RECURSIVE SEQUENCES AND FAITHFULLY FLAT EXTENSIONS

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Introduction

This note arises from an attempt to give some model-theoretic interpretation of the concept of flatness. It is well-known that a necessary condition for a ring morphism $A \to B$ to be faithfully flat, is that *any linear system of equations with coefficients from A which has a solution over B, must have already a solution over A*. In fact, if we strenghten this condition to *any solution over B comes from solutions over A by base change*, then this becomes also a sufficient condition for being faithfully flat. However, whereas the first (necessary) condition is reminiscent of the model-theoretic notion of existentially closedness, the second seems to have no model-theoretic counterpart. Recall that a subring A of a ring B is said to be *existentially closed* in B, if any (not necessarily linear) system of equations with coefficients from A which is solvable over B is already solvable over A. This is the relative version of this concept, the absolute version reads: a ring A in a class of rings $\mathcal{K}$ is called *existentially closed* or *generic* for that class, if for any overring $B \in \mathcal{K}$ we have that A is existentially closed in B.[1] So, paraphrasing this notion, one could say that if $A \subset B$ is faithfully flat, then A is existentially closed in B with respect to linear equations. But as already observed, this is not a sufficient condition to guarantee faithful flatness.

I will present a property of rings which is a consequence of faithfully flatness, but presumably stronger than existentially closedness for linear equations. The key definition is that of a (linear) recursive sequence $(x_n)_n$ over a ring A, as a sequence satisfying some fixed linear relation over A among t consecutive terms. We will show that if $A \to B$ is faithfully flat and $(x_n)_n$ is a sequence of elements in A satisfying a linear recursion relation with coefficients in B, then it already satisfies such a recursion relation (of the same length) with coefficients in A. As there is a strong connection between recursive sequences and rational power series, we obtain the following corollary. Assume, moreover, that A and B are normal domains, then any power series over A which is rational (meaning that it can be written as a quotient of two polynomials) over B, is already rational over A. Any direct attempt, however, to prove this corollary just using faithfully flatness seems to fail as far as I can tell.

1. Definition. Let A be a Noetherian ring and let $\bar{\boldsymbol{x}} = (x_n)_{n<\omega}$ be a (countable) sequence of elements of A. We say that $\bar{\boldsymbol{x}}$ is *recursive over A (of length t)*, if there

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[1]The typical example here is an algebraically closed field within the class of all fields.

exists a linear form $L(X) = r_0X_0 + \cdots + r_{t-1}X_{t-1} \in A[X]$, where $t \in \mathbb{N}$ and $X = (X_i)_{i<t}$, such that for all $n \gg 0$, we have that

$$L(x_n, \dots, x_{n+t-1}) = x_{n+t}.$$

Here we have used the notation $n \gg 0$ as an abbreviation for *there exists* n_0*, such that for all* $n \geq n_0 \dots$

2. Theorem. *Let $A \to B$ be a morphism of Noetherian rings and let $\bar{\boldsymbol{x}}$ be a sequence in A. If B is faithfully flat over A and $\bar{\boldsymbol{x}}$ is recursive over B, then $\bar{\boldsymbol{x}}$ is already recursive over A (of the same length).*

Proof. Let $\mathcal{L}_t(\bar{\boldsymbol{x}})$ be the collection of all linear forms $L(X) \in A[X]$, where $X = (X_i)_{i\leq t}$ and $t \in \mathbb{N}$, such that, for all $n \gg 0$, we have that $L(x_n, \dots, x_{n+t}) = 0$. Evidently, $\mathcal{L}_t(\bar{\boldsymbol{x}})$ carries the structure of an A-module. Let e be the $(t+1)$-tuple $(0, \dots, 0, -1)$ and let

$$\mathfrak{a}_t^A(\bar{\boldsymbol{x}}) = \{\, L(e) \mid L \in \mathcal{L}_t(\bar{\boldsymbol{x}}) \,\}\,.$$

As the latter is the image of $\mathcal{L}_t(\bar{\boldsymbol{x}})$ under the morphism $A[X] \to A$ defined by substituting e for X, it is an ideal of A.

We claim that $\bar{\boldsymbol{x}}$ is recursive over A of length t, if and only if , $\mathfrak{a}_t^A(\bar{\boldsymbol{x}}) = A$. Indeed, if $\bar{\boldsymbol{x}}$ is recursive over A of length t, then there is a linear form $L(X) \in A[X]$, with $X = (X_i)_{i<t}$, such that for all $n \gg 0$, we have that $L(x_n, \dots, x_{n+t-1}) = x_{n+t}$. Let $L'(X, X_t) = L(X) - X_t$, so that $L' \in \mathcal{L}_t(\bar{\boldsymbol{x}})$. Since $1 = L'(e)$, we proved one direction and the converse follows along the same lines.

Now, by assumption $\bar{\boldsymbol{x}}$ is recursive over B of length t, so that by the criterion we just established, $\mathfrak{a}_t^B(\bar{\boldsymbol{x}}) = B$. Hence let $L(X) = b_0X_0 + \cdots + b_tX_t \in B[X]$ be a witness to this, where $X = (X_i)_{i\leq t}$, i.e., such that for some $n_0 \in \mathbb{N}$, we have, for all $n \geq n_0$, that $L(x_n, \dots, x_{t+n}) = 0$ and $L(e) = 1$. Therefore $b_t = -1$. Put $b = (b_i)_{i<t} \in B^t$. For every $n \geq n_0$, let $M_n(Y) \in A[Y]$, where $Y = (Y_i)_{i\leq t}$, be defined as

$$M_n(Y) = x_nY_0 + \cdots + x_{n+t}Y_t.$$

By Noetherianity, there exists some $n_1 \geq n_0$, such that each M_n, for $n \geq n_0$, lies in the ideal of $A[Y]$ generated by $M_{n_0}, \dots, M_{n_1}$. In other words, there exist $p_{n,k}(Y) \in A[Y]$, such that, for all $n \geq n_0$ and all k with $n_0 \leq k \leq n_1$, we have that

$$M_n(Y) = \sum_{k=n_0}^{n_1} p_{n,k}(Y)M_k(Y). \tag{1}$$

By construction, $M_k(b, -1) = 0$, for $n_0 \leq k \leq n_1$. By flatness, we can find finitely many $a^{(j)} \in A^t$, $e^{(j)} \in A$ and $\beta^{(j)} \in B$, such that

$$M_k(a^{(j)}, e^{(j)}) = 0, \tag{2}$$

for all $j < s$ and $n_0 \leq k \leq n_1$, and

$$\begin{aligned} b &= \sum_{j<s} \beta^{(j)} a^{(j)} \\ -1 &= \sum_{j<s} \beta^{(j)} e^{(j)}. \end{aligned} \tag{3}$$

However, from (1) and (2), it then follows that $M_n(a^{(j)}, e^{(j)}) = 0$, for all $j < s$ and all $n \geq n_0$. This means that $L^{(j)}(X) = a_0^{(j)} X_0 + \cdots + a_{t-1}^{(j)} + e^{(j)} X_t$ lies in $\mathcal{L}_t(\bar{\boldsymbol{x}})$, for each $j < s$, where $a^{(j)} = (a_i^{(j)})_{i<t}$. Hence $-e^{(j)} \in \mathfrak{a}_t^A(\bar{\boldsymbol{x}})$. Together with (3), we therefore conclude that $\mathfrak{a}_t^A(\bar{\boldsymbol{x}})B = B$. But faithfully flatness then implies that $\mathfrak{a}_t^A(\bar{\boldsymbol{x}}) = A$, which by the above criterion means that $\bar{\boldsymbol{x}}$ is recursive over A of length t. ■

1.3. Proposition. *Let A be a Noetherian ring and $\bar{\boldsymbol{x}}$ a sequence in A. Let $\xi_{\bar{\boldsymbol{x}}}(T) \in A[[T]]$ be the generating series of $\bar{\boldsymbol{x}}$, i.e.,*

$$\xi_{\bar{\boldsymbol{x}}}(T) = \sum_{n=0}^{\infty} x_n T^n.$$

Then $\bar{\boldsymbol{x}}$ is recursive over A, if and only if, $\xi_{\bar{\boldsymbol{x}}}(T)$ lies in $A(T)$, where the latter ring is the localization of $A[T]$ to the multiplicative set $1 + (T)A[T]$.

Proof. Suppose that $\bar{\boldsymbol{x}}$ is recursive and let $\xi(T) = \xi_{\bar{\boldsymbol{x}}}(T)$. There is some n_0 and some $a_k \in A$, for $k < t$, such that

$$x_n = a_0 x_{n-t} + \cdots + a_{t-1} x_{n-1} \tag{1}$$

for all $n \geq n_0$. We want to show that

$$\xi(T) \equiv Q(T)\xi(T) \mod A[T] \tag{2}$$

for some polynomial $Q(T) \in (T)A[T]$. Indeed, if (2) holds, then

$$\xi(T) = Q(T)\xi(T) + P(T)$$

for some $P(T) \in A[T]$ and hence $\xi(T) = P(T)/(1 - Q(T))$ as required.

We work in the A-module $A[[T]]/A[T]$. Clearly

$$\xi(T) \equiv T^{n_0} \cdot \xi_{\bar{\boldsymbol{x}}'}(T) \mod A[T],$$

where $\bar{\boldsymbol{x}}'$ is the sequence obtained from $\bar{\boldsymbol{x}}$ by deleting the first n_0 elements. Hence in order to prove (2) we may work with this new recursive sequence and hence assume from the start that $n_0 = 0$. We have, using (1), that

$$\begin{aligned}
\xi(T) &= \sum_{n<t} x_n T^n + \sum_{n \geq t} x_n T^n \\
&\equiv \sum_{n \geq t} (a_0 x_{n-t} + \cdots + a_{t-1} x_{n-1}) T^n \mod A[T] \\
&\equiv a_0 T^t \xi(T) + \cdots + a_{t-1} T \xi(T) \mod A[T].
\end{aligned}$$

This proves (2).

The converse is an easy exercise and left to the reader. ■

1.4. Corollary. *Let $\varphi\colon A \to B$ be a morphism of Noetherian rings. If φ is faithfully flat, then*

$$A(T) = A[[T]] \cap B(T), \tag{1}$$

for T a single variable.

Proof. The $\subset$-inclusion is immediate hence take F in the right hand side of (1). Taking the coefficients of F as the members of a sequence $\bar{\boldsymbol{x}}$ over A, we have that $F(T) = \xi_{\bar{\boldsymbol{x}}}(T)$. Since $F(T) \in B(T)$, the sequence $\bar{\boldsymbol{x}}$ is recursive over B by **(1.3)**. Therefore, by faithfully flatness and **(1.2)**, $\bar{\boldsymbol{x}}$ is also recursive over A which by **(1.3)** again means that $F \in A(T)$. ■

1.5. Corollary. *Let $A \to B$ be a faithfully flat morphism of Noetherian normal domains. Then any power series with coefficients in A which is rational over B, is already rational over A.*

Proof. A Noetherian normal domain A has the Fatou property by Chabert [**1**, §3] , meaning that $A(T)$ is equal to the intersection of the fraction field of $A[T]$ with $A[[T]]$, i.e., $A(T)$ is the ring of rational power series and we can apply **(1.4)**.

References

[**1**] J.L. Chabert, *Anneaux de Fatou*, Enseign. Math. (1972), 141–144.
[**2**] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

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