follows that

$$\mathfrak{q}^\circ B + bB = (1),$$

showing that $B/\mathfrak{q}^\circ B$ is a field, as wanted. ■

Let A° be an admissible algebra with corresponding affinoid algebra $A = A^\circ \otimes_R K$. Applying this lemma to the flat map $R \to A^\circ$ and the R-rational point given by the identity morphism, shows the surjectivity of the reduction map ξ.

The following observation will be constantly used below in the proof of **(2.2)**. Let $h_1, \dots, h_s \in A^\circ$ and let Σ denote the semianalytic set of all $y \in \operatorname{Sp} A$, such that $|h_i(y)| < 1$, for $i < r$ and $|h_i(y)| \geq 1$, for $r \leq i < s$. Call such a set *special*. Let Σ° denote the locally closed subset of $\operatorname{Max}(A^\circ)$ consisting of all maximal ideals $\bar{\mathfrak{m}}$, such that $h_i \in \bar{\mathfrak{m}}$, for $i < r$, and $h_i \notin \bar{\mathfrak{m}}$, for $r \leq i < s$. Using the surjectivity of the reduction map, one easily verifies that these two sets are related to one other by

$$\xi^{-1}(\Sigma^\circ) = \Sigma \qquad \text{and} \qquad \Sigma^\circ = \xi(\Sigma). \tag{1}$$

In other words, ξ induces a bijection between the class of finite Boolean combinations of special subsets of $\operatorname{Sp} A$ and the class of constructible subsets of $\operatorname{Spec} \bar{A}$, where $\bar{A} = A^\circ/\wp A^\circ$.

A.3. *Proof of Theorem* **(2.2)**. So we are given a flat map $f\colon \operatorname{Sp} B \to \operatorname{Sp} A$ of affinoid varieties and a special set Σ of $\operatorname{Sp} B$, given as

$$\{\, y \in \operatorname{Sp} B \mid |h_i(y)| \diamond_i 1 \text{ for } i < s \,\}, \tag{1}$$

where $h_i \in B$ are of supremum norm at most one and $\diamond_i$ is either $<$ or $\geq$. We want to prove that $f(\Sigma)$ is semianalytic.

Using **(A.4)** below, we may reduce to the case that there exist admissible algebras A° and B° with $h_i \in B^\circ$, for all $i < t$, such that $A = A^\circ \otimes_R K$ and $B = B^\circ \otimes_R K$, and there exists a flat morphism of R-algebras $A^\circ \to B^\circ$ which induces the map f (after tensoring with K). By our observation above, there exists a locally closed set Σ° of $\operatorname{Max}(B^\circ)$ such that $\xi^{-1}(\Sigma^\circ) = \Sigma$. Since $\operatorname{Max}(B^\circ)$ can be identified with $\operatorname{Max}(\bar{B})$, where $\bar{B} = B^\circ/\wp B^\circ$, we can view Σ° as a locally closed subset of the latter space as well. If we also put $\bar{A} = A^\circ/\wp A^\circ$, then since both rings are finitely generated $\bar{K}$-algebras, we can invoke Chevalley's Theorem to conclude that the image of Σ° under the induced map $\bar{f}\colon \operatorname{Max}(\bar{B}) \to \operatorname{Max}(\bar{A})$ is a constructible set Ω°. Identifying $\operatorname{Max}(A^\circ)$ with $\operatorname{Max}(\bar{A})$, we may consider Ω° as a constructible set of the former space as well and as such it is the image of Σ° under the map f° induced by f. Let $\Omega = \xi^{-1}(\Omega^\circ)$, so that by our above observation Ω is semianalytic in $\operatorname{Sp} A$. Hence we proved our theorem once we showed that

$$f(\Sigma) = \Omega. \tag{2}$$

The commutative diagram $(\diamond)$ of **(A.1)** expressing the functoriality of ξ, provides the inclusion $f(\Sigma) \subset \Omega$, so we only need to deal with the opposite inclusion.

Hence let $x \in \Omega$. Let x° be the corresponding R-rational point and let $\bar{x}$ be the reduction $\xi(x)$ of x. By assumption, $\bar{x} \in \Omega^\circ$ and hence it is the image under f° of some point $\bar{y} \in \Sigma^\circ$. We can apply **(A.2)** to this situation to obtain an R-rational point y° factoring through x° and lifting $\bar{y}$. In other words, if $y \in \operatorname{Sp} B$ denotes the point corresponding to y°, then this translates into $f(y) = x$ and $\xi(y) = \bar{y}$. Since $\bar{y} \in \Sigma^\circ$, the latter implies that $y \in \Sigma$, as required. ■

A.4. Proposition. *Let $f\colon Y = \operatorname{Sp} B \to X = \operatorname{Sp} A$ be a flat map of affinoid varieties and let $h_j \in B$, for $j < t$, be of supremum norm at most one. There exist finite coverings $\{U_i = \operatorname{Sp} A_i\}_{i<s}$ of X and $\{V_i = \operatorname{Sp} B_i\}_{i<s}$ of Y by rational subdomains and R-algebra morphisms $\varphi_i^\circ\colon A_i^\circ \to B_i^\circ$ of admissible algebras, such that, for all $i < s$, we have that*

1. *$A_i = A_i^\circ \otimes_R K$ and $B_i = B_i^\circ \otimes_R K$,*
2. *the morphism φ_i° is flat and induces the map $f|_{V_i}\colon V_i \to U_i$,*
3. *$h_j \in B_i^\circ$, for all $j < t$.*

Proof. Since the h_j are of norm at most one and using [**BGR**, 6.4.3. Theorem 1], we can find an admissible algebra B° containing all h_j with $B^\circ \otimes_R K = B$, an admissible algebra A° with $A^\circ \otimes_R K = A$ and an R-algebra morphism $\varphi^\circ\colon A^\circ \to B^\circ$ inducing the map f. In general, φ° will not be flat. To remedy this, we use [**Meh**, 3.4.8], in order to find admissible coverings as asserted, for which (1) and (2) holds. Moreover, from the proof in loc. cit., it follows that B_i° is a quotient of $A_i^\circ \otimes_{A^\circ} B^\circ$. Therefore also (3) is satisfied. ■

Remark. The result in Mehlmann's paper is quite an intricate matter, using Raynaud's approach on rigid analysis through formal schemes and admissible formal blowing ups; an alternative proof can be found in [**BL**].

References

[**Ber 1**] V. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs 33, American Mathematical Society, 1990.

[**Ber 2**] ———, *Etale cohomology for non-Archimedean analytic spaces*, Publ. Math. I.H.E.S. **78** (1993), 5–161.

[**BGR**] S. Bosch, U. Güntzer and R. Remmert, *Non-archimedean analysis*, Springer-Verlag, Berlin, 1984.

[**BL**] S. Bosch and Lütkebommert, *Formal and rigid analytic geometry II*, Math. Annalen **296, 3** (1993), 403–429.

[**DvdD**] J. Denef and L. van den Dries, *p-adic and real subanalytic sets*, Ann. Math. **128** (1988), 79–138.

[**Gab**] A.M. Gabrielov, *Projections of semianalytic sets*, Functional Anal. Appl. **2** (1968), 282–291.

[**Gar**] T.S. Gardener, *Local flattening in rigid analytic geometry* (submitted).

[**Hi 1**] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. **79** (1964), 109–326.

[**Hi 2**] ———, *Introduction to real-analytic sets and real-analytic maps*, Instituto Mathematico L. Tonelli, Pisa, 1973.

[**Hi 3**] ———, *Subanalytic sets*, Number theory, algebraic geometry and commutative algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo, 1973, pp. 453–493.

[**Hi 4**] ———, *La voûte étoilée*, Singularités à Cargèse, vol. 7 et 8, Astérisque, 1973, pp. 415–440.

[**Ki**] R. Kiehl, *Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedischen Funktionentheorie*, Inventiones Math. **2** (1967), 191–214.

[**Lip**] L. Lipshitz, *Rigid subanalytic sets*, Am. J. Math. **115** (1993), 77-108.

[**LR 1**] L. Lipshitz and Z. Robinson, *Rigid subanalytic subsets of the line and the plane*, Am. J. Math. (1992).

[**LR 2**] ———, *Rigid subanalytic sets II*, Am. J. Math. (1996).

[**Mac**] A. Macintyre, *[qe for p-adics]*.

[**Mats**] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[**Meh**] F. Mehlmann, *Ein Beweis für einen Satz von Raynaud über flache Homomorphismen affinoider Algebren*, 2 Serie, vol. Heft 7, Schriftenreihe Math. Inst. Univ. Münster, 1981.

[**RG**] M. Raynaud and L. Gruson, *Critères de platitude et de projectivité*, Inventiones Math. **13** (1971), 1-89.

[**SvdP**] P. Schneider and M. van der Put, *Points and topologies in rigid geometry*, Math. Ann. **302** (1995), 81-104.

[**Sch 1**] H. Schoutens, *Rigid Subanalytic Sets*, Compositio Math. **94** (1994), 269–295.

[**Sch 2**] ———, *Uniformization of Rigid Subanalytic Sets*, Compositio Math. **94** (1994), 227–245.

[**Sch 3**] ———, *Rigid Subanalytic Sets in the Plane*, J. Algebra **170, No. 1** (1994), 266–276.

[**Sch 4**] ———, *Embedded Resolution of Singularities in Rigid Analytic Geometry* (submitted).

[**Sch 5**] ———, *Blowing Up in Rigid Analytic Geometry*, Bull. of Belg. Math. Soc. **2** (1994), 401–419.

[**Sch 6**] ———, *The closure of rigid semianalytic sets*, Am. J. Math. (1995).

[**Sch 7**] ———, *Rigid Analytic Flatificators*, Oxford Quarterly. (1997).

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