

An Introduction to Rigid Analytic Geometry

Hans Schoutens*

Ohio State University

<http://www.math.ohio-state.edu/~schoutens>

June 2002

Abstract

These notes¹ are intended to be a short course in rigid analytic geometry, without, however, providing always proofs. The excellent book [4] by BOSCH, GÜNTZER and REMMERT is an extensive introduction into rigid analytic geometry, and includes all the proofs I have omitted here.

Contents

I	Rigid Analytic Geometry	3
1	Introduction	3
2	Terminology and notations	3
3	Goal	6
4	Strictly Convergent Power Series	7
5	The algebra structure of $K\langle S \rangle$	10
6	Maximal Spectra	14
7	Affinoid Varieties	17
8	Rigid Analytic Varieties	22
II	Berkovich Analytic Spaces	28
9	Berkovich Spaces	28
10	Blowing Ups and Proper Maps	39

*Visiting Professor at Paris VII and ENS.

¹This material is taken from the second and third Chapter of [29].

III	Appendix	43
A	Blowing Up	43
B	Embedded Resolution of Singularities	50
C	Non Algebraically Closed Fields	55

Part I

Rigid Analytic Geometry

1 Introduction

We fix an algebraically closed, complete, non-archimedean normed field K , for short, an *ultrametric* field. Following [4], I start in Section 5 with the algebraic theory of the Tate rings $K\langle S \rangle$. These Tate rings consist of all *strictly convergent power series*, that is to say, all formal power series $\sum_i a_i S^i$ in several variables S , for which the general coefficient a_i tends in norm to 0. The key algebraic fact about $K\langle S \rangle$ is the Weierstrass Preparation Theorem 5.2. From Weierstrass Preparation, in conjunction with the Preparation Trick 5.4, all other algebraic properties, such as Noetherianity (Theorem 5.11), Nullstellensatz (Theorem 6.6) and Noether Normalization (Theorem 5.6), follow. An *affinoid algebra* A is then defined as a homomorphic image of a Tate ring. The K -algebra A acts as a function algebra on its maximal spectrum and the resulting space X is called an *affinoid variety*. These affinoid varieties will serve as the local models for our rigid analytic varieties. However, to even define a structure sheaf (or, equivalently, local analytic functions) on them, we have to bring in the notion of Grothendieck topology (see Definition 8.1); the canonical topology (or norm topology) is insufficient since it is totally disconnected. The basic admissible open sets are then the *rational subdomains*, given as all points $x \in X$ satisfying $|p_i(x)| \leq |p_0(x)|$ for strictly convergent power series p_0, \dots, p_n with no common zero on X ; the admissible coverings are the finite coverings by rational subdomains.

2 Terminology and notations

2.1. Definition (Ultrametric Field). Throughout the text, we will fix an algebraically closed field K , endowed with a non-archimedean *absolute value* (or, loosely speaking, *norm*) $|\cdot|$, which is complete but non-trivial. Such a field will be called *ultrametric*.

Recall that an *absolute value* or *multiplicative norm* on a ring A is a map $|\cdot| : A \rightarrow \mathbb{R}^+$, such that for any $a, b \in A$, we have that

2.1.1. $|a| = 0$ if, and only if, $a = 0$,

2.1.2. $|a - b| \leq \max\{|a|, |b|\}$,

2.1.3. $|a \cdot b| = |a| \cdot |b|$.

If Condition 2.1.3 is weakened to an \leq -inequality, together with $|1| = 1$, then we call $|\cdot|$ just a *norm* and A is correspondingly called a *normed ring*. An element $a \in A$ is then called *multiplicative*, if for every $b \in A$, Condition 2.1.3 holds (with equality). In other words, a norm is multiplicative, if each element is. If, on the contrary, Condition 2.1.1 is weakened to $|0| = 0$, then the resulting function is called a *semi-norm*. In this book, a norm on a field will always be assumed to be multiplicative, but no so on an arbitrary ring.

A (semi-)norm is called *complete*, if for any Cauchy sequence $(x_n)_n$ in A (meaning that, for any $\varepsilon > 0$, there exists an N , such that for any two $n, m \geq N$, we have that $|x_n - x_m| < \varepsilon$), we can find an element $x \in A$, such that x_n converges to x (meaning that for any $\varepsilon > 0$, there exists an N , such that for any $n \geq N$, we have that $|x - x_n| < \varepsilon$). This x is called a limit of the sequence (x_n) and when $|\cdot|$ is a norm, this limit is unique, if it exists.

The reader might be puzzled by Condition 2.1.2, which is called the *non-archimedean* inequality. This is stronger than the usual inequality for the absolute value on the reals; a fact often to our advantage. For instance, the following fact is elementary, but extremely useful and certainly not true in the reals.

2.2. Lemma. *Let K be an ultrametric field. If a_n , for $n \in \mathbb{N}$, is a sequence of elements in K , then their sum $\sum a_n$ exists if, and only if, a_n converges to zero.*

Proof. The implication from left to right is always true. To prove the other implication, recall that a series a_n converges if, and only if, the sequence of its partial sums s_n converges, where the n -th partial sum is defined as $s_n = a_0 + \dots + a_n$. Since K is complete, we only need to show that the sequence s_n is a Cauchy sequence. However, by Condition 2.1.2,

$$|s_n - s_m| = |a_{m+1} + \dots + a_n| \tag{1}$$

is at most the maximum of $|a_{m+1}|, \dots, |a_n|$, for $m < n$. Since the a_n form a zero sequence, this maximum can be made arbitrary small, by taking m (and whence also n) big enough. \square

2.3. Example. The field of the p -adic numbers \mathbb{Q}_p , with its natural norm is a complete non-archimedean normed field, as is the field of formal Laurent series $k((S))$ in one variable S over an arbitrary field k with the norm given by the order valuation.

The p -adics are defined as follows (see for instance [18]). Let p be a prime. On \mathbb{Z} we define a norm by putting $|a|_p = p^{-e}$, for a a non-zero integer, where e is the unique natural number such that a is a multiple of p^e but not of p^{e+1} . Of course we put $|0|_p = 0$. We extend this norm to \mathbb{Q} , by putting $|a/b|_p = |a|_p / |b|_p$, for $a, b \in \mathbb{Z}$ and $b \neq 0$ and one checks that this is independent of the representation of a/b as a fraction of two integers. By a theorem of OSTROWSKI, the $|\cdot|_p$, for p prime together with the usual absolute value $|\cdot|_\infty$ are the only (non-equivalent) norms on \mathbb{Q} . However, \mathbb{Q} is not complete with respect to any of these norms. The completion of \mathbb{Q} with respect to the $|\cdot|_p$ -norm is the p -adic field \mathbb{Q}_p (and of course, the completion of \mathbb{Q} with respect to the usual absolute value is \mathbb{R}).

In the definition of the p -adic norm $|a|_p$ of an integer a , twice the number p is used, once in a relevant way and once in a non-relevant one. The non-relevant instance is where we take $|a|_p$ to be the e -th power of the rational number $1/p$. We might have replaced $1/p$ here by any real number strictly smaller than 1 to obtain a norm equivalent to the original one. However, there is a good explanation for sticking to this convention of taking $1/p$ as the base of the exponential, since we then have the following cute result.

2.4. Theorem (Product Formula). *Let a be an integer, then*

$$1 = \prod_{p \in \mathbb{P} \cup \{\infty\}} |a|_p, \tag{2}$$

where \mathbb{P} is the set of all primes.

2.5. *Remark.* Although we made the assumption that K is algebraically closed, as far as the development of rigid analytic geometry goes, this is not necessary. The main disadvantage of not taking K algebraically closed is then that not every maximal ideal is rational, see Remark 5.9. This would require some notational adjustments (notation will be introduced below), such as identifying $\mathrm{Sp} K \langle S_1, \dots, S_m \rangle$ with the unit ball $R_{K^{\mathrm{alg}}}^m$ modulo the action of the absolute Galois group of K , where $R_{K^{\mathrm{alg}}}$ denotes the valuation ring of the algebraic closure K^{alg} of K . Since we will be mainly concerned in this book with the algebraically closed case (except for Section C), I avoid these complications by making this assumption from the start. For a development of rigid analytic geometry without the algebraically closedness assumption, see for instance [4].

2.6. *Remark.* There is a general procedure to extend an arbitrary non-archimedean normed field K to an ultrametric one, by first taking its completion, then its algebraic closure and then again completion. Note at that each stage, the norm uniquely extends, so that there is no ambivalence about which norm to use. The resulting field remains algebraically closed (see [4, 3.4.1. Proposition 3]) whence is ultrametric.

2.7. **Example.** The completion of the algebraic closure of \mathbb{Q}_p is an ultrametric field and will be denoted by \mathbb{C}_p . It is often called the field of *complex p -adic numbers*.

2.8. **Definition (Ball).** Let A be a (semi-)normed ring. We can extend this norm to any Cartesian power of A as follows. Let $a = (a_1, \dots, a_n) \in A^m$, then we define $|a|$ as the maximum of the $|a_i|$. This is not a canonical choice, for we could instead have defined $|a|$ as the sum of all $|a_i|$. However, the induced topology on A^m is the same, which is what really matters.

We define a closed *ball* or (*poly-*)*disk* in A^m as a set of the form

$$\mathbf{B}_{A^m}(a; \varepsilon) = \{x \in A^m \mid |x - a| \leq \varepsilon\}, \quad (3)$$

where $a \in A^m$ (the *centre* of the disk) and $\varepsilon > 0$ (the *radius*). If the context is clear, we might simply write $\mathbf{B}(a; \varepsilon)$ for $\mathbf{B}_{A^m}(a; \varepsilon)$.

When we take a strict inequality in Formula (3), then we get the definition of an open disk. This terminology is a bit misleading. Namely, due to the non-archimedean property, each closed disk is also topologically open (and vice versa). In particular, the spaces A^m are totally disconnected.

In case A is the ultrametric field K , then we write R for $\mathbf{B}_K(0; 1)$. It is the valuation ring of K . Its maximal ideal will be denoted by \wp ; it is the open unit disk consisting of all elements of norm strictly less than 1. The residue field R/\wp is denoted by \bar{R} .

A *valuation ring* A is a domain with field of fractions F , such that for any non-zero element $x \in F$ either x or its inverse x^{-1} lies in A . Such a ring A is necessarily local, that is to say, has a unique maximal ideal. The set of all ideals in a valuation ring is totally ordered (with respect to inclusion).

An example of a valuation ring is a *discrete valuation ring*. Its maximal ideal is principal and induces a natural norm on the ring (and its fraction field). However, if the maximal ideal of a valuation ring is not principal, the ring is non-Noetherian. This is the case with the ring R as defined above, since K is algebraically closed. Take any non-zero element $\pi \in \wp$, then

$$(\pi^{1/2}, \pi^{1/3}, \dots, \pi^{1/n}, \dots) = \wp, \quad (4)$$

but no finite collection of n -th roots of π can generate \wp . From this it follows immediately that R has no other prime ideals than (0) and \wp (for if \mathfrak{p} is a prime ideal containing a non-zero element π , then equality (4) shows that $\wp \subset \mathfrak{p}$).

A valuation ring A is called a *rank-one* valuation ring, if for any two non-zero elements a and b in the maximal ideal of A , we can find some $n \in \mathbb{N}$, such that $a^n \in bR$. In view of (4), we see that R is a rank-one valuation ring. A valuation ring A is a rank-one valuation ring if, and only if, its Krull dimension is 1, that is to say if, and only if, the only two prime ideals of A are the zero-ideal and the maximal ideal ([22, Theorem 10.7]; for the definition of Krull dimension, see the remarks following 5.6).

An important algebraic fact about R is the following.

2.9. Proposition. *Let A be a ring containing R and contained in K . Then either $A = R$ or $A = K$.*

Proof. It is easy to check that A is again a valuation ring. Let \mathfrak{p} be its maximal ideal. Suppose $R \subsetneq A$, so that we seek to prove that $A = K$. Choose any a in $A - R$. Since R is a valuation ring, $a^{-1} \in R$. However, it cannot be a unit in R , so $a^{-1} \in \wp$. Moreover, since $a^{-1} \in R \subset A$, it follows that a is a unit in A . Therefore, \mathfrak{p} , since it is the set of non-units in A , must lie inside R . In other words, $\mathfrak{p} = R \cap \mathfrak{p}$, so that \mathfrak{p} is a prime ideal of R , whence contained in \wp . As \mathfrak{p} does not contain a^{-1} , it follows that $\mathfrak{p} \subsetneq \wp$. Since the only other possibility for a prime ideal in R is the zero ideal, we conclude that $\mathfrak{p} = (0)$, so that A is a field, whence equal to K , as claimed.

In fact this property characterizes rank-one valuation rings (see for instance [22, Theorem 10.1 and 10.7]). \square

2.10. Definition (Truncated Division). Note that R has the pleasant property that whenever a and b are two non-zero elements of R , then one divides the other. More precisely, if $|a| \leq |b| \neq 0$, then $a/b \in R$. This enables us to define the following function \mathbf{D} , which is used in the main Quantifier Elimination Theorem of [10, 11, 29].

$$\mathbf{D}: R^2 \rightarrow R: (a, b) \mapsto \begin{cases} a/b & \text{if } |a| \leq |b| \neq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

3 Goal

3.1 (Analytic Geometry). We want to develop an analytic geometry over the base field K . Due to the total disconnectedness of the metric topology, to mimic the usual definition of an analytic function in the complex case (in other words, to call a K -valued function f on an open subset $U \subset K^n$ analytic if we can find around each point $x \in U$ a small neighborhood W and a power series F such that F converges on W and coincides with f on each point of W), would cause the existence of too many analytic functions, so that no Identity Theorem can hold. For instance, the (continuous) function

$$f: K \rightarrow K: x \mapsto \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1, \end{cases} \quad (6)$$

would be analytic according to our tentative definition, agreeing on R with the constant function 1, but not identical to the latter on K .

Hence we will require for a function to be analytic, that it converges on a bigger set. For instance on R , we would call a K -valued function f analytic, if there exists a power series F , converging on the whole unit disk R and coinciding in each point with f . Clearly, this would provide us with too few analytic functions. So we could cover R with some smaller domains and require the same thing for f on each of these domains in order to call f analytic on R . Hence a first question is, which will be *admissible* domains to this end, and equally important, which coverings by these domains are admissible (to avoid getting into the same problems as before, we can clearly not allow any covering). On the other hand, just working on disks as our analytic spaces is also too coarse. Hence a second question is, which will be our analytic spaces? I will postpone their description until Section 7.

3.2. Definition (Supremum Norm). Let X be an arbitrary set. A K -valued function $f: X \rightarrow K$ is called *bounded*, if there is an $M \in \mathbb{N}$, such that, for all $x \in X$, we have that $|f(x)| \leq M$. Denote the collection of all bounded K -valued functions on X by $\text{Func}_K(X)$. This becomes a K -algebra by pointwise addition and multiplication. On $\text{Func}_K(X)$, we consider the following norm

$$|f|_{\text{sup}} = \sup_{x \in X} \{|f(x)|\}, \quad (7)$$

where $f: X \rightarrow K$ and call it the *supremum norm* of f on X .

More generally, if G is a K -algebra of bounded K -valued functions on X , then we associate to any $f \in G$ the quantity $|f|_{\text{sup}}$ given by the same formula as in Formula (7). The thus constructed function $|\cdot|_{\text{sup}}$ is in general only a semi-norm. More precisely, the natural map $G \rightarrow \text{Func}_K(X)$ is injective if, and only if, $|\cdot|_{\text{sup}}$ is a norm.

4 Strictly Convergent Power Series

4.1. Definition (Strictly Convergent Power Series). For now, the main space of interest will be the unit disk R^m . I first want to describe all those power series p over K which converge on the whole disk R^m . These are the so called *strictly convergent power series*. Their definition is as follows. Let $S = (S_1, \dots, S_m)$ be a set of n variables and let

$$p = \sum_{\nu \in \mathbb{N}^m} a_\nu S^\nu \quad (8)$$

with $a_\nu \in K$, so that p is a formal power series over K . We say that p is strictly convergent if $|a_\nu|$ converges to 0 when $|\nu|$ goes to ∞ . This condition enables us to define the value of p on a tuple $x = (x_1, \dots, x_m) \in R^m$ by putting $p(x)$ equal to

$$\sum_{\nu \in \mathbb{N}^m} a_\nu x^\nu. \quad (9)$$

The latter sum does indeed converge by Lemma 2.2, since its general term has norm going to zero.

Convention 4.2. Here and elsewhere, I take up the following notational conventions. Given a multi-index $\nu = (\nu_1, \dots, \nu_m)$ and an m -tuple (of variables or of elements in a ring) $S = (S_1, \dots, S_m)$, I will denote by S^ν the monomial

$$S_1^{\nu_1} \cdot S_2^{\nu_2} \cdots S_m^{\nu_m}. \quad (10)$$

By $|\nu|$, I will mean the total degree $\nu_1 + \nu_2 + \cdots + \nu_m$ of this monomial. The ordering on the set of multi-indices is always the *lexicographical* ordering.

Whenever we have a finite set of variables $S = (S_1, \dots, S_m)$, we will let S' stand for (S_1, \dots, S_{m-1}) .

4.3. Definition (Free Tate Algebra). The ring of strictly convergent power series is denoted by $K\langle S \rangle$, and is called the *free Tate algebra* (or often just the Tate algebra) over K in the variables $S = (S_1, \dots, S_m)$.

Of course there is no real reason why we couldn't define strictly convergent power series over any normed ring A , and so we will encounter rings $A\langle S \rangle$ as well, which are defined likewise (although they are of minor use as long as the ring is not complete).

4.4. Definition (Gauss Norm). We define a norm on $K\langle S \rangle$, called the *Gauss norm*, by

$$|p|_{\text{Gauss}} = \max_{\nu \in \mathbb{N}^m} \{|a_\nu|\}, \quad (11)$$

where $p = \sum_{\nu} a_{\nu} S^{\nu}$. It is easy to see that $|p(x)| \leq |p|_{\text{Gauss}}$, for any $x \in R^m$, so that $K\langle S \rangle$ is a K -algebra of bounded K -valued functions on R^m . Therefore, it inherits also the supremum semi-norm as defined in Definition 3.2 and we have the following facts.

4.5 (Normalization Trick). Given $p \in K\langle S \rangle$, we see that $|p|_{\text{Gauss}} \in |K|$. In other words, there is some $\pi \in K$, such that $|p|_{\text{Gauss}} = |\pi|$. This enables us to normalize p , by dividing out π . In other words, $|\pi^{-1}p|_{\text{Gauss}} = 1$ and often there will be no harm by first adjusting the norm to one in this way before proceeding. The same holds over any normed ring A instead of K if there is some multiplicative unit u (in A or in $A\langle S \rangle$) of the same norm as the Gauss norm of p .

4.6. Theorem. *On each Tate algebra $K\langle S \rangle$, the Gauss norm is complete and multiplicative and coincides with the supremum norm.*

4.7. Theorem (Maximum Modulus Principle). *For any $p \in K\langle S \rangle$, there exists an $x \in R^m$ with $|p| = |p(x)|$.*

Proofs. By the Normalization Trick 4.5, we may assume in the statement of Theorem 4.7 that p has Gauss norm one. Let \bar{R} denote the residue field of R (that is to say, $\bar{R} = R/\wp$) and let \bar{p} denote the reduction of p , so that $\bar{p} \in \bar{R}[S]$ (recall that all but finitely many coefficients of p have small norm, whence are zero in the reduction). Since \bar{R} is algebraically closed whence in particular infinite, we can find a tuple \bar{x} in \bar{R}^m such that $\bar{p}(\bar{x}) \neq 0$. If $x \in R^m$ is a lifting of \bar{x} , then this means that $|p(x)| = 1$, proving Theorem 4.7.

Since always $|p(x)| \leq |p|_{\text{Gauss}}$, for any $x \in R^m$, we get from the previous result that supremum norm and Gauss norm coincide. To show that this norm

is complete, just observe that if $(p_n \mid n)$ is a Cauchy sequence in $K\langle S \rangle$, and if we denote by $a_{n,\nu}$ the coefficient of S^ν in p_n , for a fixed multi-index ν , then $(a_{n,\nu} \mid n)$ is a Cauchy sequence in K whence admits a limit a_ν . The reader can then easily check that $|a_\nu|$ tends to zero, so that $\sum_\nu a_\nu S^\nu$ is a strictly convergent power series, and that it is the limit of the sequence $(p_n \mid n)$. That the Gauss norm is multiplicative, is left as an exercise (see [4, 5.1.2. Proposition 1]; for more details, see [4, 5.1.4. Corollary 6] for Theorem 4.6 and [4, 5.1.4. Proposition 3] for Theorem 4.7). \square

In particular, $K\langle S \rangle$ is a subalgebra of $\text{Func}_K(R^m)$. Henceforth, I will therefore drop the subscripts in the norm symbols and just write $|p|$ for this norm. An immediate corollary of Theorem 4.6 is the Identity Theorem.

4.8. Theorem (Identity Theorem). *For any two series of $K\langle S \rangle$, if they take the same values on each point of the unit disk R^m , then they are identical (as elements of $K\langle S \rangle$).*

Proof. If $p, q \in K\langle S \rangle$, such that for each $x \in R^m$, we have that $p(x) = q(x)$, then by the definition of supremum norm, we have that $|p - q| = 0$. By the definition of the Gauss norm, this means that each coefficient in $p - q$ has norm zero, whence $p = q$. \square

4.9. Lemma. *A strictly convergent power series $p = \sum_\nu a_\nu S^\nu \in K\langle S \rangle$ is a unit in $K\langle S \rangle$ if, and only if, $|a_0| = |p|$, and for all $\nu \neq 0$, we have $|a_\nu| < |p|$.*

Proof. In order to prove the direct implication, we may assume that $a_0 = 1$ and $|a_\nu| < 1$, for all non-zero ν , after dividing by a_0 (in other words, by the Normalization Trick). Therefore, we can find $\pi \in K$ with $|\pi| < 1$ and $q \in K\langle S \rangle$, such that $p = 1 - \pi q$. The series $1 + \pi q + \pi^2 q^2 + \dots$ converges by Lemma 2.2 and is clearly the inverse of p , as required.

Conversely, again we may assume that $a_0 = 1$. Let q be the inverse of p . If $|p| > 1$, then $|q| < 1$, since $pq = 1$. However, the constant term of q has also to be 1, which contradicts that q has Gauss norm less than 1. Therefore, $|p| = 1 = |q|$, so that $p, q \in R\langle S \rangle$. Let $\bar{R} = R/\wp R$. Taking reduction modulo the valuation ideal \wp , we get that (the image of) p is an invertible polynomial in $\bar{R}[S]$ (it is a polynomial, since all but finitely many a_ν have norm strictly smaller than 1). The only invertible polynomials over a field are the constant ones, so that all $|a_\nu|$, for $\nu \neq 0$, must have norm less than 1, as required. \square

4.10. Definition (Rational Supremum Norm). There is a third candidate for a norm. Let \mathfrak{m} be a K -rational maximal ideal of $K\langle S \rangle$. By this we mean that $K\langle S \rangle/\mathfrak{m} \cong K$. Let $\pi^{\mathfrak{m}}: K\langle S \rangle \rightarrow K$ be the canonical surjection. Then we can define the norm at \mathfrak{m} of p as

$$|p|_{\mathfrak{m}} = |\pi^{\mathfrak{m}}(p)|. \quad (12)$$

We could do this for any K -rational maximal ideal of $K\langle S \rangle$ and take the supremum over all possible values in order to define the *K -rational supremum norm* of p , denoted (just for the time being) by $|p|_{\text{ratsup}}$.

One way to get a K -rational maximal ideal is to take a point $x = (x_1, \dots, x_m)$ in R^m and to consider the ideal \mathfrak{m}_x generated by all the $S_i - x_i$. More explicitly,

let

$$\pi_x: K\langle S \rangle \rightarrow K: S_i \mapsto x_i. \quad (13)$$

Then π_x is surjective with kernel \mathfrak{m}_x , so that \mathfrak{m}_x is a K -rational maximal ideal. In fact, the next Lemma shows that this is the only way.

4.11. Lemma. *There is a one-one correspondence between K -rational maximal ideals of $K\langle S \rangle$, with $S = (S_1, \dots, S_m)$, and points in R^m .*

Proof. We have seen already above how to associate a K -rational maximal ideal \mathfrak{m}_x to a point $x \in R^m$. Conversely, let \mathfrak{m} be a K -rational maximal ideal of $K\langle S \rangle$ and consider the canonical homomorphism

$$\pi^{\mathfrak{m}}: K\langle S \rangle \rightarrow K\langle S \rangle/\mathfrak{m} \cong K. \quad (14)$$

Let $x_i^{\mathfrak{m}} = \pi^{\mathfrak{m}}(S_i)$. Since $S_i - x_i^{\mathfrak{m}}$ lies in the kernel of $\pi^{\mathfrak{m}}$, it cannot be a unit. Therefore, by Lemma 4.9, we must have that $|x_i^{\mathfrak{m}}| \leq 1$. In other words, $x^{\mathfrak{m}} = (x_1^{\mathfrak{m}}, \dots, x_m^{\mathfrak{m}}) \in R^m$. It is easy to see that this correspondence $\mathfrak{m} \mapsto x^{\mathfrak{m}}$ is the inverse of the correspondence $x \mapsto \mathfrak{m}_x$. \square

By the Weak Nullstellensatz (Theorem 5.8) below, any maximal ideal of $K\langle S \rangle$ is K -rational (under the assumption that K is algebraically closed—an assumption always made in this book), so that there will be a one-one correspondence between maximal ideals of $K\langle S \rangle$ and points in R^m .

Now, returning to the K -rational supremum norm, the following equalities are immediate

$$|p|_{\mathfrak{m}_x} = |\pi_x(p)| = |p(x)|. \quad (15)$$

Yet again we see that $|p|_{\text{ratsup}} = |p|$. I brought up this third norm because it has a more intrinsic flavour to it. Rather than to refer to points (as for the supremum norm) or to power series expansions (as for the Gauss norm), we use (K -rational) maximal ideals of $K\langle S \rangle$. This third definition therefore could be extended to any K -algebra in order to provide us with a (semi-)norm. It also hints to a solution of our second problem. Which spaces might be appropriate candidates? We could take any K -algebra and take as a space the set of its K -rational maximal ideals. Of course, in this generality, one cannot expect good properties of these spaces so that some finiteness conditions will have to be imposed. First, however, we need to investigate more closely the algebra structure of $K\langle S \rangle$.

5 The algebra structure of $K\langle S \rangle$

At the base of the whole theory lies this one important theorem, the Weierstrass Preparation Theorem (Theorem 5.2). From it, many algebraic properties of the Tate rings and their homomorphic images are derived, such as Noether Normalization (Theorem 5.6) and the Nullstellensatz (Theorem 6.6). However, in order to formulate Weierstrass Preparation, I need some preliminary definitions.

5.1. Definition (Regular Series). We call a strictly convergent series p *regular* in S_m of degree d , if the following two conditions hold

5.1.1. $|p| = |p_d| > |p_k|$ for all $k > d$,

5.1.2. p_d is a unit in $K\langle S' \rangle$,

where $p = \sum_k p_k(S')S_m^k$ is the decomposition of p as a series in S_m with coefficients p_k in $K\langle S' \rangle$ (recall our convention of writing S' for (S_1, \dots, S_{m-1})).

A monic polynomial P in S_m over $K\langle S' \rangle$ (that is to say, the highest degree coefficient of P is 1, when expanded as a polynomial in S_m) which has (Gauss) norm one, will be called a *Weierstrass polynomial*. It is an example of a regular series.

These definitions could be generalized by replacing K by an arbitrary normed ring A . The one adjustment that has to be made is that in Condition 5.1.2, we need to require that p_d is a multiplicative unit in $A\langle S' \rangle$. This, of course, is no restriction when the norm happens to be multiplicative (as the supremum norm on a free Tate algebra is by Theorem 4.6).

5.2. Theorem (Weierstrass Preparation). *Let $p, q \in K\langle S \rangle$ and suppose that p is regular in S_m of degree d , where $S = (S_1, \dots, S_m)$. Then there exist unique $Q \in K\langle S \rangle$ and $r \in K\langle S' \rangle[S_m]$, such that*

$$q = pQ + r \tag{16}$$

with $\deg_{S_m}(r) < d$ (with the understanding that the zero polynomial has degree -1). Moreover, there exist a unique unit $u \in K\langle S \rangle$ and a unique Weierstrass polynomial P in S_m of degree d , such that

$$p = uP. \tag{17}$$

Proof. See [4, 5.2.1. Theorem 2 and 5.2.2. Theorem 1]. But just to explain why it works, I will prove the first statement. Without loss of generality, we may assume that $|p| = 1$ and $|q| = 1$ using the Normalization Trick 4.5. Write p as $\sum_i p_i S_m^i$ with p_i strictly convergent power series in S' . By assumption, p_d is a unit and there is no harm in dividing out the unit p_d , so that we may as well assume that the coefficient of S_m^d is one. Since $|p| = 1$, we have actually that $p \in R\langle S \rangle$. Let ϵ be the supremum of all $|p_i|$ with $i > d$, so that in particular $\epsilon < 1$. We will take residues modulo the ideal

$$\wp_\epsilon = \{x \in R \mid |x| \leq \epsilon\}. \tag{18}$$

In other words, $\wp_\epsilon = \mathbf{B}_R(0; \epsilon)$. It is easy to see that $R\langle S \rangle / \wp_\epsilon R\langle S \rangle \cong \bar{R}_\epsilon[S]$, where $\bar{R}_\epsilon = R / \wp_\epsilon$. Moreover, the regularity of p implies that its image in $\bar{R}_\epsilon[S]$ is a monic polynomial in S_m (of degree d). Now, the key idea is to use the Euclidean Division Algorithm for monic polynomials in the polynomial ring $\bar{R}_\epsilon[S]$. More precisely, let \bar{q} denote the image of q in $\bar{R}_\epsilon[S]$. Since \bar{p} is monic in S_m , we can find \bar{Q}_0 and \bar{r}_0 in $\bar{R}_\epsilon[S]$, such that

$$\bar{q} = \bar{p}\bar{Q}_0 + \bar{r}_0 \tag{19}$$

with $\deg_{S_m}(\bar{r}_0) < d$. Let Q_0 and r_0 be respective liftings in $R\langle S \rangle$ of \bar{Q}_0 and \bar{r}_0 with $\deg_{S_m}(r_0) < d$. Put $q'_1 = q - Q_0 p - r_0$, so that $|q'_1| \leq \epsilon$. Hence we can write $q'_1 = \lambda_1 q_1$ with $\lambda_1 \in \wp_\epsilon$ and q_1 a strictly convergent power series of norm

1. Repeating the same process, but this time with q_1 instead of q , we can find strictly convergent power series Q_1 and r_1 with $\deg_{S_m}(r_1) < d$, such that

$$q'_2 = q_1 - pQ_1 - r_1 \in \wp_\epsilon. \quad (20)$$

Therefore, we get that

$$q = (Q_0 + \lambda_1 Q_1)p + (r_0 + \lambda_1 r_1) + \lambda_1 \lambda_2 q_2, \quad (21)$$

where $q'_2 = \lambda_2 q_2$ with $\lambda_2 \in \wp_\epsilon$. Continuing in this way, we obtain series

$$\begin{aligned} Q &= Q_0 + \lambda_1 Q_1 + \lambda_1 \lambda_2 Q_2 + \dots \\ r &= r_0 + \lambda_1 r_1 + \lambda_1 \lambda_2 r_2 + \dots \end{aligned} \quad (22)$$

with each $\lambda_i \in \wp_\epsilon$. By Lemma 2.2, it is clear that these infinite sums converge, so that Q and r are strictly convergent power series with $\deg_{S_m}(r) < d$. Moreover $q - Qp - r$ lies in arbitrary high powers of \wp_ϵ , whence must be zero, so that $q = Qp + r$, as required. \square

This theorem has numerous applications. In fact, most algebraic properties of $K\langle S \rangle$ will be derived from it. It is clear that the Weierstrass Preparation Theorem will also enable us to say something about the homomorphic images of a free Tate algebra, so that they should not remain nameless.

5.3. Definition (Affinoid Algebra). A K -algebra A is called an *affinoid algebra*, if it is a homomorphic image of some free Tate algebra $K\langle S \rangle$. In other words, $A \cong K\langle S \rangle / \mathfrak{a}$, for some ideal \mathfrak{a} .

From now on, we will always be working in the category of K -algebras; morphisms will be homomorphisms of K -algebras, etc., without always mentioning this. Before discussing some applications of Weierstrass Preparation, I first give a recipe for converting an arbitrary strictly convergent series into a regular one. (Some people insist that it is actually this trick one should term 'Weierstrass Preparation Theorem' and what I have been calling here Weierstrass Preparation, should be called the 'Weierstrass Division Theorem').

5.4. Proposition (Preparation Trick). *Let $p = \sum_\nu a_\nu S^\nu \in K\langle S \rangle$ be a non-zero strictly convergent power series in the variables $S = (S_1, \dots, S_m)$. Then we can find a K -algebra automorphism σ of $K\langle S \rangle$, such that $\sigma(p)$ has become regular in S_m (of some degree).*

In other words, we can often assume, after applying some automorphism, that our series is regular, whence by the Weierstrass Preparation Theorem equal to the product of a unit and a monic polynomial in the last variable.

5.5. Remark. A similar trick exists if we work over an arbitrary normed ring A , instead of K . There is, however, a condition that needs to be fulfilled, namely the coefficient a_μ of p , of lexicographically largest index μ with the property that $|a_\mu| = |p|$, has to be a multiplicative unit. Note that this condition is vacuous over K .

Proof of Preparation Trick. After applying the Normalization Trick 4.5, we can assume that p has Gauss norm one. Let a_μ be the coefficient of norm one

with lexicographically maximal index (so that all coefficients of lexicographically bigger index than $\mu = (\mu_1, \dots, \mu_n)$ are of norm strictly smaller than one). Let d be bigger than $|\nu|$ for all indices ν such that $|a_\nu| = 1$ (which are finite in number by definition of strictly convergent power series). Consider the following K -algebra automorphism of $K\langle S \rangle$ given by

$$\sigma: S_i \mapsto S_i + S_m^{d^{m-i}} \quad \text{for } i < m \quad (23)$$

$$S_m \mapsto S_m. \quad (24)$$

It is an exercise to see that σ is an automorphism and that $\sigma(p)$ is regular in S_m of degree $s = \mu_m + \mu_{m-1}d + \dots + \mu_1 d^{m-1}$ (see for instance [4, 5.2.4. Proposition 1]). \square

Let me now give some corollaries to the Weierstrass Preparation Theorem.

5.6. Theorem (Noether Normalization). *Let A be an affinoid algebra, then there exists a free Tate algebra $K\langle T \rangle$ and a finite injective homomorphism*

$$\varphi: K\langle T \rangle \hookrightarrow A. \quad (25)$$

Moreover, if $T = (T_1, \dots, T_d)$, then $d = \dim A$, where \dim means the Krull dimension of a ring.

A ring homomorphism $A \rightarrow B$ is called *finite*, if B becomes a finite A -module under it. Hence to be finite as an algebra means the same as to be finite as a module, whereas we will reserve the terminology *finitely generated algebra* to express that B is a homomorphic image of some polynomial ring (in a finite number of variables) over A . Note that surjective homomorphisms are clearly finite.

The *Krull dimension* of a ring A is defined as the combinatorial dimension of $\text{Spec } A$. Or, equivalently, as the maximum of the heights of all prime ideals of A , where the *height* of a prime ideal \mathfrak{p} is the maximal length l of a chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_l = \mathfrak{p} \quad (26)$$

where all the \mathfrak{p}_i are prime ideals of A (we allow $l = \infty$ in this definition). A Noetherian local ring has always finite Krull dimension (this is an immediate consequence of KRULL's Principal Ideal Theorem, see for instance [22, Theorem 13.5]). The Krull dimension of the valuation ring R is one (as φ and (0) are the only prime ideals). The Krull dimension of $K\langle S \rangle$ equals the number m of variables $S = (S_1, \dots, S_m)$. Indeed, the chain of prime ideals

$$0 \subset (S_1) \subset (S_1, S_2) \subset \dots \subset (S_1, \dots, S_m) \quad (27)$$

shows that $K\langle S \rangle$ has Krull dimension at least m . Conversely, in [4, 7.1.1. Proposition 3], it is shown that every maximal ideal is generated by m elements, so that its height is at most m , again by KRULL's Principal Ideal Theorem. In Corollary 5.10, I will in fact show that every maximal ideal is of the form $(S_1 - r_1, \dots, S_m - r_m)$ with $r_i \in R$; this is under the assumption that K is algebraically closed. However, [4, 7.1.1. Proposition 3] holds true for non algebraically closed ground fields as well.

Proof of Noether Normalization. We will prove by induction on m , the number of variables $S = (S_1, \dots, S_m)$, the following statement.

5.7. Claim. *The theorem holds true for any affinoid algebra A which is finite over $K\langle S \rangle$.*

Note that any affinoid algebra is a finite $K\langle S \rangle$ -algebra over some free Tate algebra, since it is a homomorphic image of one.

To prove the claim, first assume that $m = 0$. But then A is a finite K -algebra and clearly K embeds in it. Hence, let $m > 0$ and assume that the claim has been proved for smaller values of m . Let $\psi: K\langle S \rangle \rightarrow A$ be the finite map, where $S = (S_1, \dots, S_m)$. If the kernel of ψ is trivial, there is nothing to prove, so take any non-zero p in the kernel of ψ . Then A is still a finite $B = K\langle S \rangle/(p)$ -algebra. By the Preparation Trick 5.4, we may assume that p is already regular and hence by the Weierstrass Preparation Theorem, a product of a unit times a monic polynomial in S_m (since automorphisms clearly preserve finiteness). We can forget the unit. So B is actually a residue ring of $K\langle S \rangle$ modulo a monic polynomial in S_m , and hence a finite $K\langle S' \rangle$ -algebra. Therefore our original A is already finite over $K\langle S' \rangle$. Induction now finishes the proof. \square

5.8. Theorem (Weak Nullstellensatz). *Every maximal ideal \mathfrak{m} of an affinoid algebra A is K -rational, that is to say, $A/\mathfrak{m} \cong K$.*

Proof. By Noether Normalization 5.6, we can find a free Tate algebra $K\langle S \rangle$ (in m variables) and a finite injective map, such that $K\langle S \rangle \hookrightarrow A/\mathfrak{m}$ (since the latter is clearly also affinoid) and such that $m = \dim A/\mathfrak{m}$. Since A/\mathfrak{m} is a field, its dimension is zero. Therefore, $m = 0$ and A/\mathfrak{m} is a finite K -algebra. Since K is algebraically closed, it has no non-trivial finite field extensions, so we must have that $K \cong A/\mathfrak{m}$. \square

5.9. Remark. If one drops the requirement that K be algebraically closed, then the above results say that for any maximal ideal \mathfrak{m} of A , the residue field A/\mathfrak{m} is a finite field extension of K .

In combination with Lemma 4.11, we therefore get the following.

5.10. Corollary. *There is a one-one correspondence between maximal ideals of $K\langle S_1, \dots, S_m \rangle$ and points in R^m . Under this correspondence, a point $x = (x_1, \dots, x_m)$ determines the maximal ideal $(S_1 - x_1, \dots, S_m - x_m)$.*

5.11. Theorem (HILBERT'S BASIS THEOREM). *An affinoid algebra A is Noetherian.*

Proof. We prove this by induction on the Krull dimension $m = \dim A$, for A an affinoid algebra. If $m = 0$, then A is a finite K -module, by Noether Normalization 5.6, whence Noetherian. So assume $m > 0$. Let us first prove the theorem for $A = K\langle S \rangle$, where $S = (S_1, \dots, S_m)$. Let \mathfrak{a} be a non-zero ideal of $K\langle S \rangle$ and take a non-zero $p \in \mathfrak{a}$. By induction, $K\langle S \rangle/(p)$ is Noetherian (note that $K\langle S \rangle/(p)$ has dimension less than m). Therefore, the image of \mathfrak{a} in $K\langle S \rangle/(p)$ is finitely generated, say, by (the images of) $p_2, \dots, p_s \in K\langle S \rangle$. Then $\mathfrak{a} = (p, p_2, \dots, p_s)$. This establishes the case $A = K\langle S \rangle$. For general A of dimension m , by Noether Normalization, A is a finite $K\langle S \rangle$ -module, whence also Noetherian. \square

6 Maximal Spectra

The Weak Nullstellensatz 5.8 and its Corollary 5.10 suggest that the *maximal spectrum* of an affinoid algebra might have various nice properties. In this section, I will explore a little these maximal spectra, with as final result, the

Nullstellensatz (Theorem 6.6). Various topologies can be put on these maximal spectra. In this and the next section, we will introduce the Zariski topology and the canonical topology. However, the 'correct' topology for doing analytic geometry, will turn out to be a Grothendieck topology and will only appear in Section 8.

6.1. Definition (Maximal Spectrum). Let us denote by $\text{Max}(A)$ the collection of all maximal ideals of a ring A , called the *maximal spectrum* of A . We could put a topology on this 'space', by taking the induced topology from the Zariski topology on $\text{Spec } A$. In case A is an affinoid algebra, the (Zariski) closed subsets of $\text{Max}(A)$ will be called *closed analytic subsets* of $\text{Max}(A)$.

Let me give an alternative description of the maximal spectrum of an affinoid algebra A . First, take $A = K\langle S \rangle$ to be a free Tate algebra in the variables $S = (S_1, \dots, S_m)$. By Corollary 5.10, there is a one-one correspondence between the points of the unit disk R^m and $\text{Max}(K\langle S \rangle)$. For a general affinoid algebra A , take a representation $A = K\langle S \rangle/\mathfrak{a}$. Then by the same correspondence, the maximal ideals of A correspond to the points of the zero-locus of \mathfrak{a} in R^m (that is to say, to the closed analytic subset $\{x \in R^m \mid p(x) = 0 \text{ for all } p \in \mathfrak{a}\}$). Thus these maximal spectra have a geometric interpretation. However, this interpretation depends on the chosen representation of A as a homomorphic image of a free Tate algebra. Nevertheless, the following lemma shows that maximal ideals behave nicely under homomorphisms between affinoid algebras.

6.2. Lemma. *Let $\varphi: A \rightarrow B$ be an arbitrary (K -algebra) homomorphism between affinoid algebras. Then each maximal ideal \mathfrak{m} of B contracts to a maximal ideal of A .*

Proof. It is standard practice in algebra to write the *contraction* in A of an arbitrary ideal \mathfrak{b} of B , as $\mathfrak{b} \cap A$. In other words, $\mathfrak{b} \cap A$ consists of all elements in A whose image in B lies in \mathfrak{b} . Now, put $\mathfrak{p} = \mathfrak{m} \cap A$. Consider the sequence of K -algebra homomorphisms

$$K \subset A/\mathfrak{p} \hookrightarrow B/\mathfrak{m} \cong K, \quad (28)$$

where the latter isomorphism is provided by the Weak Nullstellensatz 5.8. Hence the composite map must be an isomorphism, proving the claim. \square

6.3 (Affinoid Algebras as Function Algebras). Lemma 6.2 shows that any homomorphism (for the last time, we always mean K -algebra homomorphism by this) between affinoid algebras induces a map between their maximal spectra (in the reverse order, of course—this functor is contravariant) by sending a maximal ideal to its contraction. Hence the $\text{Max}(A)$ seem to get better and better candidates for our wanted analytic spaces. What is still missing is a refined notion of an analytic function. To this end, observe that an element $p \in A$ can be viewed as a function on $\text{Max}(A)$, by letting $p(\mathfrak{m})$ be the image of p under the canonical projection onto $K \cong A/\mathfrak{m}$. In other words, with notation from Definition 4.10, we define $p(\mathfrak{m})$ as $\pi^{\mathfrak{m}}(p)$. Functions arising in this way will play the role of our global analytic functions on $\text{Max}(A)$. However, we need some more local functions as well. Put differently, we need to define a sheaf of functions on $\text{Max}(A)$. This brings us back to our first problem, what should the 'admissible' opens and coverings of $\text{Max}(A)$ be? In any case, before we can even speak of 'analytic' functions, we should have some complete norm on A .

6.4 (Norms on Affinoid Algebras). By the previous argument, we can consider an affinoid algebra A as a function algebra on $\text{Max}(A)$. However, we need to be careful when doing so. It is not clear that two different elements p and q of A remain distinct as functions on $\text{Max}(A)$. And, indeed, this is not always the case. Namely, an element $p \in A$ induces the zero-map on $\text{Max}(A)$ if, and only if, it is contained in every maximal ideal of A , in other words if, and only if, it is contained in the *Jacobson radical* of A (that is, the intersection of all maximal ideals). Below in Theorem 6.6, I will prove the Nullstellensatz for affinoid algebras, which says that the Jacobson radical and the *nilradical* (that is, the collection of all nilpotent elements) of an affinoid algebra coincide.

Hence A will be truly a subalgebra of $\text{Func}_K(\text{Max}(A))$ if, and only if, A is reduced (that is to say, without nilpotent elements). When I will introduce later semianalytic and subanalytic sets, I always will make the assumption that A is reduced. For the remainder of this paragraph, I will also make this assumption. By means of Definition 3.2, since A is a function algebra on $\text{Max} A$, we can talk about the supremum norm on A , which we will denote again just by $|\cdot|$. Recall that $|p|$ is by definition the supremum of all $\pi^{\mathfrak{m}}(p)$, where \mathfrak{m} runs over all maximal ideals of A . One checks that A is complete for this norm. However, this norm is in general not multiplicative (that is to say, in general we only have an inequality $|pq| \leq |p| \cdot |q|$), though it is *power-multiplicative* (meaning that for any $p \in A$ and for any n , we have that $|p^n| = |p|^n$). Moreover, the Maximum Modulus Principle holds for this norm (see [4, 6.2.1. Proposition 4]).

There are other possibilities for defining a norm on A . Namely, choose a representation $K\langle S \rangle/\mathfrak{a} \cong A$ induced by the surjection $\alpha: K\langle S \rangle \rightarrow A$ and consider the *residue norm* $|p|_\alpha$ on A defined as

$$|p|_\alpha = \inf_{P \in \alpha^{-1}(p)} |P|. \quad (29)$$

Since each ideal \mathfrak{a} of $K\langle S \rangle$ is closed in the canonical topology ([4, 6.1.1. Proposition 3]); see the definition preceding Proposition 7.7 for the definition of canonical topology), we get a complete norm on A (which is in general neither multiplicative nor power-multiplicative). For each representation we have a (possible different) residue norm, but all these norms are equivalent (meaning that they induce the same topology on A ; see [4, 6.1.1]). In the reduced case they are even all equivalent to the supremum norm as defined above ([4, 6.2.4. Theorem 1]). These residue norms are not as intrinsic as the supremum norm, and we will therefore mainly use the supremum norm (well understood only for reduced affinoid algebras, although even for non-reduced the definition of supremum norm still makes sense, but the resulting norm is only a semi-norm).

6.5. Remark. Several constructions originally made over the base field K , can now be extended to work over any affinoid algebra A . In particular, we can define the ring of strictly convergent power series $A\langle S \rangle$ over A . Since for a reduced affinoid algebra all norms (either residue or supremum) are equivalent, we get the same collection of strictly convergent power series, regardless of the norm we start with. However, since these norms need not to be multiplicative, it does depend on the chosen norm whether a series is regular (in the sense of 5.1) or not. Nonetheless, once the norm is chosen, it is then an easy exercise to prove the Weierstrass Preparation Theorem over an arbitrary affinoid algebra (see for instance [4, 7.3.5. Proposition 8]). Note that we also have a version of the Preparation Trick over arbitrary affinoid algebras as explained in Remark 5.5.

I want to mention the following fact, for a proof of which I refer to [4, 6.4.3. Theorem 1]. If A is reduced, then there exists an epimorphism $\alpha: K\langle S \rangle \rightarrow A$, such that $|\cdot|_{\text{sup}} = |\cdot|_{\alpha}$. Since $\ker \alpha$ is closed, this implies that any function $p \in A$ of norm at most one, can be lifted to a function $P \in R\langle S \rangle$.

I conclude with the promised Nullstellensatz (see [4, 7.1.2] for the case where one does not assume K to be algebraically closed). Recall that the *radical* of an ideal \mathfrak{a} is by definition the collection of all elements a power of which lies in \mathfrak{a} . One writes $\text{rad } \mathfrak{a}$ for this radical.

6.6. Theorem (Nullstellensatz). *Let A be an affinoid algebra and \mathfrak{a} an ideal of A . The radical $\text{rad } \mathfrak{a}$ equals the intersection of all maximal ideals containing \mathfrak{a} .*

Proof. One immediately reduces (by taking reduction modulo \mathfrak{a}) to the statement that the nilradical of an affinoid algebra A equals its Jacobson radical (=intersection of all maximal ideals). Let us prove this statement by induction on $m = \dim A$. If $m = 0$, then by Noether Normalization, A is a finite K -algebra, whence an Artinian ring. By NAKAYAMA's Lemma, if \mathfrak{n} denotes its Jacobson radical, we have that $\mathfrak{n}^k \subsetneq \mathfrak{n}^{k-1}$, as long as $\mathfrak{n}^{k-1} \neq 0$. By the descending chain condition, these powers of \mathfrak{n} cannot infinitely grow smaller, whence some power is zero.

For $m > 0$, consider first the case $A = K\langle S \rangle$, for $S = (S_1, \dots, S_m)$. Let p be an element in the Jacobson radical of $K\langle S \rangle$. Hence its norm is zero (consider the (rational) supremum norm), whence p itself is zero. For the general case, let p be an element of A lying in the Jacobson radical, but suppose it is not nilpotent. Hence there exists a prime ideal \mathfrak{p} not containing p . If \mathfrak{p} has height strictly bigger than zero, we are done by induction on A/\mathfrak{p} . So we can assume that \mathfrak{p} is a minimal prime of A .

By Noether Normalization 5.6, there exists a finite and injective homomorphism

$$\varphi: K\langle S \rangle \hookrightarrow A, \quad (30)$$

with $S = (S_1, \dots, S_m)$. Since the contraction of \mathfrak{p} has to be the zero ideal ($K\langle S \rangle$ is clearly a domain), we may as well work in A/\mathfrak{p} and hence assume without loss of generality that A is a domain and that p is a non-zero element of the Jacobson radical of A . By the finiteness of φ , we can find elements $q_i \in K\langle S \rangle$, such that

$$p^d + \varphi(q_1)p^{d-1} + \dots + \varphi(q_d) = 0. \quad (31)$$

Take d to be the minimal degree of any such equation, so that in particular $\varphi(q_d) \neq 0$ (since A is a domain). Hence also $\varphi(q_d)$ lies in the Jacobson radical of A , and therefore q_d lies in the Jacobson radical of $K\langle S \rangle$, whence is zero by the above observation, contradiction. \square

7 Affinoid Varieties

I now will define and discuss the category of affinoid varieties. I will postpone the formal definition of a structure sheaf on an affinoid variety, to the next section. However, the main ingredients, the rational subdomains and their associated affinoid algebras, will be already introduced here.

7.1. Definition (Affinoid Variety). Given an affinoid algebra A , we call the pair $(\text{Max}(A), A)$ the *affinoid variety* associated to A , and we denote it by $\text{Sp } A$.

Hence the maximal ideals are to be considered as the points of this space, whereas the elements of A act as functions on these maximal ideals, as explained in 6.3. We already mentioned there that, when viewing a maximal ideal \mathfrak{m} of A as a point x of $\text{Sp } A$, then the value of $p \in A$ at \mathfrak{m} will be denoted by $p(x)$. In the notation of Definition 4.10, this means that $p(x) = \pi^{\mathfrak{m}}(p)$. This is in perfect accordance with the notation used when we view $\text{Sp } A$ as a closed analytic subset of some R^m . Namely, we then think of x as an m -tuple (x_1, \dots, x_m) and $p(x)$ has the usual meaning of substituting the x_i for the variables S_i .

7.2. Definition (Map of Affinoid Varieties). A *map of affinoid varieties* is a pair (f, φ) , where the map $f: \text{Sp } B \rightarrow \text{Sp } A$ is induced by the homomorphism $\varphi: A \rightarrow B$.

In other words, the image of a maximal ideal \mathfrak{m} of B is given by its contraction $\varphi^{-1}(\mathfrak{m})$ to A (written as $\mathfrak{m} \cap A$), which is again maximal by Lemma 6.2. Moreover, if y is the point of $\text{Sp } B$ corresponding to the maximal ideal \mathfrak{m} and x its image under f (corresponding to the maximal ideal $\mathfrak{m} \cap A$), then we have a commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\varphi} & B \\
 \downarrow & & \downarrow \\
 A/\mathfrak{m} \cap A & \xrightarrow{\quad} & B/\mathfrak{m}
 \end{array} \tag{32}$$

where the vertical maps are the canonical surjections and the bottom horizontal map is an isomorphism (since both fields are equal to K by the Weak Nullstellensatz, Theorem 5.8). Therefore, we have, for every $p \in A$, an equality

$$p(x) = \varphi(p)(y) \tag{33}$$

exhibiting the compatibility between the action of p on $\text{Sp } A$ and the action of its image $\varphi(p)$ on $\text{Sp } B$.

Most of the time we will be a bit careless and just write f for the map, rather than the pair. The thus defined objects and maps give the category of affinoid varieties (see [4, 7.1.4] for more details).

7.3 (Fibre Products). The category of affinoid varieties admits fibre products, see [4, 7.1.4. Proposition 4]. Let me start with an important special case. Recall from 6.1 that R^n can be viewed as an affinoid variety corresponding to the affinoid algebra $K\langle S \rangle$, with $S = (S_1, \dots, S_n)$. Let A be an arbitrary affinoid algebra. Then

$$\text{Sp } A \times R^n \tag{34}$$

is again an affinoid variety corresponding to the affinoid algebra $A\langle S \rangle$. This last affinoid algebra is equal to $K\langle S, T \rangle / IK\langle S, T \rangle$, where $A = K\langle T \rangle / I$ is some representation of A as a homomorphic image of a Tate ring.

In general, if $f: X = \text{Sp } A \rightarrow Z = \text{Sp } C$ and $g: Y = \text{Sp } B \rightarrow Z$ are maps of affinoid varieties, then the fibre product $X \times_Z Y$ exists and is again an affinoid variety, with affinoid algebra given by the *complete tensor product* $A \widehat{\otimes}_C B$. Let me briefly discuss the latter algebra. On each affinoid algebra, we choose some residue norm. We have already observed in 6.4, that all possible choices yield equivalent norms. Corresponding to the maps f and g , we have algebra homomorphisms $C \rightarrow A$ and $C \rightarrow B$. So we can form the (usual) tensor product $D = A \otimes_C B$ (see for instance [7, A2.2] for the construction of the tensor product). One defines from the norms on A , B and C a norm on D as follows. Let $d \in D$, so that it can be written as

$$d = \sum_{i=1}^n a_i \otimes b_i, \quad (35)$$

with $a_i \in A$ and $b_i \in B$. Let δ be the maximum of all $|a_i| \cdot |b_i|$, for $i = 1, \dots, n$. We put $|d|$ equal to the infimum of all δ for all possible representations (35). One shows that this yields indeed a norm on D (see [4, 2.1.7] for details). Finally, we let $A \widehat{\otimes}_C B$ be the completion of D with respect to this norm. This is independent from the particular choice of a norm on A , B or C . Moreover, $A \widehat{\otimes}_C B$ is again an affinoid algebra ([4, 6.1.1. Proposition 10]).

When we take $C = K$ in the above construction, we get an affinoid algebra $A \widehat{\otimes}_K B$ corresponding to the affinoid variety $X \times Y$. The example above is of this type. Indeed, if $Y = R^n$ so that $B = K\langle S \rangle$, then

$$A \widehat{\otimes}_K K\langle S \rangle \cong A\langle S \rangle, \quad (36)$$

as affinoid algebras; see [4, 6.1.1. Proposition 7].

One can generalize the construction of complete tensor product to arbitrary complete normed K -algebras. Usually, this is done via a universal property. I will formulate it only for K -affinoid algebras; the modifications for the general case are obvious.

7.4. Theorem (Universal Property of Complete Tensor Products). *If $\alpha: C \rightarrow A$ and $\beta: C \rightarrow B$ are two homomorphisms of affinoid algebras for which there exist an affinoid algebra E and two homomorphisms $a: A \rightarrow E$ and $b: B \rightarrow E$ which agree on C , meaning that the diagram*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ \beta \downarrow & & \downarrow a \\ B & \xrightarrow{b} & E \end{array} \quad (37)$$

commutes, then there exists a unique homomorphism $A \widehat{\otimes}_C B \rightarrow E$ extending the original homomorphisms. In other words, the homomorphism $a: A \rightarrow E$ equals the composition $A \rightarrow A \widehat{\otimes}_C B \rightarrow E$, and a similar statement for b .

I conclude with a useful example of the more general construction. Let L be an ultrametric field extending K , so that the norm on L is an extension of

the norm on K . Let A be a K -affinoid algebra, then $A \widehat{\otimes}_K L$ is an L -affinoid algebra. In other words, if $X = \mathrm{Sp} A$, then $X \times_K L$ is an affinoid variety over L with affinoid algebra $\mathrm{Sp} A \widehat{\otimes}_K L$.

In Proposition 7.6 below, a third important type of fibre product is discussed, the construction of an inverse image.

The above definition of the space $\mathrm{Sp} A$ is not quite satisfactory, since we would like to have a sheaf of functions rather than just an algebra of global functions (namely, A). So our next project is to define the admissible opens.

7.5. Definition (Rational Subdomain). Let $X = \mathrm{Sp} A$ be an affinoid variety (when saying this, our notation implicitly expresses that A is an affinoid algebra). A subset $U \subset X$ is called a *rational subdomain* of X , if there exist elements $p_0, p_1, \dots, p_m \in A$, generating the unit ideal, such that

$$U = \{ x \in X \mid |p_i(x)| \leq |p_0(x)| \text{ for all } i = 1, \dots, m \}. \quad (38)$$

Saying that the p_i generate the unit ideal is the same, by the Nullstellensatz 6.6, as saying that they have no common zero on X . In particular, p_0 cannot have a zero on U .

Let me now explain why a rational subdomain is itself an affinoid variety. Set

$$C = A \langle p/p_0 \rangle = A \langle T \rangle / (p_1 - p_0 T_1, \dots, p_m - p_0 T_m) \quad (39)$$

where $T = (T_1, \dots, T_m)$ and $p = (p_1, \dots, p_m)$. I claim that we have an commutative diagram

$$\begin{array}{ccc} \mathrm{Sp} C & \xrightarrow{\theta} & U \\ \downarrow & & \downarrow \\ X \times R^m & \xrightarrow{\pi} & X \end{array} \quad (40)$$

where π is the canonical projection map (induced by the embedding $A \subset A \langle T \rangle$), $\theta = \pi|_{\mathrm{Sp} C}$ is the restriction of π and the vertical maps are just inclusions. The only thing that needs to be checked is that π maps $\mathrm{Sp} C$ inside U . So let $(x, t) \in X \times R^m$ be a point in $\mathrm{Sp} C$. This means that $p_i(x) = t_i p_0(x)$. Since $|t_i| \leq 1$, we see that $|p_i(x)| \leq |p_0(x)|$, so that x lies indeed in U .

Next, I claim that θ is actually a bijection. Indeed, this is clear, since the t_i are uniquely defined from the point $x \in U$ as the quotients $p_i(x)/p_0(x)$ (note that $p_0(x)$ cannot be zero). In conclusion, we can view U as an affinoid variety (although, strictly speaking, it is only in bijection with one).

We do need something here, though, to ensure the uniqueness of C and hence of the affinoid variety structure on U . Consider the following statement: *if a map $f: \mathrm{Sp} B \rightarrow \mathrm{Sp} A$ between affinoid varieties induced by the homomorphism $\varphi: A \rightarrow B$, is bijective on the set of maximal ideals, then φ is a bijection.* If this would hold, then we can put a unique affinoid variety structure on U given by the affinoid algebra C . Unfortunately, the above statement

is in general false. Consider for instance the homomorphism $K \subset K\langle S \rangle / (S^2)$, where S is a single variable. The maximal spectrum of both affinoid algebras is the one-point space consisting of the origin (in R). Nevertheless this inclusion is clearly not an isomorphism (and hence a one-point space cannot be rational).

So remains the problem of associating a unique affinoid algebra to U . The solution is provided by a universal property. A subset $U \subset X = \mathrm{Sp} A$ is called an *affinoid subdomain*, if there exists an affinoid algebra C and a homomorphism $\varphi: A \rightarrow C$, such that the image of the associated map $f: \mathrm{Sp} C \rightarrow X$ lies inside U , and such that the pair (C, f) is universal with respect to this property. Here universality means that given any map $g: \mathrm{Sp} B \rightarrow \mathrm{Sp} A$ the image of which lies inside U , there is then a unique map $h: \mathrm{Sp} B \rightarrow \mathrm{Sp} C$, with $g = f \circ h$. By universality, C is uniquely defined by U . So we are out of trouble if we can show that a rational subdomain is in fact an affinoid subdomain. See [4, 7.2.3. Proposition 4] for a proof.

Since p_0 does not vanish on U , it must be a unit in C . Applying the Maximum Modulus Principle, to $1/p_0$, we can find a $x_0 \in U$, such that $|1/p_0(x)| \leq |1/p_0(x_0)|$, for all $x \in U$. Putting $0 \neq \delta = |p_0(x_0)|$, we get that $\delta \leq |p_0(x)|$, for all $x \in U$.

Some important properties of rational subdomains are the following, for proofs of which I have to refer to [4, 7.2. and 7.3]. The map $A \rightarrow C$ is injective and flat. In particular, this implies that if X is reduced (meaning that A is reduced), then so is U . This latter fact will justify our future restriction to work only over reduced varieties. The intersection of two rational subdomains is again a rational subdomain. A rational subdomain of an affinoid variety which itself is a rational subdomain in some bigger affinoid variety, is also a rational subdomain in the bigger variety. The following property should be thought of as the continuity of affinoid maps; this will be explained more in Definition 8.1.

7.6. Proposition. *The inverse image of a rational subdomain under a map of affinoid varieties is again a rational subdomain. More precisely, if U is a rational subdomain of $X = \mathrm{Sp} A$ with affinoid algebra C and if $f: Y = \mathrm{Sp} B \rightarrow X$ is a map of affinoid varieties, then $f^{-1}(U)$ is equal to the fibre product $Y \times_X U$ and hence has affinoid algebra $B \otimes_A C$.*

Proof. See [4, 7.2.2. Proposition 6] for rational subdomains and [4, 7.2.2. Proposition 4] for arbitrary affinoid subdomains; for the last statement, see 7.3. \square

In the next Proposition, an affinoid variety X is viewed with the topology induced by the norm. More precisely, consider X as a closed analytic subset of R^n via a representation $K\langle S \rangle / \mathfrak{a} \cong A$. If we take two different closed immersions, then the resulting topologies are nonetheless the same. This uniquely defined topology is therefore called the *canonical topology* on X . In the next section, we will replace this topology with one more suitable for defining a structure sheaf, albeit a Grothendieck topology.

7.7. Proposition. *Let X be an affinoid variety. Any rational subdomain in X is open in the canonical topology on X .*

Proof. See [4, 7.2.5. Theorem 3]. \square

I conclude this section with some discussion on rational subdomains in R and in R^2 .

Local Structure of Rational Domains

A closed disk in $X = \text{Sp } A$ is clearly a rational subdomain and likewise, so is the complement of an open disk. There is a converse, in case $X = R$. Namely, every rational subdomain U of R is a finite union of finite intersections of closed disks and complements of open disks, see [4, 9.7.2. Theorem 2].

I now turn to rational subdomains in R^2 . Their description is no longer as neat as in the one-dimensional case, however, their local structure is. Let me be more precise (and a bit more general).

7.8. Proposition. *Let $X = \text{Sp } A$ be an affinoid variety and U a rational subdomain of $X \times R^n$. There exists an $\varepsilon > 0$ and a rational subdomain V of X , such that*

$$U \cap (X \times \mathbf{B}_{R^n}(0; \varepsilon)) = V \times \mathbf{B}_{R^n}(0; \varepsilon). \quad (41)$$

In particular, if we apply this to a rational subdomain U of R^2 , then there is an $\varepsilon > 0$, such that

$$\{(x, y) \in U \mid |y| \leq \varepsilon\} = D \times \mathbf{B}_R(0; \varepsilon), \quad (42)$$

where D , by our above discussion, is a finite union of sets of the form

$$\{x \in R \mid |x - a_0| \leq |c_0| \text{ and } |x - a_i| \geq |c_i|, \text{ for } i = 1, \dots, r\}, \quad (43)$$

for some suitable a_i and c_i in R .

Proof of Proposition 7.8. See [20] or [26, Proposition 2.2]. The proof uses the following observation. Let $p(T) \in A\langle T \rangle$ be of norm smaller than or equal to one. (Recall that $A\langle T \rangle$ is the affinoid algebra of $X \times R^n$, where $T = (T_1, \dots, T_n)$). Let $a \in A$ be the constant term of p , then we have, for all (x, y) with $|y| \leq \varepsilon$, that

$$|a(x)| > \varepsilon \quad \text{implies} \quad |p(x, y)| = |a(x)|, \quad (44)$$

$$|a(x)| \leq \varepsilon \quad \text{implies} \quad |p(x, y)| \leq \varepsilon. \quad (45)$$

Now, there exist $p_i \in A\langle T \rangle$ generating the unit ideal, such that U is given as in Formula (38). We also remarked there that there exists a $\delta > 0$, such that for all $(x, y) \in U$, we have that $|p_0(x, y)| \geq \delta$. Take now ε strictly smaller than δ and use the above mentioned fact on the constant terms a_i of all the p_i , to prove that we can take

$$\{x \in X \mid |a_i(x)| \leq |a_0(x)| \text{ for all } i = 1, \dots, s\} \quad (46)$$

for V in the statement of the proposition. \square

8 Rigid Analytic Varieties

In the previous section, I introduced affinoid varieties, which will be the local models for developing rigid analytic geometry. The aim of this section is to define a structure sheaf on them and then to define the global models, the rigid analytic varieties. First, though, I need to discuss Grothendieck topologies.

8.1 (Grothendieck Topologies). We have now our candidates for the 'admissible' open subsets of an affinoid variety $X = \text{Sp } A$, namely the rational subdomains U of X . Note that by Proposition 7.7, they are, in some sense, really 'open'. By Formula (39) (and the discussion following it), to each rational subdomain is uniquely associated an affinoid algebra C , such that U carries the structure of the affinoid variety $\text{Sp } C$. We denote this by

$$\mathcal{O}_X(U) = C. \quad (47)$$

In particular, X itself is a rational subdomain of X (by taking $p_0 = p_1 = 1$ for instance) and clearly $\mathcal{O}_X(X) = A$. The notation very much suggests that this new object might well be a sheaf. If this is the case, then we have to take the rational subdomains as the opens of a topology on X . However, any disk is a rational subdomain of R^n and by considering the topology with opens all rational subdomains, the functor \mathcal{O}_X cannot be a sheaf, as the example in 3.1 showed. So we have to restrict somewhere. But we don't want to restrict further our class of opens. The only other option we have is to restrict the class of possible (read, admissible) coverings. This will obviously no longer give us a topology, but it will be a Grothendieck topology, and that is more than sufficient for defining sheaves (and their cohomology).

Grothendieck topologies also appear in algebraic geometry, for instance to define the étale topology on a scheme; see for instance [6].

For the specialists, here is a precise (though not fully general) definition of a Grothendieck topology on a set X . First, by a *covering* of a subset U of X , we mean a collection of subsets of U the union of which equals U . Now, a *Grothendieck topology* \mathbb{G} on X consists of

- a system \mathcal{S} of subsets U of X , (called *admissible open subsets*), and
- for each $U \in \mathcal{S}$, a system $\mathcal{C}_{\text{ov}} U$ of coverings of U (called *admissible coverings*), where each covering has all its members in \mathcal{S} ,

and such that the following five conditions are satisfied.

- 8.1.1. The empty set \emptyset and the whole space X are in \mathcal{S} .
- 8.1.2. If U and V are in \mathcal{S} , then so is $U \cap V$.
- 8.1.3. For every $U \in \mathcal{S}$, we have that the singleton $\{U\} \in \mathcal{C}_{\text{ov}} U$.
- 8.1.4. For every $U \in \mathcal{S}$, if $\mathfrak{U} \in \mathcal{C}_{\text{ov}} U$ and for every $V \in \mathfrak{U}$, we have a covering $\mathfrak{V}_V \in \mathcal{C}_{\text{ov}} V$, then their union

$$\bigcup_{V \in \mathfrak{U}} \mathfrak{V}_V \quad (48)$$

is a covering belonging to $\mathcal{C}_{\text{ov}} U$.

- 8.1.5. If U and V are in \mathcal{S} with $V \subset U$ and if $\mathfrak{U} \in \mathcal{C}_{\text{ov}} U$, then $V \cap \mathfrak{U} \in \mathcal{C}_{\text{ov}} V$ (where $V \cap \mathfrak{U}$ is the collection of all intersections $V \cap W$, with $W \in \mathfrak{U}$).

A set X endowed with a Grothendieck topology \mathbb{G} will be called a *G-topological space*.

Any ordinary topology is a Grothendieck topology by letting all opens be admissible opens and all coverings by opens be admissible coverings. Many notions of ordinary topology have their counterpart in Grothendieck topologies. However, apart from conditions on opens, we often have to impose conditions on coverings as well. Let me just give one typical example.

8.2. Definition (G -continuity). A map $f: X \rightarrow Y$ between two G -topological spaces is called G -continuous if the following two conditions are satisfied.

- 8.2.1. The inverse image $f^{-1}(V)$ under f of an admissible open V in Y , is an admissible open of X .
- 8.2.2. The inverse image $f^{-1}(\mathfrak{U})$ under f of an admissible covering \mathfrak{U} of an admissible open V in Y , is an admissible covering of $f^{-1}(V)$ in X .

Another notion we will come across is that of a *quasi-compact* G -topological space X . This is a space such that each admissible covering \mathfrak{U} has a *refinement* \mathfrak{V} (that is, any set out of \mathfrak{V} is contained in some set of \mathfrak{U}) which is finite and admissible.

8.3. Definition (Admissible Open). We call a subset U of X *admissible* (or, an *admissible open*), if it is a rational subdomain of X . We will call a covering of X *admissible* (or, an *admissible covering*), if it is a finite covering by rational subdomains. This constitutes a Grothendieck topology on X , so that an affinoid variety now becomes a G -topological space (which by construction is quasi-compact). We call the functor \mathcal{O}_X the *structure sheaf* of X . This is indeed a sheaf by TATE's Acyclicity Theorem (see [4, 8.2.1. Theorem 1 and Corollary 2]).

8.4. Proposition. Any map of affinoid varieties is G -continuous.

Proof. Condition 8.2.1 is just a restatement of Proposition 7.6 for general affinoid subdomains. Condition 8.2.2 is immediate by the definition of the Grothendieck topology on affinoid varieties. \square

If one were to take all affinoid subdomains as the admissible opens and all finite coverings by them as the admissible coverings, we would get another Grothendieck topology on X . However, the two are hardly any different, since by a theorem of GERRITZEN en GRAUERT, any affinoid subdomain is a finite union of rational subdomains (see [4, 7.3.5. Theorem 1]).

And here's the definition of a sheaf on a G -topological space X .

8.5. Definition (Sheaf). A *presheaf* \mathcal{F} of abelian groups or rings, etc., on X is a contravariant functor from the category of admissible opens \mathcal{S} (where the only morphisms are the inclusions) to the category of abelian groups or rings, etc. A presheaf \mathcal{F} is a *sheaf* if, for all admissible opens U of X and all admissible coverings $\mathfrak{U} = \{U_i\}_{i \in I}$ of U , the following diagram is exact

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{\sigma} \prod_{i \in I} \mathcal{F}(U_i) \begin{array}{c} \xrightarrow{\sigma'} \\ \xrightarrow{\sigma''} \end{array} \prod_{i, j \in I} \mathcal{F}(U_i \cap U_j), \quad (49)$$

where σ is induced by the restriction maps $\mathcal{F}(U) \rightarrow \mathcal{F}(U_i)$ and where σ' (respectively, σ'') is induced by the restriction maps $\mathcal{F}(U_i) \rightarrow \mathcal{F}(U_i \cap U_j)$ (respectively, $\mathcal{F}(U_j) \rightarrow \mathcal{F}(U_i \cap U_j)$). In other words, if two functions in $\mathcal{F}(U)$ agree on each open U_i of \mathfrak{U} then they are equal, and, if there is given, for each open U_i , a function in $\mathcal{F}(U_i)$, with the extra property that these functions agree on intersections of any two opens in \mathfrak{U} , then there exists (a unique) function in $\mathcal{F}(U)$ agreeing on each U_i with the given functions. In other words, the definition is the same as in the classical case, except for the fact that we only consider admissible coverings. The *stalk* of a (pre-)sheaf \mathcal{F} in a point $x \in X$ is the direct limit of all $\mathcal{F}(U)$, where U runs over all admissible opens containing x , and is denoted by \mathcal{F}_x .

The pair (X, \mathcal{O}_X) consisting of an affinoid variety X and its structure sheaf \mathcal{O}_X is a locally G -ringed space (see Definition 8.7 below) over K . These spaces are the local models of rigid analytic geometry. The global models, the rigid analytic varieties, are constructed from these local models by means of a glueing process. This can be more succinctly stated as follows.

8.6. Definition (Rigid Analytic Variety). An arbitrary locally G -ringed space (X, \mathcal{O}_X) over K is called a *rigid analytic variety (over K)*, if X admits an admissible covering \mathfrak{U} , such that each pair $(U, \mathcal{O}_X|_U)$, with $U \in \mathfrak{U}$, is an affinoid variety (or, more accurately, is isomorphic with one as locally G -ringed spaces). Such a covering will be called an *admissible affinoid covering*. A map $f: X \rightarrow Y$ of rigid analytic varieties is a morphism of locally G -ringed spaces over K .

In particular, affinoid varieties are rigid analytic varieties and a map of rigid analytic varieties between two affinoid varieties is a map of affinoid varieties as defined in Definition 7.1. Often, we will therefore suppress the clause 'of rigid analytic varieties' when referring to maps between them. It goes without saying that the above defined concepts constitute a category, the category of rigid analytic varieties.

8.7. Definition (G -Ringed Space). A pair (X, \mathcal{O}_X) consisting of a G -topological space X together with a sheaf \mathcal{O}_X of K -algebras on X , is called a *G -ringed space over K* . It is called a *locally G -ringed space (over K)*, if moreover, for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. A *morphism of G -ringed spaces over K* between two G -ringed spaces (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) is a pair (ψ, ψ^*) , where $\psi: X \rightarrow Y$ is a continuous map and where ψ^* is a collection of K -algebra homomorphisms

$$\psi_V^*: \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(\psi^{-1}(V)), \quad (50)$$

for each admissible open V of Y , such that this family is compatible with restriction homomorphisms induced by inclusions $W \subset V$. In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \xrightarrow{\psi_V^*} & \mathcal{O}_X(\psi^{-1}(V)) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y(W) & \xrightarrow{\psi_W^*} & \mathcal{O}_X(\psi^{-1}(W)) \end{array} \quad (51)$$

in which the vertical homomorphisms are induced by restricting the functions to the smaller subset.

If both spaces are moreover locally G -ringed spaces, then the morphism is called a *morphism of locally G -ringed spaces*, if each ψ_x^* is local (meaning that the maximal ideal of the former is mapped into the maximal ideal of the latter), for each $x \in X$, where $\psi_x^*: \mathcal{O}_{Y,\psi(x)} \rightarrow \mathcal{O}_{X,x}$ is the natural homomorphism between these two direct limits.

8.8 (Strong Grothendieck Topologies). Although I did not mention it, the Grothendieck topology on a rigid analytic variety X is required to satisfy also the following two conditions, for any admissible open U of X .

- 8.8.1. Let $V \subset U$. If there exists an admissible covering \mathfrak{U} of U , such that for each $W \in \mathfrak{U}$, the intersection $V \cap W$ is an admissible open of X , then V has to be already an admissible open.
- 8.8.2. Let \mathfrak{U} be a covering of U by admissible opens of X . If \mathfrak{U} has a refinement \mathfrak{V} which is admissible, then already \mathfrak{U} is admissible.

Unfortunately, the Grothendieck topology of an affinoid variety X , as defined before, does not satisfy these two requirements. But there exists a refinement of this Grothendieck topology, called the *strong* Grothendieck topology on X , which does satisfy these two conditions and remains quasi-compact (see Theorem 8.10 below). The structure sheaf \mathcal{O}_X can be extended to a sheaf of K -algebras, still denoted by \mathcal{O}_X for this new Grothendieck topology, making X now into a G -ringed space for which these two conditions do hold. See [4, 9.1.2. Proposition 2] for a general construction and [4, 9.1.4. Proposition 1 and 2] for the case of an affinoid variety. We will not be concerned too much with these two different Grothendieck topologies, although, as a rule, one should always take the strong one. The latter Grothendieck topology has obviously more admissible opens and coverings. For instance, the collection of all x in an affinoid variety $X = \mathrm{Sp} A$ for which $|f(x)| < 1$, with $f \in A$, is an admissible open in this Grothendieck topology (whereas it is neither a rational nor an affinoid subdomain). See [4, 9.1.4. Proposition 5] for some examples. Noteworthy here is also the following result concerning Zariski open sets. Recall that a *Zariski open subset* is by definition the complement of a closed analytic subset.

8.9. Theorem. *Any Zariski open subset of X is admissible and any (finite or infinite) covering by Zariski opens is admissible (in the strong topology).*

Proof. See [4, 9.1.4. Corollary 7]. □

Moreover, as already mentioned, affinoid varieties are still quasi-compact in this strong topology.

8.10. Theorem. *Any admissible affinoid covering (in the strong topology) of an affinoid variety X admits a finite subcovering.*

Proof. Apply the second statement of [4, 9.1.4. Proposition 2] to $V = X$ and φ the identity map. Note that in the terminology of [4], an *affinoid covering* is automatically finite. □

8.11. Example. Rigid analytic varieties can be glued together in order to obtain new ones. Details can be found in [4, 9.3.2. and 9.3.3], but see A.2 for a worked out example. We mention just some rigid analytic varieties that can be obtained in this way (see [4, 9.3.4] for more details).

- The affine n -space \mathbb{A}_K^n , which could be identified with the set of points in ordinary affine n -space K^n and which is not quasi-compact (hence in particular not affinoid).
- The projective n -space \mathbb{P}_K^n , which could be identified with the set of points in ordinary projective n -space and which is quasi-compact, but not affinoid.
- Any (algebraic) scheme of finite type over K carries the structure of a rigid analytic variety. By [27], this can even be extended to any scheme of finite type over an arbitrary affinoid algebra A . The functor which associates a rigid analytic variety with such a scheme, is called an *analytization*. I will elaborate further on this in the next section.

8.12. Definition (Quasi-Compact Variety). A rigid analytic variety X is called *quasi-compact*, if it admits a finite admissible affinoid covering. Therefore, the assertion in Theorem 8.10 also holds for quasi-compact rigid analytic varieties, whence the name. We call X *separated*, if the intersection of any two admissible affinoid varieties is again affinoid. Note that by the properties mentioned at the end of Definition 7.5, an affinoid variety is separated.

In [4, 9.6.1], a rigid analytic variety X is called separated, if the natural map $X \rightarrow \mathrm{Sp}K$ is separated, which means that the diagonal embedding $X \rightarrow X \times X$ is a closed immersion. More generally, a map $f: Y \rightarrow X$ of rigid analytic varieties is called separated, if the diagonal map $Y \rightarrow Y \times_X Y$ is a closed immersion. By [4, 9.6.1. Proposition 6], a separated rigid analytic variety has the property that the intersection of any two admissible affinoids is again affinoid, justifying our previous terminology.

Part II

Berkovich Analytic Spaces

In this part, I will give a quick primer to BERKOVICH's treatment of rigid analytic geometry. Given an affinoid variety X , he compactifies X by adding analytic points to it, thus obtaining the affinoid Berkovich space $\mathbb{M}(X)$. He puts a Hausdorff topology on $\mathbb{M}(X)$, so that the resulting analytic space is compact (Proposition 9.6). As a consequence, we get a new characterization for a covering of the original affinoid variety to be admissible (Proposition 9.17). I explain how to define a structure sheaf on $\mathbb{M}(X)$. The category of affinoid Berkovich spaces is then equivalent with the category of affinoid varieties (Proposition 9.9). I also briefly discuss how to obtain a Berkovich space $\mathbb{M}(X)$ associated to a separated quasi-compact rigid analytic variety X and then define an arbitrary Berkovich space as an open inside such a $\mathbb{M}(X)$. There then follows a section on Berkovich blowing ups and proper maps.

9 Berkovich Spaces

Although rigid analytic geometry provides a satisfactory analytic theory in the non-archimedean case, with rigid counterparts for all the fundamental theorems in the complex case, there is nevertheless a severe drawback in that topological arguments tend to fail in view of the absence of a Hausdorff topology. Indeed, as we have seen in the previous chapter, the analytic topology is only a Grothendieck topology, whereas the norm-topology is totally disconnected and hence even more unsuitable. To restore this, BERKOVICH developed his theory of K -analytic spaces. To an affinoid variety X he associates a compact Hausdorff space $\mathbb{M}(X)$, so that the *function theory* on X can be carried over to a function theory on $\mathbb{M}(X)$. More precisely, there is an equivalence between the category of coherent \mathcal{O}_X -modules and the category of coherent $\mathcal{O}_{\mathbb{M}(X)}$ -modules (Theorem 9.12). The following works will be used as sources of reference. Most of what we need on Berkovich spaces in this book can be found in GARDENER's paper [9] on the Voûte Etoilée, (a more extensive survey can be found in his D.Phil. thesis [8]). For an introduction to the general theory of Berkovich spaces, see BERKOVICH's own work [1, 2] or the more accessible renderings by SCHNEIDER [23] and SCHNEIDER and VAN DER PUT [24]. Below I will follow mostly SCHNEIDER's exposition, so that [23] will be the most quoted reference, although most results should be attributed to BERKOVICH.

Before I start with some more rigorous definitions, let me first motivate the definition of a Berkovich space. At least since the times of WEIL, algebraic geometers have felt the need to work over fields larger than just the base field. To be more precise, let there be given an algebraic zero-set $V \subset \mathbb{A}_k^n$, where k is some (most of the time algebraically closed) field and V is given by some equations $p_1 = \dots = p_s = 0$. Apart from the *rational* points of V , that is to say, those with coordinates over k , one should also pay attention to solutions of the system $p_1 = \dots = p_s = 0$ in extension fields l of k . Let $x = (x_1, \dots, x_n) \in \mathbb{A}_l^n$ be such a point. Then this induces a homomorphism $\varphi_x: A \rightarrow l$ given by sending the image of S_i in A to x_i , where $A = k[S]/(p_1, \dots, p_s)$ is the coordinate ring of V and where $S = (S_1, \dots, S_n)$. This does not only justify the association of the

k -algebra A to the set V , but also the custom of looking at the prime spectrum of A , since the kernel \mathfrak{p}_x of any such φ_x is a prime ideal of A (and any prime ideal arises in such way).

Perhaps more mysterious is the reason that it suffices to study just the prime spectrum; for all that matters, both π and e induce the zero ideal on $\mathbb{Q}[S]$, since they are transcendental over \mathbb{Q} , but why should we discard any differences between these two elements? Model theory provides an answer to this question. Indeed, the type of a tuple $x \in \mathbb{A}_k^n$ over k is entirely determined by the prime ideal \mathfrak{p}_x (this is a direct consequence of Algebraic Quantifier Elimination; see the end of 9.2 for more details). By the *type* of x over k , the model theorist means everything in the language of fields with parameters from the field k , that can be said about x (see for instance [15, §6.3]). The model theorist will also avoid the cumbersome keeping track of extension fields by taking a huge extension field Υ in which everything takes place; such a field is called a *universal domain* by geometers, or a *big model* by model theorists.

A big model Υ is a κ -saturated model (see [15, 10.1] for a definition), where κ is some large cardinal, exceeding the cardinality of all models one wants to study, the so-called *small models*. This exactly means that any type over such a small model is realized in Υ , meaning that there is a tuple x over Υ satisfying all the formulae in the type.

Let us mimic this procedure in our present situation. So we should take a 'big' ultrametric field Υ extending K and then look at all homomorphisms from the affinoid algebra $A = K\langle S \rangle / (p_1, \dots, p_s)$ into Υ . 'Big' should be taken here to mean the following.

9.1. Theorem. *Let K be an ultrametric field. There exists an ultrametric field extension $K \subset \Upsilon$ (that is to say, the restriction of the norm on Υ to K , coincides with the norm on K) with the following property. Each K -affinoid integral domain A together with a choice of some multiplicative norm on it, can be embedded, as a normed ring, in Υ .*

Moreover, if A and A' are two isomorphic K -affinoid integral domains embedded in Υ , then there is a K -isometry of Υ mapping A onto A' .

With a K -isometry (or, simply *isometry*), we mean a norm preserving isomorphism of K -algebras. Whenever we talk about embeddings in the context of normed rings, we always tacitly assume that the norm is also preserved.

Proof. To construct Υ , it suffices, by transfinite induction, to show that if L is an ultrametric field strictly extending K (recall that this terminology implicitly means that the norm on L extends the one on K) and if A is an affinoid integral domain with a multiplicative norm, then we can find an ultrametric field extension L_1 of L and a norm preserving embedding $A \hookrightarrow L_1$. Since L is transcendental over K , we can always choose a copy of L which is linearly disjoint from the field of fractions of A . Hence the complete tensor product $L \widehat{\otimes}_K A$ is a domain (see [22, p. 200]). Moreover, one verifies that the norm defined on this complete tensor product (see Definition 7.3) is again multiplicative. Therefore, $L \widehat{\otimes}_K A$ can be extended to an ultrametric field L_1 with the required properties.

To ensure also the final property, we take Υ uncountable and of cardinality strictly bigger than the cardinality of K . In order to verify the last assertion, we may then replace A and A' by their fraction fields and prove the following

more general statement. Let L and L' be subfields of Υ (with the induced norms) and let $\sigma: L \rightarrow L'$ be a K -isometry between them. I claim that for any $\alpha \in \Upsilon$, there exist $\alpha' \in \Upsilon$ and an isometry $\tau: L(\alpha) \rightarrow L'(\alpha')$ extending σ and sending α to α' . Given the claim, the assertion follows by transfinite induction and a back-and-forth argument, so that we only need to verify this claim.

If α is algebraic over L , with minimal polynomial $P(T)$, then we can take for α' any root of P^σ in Υ , where P^σ is the polynomial obtained by applying σ to its coefficients. The observation to make is that there is a unique way in which the norm of L extends to (the finite extension) $L(\alpha)$. Since

$$L(\alpha) \cong L[T]/(P) \cong L'[T]/(P^\sigma) \cong L'(\alpha') \quad (52)$$

there is a unique isomorphism τ extending σ and sending α to α' . Since the norms and the isomorphism τ are all uniquely determined, it follows that τ is an isometry.

So remains the case that α is transcendental over L . If L is not algebraically closed, then by the previous procedure we can first extend σ to the algebraic closure of L , and then adjoin α . In other words, we may assume that L is algebraically closed. In particular, the multiplicative subgroup $H(L)$ of \mathbb{R} consisting of all $|a|$, with a a non-zero element in L , is divisible. There are two cases to consider.

Case 1. Assume that $|\alpha|$ does not belong to $H(L)$. Since we took Υ of cardinality big enough, we can find $\alpha' \in \Upsilon$ which is transcendental over L' and such that $|\alpha| = |\alpha'|$. Let τ be the isomorphism between $L(\alpha)$ and $L'(\alpha')$ extending σ and sending α to α' . We need to check that this is an isometry. It suffices to check that if $\beta = P(\alpha)$, where $P = \sum_{i=0}^n a_i T^i$ is a polynomial in a single variable T with $a_i \in L$, then $|\beta| = |\tau(\beta)|$. Now, if for some $i < j$ with a_i and a_j both non-zero, we would have that $|a_i \alpha^i| = |a_j \alpha^j|$, then $|\alpha|$ is the $(j-i)$ -th root of $|a_i/a_j| \in H(L)$ and therefore also belongs to $H(L)$, contradiction (recall that the norm on an ultrametric field is always assumed to be multiplicative). Therefore, all non-zero terms in $\sum a_i \alpha^i$ have different norm, so that by the non-archimedean equality, we have that $|\beta| = |a_i \alpha^i|$, for some unique $i \in \{0, \dots, n\}$, and $|\beta| < |a_j \alpha^j|$ for all $j \neq i$. Since $|\alpha| = |\alpha'|$, these identities remain the same after applying τ . In particular,

$$|\tau(\beta)| = |\sigma(a_i)(\alpha')^i| = |a_i \alpha^i| = |\beta|, \quad (53)$$

as required.

Case 2. Assume next that $|\alpha| \in H(L)$. After dividing by some element of the same norm in L , we may even assume that $|\alpha| = 1$. As before, we can find $\alpha' \in \Upsilon$ transcendental over L' and of norm 1 (below, we will restrict the choice of α' even further). We define τ as before and we need to verify again that it is an isometry. It suffices to prove that $|\beta| = |\tau(\beta)|$, for $\beta = P(\alpha)$ with P as before. Moreover, we may assume that the maximum of $|a_i|$ equals 1, that is to say, that the Gauss norm of P is 1. Let us write $R_L, R_{L'}$ and R_Υ for the valuation rings of L, L' and Υ respectively, and $\bar{R}_L, \bar{R}_{L'}$ and \bar{R}_Υ for their residue fields (we will also write bars over elements to indicate their image in these residue fields). Since \bar{R}_L is also algebraically closed, we either have that $\bar{\alpha}$ lies in \bar{R}_L or is transcendental over it. We treat these two subcases differently.

Subcase 2a. Assume that $\bar{\alpha}$ is transcendental over \bar{R}_L . Choose α' in such way that its residue in \bar{R}_Υ is transcendental over $\bar{R}_{L'}$. Assume that $|\beta| < 1$. This means that $\bar{P}(\bar{\alpha}) = 0$. Since $\bar{\alpha}$ is transcendental over \bar{R}_L , we must have that $\bar{P} = 0$ and whence that P has Gauss norm strictly less than 1,

contradiction. Therefore, $|\beta| = 1$ and exactly the same argument then gives that $|\tau(\beta)| = 1$, since P^σ has also Gauss norm 1.

Subcase 2b. Finally, assume that $\bar{\alpha}$ belongs to \bar{R}_L . Therefore, we can find $b_1 \in R_L$, such that $\alpha - b_1$ has norm strictly less than 1. Let π_1 be in the valuation ideal of R_L such that $|\alpha - b_1| = |\pi_1|$. Let $\alpha_1 = (\alpha - b_1)/\pi_1$, so that it belongs to $L(\alpha)$ and its norm is 1. If $\bar{\alpha}_1$ does not belong to \bar{R}_L , then by Subcase 2a, we can construct an isometry extending σ between $L(\alpha_1)$ and $L'(\alpha'_1)$, for some $\alpha'_1 \in \Upsilon$. Since $L(\alpha) = L(\alpha_1)$, we are done also in this case. Therefore, we may assume that $\bar{\alpha}_1$ is again in \bar{R}_L and we can find $b_2, \pi_2 \in L$ with $|\pi_2| < 1$, so that $|\alpha_1 - b_2| = |\pi_2|$. Again, if $\alpha_2 = (\alpha_1 - b_2)/\pi_2$, then either $\bar{\alpha}_2$ is transcendental over \bar{R}_L , so that we are done by Subcase 2a, or it belongs to it, and we can continue the above process. If after infinitely many steps, the process has not stopped because all intermediate $\alpha_i = (\alpha_{i-1} - b_i)/\pi_i$ have residue inside \bar{R}_L , then we showed in fact that α can be written as a convergent series

$$\alpha = b_1 + \pi_1 b_2 + \pi_1 \pi_2 b_3 + \dots \quad (54)$$

and whence belongs to \hat{L} , the completion of L . However, there is a unique extension of the norm to \hat{L} and \hat{L}' and a unique isometry $\hat{\sigma}$ between these completions, as required. \square

9.2 (Universal Domains). We fix once and for all an ultrametric extension field Υ of K satisfying the statement of Theorem 9.1. Of particular interest to us therefore will be (K -algebra) homomorphisms $x: A \rightarrow \Upsilon$. Unfortunately, even if we require that x is continuous, the type of the tuple $x(S)$ over K is not entirely determined by the kernel \mathfrak{p}_x of x but also by the norm induced on the residue ring A/\mathfrak{p}_x . Another way of saying this is that there might be non-equivalent ways in which the norm of K can be extended to a multiplicative norm on A/\mathfrak{p}_x .

A note of caution on the 'big' ultrametric field Υ . There is no cardinal κ for which Υ is κ -saturated, since we took the convention that norms take their values in the reals, and this latter set is not sufficiently saturated. Formulated differently, the concept of a norm is not first order; there is no first order way of restricting the value group of a norm to be a subgroup of the reals.

As a result, the collection of all analytic points (to be defined below) does not yet constitute a *realization of the type space*. This is in contrast with the algebraic geometric case, where the prime spectrum realizes the type space in the following sense. Let \mathfrak{p} be a prime ideal in $k[S]$ with $S = (S_1, \dots, S_n)$. Let $p_{\mathfrak{p}}$ be the collection of all formulae $\varphi(S)$ with parameters from k which are a consequence (modulo the theory of fields) of the formulae $f(S) = 0$, for $f \in \mathfrak{p}$, and the formulae $g(S) \neq 0$, for $g \notin \mathfrak{p}$. Then $p_{\mathfrak{p}}$ is a complete n -type over k (of the tuple S viewed as an element in the field of fractions of $k[S]/\mathfrak{p}$). Moreover, the correspondence $\mathfrak{p} \mapsto p_{\mathfrak{p}}$ is a continuous bijection from $\text{Spec } k[S]$ to the Stone space of all n -types over k .

Although the analytic points as defined below will not realize the type space, they will nonetheless suffice for our purposes, but see [16] for a more general approach.

9.3 (Algebraic Quantifier Elimination). Let me briefly comment on Algebraic Quantifier Elimination. This states that every formula in the language of rings is equivalent with a quantifier free formula modulo the theory of algebraically closed fields (that is to say, both formulae define the same set over an arbitrary algebraically closed field). In fact, given an algebraically closed

field k and a formula φ with parameters from k (that is using polynomials over k), then we can find a quantifier free formula ψ , also with parameters from k , so that φ and ψ are equivalent modulo the theory of k . In other words, φ and ψ define the same set over any algebraically closed field l extending k . This last formulation can be rephrased in more geometric terms as follows. A quantifier free definable set in k^n , is just a constructible set in the Zariski sense (that is to say, a Boolean combination—see below—of zero-sets). A famous Theorem of CHEVALLEY, states that the image of a constructible set under a polynomial map is again constructible (see for instance [7, Corollary 14.7 and Exercise 14.7]). In particular, the projection of a constructible set is constructible, which means that every existential formula is equivalent with a quantifier free one (modulo the theory of k) and this suffices to conclude the previous Quantifier Elimination statement.

Let V be an arbitrary set. If W is a subset of V , then we will denote the *complement* of W in V , by $V - W$, or simply by $-W$ whenever the surrounding space is clear from the context. In other words, $V - W$ consists of all x in V not in W . Let \mathcal{V} be a collection of subsets of V . The *Boolean algebra* generated by \mathcal{V} is the collection of all finite unions of sets of the form $V_1 \cap \cdots \cap V_m \cap -V_{m+1} \cdots \cap -V_n$, with all $V_i \in \mathcal{V}$. In other words, all sets obtainable from \mathcal{V} by taking complements, finite unions and finite intersections. A member of the Boolean algebra generated by \mathcal{V} is called a *Boolean combination* of subsets in \mathcal{V} .

For the remainder of this section, let $X = \text{Sp } A$ be an affinoid variety.

9.4. Definition (Analytic Point). An *analytic point* x on X is a continuous K -algebra homomorphism $x: A \rightarrow \Upsilon$. Recall that Υ is a 'big' ultrametric field extending K , that is to say, an ultrametric field satisfying the statement of Theorem 9.1. Note that the kernel of an analytic point is a prime ideal \mathfrak{p} of A , but different analytic points might have the same prime ideal for kernel. An ordinary point of X corresponds to a maximal ideal of A and hence induces a surjective homomorphism $A \rightarrow K$. If we compose this with the inclusion $K \subset \Upsilon$ we get an analytic point. We will call the analytic point corresponding to an ordinary point a *geometric point*.

A rational (affinoid) subdomain $U = \text{Sp } C$ of X is called an *affinoid neighborhood* of an analytic point x , if the homomorphism $x: A \rightarrow \Upsilon$ extends to a homomorphism $C \rightarrow \Upsilon$, still denoted x (which is then necessarily unique). In other words we have a commutative diagram

$$\begin{array}{ccc}
 & A & \\
 j \swarrow & & \searrow x \\
 C & \xrightarrow{x} & \Upsilon
 \end{array} \tag{55}$$

where $j: A \rightarrow C$ is the canonical homomorphism corresponding to the inclusion $U \subset X$. Put differently, U is an affinoid neighborhood of x if, and only if, x is also an analytic point on U .

In [23], SCHNEIDER defines an analytic point more generally as a continuous K -algebra homomorphism $x: A \rightarrow F$, where F is some complete normed

field extending K (this also means that the norm $|\cdot|_F$ on F restricts to the norm on K). Of course, when $F = \Upsilon$, we recover our previous definition. Affinoid neighborhoods are defined similarly (replacing Υ by F). Two analytic points $x: A \rightarrow F$ and $x': A \rightarrow F'$ are called *congruent*, if they have the same system of affinoid neighborhoods. By [23, Corollary 10], this is equivalent with $|x(f)|_F = |x'(f)|_{F'}$, for all $f \in A$. To consolidate his definition with ours, observe the following. Suppose we have an analytic point $x: A \rightarrow F$ in his sense. Let F' be the completion of the subfield generated by $x(A)$. Then the restriction $x': A \rightarrow F'$ is an analytic point, which is congruent to x . By our assumption on Υ , we may embed $x(A)$ into Υ (preserving norms). As a consequence, also F' embeds in Υ , since Υ is complete. Let $x'': A \rightarrow \Upsilon$ be the resulting analytic point. It is again congruent to x and is now an analytic point in our sense as well. Moreover, by the already mentioned [23, Corollary 10], two analytic points x and x' in our sense, are congruent if, and only if, there is an isometry (see below) σ of Υ , such that $x = \sigma \circ x'$.

The following observation will be useful in the sequel.

9.5. Lemma. *Let $x: A \rightarrow \Upsilon$ be an analytic point. If f is an element of A of norm at most one, then $x(f)$ has also norm at most one.*

Proof. Suppose not, say $|x(f)| > |\pi|^{-1} > 1$ for some $\pi \in R$. In A , the sequence $\pi^n f^n$ converges to zero, so by continuity, the same should hold for its image under x in Υ . However, $x(\pi^n f^n) = \pi^n x(f)^n$, and this diverges in norm, contradiction. \square

As a set, the K -affinoid Berkovich space associated to X (or to A) is defined to be the set $\mathbb{M}(X)$ of all analytic points on X up to isometry. In other words, we identify two analytic points x and x' , if there is an isometry (=homeomorphism preserving norms) $\sigma: \Upsilon \rightarrow \Upsilon$ such that $x = \sigma x'$. We put the weakest topology on $\mathbb{M}(X)$ for which, for each $f \in A$, the map

$$\mathbb{M}(X) \rightarrow \mathbb{R}: x \mapsto |x(f)| \quad (56)$$

is continuous. In other words, a basis of open sets is given by the sets of the form

$$\{ x \in \mathbb{M}(X) \mid |x(p_i)| < r_i, |x(q_j)| > s_j, \text{ for } i < n \text{ and } j < m \} \quad (57)$$

where $p_i, q_j \in A$ and r_i and s_j are real numbers.

9.6. Proposition. *If X is an affinoid variety, then $\mathbb{M}(X)$ is a compact Hausdorff space. If we view X with its canonical topology, then the map $X \rightarrow \mathbb{M}(X)$ sending a (closed) point to the corresponding geometric point, is a homeomorphism of X onto an everywhere dense subset.*

Proof. For a definition of compactness, see Definition 10.2 below. For a proof, see [23, Lemma 11]. \square

Nonetheless, something strange goes on with the embedding $X \hookrightarrow \mathbb{M}(X)$. If U is a rational (affinoid) subdomain of X , that is to say, an admissible open of X , then $\mathbb{M}(U)$ is a *closed* subset of $\mathbb{M}(X)$. In fact,

$$\mathbb{M}(U) = \overline{U} \quad (58)$$

where \overline{U} denotes the closure of U in $\mathbb{M}(X)$, and, moreover, $X \cap \mathbb{M}(U) = U$. Let me just show that $\mathbb{M}(U)$ is closed. Suppose U consist of all $x \in X$ such that

$$|p_i(x)| \leq |p_0(x)|, \text{ for } i = 1, \dots, n \quad (59)$$

where the $p_i \in A$ generate the unit ideal. Then $\mathbb{M}(U)$ consist of all analytic points of X which satisfy the same inequalities (59). To be more precise, an analytic point $x: A \rightarrow \Upsilon$ belongs to $\mathbb{M}(U)$, if it extends to a continuous K -algebra homomorphism $x': C \rightarrow \Upsilon$, where C is the affinoid algebra of U . Since C is the homomorphic image of $A\langle S \rangle$ modulo the ideal generated by the $p_i - p_0 S_i$, for $i = 1, \dots, n$, this means that $x(p_i) = x'(S_i)x(p_0)$, for all i . Since S_i has norm at most one in C , so does $x'(S_i)$ in Υ by Lemma 9.5. Therefore, we have inequalities $|x(p_i)| \leq |x(p_0)|$. It is standard practice to write $x(p)$ as $p(x)$, to emphasize that x is a point and A operates as an algebra on $\mathbb{M}(X)$. With this convention, $x \in \mathbb{M}(U)$ if, and only if, inequalities (59) hold. Since these are weak inequalities, the set defined in this way is closed.

Indeed, if x_0 is an analytic point not satisfying the inequalities (59), then for some i , say $i = 1$, we have that $|p_0(x_0)| < |p_1(x_0)|$. Choose $r \in \mathbb{R}$ with $|p_0(x_0)| < r < |p_1(x_0)|$. Then the basic open of $\mathbb{M}(X)$ consisting of all x for which $|p_0(x)| < r < |p_1(x)|$ is disjoint from $\mathbb{M}(U)$ and contains x_0 , showing that the complement of $\mathbb{M}(U)$ is open.

9.7. Definition (Wide Affinoid Neighborhood). We call an affinoid neighborhood U of an analytic point x a *wide* affinoid neighborhood of x , if x lies in the (topological) interior of $\mathbb{M}(U)$.

If x is geometric, then any rational subdomain containing x is wide. Indeed, a rational (or affinoid) subdomain U is open in the canonical topology by Proposition 7.7. Therefore, we can find $\epsilon > 0$, such that the open ball $\mathbf{B}_X(x; \epsilon)$ is contained in U . Now, the open in $\mathbb{M}(X)$ consisting of all analytic points y , such that all $|y(S_i - x_i)| < \epsilon/2$ is therefore contained in $\mathbb{M}(U)$, where $x_i \in K$ are the coordinates of the geometric point x (in some embedding of X as a closed analytic subvariety of R^m), showing that x is an interior point of $\mathbb{M}(U)$.

It is perhaps more instructive to give an example of an affinoid neighborhood of an analytic point which is not wide. Let $X = R$ and $U = \mathbf{B}_R(0; |\pi|)$, with $\pi \in K$ a non-zero element of norm less than one. The affinoid algebra of U is $C = K\langle S, T \rangle / (S - \pi T)$. The supremum norm on C is a multiplicative norm, by Proposition 4.6 and the fact that $C \cong K\langle T \rangle$. Therefore, by Theorem 9.1, C with its supremum norm can be embedded in Υ . Let x be the thus defined analytic point $x: C \rightarrow \Upsilon$ (note that its kernel is the zero ideal). Since the supremum norm of $S \in C$ equals $|\pi|$, it follows that $|x(S)| = |\pi|$. Without showing in detail that x is not an interior point of $\mathbb{M}(U)$, let me just show that every basic open of the form $\mathbb{B} = \{y \in \mathbb{M}(X) \mid r < |y(S)| < s\}$ with $r < |\pi| < s$ (an annulus around x so to speak), contains at least one point not belonging to $\mathbb{M}(U)$. In fact, any analytic point y for which $|\pi| < |y(S)| < s$ does not belong to $\mathbb{M}(U)$. Indeed, since $S = \pi T$ in C , we get that

$$|y(T)| = |y(S)| / |\pi| > 1 \quad (60)$$

whereas $|T| = 1$ in C , contradicting Lemma 9.5.

This example also shows how to construct new analytic points. For instance, let us denote the analytic point x defined above by x_π , to emphasize that it comes from the supremum norm on $C = K\langle S, T \rangle / (S - \pi T)$. One can show that, if $|\pi| \neq |\pi'|$, then x_π and $x_{\pi'}$ are different analytic points. Nonetheless, the kernel of both analytic points is the zero ideal. In [1, Example 1.4.4] this technique is generalized slightly, to yield a complete list of all possible analytic points on R . See also Lemma 9.11 below, for the existence of sufficiently many 'generic points'.

Next we want to define a K -analytic structure on $\mathbb{M}(X)$. In order to do this, we need a structure sheaf $\mathcal{O}_{\mathbb{M}(X)}$. We will work a bit more general. Let \mathcal{F} be an arbitrary sheaf on X . We want to define a sheaf $\mathbb{M}(\mathcal{F})$ on $\mathbb{M}(X)$. So let \mathbb{U} be an open of $\mathbb{M}(X)$. Let $\Gamma(\mathbb{M}(\mathcal{F}), \mathbb{U})$ be the inverse limit of all $\Gamma(\mathcal{F}, U)$, where U is a finite union of rational subdomains, such that the closure of U in $\mathbb{M}(X)$ is contained in \mathbb{U} .

Strictly speaking \mathcal{F} is not defined on arbitrary finite unions of affinoid (or rational) subdomains, since these need not be affinoid subdomains anymore. In other words, we have to first give meaning to $\Gamma(\mathcal{F}, U) = \mathcal{F}(U)$, for U a finite union of affinoid subdomains U_i . This is done by requiring that diagram (49) of Chapter 1 be exact.

For $x \in \mathbb{M}(X)$, the *stalk* \mathfrak{F}_x of an arbitrary sheaf \mathfrak{F} is, as usual, defined as the direct limit of all $\mathfrak{F}(\mathbb{U})$, where \mathbb{U} runs over all opens containing x . In [23, Lemma 14], it is shown that $\mathbb{M}(\mathcal{F})_x$ is isomorphic with the direct limit of all $\mathcal{F}(U)$, where U runs over all wide affinoid neighborhoods of x .

9.8. Definition (Affinoid Berkovich Space). For structure sheaf on $\mathbb{M}(X)$, we then take $\mathbb{M}(\mathcal{O}_X)$, and we denote this sheaf by $\mathcal{O}_{\mathbb{M}(X)}$. In this way, $\mathbb{M}(X)$ becomes a locally ringed space. An easy calculation shows that the ring of global sections of the sheaf $\mathcal{O}_{\mathbb{M}(X)}$ is precisely the affinoid algebra A . In general, a locally ringed space $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ (see the commentary remarks in Definition 8.6 for a definition) is called an *affinoid Berkovich space*, if it is isomorphic as locally ringed space with a space $(\mathbb{M}(X), \mathcal{O}_{\mathbb{M}(X)})$, for some affinoid variety X . The ring of global sections of \mathbb{X} is then necessarily the affinoid algebra of X .

A map $\mathbb{Y} \rightarrow \mathbb{X}$ between affinoid Berkovich spaces is by definition a morphism of locally ringed spaces $(\mathbb{Y}, \mathcal{O}_{\mathbb{Y}}) \rightarrow (\mathbb{X}, \mathcal{O}_{\mathbb{X}})$. In this way, we get the category of affinoid Berkovich spaces.

9.9. Proposition. *There is an equivalence of categories between the category of affinoid varieties and the category of affinoid Berkovich spaces.*

Proof. Let \mathbb{X} be an affinoid Berkovich space. Let $A = \mathcal{O}_{\mathbb{X}}(\mathbb{X})$ be its ring of global sections. Recall that then $\mathbb{M}(X) \cong \mathbb{X}$, where X is the affinoid variety $\text{Sp } A$. Therefore, the functor $\mathbb{M}(\cdot)$ is bijective. We need to show that it is an equivalence of categories.

So let there be given a map $\mathbf{f}: \mathbb{Y} \rightarrow \mathbb{X}$ of affinoid Berkovich spaces. Taking global sections, we then get a homomorphism $A \rightarrow B$, where A and X are as above, and B is the affinoid algebra of the affinoid variety Y for which $\mathbb{M}(Y) = \mathbb{Y}$. In particular, we get a map of affinoid varieties $Y \rightarrow X$. Conversely, given a map of affinoid varieties $f: Y \rightarrow X$, we get a K -algebra homomorphism $\mathcal{O}_X(U) \rightarrow \mathcal{O}_Y(f^{-1}(U))$, for each rational subdomain U of X . Putting $U = X$, we get a homomorphism $\varphi: A \rightarrow B$. This induces a map $\mathbb{M}(Y) \rightarrow \mathbb{M}(X)$ given by $y \mapsto y \circ \varphi$,

for $y: B \rightarrow \Upsilon$ an analytic point. Moreover, taking inverse limits of the homomorphisms $\mathcal{O}_X(U) \rightarrow \mathcal{O}_Y(f^{-1}(U))$ yields a morphism of locally ringed spaces $(\mathbb{M}(Y), \mathcal{O}_{\mathbb{M}(Y)}) \rightarrow (\mathbb{M}(X), \mathcal{O}_{\mathbb{M}(X)})$. All these calculations show that both categories are equivalent. \square

Under this equivalence, local properties are well preserved, as the following Proposition indicates.

9.10. Proposition. *Let $X = \mathrm{Sp} A$ be an affinoid variety and $x: A \rightarrow \Upsilon$ an analytic point on X with kernel \mathfrak{p} . The natural local homomorphism $A_{\mathfrak{p}} \rightarrow \mathcal{O}_{\mathbb{M}(X),x}$ is faithfully flat. Each local ring of $\mathbb{M}(X)$ is Noetherian and Henselian. The structure sheaf $\mathcal{O}_{\mathbb{M}(X)}$ is coherent.*

Proof. See [2, Theorems 2.1.4 and 2.1.5]. In fact, by [1, §2.3], we have an isomorphism $\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{M}(X),x}$, for x a geometric point (where we write x for the point of X as well as for the geometric point in $\mathbb{M}(X)$ it determines). Using [4, 7.3.2], it follows that all local rings are Noetherian. For a definition of flat and faithfully flat homomorphisms. \square

In particular, if X is an affinoid manifold, then all local rings of $\mathbb{M}(X)$ are regular (so that we will call $\mathbb{M}(X)$ an *affinoid Berkovich manifold*). Indeed, in B.1 we said that X is a manifold, if A is regular, which means that all its localizations are regular. But a faithfully flat homomorphism preserves regularity, so that also all $\mathcal{O}_{\mathbb{M}(X),x}$ are regular. This is just one application of faithfully flatness for showing that X and $\mathbb{M}(X)$ have many local properties in common. For instance, one can also prove that a coherent \mathcal{O}_X -ideal \mathcal{I} is invertible if, and only if, $\mathbb{M}(\mathcal{I})$ is.

In fact, the morphism of locally ringed spaces $(\mathbb{M}(X), \mathcal{O}_{\mathbb{M}(X)}) \rightarrow (X, \mathcal{O}_X)$ sending an analytic point to its kernel, where $X = \mathrm{Spec} A$, induces an equivalence on the category of coherent modules by [1, p. 33].

Another advantage of affinoid Berkovich spaces over affinoid varieties, is the existence of *generic* points. For our purpose, these are points satisfying the conclusion of the following statement.

9.11. Lemma (Generic Point Lemma). *Let X be an affinoid variety and \mathbb{U} a non-empty open in $\mathbb{M}(X)$. Then there exists $x \in \mathbb{U}$, such that $\mathcal{O}_{\mathbb{M}(X),x}$ is Artinian (that is to say, has Krull dimension 0).*

Proof. We can always find a rational subdomain U of X , such that $\mathbb{M}(U) \subset \mathbb{U}$. Let C be the affinoid algebra of U . Let \mathfrak{p} be a minimal prime of C . Choose some multiplicative norm on the residue ring C/\mathfrak{p} (this is always possible by [1, Theorem 1.2.1]) and embed the resulting normed ring in Υ . This yields an analytic point x on U . In other words, $x \in \mathbb{M}(U) \subset \mathbb{U}$. From [1, Proposition 2.3.3], it follows that $\mathcal{O}_{\mathbb{U},x}$ has the same dimension as $C_{\mathfrak{p}}$, that is to say, $\mathcal{O}_{\mathbb{U},x}$ is Artinian. However, this finishes the proof, since $\mathcal{O}_{\mathbb{U}}$ is the restriction of $\mathcal{O}_{\mathbb{M}(X)}$ to \mathbb{U} , so that $\mathcal{O}_{\mathbb{U},x} \cong \mathcal{O}_{\mathbb{M}(X),x}$. \square

Let me now elaborate a bit more on how well the function theories on X and on $\mathbb{M}(X)$ correspond. Given a sheaf \mathfrak{F} on $\mathbb{M}(X)$, associate a sheaf $\tilde{\mathfrak{F}}$ on X as follows. For U a rational subdomain, we let $\Gamma(\tilde{\mathfrak{F}}, U)$ be the direct limit of all $\mathfrak{F}(\mathbb{U})$ where \mathbb{U} runs over all opens containing $\mathbb{M}(U)$. By [23, Lemma 16], we

always have an isomorphism between \mathfrak{F} and $\mathbb{M}(\tilde{\mathfrak{F}})$. Unfortunately, in most cases, we do not have an isomorphism between \mathcal{F} and $\widetilde{\mathbb{M}(\mathcal{F})}$, even not for $\mathcal{F} = \mathcal{O}_X$. However, this can easily be repaired as follows. Set

$$\mathfrak{F}^\circ = \tilde{\mathfrak{F}} \otimes_{\widetilde{\mathcal{O}_{\mathbb{M}(X)}}} \mathcal{O}_X. \quad (61)$$

I can now state the close correspondence between X and $\mathbb{M}(X)$ more precisely.

9.12. Theorem. *The functors $\mathfrak{F} \mapsto \mathfrak{F}^\circ$ and $\mathcal{F} \mapsto \mathbb{M}(\mathcal{F})$ induce an equivalence of categories of coherent sheaves. Moreover, under this equivalence, sheaf cohomology of coherent sheaves is preserved.*

Proof. The first statement reduces to showing that

$$\mathfrak{F} \cong \mathbb{M}(\mathfrak{F}^\circ) \quad \text{and} \quad \mathcal{F} \cong (\mathbb{M}(\mathcal{F}))^\circ,$$

for \mathfrak{F} and \mathcal{F} coherent sheaves on $\mathbb{M}(X)$ and X respectively. See [23, Lemma 16] and [1, p. 58] for details. The last statement is proved in [1, Theorem 3.3.4]. \square

9.13. Definition (Berkovich Space). I will not say much about the Berkovich space associated to an arbitrary rigid analytic variety. For the purposes of this book, we only need to associate a Berkovich space to a separated quasi-compact rigid analytic variety X . Recall that quasi-compactness means that X admits a finite affinoid covering $\{X_i\}$. We can glue together the affinoid Berkovich spaces $\mathbb{M}(X_i)$ along their common affinoid Berkovich spaces $\mathbb{M}(X_i \cap X_j)$ into a space denoted $\mathbb{M}(X)$. Note that since X is separated, the intersections $X_i \cap X_j$ are again affinoid. It follows from Proposition 9.6, that the space $\mathbb{M}(X)$ is compact and Hausdorff. Similarly, the structure sheaves of the $\mathbb{M}(X_i)$ glue to obtain a structure sheaf $\mathcal{O}_{\mathbb{M}(X)}$. The spaces so far obtained yield too small a category, as an arbitrary open in \mathbb{X} is not necessarily obtained from glueing finitely many affinoid Berkovich spaces. Therefore, we call a locally ringed space a *K-analytic Berkovich space* (for short, *Berkovich space*), if it is isomorphic (as a locally ringed space) with some $(\mathbb{U}, \mathcal{O}_{\mathbb{U}})$, where \mathbb{U} is an open in a space of the form $(\mathbb{M}(X), \mathcal{O}_{\mathbb{M}(X)})$ with X a separated, quasi-compact rigid analytic variety, and where $\mathcal{O}_{\mathbb{U}} = \mathcal{O}_{\mathbb{M}(X)}|_{\mathbb{U}}$.

In the literature larger, and perhaps more natural, classes of analytic spaces are introduced and still called *K-analytic*. However, since the introduction of these larger classes requires extra care, I decided to restrict the present exposition to a class sufficient for our purposes.

9.14. Definition (Berkovich Subdomain). Let \mathbb{X} be an arbitrary Berkovich space. We call a closed subset of the form $\mathbb{M}(V)$, with V an affinoid variety, a *rational* (respectively, *affinoid*) *Berkovich subdomain* of \mathbb{X} , if \mathbb{X} is an open subspace of some $\mathbb{M}(Y)$, with Y a separated, quasi-compact rigid analytic variety, such that V is a rational (respectively, affinoid) subdomain of Y , via the map $V \rightarrow Y$ induced by the map $\mathbb{M}(V) \hookrightarrow \mathbb{X} \hookrightarrow \mathbb{M}(Y)$.

More generally, we call a closed subset $\mathbb{M}(V)$, with V a separated, quasi-compact rigid analytic variety, a *Berkovich subdomain* of \mathbb{X} , if \mathbb{X} is an open subspace of some $\mathbb{M}(Y)$, with Y a separated, quasi-compact rigid analytic variety, such that V is an admissible open of Y .

In [1, Proposition 2.2.3], it is shown that an (affinoid) Berkovich domain satisfies a universal property similar to the one for affinoid subdomains (see the commentary remarks following Definition 7.5).

Although a (rational) Berkovich subdomain is a closed subset, it has many features of an open immersion (since it comes from an open immersion of rigid analytic varieties), as the following lemma shows.

9.15. Lemma. *Let $\mathbb{M}(V)$ be a Berkovich subdomain of \mathbb{X} and let $x \in \mathbb{M}(V)$. Then the local rings $\mathcal{O}_{\mathbb{M}(V),x}$ and $\mathcal{O}_{\mathbb{X},x}$ are isomorphic.*

Proof. In view of the local nature of the problem, we may assume without loss of generality, that $\mathbb{M}(V)$ is affinoid. By definition, \mathbb{X} is an open subspace of some $\mathbb{M}(Y)$, with Y a separated, quasi-compact rigid analytic variety, such that V is an affinoid subdomain of Y . In particular, since \mathbb{X} is an open subspace, $\mathcal{O}_{\mathbb{X},x} \cong \mathcal{O}_{\mathbb{M}(Y),x}$. We already quoted the result from [23, Lemma 14], that $\mathcal{O}_{\mathbb{M}(Y),x}$ is isomorphic to the direct limit of all $\mathcal{O}_Y(U)$, where $U \subset Y$ runs over all wide affinoid neighborhoods of x . Since $V \subset Y$ is an affinoid subdomain and since the intersection of a wide affinoid neighborhood with an affinoid subdomain is again wide (use that $\mathbb{M}(U \cap V) = \mathbb{M}(U) \cap \mathbb{M}(V)$; see [23, Lemma 12]), it follows that the local ring $\mathcal{O}_{\mathbb{M}(V),x}$ is the direct limit of all $\mathcal{O}_V(U)$, where $U \subset V$ runs over all wide affinoid neighborhoods of x in V . Therefore, since $\mathcal{O}_V = \mathcal{O}_Y|_V$, both rings are the same, as required. \square

All what we have said for affinoid Berkovich spaces generalizes with little effort to these more general Berkovich spaces. In particular, any map $f: Y \rightarrow X$ between separated quasi-compact rigid analytic varieties uniquely determines a map $\mathbb{M}(f): \mathbb{M}(Y) \rightarrow \mathbb{M}(X)$ between the corresponding Berkovich spaces.

9.16. Proposition. *Let $f: Y \rightarrow X$ be a map between separated quasi-compact rigid analytic varieties. Then f is flat if, and only if, $\mathbb{M}(f)$ is.*

Proof. The map f is called *flat* in a point $y \in Y$ if the homomorphism of local rings $\mathcal{O}_{X,f(y)} \rightarrow \mathcal{O}_{Y,y}$ is flat. Similarly, $\mathbb{M}(f)$ is flat in $y \in \mathbb{M}(Y)$, if $\mathcal{O}_{\mathbb{M}(X),\mathbb{M}(f)(y)} \rightarrow \mathcal{O}_{\mathbb{M}(Y),y}$ is flat. The result now follows easily from either [1, §2.3] and the fact that we only need to check flatness for geometric points, or from [4, 7.3.2] and Proposition 9.10. \square

The following rigid analytic consequence of the compactness of Berkovich spaces will be useful.

9.17. Proposition. *Let X be an affinoid variety. If $\{U_i\}_i$ is a collection of affinoid subdomains of X , such that, for each analytic point x on X , one of the U_i is a wide affinoid neighborhood of x , then the covering is admissible (in the Grothendieck topology on X) whence, by Theorem 8.10, already finitely many U_i cover X .*

Proof. See [2, Lemma 1.6.2]. \square

Beware that the above condition is stronger than just requiring $\{U_i\}_i$ to be a covering of X , for this would only mean that the geometric points are covered.

Convention 9.18. Sometimes I will drop the name *Berkovich* when referring to a Berkovich space. For instance, a closed analytic subset carrying the structure of a Berkovich space (defined by a coherent ideal), will simply be called a *closed*

analytic subspace. I reserve the designation *variety* for objects in the rigid analytic category and *space* for objects in the Berkovich category. Throughout the whole book the notational distinction between rigid analytic varieties and Berkovich spaces will be maintained, where the former are denoted by capital letters X, Y, Z, \dots and the latter by blackboard capital letters $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \dots$

10 Blowing Ups and Proper Maps

This section continues with expanding on the similarity between the category of rigid analytic varieties and Berkovich spaces. In particular, blowing ups are defined in a similar way and have similar properties (see Appendix A for more details on blowing ups in rigid analytic geometry). I also discuss the connection between the seemingly unrelated notions of properness in both categories.

10.1. Theorem. *Let \mathbb{X} be a Berkovich space and \mathbb{Z} a closed analytic subspace. Then the blowing up $\pi: \tilde{\mathbb{X}} \rightarrow \mathbb{X}$ of \mathbb{X} with centre \mathbb{Z} exists. This blowing up induces an isomorphism between $\tilde{\mathbb{X}} - \pi^{-1}(\mathbb{Z})$ and $\mathbb{X} - \mathbb{Z}$.*

Proof. To say that the blowing up of \mathbb{X} with centre \mathbb{Z} (given by a coherent ideal \mathfrak{J}) exists, means that there exists a map $\pi: \tilde{\mathbb{X}} \rightarrow \mathbb{X}$ of Berkovich spaces such that the following universal property holds. The inverse image $\mathfrak{J}\mathcal{O}_{\tilde{\mathbb{X}}}$ is invertible and, moreover, if $f: \mathbb{Y} \rightarrow \mathbb{X}$ is any other map of Berkovich spaces such that $\mathfrak{J}\mathcal{O}_{\mathbb{Y}}$ is invertible, then there exists a unique map $g: \mathbb{Y} \rightarrow \tilde{\mathbb{X}}$ such that f factors as $\pi \circ g$.

I claim that it suffices to show the theorem for \mathbb{X} affinoid, that is to say, $\mathbb{X} = \mathbb{M}(X)$, with $X = \text{Sp } A$ an affinoid variety. Indeed, for the general case, \mathbb{X} is some open in a Berkovich space of the form $\mathbb{M}(Y)$, with Y a separated, quasi-compact rigid analytic variety. If \mathbb{F} is a closed analytic subspace of $\mathbb{M}(Y)$, such that $\mathbb{F} \cap \mathbb{X} = \mathbb{Z}$, then, in view of the local nature of blowing ups, the blowing up of \mathbb{X} with centre \mathbb{Z} exists, if the blowing up of $\mathbb{M}(Y)$ with centre \mathbb{F} exists. More precisely, if $\theta: \tilde{\mathbb{Y}} \rightarrow \mathbb{M}(Y)$ is this latter blowing up, then the restriction of θ to $\theta^{-1}(\mathbb{X})$ is the required blowing up $\pi: \tilde{\mathbb{X}} \rightarrow \mathbb{X}$ (this follows from the Berkovich analog of the last statement in Proposition A.8). In particular, $\tilde{\mathbb{X}}$ is the open $\theta^{-1}(\mathbb{X})$ in $\tilde{\mathbb{Y}}$, whence is itself a Berkovich space. Finally, since Y is a finite union of affinoid varieties Y_i , we then reduce the existence of the blowing up θ to the existence of the blowing up of $\mathbb{M}(Y_i)$ with centre $\mathbb{F} \cap \mathbb{M}(Y_i)$, for each i . This follows from the Berkovich equivalent of [25, Proposition 1.4.4], which admits an almost identical proof.

So, we may assume that $\mathbb{X} = \mathbb{M}(X)$, with $X = \text{Sp } A$ an affinoid variety. If Z is the closed analytic subvariety of X defined by \mathfrak{J}° , then $\mathbb{M}(Z) = \mathbb{Z}$ (see Theorem 9.12). Let $\pi: \tilde{X} \rightarrow X$ be the blowing up of X with centre Z , which exists by Theorem A.4. I claim that $\mathbb{M}(\pi): \mathbb{M}(\tilde{X}) \rightarrow \mathbb{M}(X)$ is the blowing up of $\mathbb{X} = \mathbb{M}(X)$ with centre \mathbb{Z} . The claim follows easily from Theorem 9.12; see [9, Lemma 2.1] for details. The last assertion follows easily from its rigid analytic analogue Proposition A.5 \square

Before I discuss further properties of these blowing ups, I need to recall the topological notions of compactness and properness.

10.2. Definition (Compact Set). Let W be an arbitrary Hausdorff topological space. A subset V of W is called *compact*, if any open covering of V admits

a finite subcovering, that is to say, if the U_i , for i in some index set I , are opens of W such that V lies in their union, then V lies already in the union of finitely many of the U_i . We say that V is *relatively compact*, if its closure is compact. In other words, V is relatively compact if, and only if, it is contained in a compact set.

Any compact set is necessarily closed. Indeed, let V be compact subset of W and $x \in -V$. Since W is assumed to be Hausdorff, we can find for each $v \in V$ disjoint open sets V_v and W_v , such that $v \in V_v$ and $x \in W_v$. The V_v cover V , when v runs over all points of V , whence already finitely many V_{v_1}, \dots, V_{v_s} do so. Then $W_{v_1} \cap \dots \cap W_{v_s}$ is an open containing x and disjoint from V , proving that V is closed.

10.3. Proposition. *A Berkovich space is locally compact and Hausdorff.*

A Hausdorff topological space is called *locally compact*, if each point admits a system of compact neighborhoods (a *neighborhood* of a point x is a set containing an open U with $x \in U$).

Proof. By Proposition 9.6, any affinoid Berkovich space is compact. Since an arbitrary Berkovich space (according to our definition) is an open inside a finite union of affinoid Berkovich spaces, the statement follows. \square

We will often invoke compactness through the following property. Recall that a collection of subsets \mathcal{V} of a set W is said to have the *finite intersection property*, if every intersection of finitely many members of \mathcal{V} is non-empty.

10.4. Theorem (Intersection Property). *Let W be a Hausdorff topological space and V a closed subset of W . Then V is compact if, and only if, for every collection \mathcal{V} of closed subsets among which is V and having the finite intersection property, we have that the intersection of all members of \mathcal{V} is non-empty.*

Proof. Let \mathcal{V}' be the set of all $V \cap W$, where W runs over all members of \mathcal{V} . Clearly, also \mathcal{V}' has the finite intersection property and if the conclusion of the theorem holds for the collection \mathcal{V}' in the space V , then it also holds for collection \mathcal{V} in the space W . In other words, replacing W and \mathcal{V} by V and \mathcal{V}' respectively, we may assume that $V = W$.

Assume first that all collections \mathcal{V} of closed subsets of W with the finite intersection property have non-empty intersection. We need to prove that W is compact. Towards a contradiction, assume that there exists an open covering \mathcal{U} of W , which admits no finite subcovering. Let \mathcal{V} be the collection of all complements of members of \mathcal{U} . Since no finite number of opens in \mathcal{U} cover W , the collection \mathcal{V} has the finite intersection property. Therefore the intersection of all members of \mathcal{V} is non-empty, contradicting that their complements cover W .

Conversely, assume that W is compact and that \mathcal{V} is a collection of closed subsets with empty intersection. We need to show that already finitely many members of \mathcal{V} have empty intersection. Let \mathcal{U} consist of the complements of all members of \mathcal{V} . Our assumption implies that \mathcal{U} is an open covering of W . By compactness already finitely many cover W , proving that the intersection of their complements is empty. \square

10.5. Definition (Proper Map). A continuous map $f: V \rightarrow W$ of Hausdorff topological spaces is called *proper*, if the inverse image of an arbitrary compact set is again compact.

Normally, an arbitrary continuous map $f: V \rightarrow W$ of topological spaces is called proper if it satisfies the above condition and is moreover *separated*, meaning that the image of V via the diagonal embedding $\Delta: V \hookrightarrow V \times_W V$ is closed. However, if V is a Hausdorff space, then any continuous map with source V is separated. Indeed, take any pair (x, x') in $V \times_W V$ outside $\Delta(V)$. Since then $x \neq x'$ and using the Hausdorff assumption, we can find disjoint opens U and U' containing x and x' respectively. The open $(U \times U') \cap (V \times_W V)$ contains (x, x') and is disjoint from $\Delta(V)$, as required.

Since we will always deal with Hausdorff spaces, there is no need for including the separatedness condition.

Note that the notion of a proper map in rigid analytic geometry or in algebraic geometry is different from the above characterization. In the algebraic geometric case a map is called *proper*, if it is separated and universally closed (with respect to the Zariski topology). Recall that a map is called *closed*, if the image of a closed set is again closed, and *universally closed*, if any base change is closed. In the rigid analytic case a more complicated definition is used ([4, 9.6.2]) and it is then a non-trivial theorem due to KIEHL ([4, 9.6.3. Proposition 3]) that the image of a closed analytic subset under a proper map is again a closed analytic subset.

10.6. Definition (Rigid Analytic Proper Map). Let $Y = \mathrm{Sp} B \rightarrow X = \mathrm{Sp} A$ be a map of affinoid varieties and let U be a rational subdomain of Y . We say that U is *relatively compact* in Y with respect to X , if we can find $g_i \in B$ of supremum norm at most one, such that the A -algebra homomorphism $A\langle S_1, \dots, S_n \rangle \rightarrow B$ given by $S_i \mapsto g_i$ is surjective and such that the supremum norm of each g_i on U is strictly less than 1. We denote this by $U \Subset_X Y$. An arbitrary map $f: Y \rightarrow X$ of rigid analytic varieties is called *proper*, if it is separated (see Definition 8.12) and there exists an admissible affinoid covering $\{X_i\}$ of X , and, for each i , admissible affinoid coverings $\{U_{i,j}\}$ and $\{Y_{i,j}\}$ of $f^{-1}(X_i)$, such that $U_{i,j} \Subset_{X_i} Y_{i,j}$, for all i and j .

Some further links between these different notions of properness are provided by the following two results.

10.7. Theorem. *If $f: Y \rightarrow X$ is a proper map of separated quasi-compact rigid analytic varieties, then the corresponding Berkovich map $\mathbb{M}(f)$ is (topologically) proper.*

Proof. See [1, Proposition 3.3.2]. In fact, BERKOVICH has a slightly less general definition of proper map, for which the converse to the theorem also holds. \square

It follows that closed immersions, or more generally, finite maps in the Berkovich category are proper (as they are in the rigid analytic or the algebraic geometric case). Recall the definition of a separated map of rigid analytic varieties from Definition 8.12. In particular, if X is separated (and quasi-compact), then $\mathbb{M}(X)$ is Hausdorff, as we already observed in Proposition 10.3.

10.8. Theorem. *If $f: V \rightarrow W$ is a proper map of locally compact Hausdorff spaces, then it is closed.*

Proof. Let F be a closed subset of V . We need to show that $f(F)$ is closed. To this end, let x be a point in the closure of $f(F)$. Let \mathcal{K} be the collection of all compact neighborhoods of x . By local compactness and the

Hausdorff property, their intersection is the singleton $\{x\}$. Furthermore, since x lies in the closure of $f(F)$, every $K \in \mathcal{K}$ has a non-empty intersection with $f(F)$. Hence all the $f^{-1}(K) \cap F$ are non-empty and the finite intersection property holds for them. By properness, each $f^{-1}(K) \cap F$ is compact. Therefore, by Theorem 10.4, the intersection of all $f^{-1}(K) \cap F$ is non-empty. Pick some point y inside all the $f^{-1}(K) \cap F$. In particular, $f(y) \in K \cap f(F)$, for all $K \in \mathcal{K}$. Therefore, $f(y) = x$ so that $x \in f(F)$, as required. \square

Returning to our Berkovich blowing ups, the fact that they are obtained from their rigid analytic counterparts, implies that they share similar properties. For sake of reference, I state the following Berkovich analogue of Proposition A.6.

10.9. Proposition. *Let $\pi: \tilde{\mathbb{X}} \rightarrow \mathbb{X}$ be the blowing up of the Berkovich space \mathbb{X} with centre \mathbb{Z} . Then the following holds.*

- 10.9.1. *The map π is proper.*
- 10.9.2. *If \mathbb{Z} is nowhere dense, then π is surjective and $\pi^{-1}(\mathbb{Z})$ is also nowhere dense.*
- 10.9.3. *If \mathbb{X} is reduced, then so is $\tilde{\mathbb{X}}$. Similarly, if \mathbb{X} is irreducible, then so is $\tilde{\mathbb{X}}$ provided \mathbb{Z} as a set is strictly smaller than \mathbb{X} .*

Proof. The proof is basically an application of the equivalence of categories given in Theorem 9.12 and the analog properties listed in Proposition A.6. To prove 10.9.1, use Theorem 10.7. We call \mathbb{X} *reduced*, if all its local rings are reduced, and *irreducible*, if it is not the union of two proper closed analytic subspaces. For the reduced case of 10.9.3, use also Proposition 9.10. \square

Part III

Appendix

A Blowing Up

For the discussion on Embedded Resolution of Singularities in Appendix B below, we need to introduce first blowing up maps. This section is meant as a brief review of their elementary theory in the rigid analytic category. A detailed treatment of rigid analytic blowing up can be found in [25].

A.1. Definition (Invertible Ideal). Let $X = \mathrm{Sp} A$ be an affinoid variety. The purpose of blowing up is to make an arbitrary ideal principal. Let me be a bit more specific. Let \mathfrak{a} be an ideal of A . We say that \mathfrak{a} is *principal*, if it is generated by a single element and *locally principal*, if it is principal in each localization $A_{\mathfrak{p}}$, for \mathfrak{p} a prime in A . The ideal \mathfrak{a} is called *invertible*, if it is locally principal and, moreover, the generator in each localization is not a zero divisor. Another way of saying this is that \mathfrak{a} is locally principal with $\mathrm{Ann}_A(\mathfrak{a}) = 0$ (recall that $\mathrm{Ann}_A(\mathfrak{a})$ is the ideal of all $x \in A$ for which $x\mathfrak{a} = 0$).

More generally, let X be a rigid analytic variety and let \mathcal{I} be a coherent \mathcal{O}_X -ideal. We say that \mathcal{I} is *invertible*, if there exists an admissible affinoid covering $\{X_i = \mathrm{Sp} A_i\}_i$ of X , such that $\mathcal{I}A_i$ is generated by a single non-zero divisor. (By taking a small enough covering, we may indeed assume that the ideal is principal on each admissible affinoid).

Let X be a rigid analytic variety. A sheaf \mathcal{F} on X is called *coherent* (often, also called a *coherent \mathcal{O}_X -module*), if there exists an admissible affinoid covering $\{X_i = \mathrm{Sp} A_i\}$ and exact sequences

$$\mathcal{O}_{X_i}^m \rightarrow \mathcal{O}_{X_i}^n \rightarrow \mathcal{F}|_{X_i} \rightarrow 0. \quad (62)$$

Another way to express this is by claiming the existence of finite A_i -modules M_i , such that for each $f \in A_i$, we have compatible isomorphisms

$$\mathcal{F}(X_{i,f}) \cong (M_i)_f, \quad (63)$$

where $X_{i,f}$ is the admissible open in X_i given by $f \neq 0$. Compatible means that for any $f, g \in A_i$, these isomorphisms localize to the isomorphism determined by the element fg on $X_{i,fg} = X_{i,f} \cap X_{i,g}$. If, moreover, \mathcal{F} is a subsheaf of \mathcal{O}_X , then we call it a *coherent \mathcal{O}_X -ideal*. In particular, if $X = \mathrm{Sp} A$ is an affinoid variety then there is a one-one correspondence (in fact, an equivalence of categories) between the category of finite A -modules and the category of coherent \mathcal{O}_X -modules, on the one hand, and between the set of ideals and the set of coherent \mathcal{O}_X -ideals, on the other. Moreover, to any ideal \mathfrak{a} of A corresponds an affinoid algebra A/\mathfrak{a} , and whence an affinoid variety $Z = \mathrm{Sp} A/\mathfrak{a}$. There is a natural map $\iota: Z \rightarrow X$ which is in fact injective and any such map is called a *closed immersion*. It is easy to verify that the image of ι is a closed analytic subset of X , whence the terminology. Another frequently used terminology is to call Z a *closed analytic subvariety* of X ; here the intention is to 'forget' the map ι and view Z as a subspace of X . Again, we can extend this notion to an arbitrary rigid analytic variety X , and hence establish a one-one correspondence between the closed analytic subvarieties of X and the coherent \mathcal{O}_X -ideals.

A.2 Blowing Up of the Plane

The blowing up of A with respect to \mathfrak{a} will be a rigid analytic variety \tilde{X} with the property that \mathfrak{a} becomes an invertible ideal on \tilde{X} and \tilde{X} is in some sense the smallest rigid analytic variety for which this happens. Before I give the precise definition, here is an example of what a blowing up might look like. Consider $A = K\langle S_1, S_2 \rangle$ the free Tate algebra in two variables and let $\mathfrak{a} = (S_1, S_2)$ be the maximal ideal defining the origin. This ideal is clearly not principal. In order for it to be principal, we would need, for instance, that there exists some element t_1 (in an extension of A) such that $S_2 = t_1 S_1$. Moreover, we want to do this in such way that any other ring extension factors through this. Put differently, we want to construct a generic extension of A in which S_1 divides S_2 . Of course, we want to stay within the rigid analytic category and hence the natural candidate for such an extension is

$$\tilde{A}_1 = K\langle S_1, S_2, T_1 \rangle / (S_2 - T_1 S_1). \quad (64)$$

Clearly \tilde{A}_1 is isomorphic with $K\langle S_1, T_1 \rangle$ and the extension $A \subset \tilde{A}_1$ is given by the homomorphism $S_1 \mapsto S_1$ and $S_2 \mapsto S_1 T_1$. Under this homomorphism \mathfrak{a} has indeed become invertible, namely, $\mathfrak{a}\tilde{A}_1 = S_1\tilde{A}_1$. However, we could instead have opted to make S_2 our single generator of \mathfrak{a} , by finding a t_2 so that $S_1 = t_2 S_2$. In other words, the homomorphism

$$K\langle S_1, S_2 \rangle \rightarrow \tilde{A}_2 = K\langle S_2, T_2 \rangle: S_1 \mapsto S_2 T_2, \quad S_2 \mapsto S_2 \quad (65)$$

is a second solution to our problem. Apparently this is not an extension of the first example. So somehow we have to combine these two examples to obtain the 'right' blowing up. In other words, we will glue together two copies of R^2 , one, $\tilde{X}_1 = \text{Sp } \tilde{A}_1$, with coordinates (S_1, T_1) , the other, $\tilde{X}_2 = \text{Sp } \tilde{A}_2$ with coordinates (S_2, T_2) , as follows. We have from our above discussion two maps $\pi_i: \tilde{X}_i \rightarrow R^2$, for $i = 1, 2$, given by $(s_1, t_1) \mapsto (s_1, s_1 t_1)$ and $(s_2, t_2) \mapsto (s_2 t_2, s_2)$. Consider the rational subdomain

$$\tilde{Y}_i = \left\{ (s_i, t_i) \in \tilde{X}_i \mid |t_i| = 1 \right\} \quad (66)$$

of \tilde{X}_i , for $i = 1, 2$. The map θ given by

$$(s_1, t_1) \mapsto (s_1 t_1, 1/t_1) \quad (67)$$

is an isomorphism between $\tilde{Y}_1 \rightarrow \tilde{Y}_2$, compatible with the maps π_i to R^2 . In other words, we have a commutative diagram

$$\begin{array}{ccc} & \tilde{Y}_1 & \\ \theta \swarrow & & \searrow \pi_1 \\ \tilde{Y}_2 & \xrightarrow{\pi_2} & R^2 \end{array} \quad (68)$$

This enables us to glue \tilde{X}_1 and \tilde{X}_2 along the isomorphic rational subdomains \tilde{Y}_1 and \tilde{Y}_2 , to obtain a rigid analytic variety \tilde{X} (containing both \tilde{X}_i as an admissible open). This glueing process also provides us with a map $\pi: \tilde{X} \rightarrow X$, agreeing with π_i on \tilde{X}_i . This map will be the *blowing up at the origin*.

One can describe the above rigid analytic variety \tilde{X} as the closed analytic subvariety of $R^2 \times \mathbb{P}^1$ given by $s_1\xi_2 = s_2\xi_1$, where $(s_1, s_2; \xi_1 : \xi_2)$ are the coordinates on $R^2 \times \mathbb{P}^1$ and where \tilde{X}_1 corresponds to the subset given by $|\xi_2| \leq |\xi_1|$, by letting $t_1 = \xi_2/\xi_1$, and similar for the second chart. Note that π is simply the canonical projection $R^2 \times \mathbb{P}^1 \rightarrow R^2$ restricted to \tilde{X} . Consequently, π is an isomorphism above each point of R^2 except above the origin. In fact, the fibre at the origin is isomorphic to projective 1-space. This is typical for a blowing up map, as we will see shortly. This last form allows us to define more generally the blowing up of R^n with respect to the maximal ideal corresponding to the origin as follows. It is the closed analytic subvariety \tilde{X} of $R^n \times \mathbb{P}^{n-1}$, given by $s_i\xi_j = s_j\xi_i$, for all $i, j = 1, \dots, n$, where $(s_1, \dots, s_n; \xi_1 : \dots : \xi_n)$ are the coordinates on $R^n \times \mathbb{P}^{n-1}$. The associated blowing up map $\pi: \tilde{X} \rightarrow R^n$ is just projection on the first n -coordinates.

Now that I have constructed an example, let me give the precise definition.

A.3. Definition (Blowing Up). Let X be a rigid analytic variety and let \mathcal{I} be a coherent \mathcal{O}_X -ideal. Such an ideal uniquely defines and is defined by a closed analytic subvariety Z of X . A map $\pi: \tilde{X} \rightarrow X$ of rigid analytic varieties is called the *blowing up of X with centre Z* , or, *with respect to \mathcal{I}* , if $\mathcal{I}\mathcal{O}_{\tilde{X}}$ is invertible and π is universal with respect to this property. With the latter condition we mean that, given any map $f: Y \rightarrow X$ of rigid analytic varieties, such that $\mathcal{I}\mathcal{O}_Y$ is invertible, then there exists a unique map $g: Y \rightarrow \tilde{X}$ which factors through f , that is to say, such that the following diagram is commutative

$$\begin{array}{ccc}
 & Y & \\
 g \swarrow & & \searrow f \\
 \tilde{X} & \xrightarrow{\pi} & X
 \end{array} \tag{69}$$

Since a blowing up map is defined by a universal property, it is unique, if it exists.

Given an arbitrary map $f: Y \rightarrow X$ of rigid analytic varieties and an \mathcal{O}_X -ideal \mathcal{I} , we define the *inverse image* of \mathcal{I} on Y , denoted by $\mathcal{I}\mathcal{O}_Y$ as the image of the canonical morphism of sheaves $f^*\mathcal{I} \rightarrow \mathcal{O}_Y$. In other words, if $X = \text{Sp } A$ and $Y = \text{Sp } B$ are affinoid and \mathcal{I} is given by the ideal \mathfrak{a} of A , then $\mathcal{I}\mathcal{O}_Y$ is given by $\mathfrak{a}B$. Moreover, if Z is the closed analytic subvariety defined by \mathcal{I} , then by $f^{-1}(Z)$ we will mean the closed analytic subvariety of Y given by $\mathcal{I}\mathcal{O}_Y$. In particular, $f^{-1}(Z)$ is isomorphic with $Y \times_X Z$ (compare with Proposition 7.6).

Let us verify that the universal property is satisfied in the above example of the blowing up of R^2 with centre the origin. Therefore, let $f: Y \rightarrow R^2$ be a map of rigid analytic varieties so that (S_1, S_2) becomes an invertible ideal on Y . Since this is a local question, we may assume that $Y = \text{Sp } B$ is affine. Let $b_i \in B$ be the image of S_i under the homomorphism $K\langle S_1, S_2 \rangle \rightarrow B$ corresponding to f . Hence by assumption the ideal generated by b_1 and b_2 is invertible. Replacing Y by one of its rational subdomains, we may already assume that the latter ideal is generated by a single non-zero divisor and without loss of generality we may assume that this is b_1 . Therefore, there is some $c \in B$, such that $b_2 = cb_1$. Moreover, c is unique, since b_1 is a non-zero

divisor. Define a homomorphism $K\langle S_1, T_1 \rangle \rightarrow B$ by sending S_1 to b_1 and T_1 to c . This gives a map $Y \rightarrow \tilde{X}_1$ which composed with π gives the original f . Hence let g just be this map followed by the open immersion $\tilde{X}_1 \hookrightarrow \tilde{X}$. This is indeed the only possible map due to the fact that c is uniquely defined. This shows that $\pi: \tilde{X} \rightarrow R^2$ is the blowing up of R^2 with centre the origin.

Exactly the same proof shows that the blowing up of R^n with centre the origin, constructed above, satisfies the defining universal property of blowing up maps. This is the point of departure for proving that a general blowing up always exists.

A.4. Theorem. *Let X be a rigid analytic variety and let Z be a closed analytic subvariety, then the blowing up $\pi: \tilde{X} \rightarrow X$ with centre Z exists.*

Sketch of proof. We already observed that the blowing up of R^n with centre the origin exists. One then shows that the blowing up of R^{n+m} with centre R^m exists, as the base change of the former blowing up with $\{0\} \times R^m$. Next, we treat the affinoid case $X = \text{Sp } A$. We can embed X in some R^{n+m} in such way that $Z = X \cap (\{0\} \times R^m)$ in this embedding. So one needs only to show that given a blowing up π as in the statement and a closed analytic subvariety F of X , then the *strict transform* of F under π exists (as a closed analytic subvariety of \tilde{X}) and equals the blowing up of F with centre $F \cap Z$. See A.7 below for the definition of strict transform and Proposition A.9 for the construction of the strict transform of F . This finishes the affinoid case and the general case now follows by glueing together; for a detailed proof, see [25, Theorem 2.2.2]. \square

A.5. Proposition. *Let $\pi: \tilde{X} \rightarrow X$ be the blowing up of the rigid analytic variety X with centre Z . Then away from the centre Z , the map π is an isomorphism. In other words, π induces an isomorphism between $\tilde{X} - \pi^{-1}(Z)$ and $X - Z$.*

On the other hand, $\pi^{-1}(Z)$ is a closed analytic subvariety of $Z \times \mathbb{P}^n$, for some n , and the restriction of π to $\pi^{-1}(Z)$ is just the projection onto Z .

Proof. We already observed this for the blowing up of the plane at the origin. For the general case, see [25, Corollary 1.4.5].

For the last statement, this is clear for the blowing up π of R^n with centre the origin from its description at the end of A.2, since then $\pi^{-1}(0) \cong \mathbb{P}^{n-1}$. On the other hand, if π is the blowing up of R^{n+m} with centre $\{0\} \times R^m$, then after taking base change, we get that

$$\pi^{-1}(\{0\} \times R^m) \cong \mathbb{P}^{n-1} \times R^m. \quad (70)$$

The general case is given as some strict transform of this last blowing up (see the proof of Theorem A.4), so that the assertion holds in general by Proposition A.9. \square

The following properties of a blowing up in the rigid analytic category are just the analogues of their algebraic-geometric counterparts—see for instance [13, Chapter II, Section 7].

A.6. Proposition. *Let $\pi: \tilde{X} \rightarrow X$ be the blowing up of the rigid analytic variety X with centre Z . Then the following holds.*

A.6.1. *The map π is proper.*

A.6.2. If Z is nowhere dense (in the Zariski topology), then π is surjective and $\pi^{-1}(Z)$ is also nowhere dense.

A.6.3. If X is reduced, then so is \tilde{X} . Similarly, if X is irreducible, then so is \tilde{X} provided Z as a set is strictly smaller than X .

A.6.4. If both X and Z are manifolds, then so is \tilde{X} .

Proof. For Properties A.6.1 and A.6.2, see [25, Theorem 3.2.1 and Corollaries 3.2.2. and 3.2.3]. For Property A.6.3, the irreducible case is proved in [25, Corollary 3.2.3] and the reduced case in [28, Corollary 5.6]. The last property is proved in [27, Corollary 2.2.3]. \square

A subset Z in a topological space X is called *dense*, if any non-empty open subset of X has non-empty intersection with Z . A point $z \in Z$ is called an *interior point*, if there exists an open U containing z and contained in Z . A subset Z is called *nowhere dense*, if its closure has no interior points. An affinoid variety $X = \text{Sp } A$ is called *reduced*, if A is reduced, that is to say, has no non-trivial nilpotent elements; it is called a *manifold*, if A is a regular domain (for a definition of a regular ring, see Definition B.1 below). A rigid analytic variety X is called reduced or a manifold if it admits an admissible affinoid covering each of its members is reduced or a manifold respectively.

The least obvious of the properties listed in Proposition A.6 is perhaps the first, since properness has a different definition in the rigid analytic category; see 10.5 below.

Let me just show that our example of the blowing up of the plane with centre the origin is proper. This follows immediately from its description as a closed analytic subvariety in $R^2 \times \mathbb{P}^1$ followed by the projection on the first two coordinates. The latter projection is proper, since it is the base change of the proper map $\mathbb{P}^1 \rightarrow \{0\}$ (any map with source a projective space is proper). Therefore its composition with a closed immersion is still proper, as desired.

A.7. Definition (Strict Transform). Let $\pi: \tilde{X} \rightarrow X$ be a blowing up map with centre Z and let $f: Y \rightarrow X$ be an arbitrary map of rigid analytic varieties. The *strict transform* of f under π is defined as follows. Let $\theta: \tilde{Y} \rightarrow Y$ be the blowing up of Y with centre $f^{-1}(Z)$. In other words, if \mathcal{I} is the coherent \mathcal{O}_X -ideal defining Z , then $\mathcal{I}\mathcal{O}_Y$ defines $f^{-1}(Z)$. Hence, by definition of blowing up, the coherent ideal $\mathcal{I}\mathcal{O}_{\tilde{Y}}$ is invertible and therefore there must be a unique map $\tilde{f}: \tilde{Y} \rightarrow \tilde{X}$ making the following diagram commute

$$\begin{array}{ccc}
 \tilde{Y} & \xrightarrow{\theta} & Y \\
 \tilde{f} \downarrow & & \downarrow f \\
 \tilde{X} & \xrightarrow{\pi} & X
 \end{array} \tag{71}$$

The map \tilde{f} is called the *strict transform* of f under π and Diagram (71) is referred to as the *diagram of strict transform*.

A.8. Proposition. Let $\pi: \tilde{X} \rightarrow X$ be the blowing up of X with centre Z and let $f: Y \rightarrow X$ be an arbitrary map of rigid analytic varieties. Let

$$\begin{array}{ccc}
 \tilde{Y} & \xrightarrow{\theta} & Y \\
 \tilde{f} \downarrow & & \downarrow f \\
 \tilde{X} & \xrightarrow{\pi} & X
 \end{array} \tag{72}$$

be the diagram of the strict transform of f under π . Then \tilde{Y} is a closed analytic subvariety of the fibre product $\tilde{X} \times_X Y$. If f is a closed, locally closed or open immersion, then so is \tilde{f} . In fact, if f is an open immersion, then $\tilde{Y} = \pi^{-1}(Y)$ and θ is just the restriction of π .

Proof. See [25, Proposition 3.1.1]. For the discussion of fibre products, see 7.3. For future reference, I formulate the case that f is a closed immersion in a separate proposition and give a more detailed proof. \square

A.9. Proposition. Let $\pi: \tilde{X} \rightarrow X$ be the blowing up of X with centre Z and let $Y \subset X$ be a closed analytic subvariety. Let \mathcal{I} be the coherent \mathcal{O}_X -ideal defining Z and let \mathcal{J} be the coherent \mathcal{O}_X -ideal defining Y . Then the strict transform \tilde{Y} of Y under π is the closed analytic subvariety of \tilde{X} given by the coherent $\mathcal{O}_{\tilde{X}}$ -ideal

$$\mathcal{H} = \sum_{m=1}^{\infty} (\mathcal{J}\mathcal{O}_{\tilde{X}} : \mathcal{I}^m \mathcal{O}_{\tilde{X}}). \tag{73}$$

Proof. Recall, that the colon ideal $(\mathfrak{a} : \mathfrak{b})$ of two ideals $\mathfrak{a}, \mathfrak{b}$ in a ring B , is the ideal of all $x \in B$ for which $x\mathfrak{b} \subset \mathfrak{a}$.

As observed in A.7, the coherent ideal defining $\pi^{-1}(Y)$ in \tilde{X} is $\mathcal{J}\mathcal{O}_{\tilde{X}}$. Let us put $T = \pi^{-1}(Y)$, so that $\mathcal{O}_T = \mathcal{O}_{\tilde{X}}/\mathcal{J}\mathcal{O}_{\tilde{X}}$. Therefore, $\mathcal{H}\mathcal{O}_T$ is the same as

$$\sum_{m=1}^{\infty} \text{Ann}_{\mathcal{O}_T}(\mathcal{I}^m \mathcal{O}_T). \tag{74}$$

In [25, Proposition 2.2.1] it is shown that the latter ideal is indeed coherent whence so is \mathcal{H} . Let \tilde{Y} be the closed analytic variety defined by \mathcal{H} (which is then also a closed analytic variety of $T = \pi^{-1}(Y)$). I claim that

$$\text{Ann}_{\mathcal{O}_{\tilde{Y}}}(\mathcal{I}\mathcal{O}_{\tilde{Y}}) = 0. \tag{75}$$

Indeed, we need to check this only on stalks. So let \tilde{x} be a point in \tilde{Y} and let $p \in \mathcal{O}_{\tilde{Y}, \tilde{x}}$ such that

$$p\mathcal{I}\mathcal{O}_{\tilde{Y}, \tilde{x}} = 0. \tag{76}$$

Since $\mathcal{O}_{\tilde{Y}, \tilde{x}} = \mathcal{O}_{\tilde{X}, \tilde{x}}/\mathcal{H}\mathcal{O}_{\tilde{X}, \tilde{x}}$, this means that there is some $m \geq 1$, such that

$$p\mathcal{I}\mathcal{O}_{\tilde{X}, \tilde{x}} \subset (\mathcal{J}\mathcal{O}_{\tilde{X}, \tilde{x}} : \mathcal{I}^m \mathcal{O}_{\tilde{X}, \tilde{x}}). \tag{77}$$

But then

$$p\mathcal{I}\mathcal{O}_{\tilde{X},\tilde{x}}^{m+1} \subset \mathcal{J}\mathcal{O}_{\tilde{X},\tilde{x}} \quad (78)$$

so that in fact p belongs to $\mathcal{H}\mathcal{O}_{\tilde{X},\tilde{x}}$ whence is zero in $\mathcal{O}_{\tilde{Y},\tilde{x}}$, proving the claim.

Since $\mathcal{I}\mathcal{O}_{\tilde{X}}$ is by construction invertible, its image under the surjection $\mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{Y}}$ is locally principal. Together with (75), it follows that $\mathcal{I}\mathcal{O}_{\tilde{Y}}$ is invertible as well. Note that the coherent \mathcal{O}_X -ideal defining $Y \cap Z$ is $\mathcal{I} + \mathcal{J}$. In other words, $\mathcal{I}\mathcal{O}_Y$ is the coherent \mathcal{O}_Y -ideal defining $Y \cap Z$, viewed as a closed analytic subvariety of Y , since $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$. To conclude that \tilde{Y} is indeed the blowing up of Y with respect to $Y \cap Z$, we therefore only need to show that the universal property holds for the restriction $\pi': \tilde{Y} \rightarrow Y$. This is explained in more detail in [25, Proposition 2.2.1]; I will repeat here just the main points. Let $g: W \rightarrow Y$ be an arbitrary map of rigid analytic varieties such that $\mathcal{I}\mathcal{O}_W$ is invertible. By the universal property of blowing up applied to the blowing up π and the composition $W \xrightarrow{g} Y \hookrightarrow X$, we can find a map $h: W \rightarrow \tilde{X}$ making the following diagram commute

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ h \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\pi} & X \end{array} \quad (79)$$

From the commutativity of this diagram, it follows that $h(W) \subset T = \pi^{-1}(Y)$, as sets. Moreover, h will factor over \tilde{Y} (that is to say, h induces a map $h': W \rightarrow \tilde{Y}$), provided $\mathcal{H}\mathcal{O}_W = 0$ (reason on stalks or take an admissible affinoid covering and work with the affinoid algebras). The latter condition is easily seen to be fulfilled, since $\mathcal{I}\mathcal{O}_W$ is invertible and since $\mathcal{H}\mathcal{O}_T$ is given by (74). The uniqueness of h' follows easily from the uniqueness of h , so that we have indeed verified the universal property for π' . \square

A.10. Definition (Local Blowing Up). A map $\pi: \tilde{X} \rightarrow X$ is called a *local blowing up*, if it is the composition of a blowing up $\tilde{X} \rightarrow U$ (with a certain centre Z in U) followed by the open immersion $U \hookrightarrow X$, where U is an admissible affinoid U in X . Hence Z is then a *locally closed* analytic subvariety of X and we will often say that π is the local blowing up with centre Z , without reference to U , (in spite of the fact that different choices for U give clearly rise to different local blowing up maps; this is justified in part by the last assertion of Proposition A.8).

The reader should check that the strict transform of a map under a local blowing up map is equally well-defined and, moreover, the same holds true for any composition of local blowing up maps. Compositions of local blowing ups will play a crucial role in the next Chapter, when I define the Voûte Etoilée.

B Embedded Resolution of Singularities

The aim of this section is to present a rigid analytic analogue of HIRONAKA's famous Embedded Resolution of Singularities. Since this latter theorem is presently only known to hold for zero characteristic, we will assume in this section that the characteristic of K is zero. But once somebody is able to prove Embedded Resolution of Singularities for positive characteristic as well, the techniques described here will enable us to extend the rigid version to general characteristic. (In fact, Embedded Resolution of Singularities for surfaces in positive characteristic is known, due to a result of ABHYANKAR, and accordingly we can obtain this also in the rigid case. It should also be mentioned that DE JONG's result on alterations is apparently too weak to derive the Uniformization Theorem.) The details of what I present below can be found in [27].

B.1. Definition (Normal Crossing). Let us call $X = \mathrm{Sp} A$ an *affinoid manifold*, if A is a regular ring. Alternatively, we may call X *smooth*. A non-zero element p of A will be said to have *normal crossings*, if in each point we can find a local coordinate system such that p is (locally) a unit times a monomial in that coordinate system. Note that p determines a closed analytic subvariety H of X (namely, $\mathrm{Sp} A/pA$), of codimension one. Such a closed analytic subvariety of codimension one, will be called a *hypersurface*.

Likewise, we call a rigid analytic variety X a manifold, if it admits an admissible covering by affinoid manifolds. We say that a hypersurface H of X is said to have normal crossings, if it has so locally. Note that since X is assumed to be a manifold, any hypersurface of X is locally given by a single equation p , so that the above definition makes sense.

A Noetherian local ring A with maximal ideal \mathfrak{m} , is called *regular*, if \mathfrak{m} can be generated by d elements, where d is the Krull dimension of A . Such a set $x = (x_1, \dots, x_d)$ of generators is called a *regular system of parameters* or *local coordinate system*. A non-zero element $p \in A$ is said to have *normal crossings*, if there exists a regular system of parameters x , such that $p = ux^\nu$, for some multi-index $\nu \in \mathbb{N}^d$ and some unit u of A .

An arbitrary Noetherian ring A is called *regular*, if all its localizations at maximal ideals are. If this is the case, then each localization at an arbitrary prime ideal is a regular local ring ([22, Theorem 19.3]). Let A be a regular ring. A non-zero element p of A is said to have *normal crossings at a prime ideal* \mathfrak{p} , if its image in the local ring $A_{\mathfrak{p}}$ has. It is said to have *normal crossings*, if it has so at each maximal ideal of A , and if this holds, then it has normal crossings at any prime ideal. The locus of maximal ideals \mathfrak{m} , such that p has normal crossings at \mathfrak{m} is Zariski open (see for instance [3]). Applying the above definitions to an affinoid algebra, gives the corresponding notions of regularity and normal crossings for an affinoid variety. One might object that it is rather artificial to define local concepts such as these for an affinoid variety $X = \mathrm{Sp} A$ by defining them through the localizations $A_{\mathfrak{m}}$ rather than through the local rings $\mathcal{O}_{X,x}$. However, since the maps $A \rightarrow C$ are flat, for $\mathrm{Sp} C$ a rational subdomain of X , we obtain that the natural map of local rings $A_{\mathfrak{m}} \rightarrow \mathcal{O}_{X,x}$ (where \mathfrak{m} is the maximal ideal corresponding to $x \in X$), is faithfully flat (see [4, 7.3.2. Proposition 3 and Corollary 6]). Therefore, by faithfully flat descent (see for instance [22, §23]), many local properties can equally well be defined on either local ring.

Here is the main theorem of this section.

B.2. Theorem (Embedded Resolution of Singularities). *Let $X = \text{Sp } A$ be an affinoid manifold. Assume that the characteristic of K is zero. Let p be a non-zero element of A and H the hypersurface determined by it. Then there exist an admissible covering $\{X_i\}_i$ of X , and, for each i , a rigid analytic manifold \tilde{X}_i and a map $h_i: \tilde{X}_i \rightarrow X_i$ of rigid analytic varieties, such that*

B.2.1. *h_i is a composition of finitely many blowing up maps with smooth centres of codimension at least two,*

B.2.2. *$h_i^{-1}(H \cap X_i)$ has normal crossings in \tilde{X}_i .*

For the notions of dimension and codimension.

This is definitely an important result which has a highly non-trivial proof since it is based on Theorem B.3 below. As the latter is formulated for schemes (which will appear sporadically on other occasions throughout the text as well) I reserve a special script ($\mathbf{X}, \mathbf{Y}, \dots$) to distinguish them from rigid analytic varieties.

B.3. Theorem (Hironaka's Embedded Resolution of Singularities). *Let A be an excellent regular local ring which contains a field of characteristic zero, X a regular integral scheme of finite type over $\text{Spec } A$ and H a hypersurface of X . Then there exist a regular integral scheme \tilde{X} of finite type over $\text{Spec } A$ and a map $h: \tilde{X} \rightarrow X$, such that*

B.3.1. *h is a composition of finitely many blowing up maps with smooth centres of codimension at least two,*

B.3.2. *$h^{-1}(H)$ has normal crossings.*

Proof. See [14, p.146 Corollary 3 and p.161 Remark]. □

B.4. Remark. For a definition of an excellent ring see for instance [22, p. 260] or, for a more detailed treatment, including proofs, see [21, §34] or [12, Chap. IV, §7.8]. Complete local rings are excellent. Any localization and any homomorphic image of an excellent ring is again excellent, and so is any finitely generated algebra over an excellent ring.

An affinoid algebra A is excellent. By the above, it suffices to prove this for A a Tate algebra $K\langle S \rangle$. In characteristic zero this follows from an application of the Jacobi Criterion, see [21, Theorem 102] and in positive characteristic (assuming that K is algebraically closed) from a theorem of KUNZ, see [21, Theorem 108]. In fact, KIEHL proved excellence of affinoids in [17] without assuming K to be algebraically closed. The local rings $\mathcal{O}_{X,x}$ of an affinoid variety X are also excellent; see [5, Theorem 1.1.3].

Sketch of proof of Theorem B.2. I will not give all the details of the proof of the rigid analytic Embedded Resolution of Singularities, but merely provide a sketch. For a full proof see [27].

We call a scheme \mathbf{X} *analytic*, if it is of finite type over an affinoid algebra. In [27] or [19], it is shown that there exists a functor \mathfrak{F} from the full subcategory of analytic schemes to the category of rigid analytic varieties, called the *analytization* functor. If \mathbf{X} is an analytic scheme, then there is a one-one correspondence between the closed points of \mathbf{X} and the points of $\mathfrak{F}(\mathbf{X})$. Moreover, \mathbf{X} is regular or reduced if, and only if, $\mathfrak{F}(\mathbf{X})$ is. If \mathbf{H} is a hypersurface in \mathbf{X} , then $\mathfrak{F}(\mathbf{H})$ is a hypersurface in $\mathfrak{F}(\mathbf{X})$ and the former has normal crossings if, and only if, the latter has. One particular preservation property of \mathfrak{F} is that it commutes with blowing up maps.

Here is the full definition of an *analytization*. Let \mathbf{X} be a scheme over K and X a rigid analytic variety. We call X an analytization of \mathbf{X} , if there exists a morphism of locally G -ringed spaces

$$(\eta, \eta^*): (X, \mathcal{O}_X) \rightarrow (\mathbf{X}, \mathcal{O}_{\mathbf{X}}) \quad (80)$$

such that, given any rigid analytic variety Y , and any morphism

$$(\theta, \theta^*): (Y, \mathcal{O}_Y) \rightarrow (\mathbf{X}, \mathcal{O}_{\mathbf{X}}), \quad (81)$$

of locally G -ringed spaces over K , there exists a unique map of rigid analytic varieties $f: Y \rightarrow X$ making following diagram commute

$$\begin{array}{ccc}
 & Y & \\
 f \swarrow & & \searrow \theta \\
 X & \xrightarrow{\eta} & \mathbf{X}
 \end{array} \quad (82)$$

The Grothendieck topology on \mathbf{X} is nothing else than the Zariski topology, whereas we have to take the strong topology on X , in order to make η continuous. Note that, since an analytization is defined by a universal problem, we have that, if an analytization exists, it must be unique (up to a unique isomorphism). Hence if \mathbf{X} is analytic, then $\mathfrak{F}(\mathbf{X}) = X$.

Now, let there be given a regular affinoid algebra A and a non-zero element p in it. Let H denote the hypersurface defined by p . Take a point $x \in X = \mathrm{Sp} A$ and let \mathfrak{m} be the corresponding maximal ideal of A . Consider the localization $A_{\mathfrak{m}}$. Note that $\mathbf{X} = \mathrm{Spec}(A_{\mathfrak{m}})$ is not an analytic scheme (in general), since $A_{\mathfrak{m}}$ is not finitely generated over A . We can apply HIRONAKA's Embedded Resolution of Singularities to \mathbf{X} , in order to obtain a regular integral scheme $\tilde{\mathbf{X}}$ of finite type over \mathbf{X} and a map $\mathfrak{h}: \tilde{\mathbf{X}} \rightarrow \mathbf{X}$, such that

B.4.1. \mathfrak{h} is a composition of finitely many blowing up maps with smooth centres of codimension at least two,

B.4.2. $\mathfrak{h}^{-1}(H)$ has normal crossings,

where H denotes the hypersurface given by $p = 0$ in \mathbf{X} . We can take a small (Zariski) neighborhood Y of x in \mathbf{X} , such that Y is an analytic scheme. Indeed, take any $s \notin \mathfrak{m}$, then $Y = \mathrm{Spec}(A_s)$ will be such a neighborhood. We can also extend all the blowing up maps occurring in \mathfrak{h} , in order to obtain a composition of finitely many blowing up maps $\mathfrak{g}: \tilde{Y} \rightarrow Y$. Since both the regular locus and the locus of normal crossings are open, we can take Y small enough, such that Conditions B.4.1 and B.4.2 still hold. Let $g: \tilde{Y} \rightarrow Y$ be the analytization of \mathfrak{g} , then Y and \tilde{Y} are rigid analytic manifolds and Conditions B.4.1 and B.4.2 hold for g and H .

Note that Y is an admissible open (in the strong topology) of X containing x by Theorem 8.9. Moreover, this theorem also tells us that the collection of all these Y , for all points x , is an admissible covering of X . Hence we have found the desired blowing up, at least locally in the strong topology on X . To conclude, we choose, for each point x , an admissible covering by rational subdomains

of the corresponding Y . Note that the restriction of g to each such rational subdomain satisfies again Conditions B.4.1 and B.4.2. By Condition 8.1.4 in Definition 8.1, we get an admissible covering of X by putting together all these rational subdomains for all points x . Since X is quasi-compact by Theorem 8.10, already finitely many of these rational subdomains cover X , so that we get the desired finite covering. \square

The reader might wonder why these theorems are called resolutions of singularities. Let me comment a little on this. Assume that H is moreover irreducible. If we were to blow up only $H \cap X_i$, though with the same centres as used in h_i , the resulting space is a closed analytic subvariety of \tilde{X}_i , called the *strict transform* of H under h_i (see Definition A.7). The normal crossings condition implies that this strict transform is regular. Namely, its equation is given locally by a divisor of the equation of the inverse image of H . By assumption, the equation of the inverse image of H is a unit times a monomial. But the strict transform must also be irreducible and reduced (since it is obtained by blowing up the irreducible and reduced space H ; use Proposition A.6). Therefore, the equation of the strict transform is given by a monomial of degree one, that is to say, a regular equation.

By using this desingularization result for the codimension one case, one can give a proof of a general desingularization theorem, see [27] for more details.

Let us have a closer look at the maps that occur in the statement of the Embedded Resolution of Singularities. First we have an *open immersion* (that is, the inclusion map of the admissible open X_i in X), followed by h_i , which is a composition of finitely many blowing up maps. Therefore, we will be dealing in the sequel with maps $h: \tilde{X} \rightarrow X$ of rigid analytic varieties which are compositions of (finitely many) local blowing up maps. Such a map h is called *affinoid*, if \tilde{X} is affinoid.

By Proposition A.6, the blowing up of an admissible open in an affinoid manifold with centre of codimension at least two (whence nowhere dense), is surjective. In other words, the h_i in Theorem B.2 are surjective (onto X_i) and the collection of all of them constitutes a surjective family onto X . This suggests to make the following definition.

B.5. Definition (Blowing Up Tree). A *blowing up tree* (for short, *bu-tree*) on X is a finite collection $e = \{h_1, \dots, h_s\}$, with each h_i a composition of finitely many local blowing up maps, which forms a surjective family onto X . If all centres involved in a bu-tree e are smooth and nowhere dense, we call e a *smooth* bu-tree. If, on the other hand, all $h \in e$ are affinoid (that is to say, the source space of h is affinoid), then we call the bu-tree e *affinoid*.

Here is the precise definition in the more general situation of a *quasi-compact* rigid analytic variety X (recall from Definition 8.12 that this means that X admits a finite admissible affinoid covering). A finite collection e of compositions of (finitely many) local blowing up maps is a bu-tree, if it can be obtained by successive applications of the following two rules.

- B.5.1. The singleton consisting of the identity map 1_X is a bu-tree on X .
- B.5.2. Suppose $e = \{h_1, \dots, h_s\}$ is a bu-tree. Hence each $h_i: X_i \rightarrow X$ is a composition of blowing up maps with nowhere dense centre. Let \mathcal{U}_i be a finite admissible affinoid covering of X_i (which exists by quasi-compactness) and, for each $V \in \mathcal{U}_i$, let $\pi'_V: \tilde{V} \rightarrow V$ be a blowing up

map with nowhere dense centre and let π_V denote the composite map $\tilde{V} \rightarrow X_i$, so that π_V is a local blowing up. Then the collection of all local blowing up maps $\{\pi_V \circ h_i\}$ for all $i = 1, \dots, s$ and $V \in \mathfrak{U}_i$, is a bu-tree on X .

Note that an open immersion $U \subset X$ can be viewed as the local blowing up with empty centre. In particular, a finite affinoid covering is a bu-tree and therefore any bu-tree can easily be turned into an affinoid one.

We can now rephrase Embedded Resolution of Singularities as follows. Given an affinoid manifold X and a hypersurface H in it, then there exists a smooth bu-tree e on X , such that for each $h \in e$, the inverse image $h^{-1}(H)$ has normal crossings. We can even take e to be affinoid, by the above remarks.

The next theorem shows how Embedded Resolution of Singularities can be used to obtain nice division properties.

B.6. Theorem. *Assume that the characteristic of K is zero. Let $X = \mathrm{Sp} A$ be an affinoid manifold and let $p, q \in A$. Then there exists a smooth affinoid bu-tree e on X , such that for each map $h: Y \rightarrow X$ belonging to e , either $p \circ h$ divides $q \circ h$, or vice versa, $q \circ h$ divides $p \circ h$ in $\mathcal{O}(Y)$.*

Proof. Here is how it works. There is not much to prove if either p or q is zero or if both are equal. So we may assume that $pq(p-q)$ is non-zero. Let H be the hypersurface defined by $pq(p-q)$. By Embedded Resolution of Singularities, we can find a smooth affinoid bu-tree e on X , such that for each $h \in e$, the inverse image $h^{-1}(H)$ has normal crossings. Since we are proving things modulo such bu-trees, we may already assume that $pq(p-q)$ has normal crossings in X . Take a point $x \in X$ and let \mathfrak{m} be the corresponding maximal ideal of A . By assumption, there exists a regular system of parameters $\xi = (\xi_1, \dots, \xi_d)$ of $A_{\mathfrak{m}}$, where d is the dimension of X , such that $pq(p-q)$ is a unit times a monomial in $A_{\mathfrak{m}}$. Hence the same holds true for its three factors p , q and $p-q$. Therefore, we get an equation in $A_{\mathfrak{m}}$ of the form

$$p - q = u\xi^\alpha - v\xi^\beta = w\xi^\gamma, \quad (83)$$

where u, v, w are units in the local ring $A_{\mathfrak{m}}$. Since a regular local ring is a unique factorization domain, both sides of Equation (83) must have the same irreducible factors. But all the ξ_i are irreducible, so ξ^γ must divide both ξ^α and ξ^β . Therefore Equation (83) reduces to

$$u\xi^{\alpha-\gamma} - v\xi^{\beta-\gamma} = w, \quad (84)$$

with $\alpha - \gamma$ and $\beta - \gamma$ in \mathbb{N}^d . For the left hand side to be a unit, at least one of α or β must be equal to γ , implying that either p divides q or vice versa.

So we proved the statement of the theorem, at least locally at each point. However, in view of the local nature of the properties involved, we can find, for each point x with maximal ideal \mathfrak{m} , a Zariski open U_x containing x , such that the relative division of p and q holding in $A_{\mathfrak{m}}$ remains valid in $\mathcal{O}(U_x)$. We then finish in the same way as in the proof of Theorem B.2. Namely, for each point x , we choose an admissible covering of U_x by rational subdomains. Note that the relative division still holds on each of these affinoid subdomains. By Theorem 8.9 and Condition 8.1.4 in Definition 8.1, we get an admissible covering of X by putting together all these rational subdomains for all points x . Since X is quasi-compact by Theorem 8.10, already finitely many of these rational subdomains cover X , so that we get the desired finite covering. \square

C Non Algebraically Closed Fields

In this section, I briefly discuss how one can develop rigid analytic geometry if one drops the requirement on K to be algebraically closed.

Convention C.1. In the remainder of this section K is an arbitrary non-archimedean complete normed field. We keep denoting its valuation ring by R and the valuation ideal by \wp . The residue field is still denoted by \bar{R} ; it is in general no longer algebraically closed. More generally, if L is any normed field extending K (and its norm), then we denote the valuation ring of L by R_L , its valuation ideal by \wp_L and its residue field R_L/\wp_L by \bar{R}_L . In other words, $R \subset R_L$ is a local homomorphism and $\bar{R} \subset \bar{R}_L$ an extension of fields.

We fix once and for all an algebraic closure K^{alg} of K . Note that K^{alg} carries a canonically defined norm, but in general it is no longer complete with respect to this norm (see Remark 2.6).

If K is algebraically closed, then the maximal spectrum of $K\langle S \rangle$ with $S = (S_1, \dots, S_m)$, can be identified with R^m by Corollary 5.10. This is no longer true for arbitrary K . We still have a Weak Nullstellensatz (Theorem 5.8), but its formulation should be adjusted thus: *any maximal ideal \mathfrak{m} of $K\langle S \rangle$ has residue field a finite extension L of K .* In particular, we may choose L inside K^{alg} . Writing x_i for the image of S_i in $K\langle S \rangle/\mathfrak{m} \hookrightarrow L$, we get an m -tuple $x = (x_1, \dots, x_m) \in R_L^m$, such that

$$\mathfrak{m} = (S_1 - x_1, \dots, S_m - x_m)L\langle S \rangle \cap K\langle S \rangle. \quad (85)$$

In other words, $p \in K\langle S \rangle$ belongs to \mathfrak{m} if, and only if, $p(x) = 0$. However, if σ is an element of the absolute Galois group $G(K)$ of K (consisting of all K -algebra automorphisms of K^{alg}), then the tuple $\sigma(x) = (\sigma(x_1), \dots, \sigma(x_m))$ also determines \mathfrak{m} as in (85). If L_1 is a finite Galois extension of K containing L , then $\sigma(x)$ belongs again to L_1 . In particular, this shows that there are only finitely many possibilities for the point $\sigma(x)$. In other words, the orbits of the action of the absolute Galois group on $R_{K^{\text{alg}}}^m$ are all finite. In conclusion, we have a one-one correspondence between $\text{Sp } K\langle S \rangle$ and the orbits of $G(K)$ on $R_{K^{\text{alg}}}^m$. Noteworthy is also that any $\sigma \in G(K)$ preserves the norm, whence in particular is an automorphism of $R_{K^{\text{alg}}}$. All this easily extends to arbitrary affinoid varieties, so that we have shown basically the following result.

C.2. Proposition. *Let A be a K -affinoid algebra and let \mathfrak{m} be a maximal ideal of A . If we write A as a homomorphic image of $K\langle S \rangle$ modulo an ideal (f_1, \dots, f_s) , with $S = (S_1, \dots, S_m)$, then there exists $x \in R_{K^{\text{alg}}}^m$ such that all $f_i(x) = 0$, and such that \mathfrak{m} consists of all $p \in A$, such that $p(x) = 0$.*

If x' is another m -tuple characterizing \mathfrak{m} in the same way, then x' is a conjugate of x under the action of the absolute Galois group $G(K)$ on $R_{K^{\text{alg}}}^m$. Moreover, there are only finitely many possibilities for a tuple x' characterizing \mathfrak{m} .

Let us denote the *unit ball* $\text{Sp } K\langle T \rangle$, for T a single variable, by \mathbf{B} . We denote the *origin* simply by 0 ; it is the point corresponding to the maximal ideal (T) . As far as the development of rigid analytic geometry over K is concerned, the one adjustment to make is to replace everywhere the affinoid variety R^m by $\mathbf{B}^m = \text{Sp } K\langle S_1, \dots, S_m \rangle$. The main observation to make is that everything in Part 1 is, or at least, can be, expressed in terms of maximal ideals rather than in terms of R -tuples. A similar assertion holds for the theory of Berkovich spaces.

References

- [1] V. Berkovich, *Spectral theory and analytic geometry over non-archimedean fields*, Mathematical Surveys and Monographs, vol. 33, Amer. Math. Soc., 1990.
- [2] ———, *Étale cohomology for non-archimedean analytic spaces*, Inst. Hautes Études Sci. Publ. Math. **78** (1993), 5–161.
- [3] E. Bierstone and P.D. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math. **67** (1988), 5–42.
- [4] S. Bosch, U. Güntzer, and R. Remmert, *Non-archimedean analysis*, Springer-Verlag, Berlin, 1984.
- [5] B. Conrad, *Irreducible components of rigid spaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), 473–541.
- [6] P. Deligne, *Cohomologie étale*, Lect. Notes in Math., vol. 269, Springer Verlag, 1977.
- [7] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.
- [8] T.S. Gardener, *Logical aspects of rigid analytic geometry and of classical groups*, Ph.D. thesis, University of Oxford, 1995.
- [9] ———, *Local flattening in rigid analytic geometry*, Proc. London Math. Soc. **80** (2000), 179–197.
- [10] T.S. Gardener and H. Schoutens, *Flattening and subanalytic sets in rigid analytic geometry*, Proc. London Math. Soc. **83** (2001), 681–707.
- [11] ———, *Rigid subanalytic sets*, Valuation Theory and its Applications (F.-V. Kuhlmann, S. Kuhlmann, and M. Marshall, eds.), Fields Institute Communications Series, Amer. Math. Soc., to appear.
- [12] A. Grothendieck and J. Dieudonné, *Elements de géométrie algébrique I-IV*, Inst. Hautes Études Sci. Publ. Math., vol. 4, 8, 11, 17, 20, 24, 28, Presses universitaires de France, Paris, 1960-1965.
- [13] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977.
- [14] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. **79** (1964), 109–326.
- [15] W. Hodges, *Model theory*, Cambridge University Press, Cambridge, 1993.
- [16] R. Huber, *On semianalytic subsets of rigid analytic varieties*, Geom. Dedicata **58** (1995), 291–311.
- [17] R. Kiehl, *Ausgezeichnete Ringe in der nichtarchimedischen analytischen Geometrie*, Journal für Mathematik **234** (1969), 89–98.
- [18] N. Koblitz, *p-adic numbers, p-adic analysis and zeta-functions*, Springer-Verlag, New York, 1977.
- [19] U. Köpf, *Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen*, 2, vol. 7, Schriftenreihe Math. Inst. Univ. Münster, 1974.
- [20] L. Lipshitz and Z. Robinson, *Rigid subanalytic subsets of the line and the plane*, Amer. J. Math. **118** (1996), 493–527.
- [21] H. Matsumura, *Commutative algebra*, W.A. Benjamin, 1970.
- [22] ———, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.
- [23] P. Schneider, *Points of rigid analytic varieties*, J. Reine Angew. Math. **434** (1993), 127–157.

- [24] P. Schneider and M. van der Put, *Points and topologies in rigid geometry*, Math. Ann. **302** (1995), 81–104.
- [25] H. Schoutens, *Blowing up in rigid analytic geometry*, Bull. Soc. Math. Belg. **2** (1994), 401–419.
- [26] ———, *Rigid subanalytic sets in the plane*, J. Algebra **170** (1994), no. 1, 266–276.
- [27] ———, *Embedded resolution of singularities in rigid analytic geometry*, Ann. Fac. Sci. Toulouse **VIII** (1999), 297–330.
- [28] ———, *Rigid analytic flatifiers*, Quart. J. Math. Oxford Ser. **50** (1999), 321–353.
- [29] ———, *Rigid subanalytic sets*, will appear in Astérisque, 1999.

Index

- $(\mathfrak{a} : \mathfrak{b})$ (colon ideal), 48
- $\mathfrak{a} \cap A$ (contraction of \mathfrak{a} to A), 15
- $A \widehat{\otimes}_C B$ (complete tensor product), 19
- \mathbb{A}_K^n (affine n -space over K), 26
- $\text{Ann}_A(\mathfrak{a})$ (annihilator of ideal \mathfrak{a}), 43
- $A\langle S \rangle$ (ring of strictly convergent power series over A), 16
- $A\langle f/f_0 \rangle$ (affinoid algebra of rational subdomain), 20
- $\mathbf{B}_{A^m}(a; \varepsilon)$ (ball in A^m with centre a and radius ε), 5
- \mathbf{B} (unit ball), 55
- \mathbf{D} (truncated division function), 6
- $\dim A$ (Krull dimension of ring A), 13
- \mathcal{F}_x (stalk of \mathcal{F} in x), 24
- $\mathfrak{F}(\mathbb{X})$ (analytization of analytic scheme \mathbb{X}), 51
- \mathfrak{F}° (Berkovich sheaf associated to \mathfrak{F}), 37
- $\tilde{\mathfrak{F}}$ (rigid analytic sheaf induced by \mathfrak{F}), 36
- $\text{Func}_K(X)$ (function algebra on X), 7
- $\mathcal{I}\mathcal{O}_Y$ (inverse image of \mathcal{O}_X -ideal \mathcal{I}), 45
- K (ultrametric field), 3
- K^{alg} (algebraic closure of K), 5
- $K\langle S \rangle$ (ring of strictly convergent power series over K), 8
- $\mathbb{M}(f)$ (map of Berkovich spaces associated to f), 38
- $\mathbb{M}(X)$ (Berkovich space associated to X), 33
- $\mathbb{M}(\mathcal{F})$ (Berkovich sheaf associated to \mathcal{F}), 35
- \mathfrak{m}_x (maximal ideal associated to point x), 9
- $\text{Max}(A)$ (maximal spectrum of A), 15
- $|f|_\alpha$ (residue norm of f), 16
- $|f|_{\text{Gauss}}$ (Gauss norm of f), 8
- $|f|_{\text{sup}}$ (supremum norm of f), 7
- $\mathcal{O}_X(U)$ (structure sheaf on X), 23
- $\mathcal{O}_{\mathbb{X}}$ (structure sheaf on \mathbb{X}), 35
- \wp (valuation ideal of K), 5, 8
- \mathbb{P}_K^n (projective n -space over K), 26
- \mathbb{Q}_p (field of p -adics), 4
- R (valuation ring of K), 5
- R_L (valuation ring of L), 30
- $\text{rad } \mathfrak{a}$ (radical of ideal \mathfrak{a}), 17
- \bar{R} (residue field of K), 5, 8
- \bar{R}_L (residue field of L), 30
- S' (first $n - 1$ variables of S), 8
- S^ν (monomial in S), 8
- $\text{Sp } A$ (affinoid variety associated to A), 18
- $U \Subset_X Y$ (relatively compact with respect to X), 41
- Υ (universal domain), 31
- $V - W$ (complement of W in V), 32
- $-W$ (complement of W), 32
- \overline{W} (topological closure of W), 34
- $X \times Y$ (fibre product of rigid analytic varieties), 18
- $X \times_K L$ (extension of scalars of X to L), 20
- $X \times_Z Y$ (fibre product of rigid analytic varieties over Z), 19
- absolute Galois group, 5, 55
- absolute value, *see* norm
- Acyclicity Theorem, 24
- admissible
 - \sim algebra, *see* algebra
 - \sim covering, *see* covering
 - \sim open, *see* subset
- affinoid
 - \sim algebra, *see* algebra
 - \sim map, *see* map
 - \sim subdomain, *see* subdomain
 - \sim variety, *see* variety
- algebra
 - affinoid \sim , 3, 12
 - Boolean \sim , 32
 - finitely generated \sim , 13
 - Tate \sim , 3, 8
- analytic point, *see* point
- ball, 5
- Berkovich, 28
 - \sim space, 28, 37
 - affinoid \sim space, 33
 - affinoid \sim spaces are compact, 38, 40
- blowing up, 43–49
 - \sim and immersions, 47
 - \sim at the origin, 44, 45
 - \sim of the plane, 44
 - \sim tree, 53
 - affinoid \sim tree, 53
 - centre of a \sim , 45
 - local \sim , 49
 - properness of the \sim , 42, 46
 - smooth centre of a \sim , 51, 52
 - smooth \sim tree, 53
 - universal property of the \sim , 45, 49

Boolean combination, 32
 bu-tree, *see* blowing up

Cauchy sequence, 4
 centre, *see* blowing up
 Chevalley's Theorem, 32
 closed analytic subvariety, *see* variety
 closed immersion, *see* immersion
 compact, 28, 33, 37, 39

- locally \sim , 40
- quasi- \sim , 24
- relatively \sim , 40
- relatively \sim with respect to X , 41

convergence, 4, 7, 12, 31, 33
 covering, 23

- admissible affinoid \sim , 25
- admissible \sim , 3, 23–25
- refinement of a \sim , 24

dense, *see* subset
D-function, *see* function
 dimension

- \sim of a variety, 51
- co- \sim , 51
- Krull \sim , 6, 13

disk, 5

- closed \sim s are open, 5

D-semianalytic, *see* semianalytic

Embedded Resolution of Singularities,
see Resolution of Singularities

excellent, *see* ring
 existentially definable, *see* subset

faithfully flat, *see* map
 fibre

- \sim product, 18, 21

field

- \sim of p -adic complex numbers, 5
- \sim of the p -adics, 4
- algebraic closure of a complete \sim ,
5
- ultrametric \sim , 3

flat, *see* map
 faithfully \sim , *see* map
 flatificator

- Berkovich \sim , *see* Berkovich

formula

- existential \sim , 32
- quantifier free \sim , 31

founded, *see* module
 free Tate algebra, *see* algebra
 function

- D**- \sim , 6
- analytic \sim , 15
- bounded \sim , 7

G -continuous map, *see* map
 geometric point, *see* point
 glueing, *see* variety
 G -ringed space, *see* space
 Grothendieck topology, *see* topology
 G -topological space, *see* space

height, 13
 Hilbert's Basis Theorem, 3, 14
 Hironaka, 50
 homomorphism

- finite \sim , 13

hypersurface, 50

ideal

- K -rational \sim , 9, 14
- colon \sim , 48
- contraction of an \sim , 15, 18
- invertible \sim , 43
- locally principal \sim , 43
- \mathcal{O}_X - \sim , *see* \mathcal{O}_X -ideal
- principal \sim , 43

Identity Theorem, 6, 9

immersion

- closed \sim , 41, 43, 47, 48
- locally closed \sim , 48
- open \sim , 38, 48, 53

Intersection Property, 40, 42

- finite \sim , 40

inverse image, *see* \mathcal{O}_X -ideal
 isometry, 29, 33

Jacobson radical, *see* radical

Krull dimension, *see* dimension

manifold, 47

- affinoid \sim , 36, 50

map

- \sim of affinoid varieties, 18
- \sim of Berkovich spaces, 35
- \sim of rigid analytic varieties, 25
- affinoid \sim , 53
- closed \sim , 41
- faithfully flat \sim , 50
- flat \sim , 21, 38, 50
- G -continuous \sim , 24
- proper \sim , 40, 41, 47
- separated \sim , 27, 41
- universally closed \sim , 41

Maximum Modulus Principle, 8, 16, 21
 model

- big \sim , 29
- model theory, 29
- morphism
 - \sim of G -ringed spaces, 25
 - \sim of locally G -ringed spaces, 25
 - \sim of locally ringed spaces, 36
- neighborhood, 40
 - affinoid \sim , 32
 - wide affinoid \sim , 34
- nilradical, *see* radical
- Noether Normalization, 3, 10, 13, 14, 17
- non-archimedean inequality, 4
- norm, 3
 - \sim on a Cartesian power, 5
 - complete \sim , 4
 - equivalent \sim , 4, 16
 - Gauss \sim , 8
 - Gauss \sim equals supremum \sim , 8
 - multiplicative \sim , 3, 8, 16, 29, 30, 34
 - power-multiplicative \sim , 16
 - rational supremum \sim , 9
 - residue \sim , 16, 19
 - supremum \sim , 7, 16, 17
- normal crossings, 50
 - \sim at a prime ideal, 50
- Normalization Trick, 8, 11, 12
- nowhere dense, *see* subset
- nuclear, *see* filter
- Nullstellensatz, 3, 10, 15–17, 20
 - Weak \sim , 10, 14, 15, 18, 55
- open immersion, *see* immersion
- \mathcal{O}_X -ideal, 43, 45
 - coherent \sim , 43
 - inverse image of an \sim , 45
 - invertible \sim , 36, 43
- p -adic, *see* field
- p -component, *see* function
- point
 - analytic \sim , 28, 32
 - congruence of analytic \sim s, 33
 - generic \sim , 36
 - geometric \sim , 32
 - interior \sim , 47
- polynomial
 - monic \sim , 11
 - Weierstrass \sim , 11
- power series
 - formal \sim , 7
 - regular \sim , 10, 12, 16
 - strictly convergent \sim , 3, 7, 8, 16
- Preparation Trick, 3, 12, 14, 16
- presheaf, *see* sheaf
- Product Formula, 4
- proper, *see* map
- Quantifier Elimination, 6
- radical
 - \sim of an ideal, 17
 - Jacobson \sim , 16, 17
 - nil \sim , 16, 17
- rank-one valuation, *see* ring
- rational subdomain, *see* subdomain
- reduct, *see* language
- regular power series, *see* power series
- regular system of parameters, 50, 54
- Resolution of Singularities, 53
 - Embedded \sim , 50–53
- rigid analytic geometry, 1, 25, 28
- rigid analytic variety, *see* variety
- ring
 - discrete valuation \sim , 5
 - excellent \sim , 51
 - Henselian \sim , 36
 - local \sim , 5
 - Noetherian \sim , 14, 36
 - normed \sim , 3
 - rank-one valuation \sim , 6
 - reduced \sim , 16
 - regular \sim , 50
 - valuation \sim , 5
- scheme, 51
 - analytic \sim , 51
 - analytization of a \sim , 26, 52
- separated power series, *see* power series
- sheaf, 23, 24
 - \sim cohomology, 23
 - coherent \sim , 43
 - pre \sim , 24
 - stalk of a \sim , 24, 35
 - structure \sim , 24
- smooth, *see* variety
- space, 39
 - affine \sim , 26
 - Berkovich \sim , *see* Berkovich
 - closed analytic sub \sim , 39
 - G -ringed \sim , 25
 - G -topological \sim , 23, 24
 - locally G -ringed \sim , 25
 - projective \sim , 26
 - quasi-compact \sim , 24, 26, 53, 54

- spectrum
 - maximal \sim , 14, 15
 - prime \sim , 13, 15, 29
- Stone space, 31
- strict transform, 47, 53
 - \sim of a closed analytic subvariety, 46
 - \sim of a map, 47
 - \sim under local blowing up, 49
 - diagram of \sim , 47
- subanalytic
 - dimension of a \sim set, *see* dimension
- subdomain
 - a rational \sim is open in the canonical topology, 21
 - affinoid Berkovich \sim , 37
 - affinoid \sim , 21
 - an affinoid \sim is a finite union of rational \sim s, 24
 - Berkovich \sim , 37
 - intersection of rational \sim s is a rational \sim , 21
 - local structure of a rational \sim , 22
 - rational Berkovich \sim , 37
 - rational \sim , 3, 20–24
- subset
 - admissible open \sim , 3, 23
 - closed analytic \sim , 15
 - constructible \sim , 32
 - dense \sim , 47
 - nowhere dense \sim , 47, 53
 - Zariski closed \sim , 15
 - Zariski open \sim , 26
- surjective family, 53

- Tate algebra, *see* algebra
- tensor product, 19
 - complete \sim , 19
 - universal property of complete \sim , 19
- topology
 - canonical \sim , 3, 15, 16, 21
 - every \sim is a Grothendieck \sim , 23
 - Grothendieck \sim , 3, 15, 23–25, 28
 - refinement of a Grothendieck \sim , 26
 - strong Grothendieck \sim , 26
 - strong Grothendieck \sim and Zariski open subsets, 26, 52
 - strong Grothendieck \sim is quasi-compact, 26, 38, 53, 54
 - totally disconnected \sim , 5, 6, 28
 - Zariski \sim , 15

- Truncated Division, *see* function type
 - n - \sim , 31
 - \sim of an element, 29
 - \sim of an element and prime ideals, 29
 - realizing a \sim , 29
 - realizing the \sim space, 31

- unique factorization domain, 54
- unit
 - multiplicative \sim , 8
- universal domain, 29

- valuation, *see* ring
- variety, 39
 - affinoid \sim , 3, 18
 - closed analytic sub \sim , 43
 - coherent \mathcal{O}_X -ideal of a closed analytic sub \sim , 43
 - glueing together \sim s, 26, 37, 44
 - locally closed analytic sub \sim , 49
 - quasi-compact \sim , 27, 53
 - reduced \sim , 47
 - rigid analytic \sim , 3, 25
 - separated \sim , 27, 37
 - smooth \sim , 50
- Voûte Etoilée, 49

- Weierstrass Division, *see* Weierstrass Preparation
- Weierstrass Preparation
 - \sim Theorem, 3, 10–13, 16
 - Euclidean Division and \sim , 11

- Zariski topology, *see* topology