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Rigid Subanalytic Sets

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Abstract. Let K be an algebraically closed field endowed with a complete non-archimedean norm. Let $f\colon Y \to X$ be a map of K-affinoid varieties. In this paper we study the analytic structure of the image $f(Y) \subset X$; such an image is a typical example of a subanalytic set. We show that the subanalytic sets are precisely the $\mathbf{D}$-semianalytic sets, where $\mathbf{D}$ is the truncated division function first introduced by Denef and van den Dries.

To prove this we establish a Flattening Theorem for affinoid varieties in the style of Hironaka, which allows a reduction to the study of subanalytic sets arising from flat maps. More precisely, we show that a map of affinoid varieties can be rendered flat by using only finitely many local blowing ups. The case of a flat map is then dealt with by a small extension of a result of Raynaud and Gruson showing that the image of a flat map of affinoid varieties is open in the Grothendieck topology.

Using Embedded Resolution of Singularities, we derive in the zero characteristic case a Uniformization Theorem for subanalytic sets: a subanalytic set can be rendered semianalytic using only finitely many local blowing ups with smooth centres. As a corollary we obtain that any subanalytic set in the plane is semianalytic.

Let K be an algebraically closed field with a complete non-archimedean norm. The *free Tate algebra* of all formal power series $f = \sum_\nu a_\nu S^\nu$ over K for which $|a_\nu|$ tends to zero as ν goes to infinity, is denoted by $K\langle S_1, \ldots, S_n\rangle$ and its elements

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are called *strictly convergent power series*. With a *rigid analytic variety*, we always mean a separated quasi-compact K-rigid analytic variety as defined in [4] (quasi-compact means admitting a finite admissible affinoid covering). With a *Berkovich space*, we always mean a Hausdorff paracompact strictly K-analytic space as defined in [2] or [3]. We will denote rigid analytic varieties by letters $X, Y, \dots$, and reserve the letters $\mathbb{X}$, $\mathbb{Y}$, $\dots$, for Berkovich spaces; there is functor taking a rigid analytic variety X into its corresponding Berkovich space $\mathbb{M}(X)$ (or just $\mathbb{X}$). For sake of convenience, we assume the existence of a universal domain, that is to say an algebraically closed field $\mathbf{K}$ with a complete non-archimedean norm which is of sufficiently large cardinality as to contain any free Tate algebra over K. Under this assumption, an *analytic point* of an affinoid variety $X = \operatorname{Sp} A$ is a continuous K-algebra morphism $A \to \mathbf{K}$ and the corresponding Berkovich space $\mathbb{M}(X)$ is defined as the collection of all analytic points up to congruence. Any such space $\mathbb{M}(X)$ is called an *affinoid* Berkovich space. With a *local blowing up*, we mean the composition of a blowing up followed by an open immersion.

Subanalytic Sets. In the study of the rigid analytic site (or its Berkovich extension), one immediately encounters objects which are no longer themselves analytic spaces, viz., images of arbitrary analytic maps, projections of analytic spaces, ... These are examples of *subanalytic sets*. In fact, (finite) Boolean combinations of either class of sets just mentioned constitute the class of subanalytic sets. As subanalytic subsets of a rigid analytic variety X do not carry a priori any analytic structure, they remain the same when replacing X by its reduction and hence in order to study subanalytic sets we can restricted ourselves to the case that X is already reduced. A more conventional definition of a subanalytic subset in a reduced rigid analytic variety X is then as the projection on to X of a semianalytic subset of $X \times \operatorname{Sp} K\langle S_1, \dots, S_n\rangle$. A *globally semianalytic subset* of a affinoid variety $X = \operatorname{Sp} A$ is a finite Boolean combination of sets of the form

$$\{\, x \in X \mid |f(x)| \leq |g(x)| \,\}, \tag{1}$$

where f and g are elements in A, viewed as functions on X. (Note that these include (closed) analytic sets, by taking $g = 0$ in (1). A *semianalytic subset* Σ of an arbitrary rigid analytic variety X is then any subset for which there exists a finite admissible affinoid covering $\{X_i\}_i$ of X, such that each $\Sigma \cap X_i$ is globally semianalytic in X_i. Note that this in nature is a local concept and that in general a semianalytic set in an affinoid variety will not necessarily be globally semianalytic. (By virtue of our Main Structure Theorem below, this distinction between local and global will be obsolete in the subanalytic case).

In order to study subanalytic sets, we need a notion similar to semianalyticity, except that we will allow more functions on an affinoid variety $X = \operatorname{Sp} A$ than just strictly convergent power series. Roughly speaking, we will allow division as well. To be more precise, we define a special function $\mathbf{D}$, first introduced by Denef and van den Dries in their paper [5], in which they describe p-adic and real subanalytic sets. Let $\mathbf{D}(a, b)$, for $a, b \in K$ be defined as their quotient a/b, provided $|a/b| \leq 1$ (and of course $b \neq 0$), in all other cases we put $\mathbf{D}(a, b)$ simply equal to zero (one might say that $\mathbf{D}$ is a *truncated division*). We define the *algebra* $A^{\mathbf{D}}$ *of* $\mathbf{D}$*-functions* on X, as the smallest K-algebra of K-valued functions on X containing A and closed under the following two operations.

- If $p, q \in A^{\mathbf{D}}$, then also $\mathbf{D}(p, q) \in A^{\mathbf{D}}$.

- If $p \in A\langle T_1, \dots, T_N\rangle$ and $q_i \in A^{\mathbf{D}}$ with $|q_i| \leq 1$, for $i = 1, \dots, N$, then also $p(q_1, \dots, q_N) \in A^{\mathbf{D}}$.

Here, the function $\mathbf{D}(p, q)$ is to be considered as a pointwise division, that is to say, defined by $x \mapsto \mathbf{D}(p(x), q(x))$. Note also that if $p \in A^{\mathbf{D}}$ then p defines a bounded function on X and hence it makes sense to define $|p| = \sup_{x\in X} |p(x)|$. If we allow in the definition of (globally) semianalytic sets also $\mathbf{D}$-functions rather than just elements of A, we may now formulate the definition of *(globally)* $\mathbf{D}$-*semianalytic* sets: the functions appearing in (1) may be elements of $A^{\mathbf{D}}$. Affinoid subdomains are clearly semianalytic sets by the GERRITZEN-GRAUERT theorem [4, 7.3.5. Corollary 3]; using $\mathbf{D}$-functions, any globally semianalytic set in an affinoid domain U of X is in fact globally $\mathbf{D}$-semianalytic in X (not just in U). Therefore, the class of globally $\mathbf{D}$-semianalytic sets coincides with the class of $\mathbf{D}$-semianalytic sets on an affinoid variety. Our main result is now:

1. Theorem (Main Structure Theorem) *Let X be a reduced rigid analytic variety, then the subanalytic subsets of X are precisely the $\mathbf{D}$-semianalytic subsets of X.*

In the sequel, we will describe our strategy in proving the above result. Full details can be found in either [7] or [18].

Note that any $\mathbf{D}$-semianalytic set is subanalytic, so only the converse requires proof. An immediate corollary of the Main Structure Theorem is the Theorem of the Complement: the complement of a subanalytic set is again subanalytic. Indeed, by definition, the complement of a $\mathbf{D}$-semianalytic set is again $\mathbf{D}$-semianalytic. However, we will make already use of this result on the complement in our current proof. Fortunately, an independent proof is given by LIPSHITZ and ROBINSON in [12], using Quantifier Elimination and their theory of *separated power series.* In fact, another consequence of their work will be useful in the present proof. In [11], they show that there is a good notion of dimension of a subanalytic set. More precisely, any subanalytic set Σ in a rigid analytic variety X can be written as the disjoint union of rigid analytic manifolds Y_i, where each $Y_i \subset X$ is embedded itself as a subanalytic set of X. The dimension of Σ is then the maximum of the dimensions of the manifolds Y_i (viewed merely as rigid analytic varieties). This dimension is proven to be independent from the particular partioning in submanifolds and satisfies the expected properties. To name few: a subanalytic set has dimension zero, if and only if, it is finite; the dimension of the closure $\bar{\Sigma}$ (in the norm topology) of a subanalytic set Σ is equal to the dimension of Σ and the difference $\bar{\Sigma} \setminus \Sigma$ has strictly smaller dimension.

Reduction to the image of a special set. So, let Σ be a subanalytic subset of X. As the question is local, we may assume that X is affinoid and that Σ is the projection of a globally semianalytic set in some $X \times \operatorname{Sp} K\langle S_1, \dots, S_n\rangle$. By induction on the dimension of Σ and using the above mentioned results of LIPSHITZ and ROBINSON, we may further assume that Σ is closed (in the norm topology). It then easily follows that Σ can actually be realised as the image of a *globally special* set $\Omega \subset Y = \operatorname{Sp} B$ under a map of affinoid varieties $f\colon Y \to X$. With a globally special set we mean a (finite) Boolean combination of sets of the form

$$\Omega = \{\, y \in Y \mid |h(y)| < 1 \,\}, \tag{2}$$

where $h \in B$ has supremum norm at most one. A special set is then a subset which is locally (in the Grothendieck topology) globally special. Our whole proof relies on the key fact, first communicated to us by Denef, that the image of a flat affinoid map is always a special set. This fact is due to Raynaud and Gruson (a full account by Mehlmann appeared in his Ph.D. Dissertation [13]). Inspection of the proof shows that in fact the following slightly more general result can be deduced:

2. Theorem (Raynaud-Gruson-Mehlmann) *The image of a special set under a flat map of affinoid varieties is again special.*

Flattening. In other words, if the above map f were to be moreover flat, we would have showed that $\Sigma = f(\Omega)$ is semianalytic (and in fact special). Therefore, remains to find a way to *flatten* an arbitrary map of affinoid varieties in a controlable way. With this we mean a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\theta} & Y \\ {\scriptstyle \tilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \tilde{X} & \xrightarrow[\pi]{} & X \end{array} \tag{3}$$

in which $\tilde{f}$ is flat, the horizontal maps θ and π are surjective and the bottom map π takes special sets into **D**-semianalytic sets. Indeed, whatever the nature of the map θ, we always have that $\theta^{-1}(\Omega)$ is again special and so by the Raynaud-Gruson-Mehlmann Theorem also the set

$$\tilde{\Sigma} = \tilde{f}(\theta^{-1}(\Omega)) \tag{4}$$

is special. Whence its image $\pi(\tilde{\Sigma})$ is **D**-semianalytic and surjectivity of θ and π implies that the latter set is equal to $\Sigma = f(\Omega)$.

Firstly, we cannot hope to find just a single diagram (3), but in view of the local nature of the matter, it suffices to find finitely many diagrams (3) such that the horizontal maps form a surjective family (that is to say, the union of their images covers Y and X respectively). [1] Secondly, even then we cannot always guarantee that the upper horizontal θ-maps form a surjective family, so that it might well be that the difference

$$\Sigma \setminus \pi(\tilde{\Sigma}) \tag{5}$$

is non-empty. We will indicate below how these 'missing' points will be dealt with.

Remains the problem of the exact nature of the bottom maps so that special (or more generally, **D**-semianalytic) sets are sent to **D**-semianalytic ones. It turns out that local blowing up maps and their compositions, have the required property (where then the maps $\tilde{f}$ in (3) are strict transforms): just observe that on an affinoid chart, a (local) blowing up is essentially described by certain divisions. In fact, this holds only true outside the centre of blowing up. Therefore, we will allow only nowhere dense centres, so that by induction on the dimension we can deal with the contribution to $f(\Omega)$ from the centre as well. In conclusion, what is required to finish the proof of the Main Structure Theorem is the following Flattening Theorem (see the subsequent remark on 'missing' points).

3. Theorem (Flattening Theorem) *Let $f\colon Y \to X$ be a map of affinoid varieties with X reduced. Then there exists a finite collection E of maps $\pi\colon X_\pi \to$*

[1] In fact, we only will require that the union of the images of all π-maps covers the image of f, which clearly suffices as we only seek to describe $\Sigma = f(\Omega)$.

X, with each X_π again affinoid such that the following properties hold. (We set $f\colon Y \to X$ equal to $f_0\colon Y_0 \to X_0$).

3.i. *Each $\pi \in E$ is the composition $\psi_1 \circ \cdots \circ \psi_m$ of finitely many local blowing up maps ψ_i with locally closed nowhere dense centres $Z_{i-1} \subset X_{i-1}$, for $i = 1, \dots, m$.*

3.ii. *For each $\pi \in E$, let f_i be inductively defined as the strict transform of f_{i-1} under the local blowing up ψ_i. Then $f_i^{-1}(Z_i) \to Z_i$ is flat, for $i = 1, \dots, m$.*

3.iii. *The strict transform $f_\pi\colon Y_\pi \to X_\pi$ of f under the whole map π (which is f_m according to our enumeration) is flat.*

3.iv. *The union of all the $\mathrm{Im}(\pi)$, for $\pi \in E$, contains the image $\mathrm{Im}\, f$.*

Note that although the centres Z_i are nowhere dense in X_i, it is not necessarily the case for their inverse images $f_i^{-1}(Z_i)$ in Y_i and whence the θ_π need not form a surjective family. However, the 'missing' points as given by (5) all are images of points lying in one of the centres Z_i. As condition (3.ii) guarantees that above these centres the maps f_i are also flat, an application of the Raynaud-Gruson-Mehlmann Theorem takes care of the set (5). Therefore the Flattening Theorem implies the Main Structure Theorem.

Outline of a proof of the Flattening Theorem. Following ideas of Hironaka in [8] or [10], we introduce the notion of flatificator in a point $x \in X$ of a map $f\colon Y \to X$, which serves to measure the flatness defect in x. Taking such a flatificator as centre of a blowing up will then exhibit the lack of flatness as torsion. More precisely, if $\pi\colon \tilde{X} \to X$ denotes this blowing up (which in fact will only be a local blowing up, as a flatificator only exists locally, for more details see below) and $\tilde{f}$ is the strict transform of f, then each closed immersion of fibres

$$\tilde{f}^{-1}(\tilde{x}) \subset f^{-1}(x) \tag{6}$$

is proper, for any point $\tilde{x} \in \tilde{X}$ lying above x, provided that f was not already flat in each point of $f^{-1}(x)$. If we are still not in a flat situation, then we can again take the flatificator of $\tilde{f}$ in one such point $\tilde{x}$ as a new centre of blowing up and a further shrinking of the fibres as in (6) will occur. In this way we build a 'tree of blowing ups' in which each branch must be finite by Noetherianity.

However, to ensure that in fact only finitely many branches of this 'tree' are required in order for the images of the various compositions of blowing ups to form a neighbourhood of the original point x, some compactness argument on the space of blowing ups is required. In Hironaka's work [8] or [9] this is established by means of the Voûte Etoilée. To make the construction of the Voûte Etoilée in the present situation, the rigid analytic topos lacks the required topological properties and instead we will work with Berkovich spaces. Therefore we will only present the details of the theory of flatificators in this setting, where the reader has to bear in mind that in nature a flatificator is an algebraic object and whence we actually have developed in [17] the theory first in the affinoid case and only then the required properties were translated in terms of Berkovich spaces.

Flatificators. Let $f\colon \mathbb{Y} \to \mathbb{X}$ be a map and let $x \in \mathbb{X}$. Recall that x comes from a continuous morphism $A \to \mathbf{K}$, where $\mathbb{M}(\mathrm{Sp}\, A)$ is an affinoid neighbourhood of x in $\mathbb{X}$ and $\mathbf{K}$ is our universal domain. This gives the collection $f^{-1}(x)$ of all points $y \in \mathbb{Y}$ with $f(y) = x$, a natural structure of $\mathbf{K}$-analytic space, called the *fibre* of f in the point x. Base change preserves fibres: let $\pi\colon \mathbb{T} \to \mathbb{X}$ be another map and choose a point t of $\mathbb{T}$ with $\pi(t) = x$, then $g^{-1}(t) \subset f^{-1}(x)$, where $g\colon \mathbb{T} \times_{\mathbb{X}} \mathbb{Y} \to \mathbb{T}$ is the base change of f. If π is moreover a local blowing up, then we have a closed immersion of fibres $\tilde{f}^{-1}(t) \subset f^{-1}(x)$, where $\tilde{f}$ is the strict transform of f. This holds because the strict transform under a (local) blowing up is a closed immersion inside the base change.

A *flatificator* of f at x is a locally closed subspace $\mathbb{Z}$ of $\mathbb{X}$ containing x, such that f is flat over it (that is to say, the restriction $f^{-1}(\mathbb{Z}) \to \mathbb{Z}$ is flat), and such that, whenever $\mathbb{V}$ is a second locally closed subspace containing x over which f is also flat, at least locally around x, then $\mathbb{V}$ is a subspace of $\mathbb{Z}$ locally around x. In other words, a flatificator is a largest locally closed subspace over which f becomes flat in a neighbourhood of x. Such a flatificator is called *universal*, if it is stable under base change (that is to say, if $g\colon \mathbb{X}' \to \mathbb{X}$ is arbitrary, then $g^{-1}(\mathbb{Z})$ is the flatificator of the base change $\mathbb{Y} \times_{\mathbb{X}} \mathbb{X}' \to \mathbb{X}'$ at x', for any x' in the fibre above x). In [17, Theorem A.2] it is shown that any map $f\colon \mathbb{Y} \to \mathbb{X}$ admits a universal flatificator $\mathbb{Z}$ in each point x of $\operatorname{Im} f$. If $\mathbb{X}$ is moreover reduced then we can detect flatness via the flatificator: blowing up the flatificator exhibits some non-trivial portion of non-flatness as torsion. More precisely, it is shown in ([17, Theorem A.6]) that whenever f is not flat in some point of $f^{-1}(x)$, then there exists a nowhere dense subspace $\mathbb{Z}_0$ of $\mathbb{Z}$, such that the local blowing up $\psi\colon \tilde{\mathbb{X}} \to \mathbb{X}$ with centre $\mathbb{Z}_0$ *renders the fibre above x smaller*. With this we mean the following. Let $\tilde{f}$ be the strict transform of f. Then, for every $\tilde{x} \in \tilde{\mathbb{X}}$ lying above x, we have a non-trivial embedding of closed subspaces

$$\tilde{f}^{-1}(\tilde{x}) \subsetneq f^{-1}(x). \tag{7}$$

We refer to this result as the *Fibre Lemma*.

Voûte Etoilée. Next we explain the construction of the Voûte Etoilée; details can be found in [6]. Let $\mathrm{BU}(\mathbb{X})$ denote the collection of all maps $\pi\colon \mathbb{X}' \to \mathbb{X}$ which are finitely many compositions of local blowing up maps. One can define a partial order relation on $\mathrm{BU}(\mathbb{X})$ by calling $\psi\colon \mathbb{X}'' \to \mathbb{X}$ smaller than $\pi\colon \mathbb{X}' \to \mathbb{X}$, if ψ factors as πq, for some $q\colon \mathbb{X}'' \to \mathbb{X}'$. We denote this by $\psi \leq \pi$. Such a q is then necessarily unique and must belong to $\mathrm{BU}(\mathbb{X}')$ ([6, Proposition 3.2]). If, moreover, the image $q(\mathbb{X}'')$ of q is relatively compact (that is to say, its closure is compact), then we denote this by $\psi \ll \pi$. Any two maps $\pi_1, \pi_2 \in \mathrm{BU}(\mathbb{X})$ admit a unique minimum or *meet* $\pi_3 \in \mathrm{BU}(\mathbb{X})$ with respect to the order $\leq$ ([6]Lemma 3.3), denoted by $\pi_1 \wedge \pi_2$. This meet π_3 is just the strict transform of π_2 under π_1 (or vice versa). The set $\mathrm{BU}(\mathbb{X})$ then becomes a semi-lattice with smallest element the empty map $\emptyset\colon \emptyset \to \mathbb{X}$. A subset e of $\mathrm{BU}(\mathbb{X})$ is called a *filter*, if (1) it does not contain $\emptyset$; (2) it is closed under meets; and (3), for any $\psi \in e$ and $\pi \in \mathrm{BU}(\mathbb{X})$, with $\psi \leq \pi$, we have that also $\pi \in e$. An *étoile* e on X is now defined as a maximal element among the collection of all filters on the semi-lattice $\mathrm{BU}(\mathbb{X})$ subject to the extra condition that for any $\pi \in e$ we can find $\psi \in e$, with $\psi \ll \pi$.

The collection of all étoiles on $\mathbb{X}$ is called the Voûte Etoilée of $\mathbb{X}$ and is denoted by $\mathcal{E}_{\mathbb{X}}$. This space is topologised by taking for opens the sets of the form $\mathcal{E}_\pi$ given as the collection of all étoiles on $\mathbb{X}$ containing $\pi\colon \mathbb{X}' \to \mathbb{X}$, for some $\pi \in \mathrm{BU}(\mathbb{X})$. In fact, $\mathcal{E}_\pi$ is isomorphic with $\mathcal{E}_{\mathbb{X}'}$ via the map $J_\pi\colon \mathcal{E}_{\mathbb{X}'} \to \mathcal{E}_{\mathbb{X}}$, sending e' to the collection of all $\theta \in \mathrm{BU}(\mathbb{X})$ for which there exists some $\psi \in e'$ such that $\pi \circ \psi \leq \theta$ ([6, Proposition 3.6]). The Voûte Etoilée is Hausdorff in this topology ([6, Theorem 3.11]). Moreover, for any étoile $e \in \mathcal{E}_{\mathbb{X}}$, the intersection of all $\operatorname{Im} \pi$, where π runs through the maps in e, is a singleton $\{x\}$ and any open immersion $1|_{\mathbb{U}}\colon \mathbb{U} \hookrightarrow \mathbb{X}$ with $x \in \mathbb{U}$, belongs to e ([6, Proposition 3.9]). We denote the thus defined map $e \mapsto x$ by $p_{\mathbb{X}}\colon \mathcal{E}_{\mathbb{X}} \to \mathbb{X}$. It is a continuous and surjective map (see remark in loc. cit.). It is a highly non-trivial result that this map is also proper in the sense that the inverse image of a compact is compact ([6, Theorem 3.13]).

The Fibre Lemma together with the properness of the Voûte Etoilée map $p_{\mathbb{X}}$ yields the following local version of the Flattening Theorem for Berkovich spaces.

4. Theorem (Local Flattening of Berkovich Spaces) *Let $f\colon \mathbb{Y} \to \mathbb{X}$ be a map of Berkovich spaces with $\mathbb{X}$ reduced. Pick $x \in \operatorname{Im}(f)$ and let $\mathbb{L}$ be a non-empty compact subset of $f^{-1}(x)$. There exists a finite collection E of maps $\pi\colon \mathbb{X}_\pi \to \mathbb{X}$, with each $\mathbb{X}_\pi$ affinoid, such that the following four properties hold, where we put $\mathbb{X}_0 = \mathbb{X}$, $\mathbb{Y}_0 = \mathbb{Y}$ and $f_0 = f$.*

4.i. *Every map $\pi \in E$ is a composition $\psi_1 \circ \cdots \circ \psi_m$ of finitely many local blowing up maps $\psi_i\colon \mathbb{X}_i \to \mathbb{X}_{i-1}$ with locally closed nowhere dense centres $\mathbb{Z}_i \subset \mathbb{X}_i$, for $i = 1, \dots, m$.*

4.ii. *Let f_i be defined inductively as the strict transform of f_{i-1} under the local blowing up ψ_i. Then $f_i^{-1}(\mathbb{Z}_i) \to \mathbb{Z}_i$ is flat, for $i = 1, \dots, m$.*

4.iii. *The final strict transform $f_m\colon \mathbb{Y}_m \to \mathbb{X}_m$ of f under the whole map π, is flat at each point of $\mathbb{Y}_m$ lying above a point of $\mathbb{L}$.*

4.iv. *The union of all the $\operatorname{Im}\pi$, for $\pi \in E$, is a neighbourhood of x.*

To obtain the full (rigid analytic) Flattening Theorem 3 one starts with the corresponding map of affinoid Berkovich spaces $f\colon \mathbb{Y} = \mathbb{M}(Y) \to \mathbb{X} = \mathbb{M}(X)$ and a point x in the image of f (note that a point of $\mathbb{X}$ corresponds to an analytic point of X). Let $\mathbb{L} = f^{-1}(x)$, which is compact as $\mathbb{Y}$ is, and apply the above theorem. By the following three observations, the Flattening Theorem then follows.

5. Remark We can improve Condition (4.iii) in Theorem 4 to ensure that the strict transform f_m is everywhere flat, using the compactness of affinoid Berkovich spaces and the fact that flatness is local in the source, after possibly taking one more open immersion (which is of course also a local blowing up).

6. Remark Let X be an affinoid variety and let $\mathbb{X} = \mathbb{M}(X)$ be the corresponding affinoid Berkovich space. Suppose $\pi\colon \tilde{\mathbb{X}} \to \mathbb{U} \hookrightarrow \mathbb{X}$ is a local blowing up with centre $\mathbb{Z}$, where the latter is a closed subspace of the open $\mathbb{U}$. We can find a wide affinoid neighbourhood V of some analytic point of X, such that its closure $\mathbb{M}(V)$ in $\mathbb{X}$ is contained inside $\mathbb{U}$. Hence there exists a closed analytic subvariety Z of V, such that $\mathbb{M}(Z) = \mathbb{Z} \cap \mathbb{M}(V)$. Let $p\colon \tilde{X} \to V$ be the blowing up of V with this centre Z, then $\mathbb{M}(\tilde{X}) \subset \tilde{\mathbb{X}}$ (see [6, Lemma 2.2] for the details).

7. Remark Moreover, if $\mathbb{W}$ is an open inside $\mathbb{U}$ such that its closure $\overline{\mathbb{W}}$ is still contained in $\mathbb{U}$, then by a compactness argument, we may furthermore assume that $\mathbb{W} \subset \mathbb{M}(V) \subset \mathbb{U}$, after possibly replacing V by a finite union of affinoid neighbourhoods. In particular, we have that

$$\pi^{-1}(\mathbb{W}) \subset \mathbb{M}(\tilde{X}) \subset \tilde{\mathbb{X}}. \tag{8}$$

Note that the local blowing up $\tilde{\mathbb{W}} \to \mathbb{W} \hookrightarrow \mathbb{X}$ of $\mathbb{X}$ with centre $\mathbb{Z} \cap \mathbb{W}$ coincides with the restriction $\pi^{-1}(\mathbb{W}) \to \mathbb{X}$, so that the rigid analytic local blowing up $\tilde{X} \to X$ is sandwiched by the Berkovich local blowing ups $\pi^{-1}(\mathbb{W}) \to \mathbb{X}$ and $\tilde{\mathbb{X}} \to \mathbb{X}$. The picture is

$$
\begin{tikzcd}
\tilde{\mathbb{W}} \arrow[r] \arrow[d] & \mathbb{W} \arrow[r] \arrow[d] & \mathbb{X} \arrow[d, equal] \\
\mathbb{M}(\tilde{X}) \arrow[r] \arrow[d] & \mathbb{M}(V) \arrow[r] \arrow[d] & \mathbb{X} \arrow[d, equal] \\
\tilde{\mathbb{X}} \arrow[r] & \mathbb{U} \arrow[r] & \mathbb{X},
\end{tikzcd}
\tag{9}
$$

where the composite vertical maps are open immersions and the outer composite horizontal maps are local blowing ups.

Therefore, to every local blowing up $\psi_i\colon \mathbb{X}_i \to \mathbb{X}_{i-1}$ appearing in the above theorem corresponds as described in Remark 6 a local blowing up $X_i \to X_{i+1}$ of rigid analytic varieties, and we can choose its centre in such way that Condition (4.ii) of Theorem 4 and the improved Condition (4.iii) mentioned in Remark 5, remain valid. Moreover, by Remark 7, any covering property of the original blowing up can be preserved, that is to say, we can ensure that the union of the images of all the π is a wide affinoid neighbourhood of the analytic point x of X.

8. Remark In view of the previous observations, we have now a local Flattening Theorem for affinoid varieties. We then invoke the following compactness property for affinoid varieties: any collection of affinoid subdomains $\{U_i\}_i$ of an affinoid variety X with the property that for every analytic point x of X at least one of the U_i is a wide affinoid neighbourhood of x, can be refined to a finite covering of X.

This concludes the proof of the Flattening Theorem 3.

Uniformization. In [15, Theorem 4.4] it was proved that for any strongly subanalytic set Σ (=image of a semianalytic set under a proper map) in an affinoid manifold X, there exists a finite covering family of compositions π of finitely many local blowing ups with smooth and nowhere dense centre, such that the preimage $\pi^{-1}(\Sigma)$ is semianalytic, provided the characteristic of K is zero. The restriction to zero characteristic is entirely due to the lack of an Embedded Resolution of Singularities in positive characteristic. A proof of this rigid analytic Embedded Resolution of Singularities for zero characteristic can be found in [16, Theorem 3.2.5]. In the present paper, we extend the above Uniformization Theorem to the class of all subanalytic sets. The proof is entirely the same as for the strongly subanalytic case, in that we only make use of the fact that a subanalytic set is **D**-semianalytic. Recall that an affinoid variety is called a *manifold*, if all of its local rings are regular.

9. Theorem (Uniformization Theorem) *Assume K has characteristic 0. Let X be an affinoid manifold and let Σ be a subanalytic subset of X. Then there exists a finite collection E of maps $\pi\colon X_\pi \to X$, with each X_π again affinoid, such that the following properties hold.*

- *Each $\pi \in E$ is the composition $\psi_1 \circ \cdots \circ \psi_m$ of finitely many local blowing up maps ψ_i with nowhere dense and smooth centre, for $i = 1, \dots, m$.*
- *The union of all the $\mathrm{Im}(\pi)$, for $\pi \in E$, equals X.*
- *For each $\pi \in E$, we have that $\pi^{-1}(\Sigma)$ is semianalytic in X_π.*

The following corollary follows along the same lines as [14, Theorem 3.2], where the strongly subanalytic case is treated.

10. Theorem (Subanalytic Sets in the Plane) *Assume K has characteristic* 0 *and let $\Sigma \subset \operatorname{Sp} K\langle S, T\rangle$, where S and T are two variables. If Σ is subanalytic, then in fact it is semianalytic.*

11. Remark Using ABHYANKAR's Embedded Resolution of Singularities [1] in positive characteristic for excellent local rings of dimension two, one can remove the assumption on the characteristic in the above Corollary.

Some final remarks.

12. Remark Although rigid analytic geometry works over any complete non-archimedean ordered field, it is most convenient to take the field to be algebraically closed as well, so that all points are rational. Since it seems easier to make such an extra assumption, we have done so in the present paper. However, the Flattening Theorem remains true for arbitrary complete non-archimedean fields.

13. Remark The authors have restricted their attention only to the rigid analytic case, but a treatment of Berkovich subanalytic sets seems now to be accessible, using the same methods. To this end, a uniform version of the Theorem of the Complement of LIPSHITZ and ROBINSON is required.

The study of Berkovich subanalytic sets would be desirable since then topological properties of subanalytic sets can be studied, such as the finiteness of connected components or other homotopic invariants, triangularization, ...

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