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SCHEMIC GROTHENDIECK RINGS I: MOTIVIC SITES

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Abstract

We propose a suitable substitute for the classical Grothendieck ring of an algebraically closed field, in which any quasi-projective scheme is represented with its non-reduced structure. This yields a more subtle invariant, called the schemic Grothendieck ring. In order to include open subschemes and their complements, we introduce formal motives. Although originally cast in terms of definability, everything in this paper has been phrased in a topos-theoretic framework.

1. Introduction

Modeled on p-adic integration, Kontsevich [7] formulated a general integration technique for smooth varieties over an algebraically closed field κ, called *motivic integration*. This was extended by Denef and Loeser [1, 2, 3] to arbitrary varieties to achieve *motivic rationality*, by which they mean the fact that the rationality of a certain generating series from geometry or number-theory, like the Igusa-zeta series, is "motivated" by the rationality of its motivic counterpart. Here, the motivic counterpart is supposed to specialize to the given classical series via some multiplicative function, like a counting function or Euler characteristic. The two main ingredients of this construction are the Grothendieck ring of varieties over κ, in which the integration takes its values, and the *truncated arc space* $L(X)$ of a variety X, that is to say, the reduced Hilbert scheme classifying all jets $\operatorname{Spec} \kappa[[\xi]] \to X$. Our aim is to extend this by replacing varieties by schemes (of finite type), in such a way that killing the nilpotent structure reverts to the old theory.

In the present paper, we will deal with the first ingredient only, leaving the definitions of jet schemes (=truncated arc schemes) and motivic integrals to [11]. The classical Grothendieck ring $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$ of an algebraically closed field κ is designed to encode both combinatorial and geometric properties of varieties. It is defined as the quotient of the free Abelian group on varieties over κ, modulo the relations $[X] - [X']$, if $X \cong X'$, and

$$[X] = [X - Y] + [Y], \tag{1}$$

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if Y is a closed subvariety, for Y, X, X' varieties (=reduced, separated schemes of finite type over κ). We will refer to the former relations as *isomorphism relations* (or more correctly, *homeomorphism relations*) and to the latter as *scissor relations*, in the sense that we "cut out Y from X." Multiplication on $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$ is then induced by the fiber product. In sum, the three building blocks for a Grothendieck ring are: scissor relations, isomorphism relations, and products. Only the former causes problems if one wants to extend the construction of the Grothendieck ring from varieties to arbitrary finitely generated schemes. Put bluntly, we cannot cut a scheme in two, as there is no notion of a scheme-theoretic complement, and so we ask what new objects we should add to make this work. Let us call these new objects tentatively *motives*,[1] in the sense that their existence is motivated by a combinatorial necessity. There are now two approaches to construct these.

The first one, discussed at length in [10], is based on definability, and was the original approach. The point of departure is the representation of a scheme by an equational (first-order) formula modulo the theory of local Artin algebras. The logical operations of conjunction, disjunction, and negation are then used to form the required new objects: motives arise as Boolean combinations of equational formulae. Scissor relations are now easily expressed in this formalism, whereas products are given by conjunction with respect to distinct variables, and isomorphism relations are phrased in terms of definable isomorphisms, cumulating in the construction of the *schemic Grothendieck ring* over κ. To obtain geometrically more significant motives, one is forced not only to introduce quantifiers, but also to resort to some infinitary logic via formularies, leading to larger Grothendieck rings, all of which still admit a natural homomorphism onto the classical Grothendieck ring. To define the analogue of jet spaces or truncated arc spaces, jet formulae are obtained by interpreting the theory of a local Artin algebra in that of its residue field. The resulting jet operator is compatible with Boolean combinations and hence induces an endomorphism on the Grothendieck rings. This was the first striking success of the new theory: no such operation exists in the classical Grothendieck ring. Other main advantages of this approach are (i) the presence of negation, allowing one to "cut up" a scheme into motives, and (ii) the uniformity inherent in model-theory, allowing one to use parameters, and hence to work over an arbitrary base ring rather than just an algebraically closed field. The main disadvantage stems from non-functoriality, in particularly when dealing with morphisms. Nonetheless, the theory in [10] has led to new motivic rationality results in certain cases, even the thus far illusive, positive characteristic case.

However, as research on the topic progressed, I came to realize that sacrificing definability for functoriality has its own benefits. In this approach, which forms the

[1] This is **not** the usual meaning of motif in algebraic geometry, and so perhaps the more appropriate nomenclature should have been *combinatorial motif*.

content of this paper and is essentially topos-theoretical, schemes are viewed as (contravariant) functors. Traditionally, one views them as functors, called *representable functors*, from the category of κ-algebras to the category of sets, but the power of the present approach comes from narrowing down the former category to that of *fat points*, consisting only of one-point schemes over κ. Thus, given a scheme X of finite type over κ and a fat point $\mathfrak{z}$, we let $X(\mathfrak{z})$ be the set of all $\mathfrak{z}$-*rational points*, that is to say, κ-morphisms $\mathfrak{z} \to X$. The functor $\mathfrak{z} \mapsto X(\mathfrak{z})$ uniquely determines the scheme structure on X. Motives are then certain subfunctors of these representable functors, with morphisms between them given by certain natural transformations. Since these functors take values in the category of sets, all set-theoretic operations are available to us, such as union, intersection, and complement. However, complementation does not behave functorially, and so motives now only form a lattice, leading to the notion of a *motivic site*: apart from a Grothendieck topology inherited from the Zariski topology on the underlying schemes, we also require a categorical lattice structure in order to formulate scissor relations. Defining multiplication by means of fiber products, we thus get the Grothendieck ring of a motivic site. In a sequel paper [11], using some tools from category theory and topos theory, in particular, adjunction, we will generalize the construction of truncated arc spaces to the present setup, by introducing for each fat point $\mathfrak{z}$ a jet morphism $\nabla_{\mathfrak{z}}$ operating on the corresponding Grothendieck ring.

Let me now briefly discuss in more detail the content of the present paper. In §2, we discuss the functors that will play the role of motives. Borrowing terminology from topos theory, on the category of fat points, a subfunctor $\mathfrak{X}$ of a representable functor given by a scheme X is called a *sieve* on X, and X is called an *ambient space* of $\mathfrak{X}$. We may do this over an arbitrary Noetherian base scheme V. For the sake of this introduction, I will only treat the case of greatest interest to us, namely, when V is the spectrum of an algebraically closed field κ. To define a morphism of sieves, we cannot just allow any natural transformation, see §2.14 for details. More often than not, such a morphism is induced by a morphism of the ambient schemes, in which case we call it *rational*. We turn this into a true topos in §3, by defining an admissible open of a sieve $\mathfrak{X}$ to be its restriction to an open in its ambient space. We define a *global section* on a sieve $\mathfrak{X}$ to be any morphism of sieves into the affine line. We establish an acyclicity result for global sections, allowing us to define the *structure sheaf* $\mathcal{O}_{\mathfrak{X}}$ of $\mathfrak{X}$.

In the next four sections, §§4–7, we introduce the Grothendieck ring of a motivic site, and discuss the three main cases. As already mentioned, a motivic site is for each scheme, a choice of lattice of sieves on that scheme, called the *motives* of the site. The associated Grothendieck ring is then defined as the free Abelian group on motives in the site modulo the isomorphism relations and the scissor relations, where the latter take the lattice form

$$[\mathfrak{X}] + [\mathfrak{Y}] = [\mathfrak{X} \cup \mathfrak{Y}] + [\mathfrak{X} \cap \mathfrak{Y}], \tag{2}$$

for any two motives $\mathfrak{X}$ and $\mathfrak{Y}$ on the same ambient space. The first motivic site of interest consists of the *schemic* motives, given on each scheme as the lattice of its closed subschemes (viewed as representable subfunctors). The resulting Grothendieck ring is too coarse, as it is freely generated as an additive group by the classes of irreducible schemic motives (Theorem 5.7). A larger, more interesting site is given by the sub-schemic motives, where we call a sieve $\mathfrak{X}$ on X *sub-schemic*, if there is a morphism $\varphi\colon Y \to X$ such that at each fat point $\mathfrak{z}$, the set $\mathfrak{X}(\mathfrak{z})$ consists of all $\mathfrak{z}$-rational points $\mathfrak{z} \to X$ that factor through φ, that is to say, $\mathfrak{X}(\mathfrak{z})$ is the image of the induced map $\varphi(\mathfrak{z})\colon Y(\mathfrak{z}) \to X(\mathfrak{z})$. Any locally closed subscheme is sub-schemic. Moreover, any morphism of sieves with domain a sub-schemic motif is rational (Theorem 3.15), from which it follows that the sub-schemic Grothendieck ring admits a natural homomorphism into the classical Grothendieck ring.

Whereas in general the complement of a sieve is no longer a sieve (as functoriality fails), this does hold for any open subscheme. However, such a complement is in general no longer sub-schemic, but only what we will call a *formal motif*, that is to say, a sieve $\mathfrak{X}$ that can be approximated by sub-schemic submotives in the sense that for each fat point $\mathfrak{z}$, one of its sub-schemic approximations has the same $\mathfrak{z}$-rational points as $\mathfrak{X}$. In case of an open $U = X - Y$, the complement is represented by the formal completion $\widehat{X}_Y$, whose approximations are the *co-jet spaces* $J^n_Y X := \operatorname{Spec}(\mathcal{O}_X/\mathcal{I}^n_Y)$. If $Y = \{P\}$ is a single closed point, then this is also represented by the local scheme $\operatorname{Spec}(\mathcal{O}_{X,P})$. A morphism in the site of formal motives $\mathbb{Form}_\kappa$ is approximated by rational morphisms, and therefore, the ensuing Grothendieck ring $\mathbf{Gr}(\mathbb{Form}_\kappa)$ still admits a canonical homomorphism onto the classical Grothendieck ring $\mathbf{Gr}(\mathbb{Var}_\kappa)$. We conclude with a brief discussion on how to circumvent the lack of complements in this setup by restricting the morphisms in the category of fat points to only those that are split; we will use this to define motivic integration in [11].

Notation and terminology. Varieties are assumed to be reduced, but not necessarily irreducible. Given a scheme X, we let X^{red} denote its *underlying variety* or *reduction*. We often denote a morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ by the same letter as the corresponding ring homomorphism $A \to B$, whenever this causes no confusion. By a *germ* (X, Y) we mean a scheme X together with a closed subscheme $Y \subseteq X$. Most of the time Y is an irreducible subvariety, that is to say, the closure of a point $y \in X$, and we simply write (X, y) for this germ. If Y is a closed point, we call the germ *closed*. The n*-th co-jet* $J^n_Y X$ of a germ (X, Y) is the closed subscheme defined by $\mathcal{I}^n_Y$, where $\mathcal{I}_Y$ is the ideal of definition of Y.[2] The *formal completion* $\widehat{X}_Y$ of the germ (X, Y) is the locally ringed space obtained as the

[2]Note that many authors take instead the $n+1$-th power.

direct limit of the $J^n_Y X$ (see [6, II.§9]). For instance, if $Y = P$ is a closed point with maximal ideal $\mathfrak{m}_P$, then the ring of global sections of $\widehat{X}_P$ is the $\mathfrak{m}_P$-adic completion $\widehat{\mathcal{O}}_{X,P}$ of $\mathcal{O}_{X,P}$.

We denote the affine line $\mathbb{A}^1_V := \mathbb{A}^1_{\mathbb{Z}} \times V$ over a base scheme V by $\mathbb{L}_V$, or simply $\mathbb{L}$, and also use this notation for its class in any of the Grothendieck rings. The formal completion of the germ $(\mathbb{L}, O)$, where O is the origin, is denoted $\widehat{\mathbb{L}}$, and the *punctured line* $\mathbb{L}\backslash O$, that is to say, the open subscheme obtained by removing the origin, is denoted $\mathbb{L}_*$. Whereas in the classical Grothendieck ring, $\mathbb{L} = \mathbb{L}_* + 1$ (and $\widehat{\mathbb{L}}$ is undefined), in the formal Grothendieck ring, we have $\mathbb{L} = \mathbb{L}_* + \widehat{\mathbb{L}}$ (see Proposition 7.1), which over a field κ, after taking global sections, takes the suggestive form

$$\text{“}\kappa[x]\text{”} = \text{“}\kappa[x, 1/x]\text{”} + \text{“}\kappa[[x]]\text{”}. \tag{3}$$

The n-th co-jet of $(\mathbb{L}, O)$ will be denoted $\mathfrak{l}_n := \operatorname{Spec}(\kappa[x]/(x^n))$.

2. Sieves

Fix a Noetherian scheme V, to be used as our base space. Most often, V is just the spectrum of an algebraically closed field κ. By a *V-scheme* X, we mean a separated scheme X together with a morphism of finite type $X \to V$, called the *structure morphism*. We call X a *fat V-point*, if $X \to V$ is finite and X has a unique point. In other words, $X = \operatorname{Spec} R$, for some Artinian local ring R which is finite as an $\mathcal{O}_V$-module. We call the length of R the *length* of the fat point and denote it $\ell(\mathfrak{z})$. We denote the subcategory of fat V-points by $\mathbb{F}\mathtt{at}_V$, and we will use letters $\mathfrak{z}, \mathfrak{v}, \dots$ to denote fat points. An important example of fat points are the co-jets of a closed germ (X, P), that is to say, given a V-scheme X and a closed point P on X with corresponding maximal ideal $\mathfrak{m}_P$, we let $J^n_P X$, called the *n-th co-jet of X along P*, be the fat point with coordinate ring $\mathcal{O}_{X,P}/\mathfrak{m}^n_P$.

Let X be a V-scheme and $\mathfrak{z}$ a fat point. A morphism of V-schemes $a\colon \mathfrak{z} \to X$ will be called a *$\mathfrak{z}$-rational point* on X (over V). The set of all $\mathfrak{z}$-rational points on X will be denoted by $X(\mathfrak{z})$, or sometimes by $X(R)$, where R is the coordinate ring of $\mathfrak{z}$. The image of the unique point of $\mathfrak{z}$ under a is a closed point on X, called the *center* (or *origin*) of a. Indeed, since the composition $\mathfrak{z} \to X \to V$ is the structure map whence finite, so is its first component $a\colon \mathfrak{z} \to X$. As finite morphisms are proper, closed points are mapped to closed points. Let x be the center of a $\mathfrak{z}$-rational point a on a V-scheme X. We will denote the residue field of x and $\mathfrak{z}$ respectively by $\kappa(x)$ and $\kappa(\mathfrak{z})$. The $\mathfrak{z}$-rational point a induces a homomorphism of residue fields $\kappa(x) \to \kappa(\mathfrak{z})$. By the Nullstellensatz this a is a finite extension, and its degree will be called the *degree* of a.

The category of contravariant functors (with morphisms given by natural transformations) from $\mathbb{F}\mathtt{at}_V$ to the category of sets will be called the category of *pre-sieves* over V. The *product* $\mathfrak{X} \times \mathfrak{Y}$ (respectively, the *disjoint union* $\mathfrak{X} \sqcup \mathfrak{Y}$) of two pre-sieves $\mathfrak{X}$ and $\mathfrak{Y}$ is defined point-wise by the rule that a fat point $\mathfrak{z}$ is mapped to the Cartesian product $\mathfrak{X}(\mathfrak{z}) \times \mathfrak{Y}(\mathfrak{z})$ (respectively, to the disjoint union $\mathfrak{X}(\mathfrak{z}) \sqcup \mathfrak{Y}(\mathfrak{z})$). Similarly, we say that $\mathfrak{X}$ is a *sub-pre-sieve* of $\mathfrak{Y}$, symbolically $\mathfrak{X} \subseteq \mathfrak{Y}$, if for every fat point $\mathfrak{z}$, we have an inclusion $\mathfrak{X}(\mathfrak{z}) \subseteq \mathfrak{Y}(\mathfrak{z})$, and this inclusion is a natural transformation, meaning that for any morphism $j\colon \tilde{\mathfrak{z}} \to \mathfrak{z}$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{X}(\mathfrak{z}) & \xrightarrow{\quad\subseteq\quad} & \mathfrak{Y}(\mathfrak{z}) \\
{\scriptstyle \mathfrak{X}(j)}\downarrow & & \downarrow{\scriptstyle \mathfrak{Y}(j)} \\
\mathfrak{X}(\tilde{\mathfrak{z}}) & \xrightarrow[\quad\subseteq\quad]{} & \mathfrak{Y}(\tilde{\mathfrak{z}})
\end{array}
\tag{4}
$$

where the downward arrows are the maps induced functorially by j. We call a pre-sieve $\mathfrak{X}$ on $\mathbb{F}\mathtt{at}_V$ *representable* (respectively, *pro-representable*), if there exists a V-scheme X (respectively, a scheme X which is not necessarily of finite type over V) such that $X(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z})$, for all fat points $\mathfrak{z}$. To emphasize that we view the (V-)scheme X as a contravariant functor on $\mathbb{F}\mathtt{at}_V$, we will denote it by $X^\circ := \operatorname{Mor}_V(\cdot, X)$. Any morphism $\varphi\colon Y \to X$ of (V-)schemes induces, by composition, a natural transformation $\varphi^\circ\colon Y^\circ \to X^\circ$, that is to say, a morphism in the category of pre-sieves on $\mathbb{F}\mathtt{at}_V$. More precisely, given a fat point $\mathfrak{z}$ and a $\mathfrak{z}$-rational point $b\colon \mathfrak{z} \to Y$, let $\varphi^\circ(\mathfrak{z})(b) := \varphi \circ b$. Instead of $\varphi^\circ(\mathfrak{z})$, we will simply write $\varphi(\mathfrak{z})$ for the induced map $Y(\mathfrak{z}) \to X(\mathfrak{z})$ if there is no danger for confusion.

2.1. Example. Suppose $V = \operatorname{Spec} \kappa$ with κ an algebraically closed field. Since $\kappa = \kappa(\mathfrak{z})$, any $\mathfrak{z}$-rational point has then degree zero. After identifying the closed points of X with the set $X(\kappa)$ of κ-rational points, the map sending a $\mathfrak{z}$-rational point a to its center x is none other than the map $X(\mathfrak{z}) \to X(\kappa)$ induced by the residue morphism $\operatorname{Spec} \kappa \to \mathfrak{z}$ and will be denoted $\rho_{\mathfrak{z}}$. More generally, functoriality yields, for every pre-sieve $\mathfrak{X}$, a (residue) map

$$
\rho_{\mathfrak{z}} (= \rho_{\mathfrak{z}}^{\mathfrak{X}})\colon \mathfrak{X}(\mathfrak{z}) \to \mathfrak{X}(\kappa). \tag{5}
$$

2.2. Lemma. *Two closed closed subschemes X and Y of a V-scheme Z are distinct if and only if there is a fat point $\mathfrak{z}$ such that $X(\mathfrak{z})$ and $Y(\mathfrak{z})$ are distinct subsets of $Z(\mathfrak{z})$.*

Proof. One direction is immediate, so assume X and Y are distinct. Then their restriction to some affine open of Z remains distinct, and hence we may assume $Z = \operatorname{Spec} A$ is affine. Let I and J be the ideals in A defining X and Y respectively.

Since $I \neq J$, there is a maximal ideal $\mathfrak{m} \subseteq A$ such that $IA_\mathfrak{m} \neq JA_\mathfrak{m}$. Hence, by Krull's Intersection Theorem in the Noetherian local ring $A_\mathfrak{m}$, there is some n such that I and J remain distinct ideals in $A/\mathfrak{m}^n$. In particular, upon replacing I by J if necessary, $\mathfrak{z} := \operatorname{Spec}(A/(I + \mathfrak{m}^n))$ is a closed subscheme of X, but not of Y, showing that the closed immersion $\mathfrak{z} \subseteq Z$ lies in $X(\mathfrak{z}) \setminus Y(\mathfrak{z})$. □

The resulting map from the category of V-schemes to the category of pre-sieves on $\mathbb{F}\mathtt{at}_V$ is an embedding by Yoneda's Lemma and Lemma 2.2. Note that this is no longer true for schemes not of finite type, an obvious reason for this failure being that there might be no rational points at all: for instance, $\operatorname{Spec}(\mathbb{C})$ has no rational points over any fat point defined over the algebraic closure $\bar{\mathbb{Q}}$ (for another example, see (2.3.iv) below). The product of any two (pro-)representable pre-sieves is again (pro-)representable. More explicitly, the product $X^\circ \times Y^\circ$ is the same as $(X \times_V Y)^\circ$, where $X \times_V Y$ is the usual fiber product in the category of V-schemes. If $s\colon \mathfrak{X} \to \mathfrak{Y}$ is a morphism of pre-sieves, then we define its *image* $\mathtt{Im}(s)$ and its *graph* $\Gamma(s)$ as the sub-pre-sieve of respectively $\mathfrak{Y}$ and $\mathfrak{X} \times \mathfrak{Y}$, given at each fat point $\mathfrak{z}$ as respectively the image and the graph of the map $s(\mathfrak{z})\colon \mathfrak{X}(\mathfrak{z}) \to \mathfrak{Y}(\mathfrak{z})$.

2.3. Sieves. By a *sieve*, we mean a sub-pre-sieve $\mathfrak{X}$ of some representable X°. If we want to emphasize the underlying V-scheme X, we say that $\mathfrak{X}$ is *a sieve on* X, or that X is an *ambient space* of $\mathfrak{X}$. We say that $\mathfrak{X}$ is *affine*, if it admits some affine ambient space. Some examples of sieves:

(2.3.i) If $Z \subseteq X$ is a closed subscheme, then $Z(\mathfrak{z})$, for a fat point $\mathfrak{z}$, consists precisely of those $\mathfrak{z}$-rational points $\mathfrak{z} \to X$ that factor through Z, and hence Z° is a subsieve of X°, called a *closed subsieve*;

(2.3.ii) If $\varphi\colon Y \to X$ is a morphism of V-schemes, then we let $\mathtt{Im}(\varphi)$ be the image pre-sieve of the corresponding natural transformation $\varphi^\circ\colon Y^\circ \to X^\circ$, that is to say, $\mathtt{Im}(\varphi)(\mathfrak{z})$ consists of all $\mathfrak{z}$-rational points on X that lift to a $\mathfrak{z}$-rational point on Y, meaning that $\mathfrak{z} \to X$ factors through Y. Any sieve of the form $\mathtt{Im}(\varphi)$ for some morphism $\varphi\colon Y \to X$ of V-schemes is called *sub-schemic* (we will study these more extensively in §6 below).

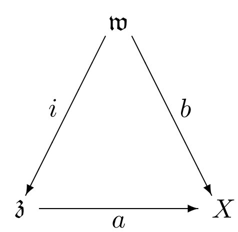
(2.3.iii) If φ is a (locally) closed or open immersion, then $\mathtt{Im}(\varphi)$ is equal to Y°, whence is itself representable, and we call $\mathtt{Im}(\varphi) \cong Y^\circ$ respectively a *(locally) closed* or *open subsieve* on X.

(2.3.iv) If P is a closed point on X, then the pro-representable pre-sieve given by the localization $X_P := \operatorname{Spec}(\mathcal{O}_{X,P})$ is a sieve on X. For each fat point $\mathfrak{z}$, a $\mathfrak{z}$-rational point on X belongs to $X_P(\mathfrak{z})$ if and only if its center is P; we will see shortly that this is an example of a *formal motif* (see Corollary 7.3). When P is not a closed point, the pre-sieve represented by the local scheme X_P is empty: indeed, suppose we are given a $\mathfrak{z}$-rational point $\mathfrak{z} \to X_P$, composition with $X_P \to X$ yields a $\mathfrak{z}$-rational point on X, and hence its center is a closed point Q. Since P is not closed, Q does not lie on X_P, contradiction.

2.4. Lemma. *Let $\mathfrak{X}$ be a sieve on X and $\mathfrak{z} = \operatorname{Spec} R$ a fat point. A $\mathfrak{z}$-rational point $a\colon \mathfrak{z} \to X$ belongs to $\mathfrak{X}(\mathfrak{z})$ if and only if $\mathtt{Im}(a) \subseteq \mathfrak{X}$. If $\mathfrak{X} = Y^\circ$ is a closed subsieve, given by a closed subscheme $Y \subseteq X$, then this is also equivalent with $\mathfrak{z} \cong \mathfrak{z} \times_X Y$ and also with $a^*\mathcal{I}_Y = 0$, where $\mathcal{I}_Y$ is the ideal sheaf of Y and $a^*\mathcal{I}_Y$ its image in R.*

Proof. Suppose $a \in \mathfrak{X}(\mathfrak{z})$. Let $\mathfrak{w}$ be a fat point and $b \in \mathtt{Im}(a)(\mathfrak{w})$. Hence $b\colon \mathfrak{w} \to X$ factors through a, that is to say, we can find a morphism $i\colon \mathfrak{w} \to \mathfrak{z}$ such that the diagram

(6)



commutes. By functoriality, i induces a map $\mathfrak{X}(\mathfrak{z}) \to \mathfrak{X}(\mathfrak{w})$, sending a to b, proving that $b \in \mathfrak{X}(\mathfrak{w})$. Since this holds for all $\mathfrak{w}$ and b, we showed $\mathtt{Im}(a) \subseteq \mathfrak{X}$. Conversely, assume the latter inclusion of sieves holds. In particular, the identity $1_\mathfrak{z}$ is a $\mathfrak{z}$-rational point whose image under $a(\mathfrak{z})$ is just a, proving that $a \in \mathtt{Im}(a)(\mathfrak{z}) \subseteq \mathfrak{X}(\mathfrak{z})$.

To see the equivalence with the last two conditions if $\mathfrak{X} = Y^\circ$, we may work locally and assume $X = \operatorname{Spec} A$. Let I be the ideal defining Y, and let $A \to R$ be the homomorphism corresponding to a. Then $a \in Y(\mathfrak{z})$ if and only if $IR = 0$ if and only if $A/I \otimes_A R \cong R$, proving the desired equivalences. □

We may generalize the notion of an image sieve as follows. Given a sieve $\mathfrak{Y}$ on an V-scheme Y and a morphism $\varphi\colon Y \to X$ of V-schemes, we define the *push-forward* $\varphi_*\mathfrak{Y}$ as the sieve on X given at a fat point $\mathfrak{z}$ as the image of $\mathfrak{Y}(\mathfrak{z}) \subseteq Y(\mathfrak{z})$ under the map $\varphi(\mathfrak{z})\colon Y(\mathfrak{z}) \to X(\mathfrak{z})$. In particular, $\varphi_*Y^\circ = \mathtt{Im}(\varphi)$. Similarly, given a sieve $\mathfrak{X}$ on X, we define its *pull-back* $\varphi^*\mathfrak{X}$ as the sieve on Y given at a fat point $\mathfrak{z}$ as the pre-image of $\mathfrak{X}(\mathfrak{z})$ under the map $\varphi(\mathfrak{z})\colon Y(\mathfrak{z}) \to X(\mathfrak{z})$. In other words, $\varphi^*\mathfrak{X}(\mathfrak{z})$ consists of those rational points $\mathfrak{z} \to Y$ such that the composition $\mathfrak{z} \to Y \to X$ lies in $\mathfrak{X}(\mathfrak{z})$. The pull-back of a closed subsieve is again a closed subsieve: if $\bar{X} \subseteq X$ is a closed immersion and $\varphi\colon Y \to X$ a morphism, then $\varphi^*\bar{X}^\circ$ is the closed subsieve given by $\varphi^{-1}(\bar{X}) = Y \times_X \bar{X}$. More generally, if $\bar{X} \to X$ is an arbitrary morphism, then the pull-back of the sub-schemic sieve $\mathtt{Im}(\bar{X} \to X)$ is the sub-schemic sieve $\mathtt{Im}(Y \times_X \bar{X} \to Y)$ of the base change.

2.5. The lattice of sieves on a fixed scheme. Below we will consider the category of all sieves. For now, we restrict to sieves on a fixed V-scheme X, with morphisms given by inclusion. They form a *lattice* in the following sense: given sieves $\mathfrak{X}$ and $\mathfrak{Y}$ on X, we define their *intersection* $\mathfrak{X} \cap \mathfrak{Y}$ and *union* $\mathfrak{X} \cup \mathfrak{Y}$ as the sieves given respectively by point-wise intersection and union, that is to say, $(\mathfrak{X} \cap \mathfrak{Y})(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z}) \cap \mathfrak{Y}(\mathfrak{z}) \subseteq X(\mathfrak{z})$ and $(\mathfrak{X} \cup \mathfrak{Y})(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z}) \cup \mathfrak{Y}(\mathfrak{z}) \subseteq X(\mathfrak{z})$. One easily checks that they are again sieves, that is to say, (contravariant) functors. Clearly, the intersection of two closed subsieves is again a closed subsieve: $V^\circ \cap W^\circ = (V \cap W)^\circ$, for $V, W \subseteq X$ closed subschemes, but this is no longer true for their union.

By the *Zariski closure* of $\mathfrak{X}$ in X, denoted $\bar{\mathfrak{X}}$, we mean the intersection of all closed subschemes $Y \subseteq X$ such that $\mathfrak{X} \subseteq Y^\circ$. By Noetherianity, $\mathfrak{X}$ is a sieve on its Zariski closure $\bar{\mathfrak{X}}$, and the latter is the smallest closed subscheme on which $\mathfrak{X}$ is a sieve. We say that $\mathfrak{X}$ is *Zariski dense* in X if $\bar{\mathfrak{X}} = X$. For instance, given a morphism $\varphi \colon Y \to X$ of V-schemes, the Zariski closure of $\mathtt{Im}(\varphi)$ is the so-called *scheme-theoretic image* of φ, that is to say, the closed subscheme of X given by the kernel of the induced morphism $\mathcal{O}_X \to \varphi_*\mathcal{O}_Y$. Let us call φ *strongly dominant* if $\mathtt{Im}(\varphi)$ is Zariski dense.

2.6. Lemma. *If $\mathfrak{w} \to \mathfrak{v}$ is a strongly dominant morphism of fat points, then the induced map $\mathfrak{X}(\mathfrak{v}) \to \mathfrak{X}(\mathfrak{w})$ is injective, for any sieve $\mathfrak{X}$.*

Proof. This is a form of duality: strongly dominant morphisms are epimorphisms and so under a contravariant functor they become monomorphisms. More explicitly, since $\mathfrak{X} \subseteq X^\circ$, for some V-scheme X, it suffices to show injectivity for the latter, that is to say, we may assume $\mathfrak{X} = X^\circ$ is representable, and then since the problem is local, we may assume X is affine with coordinate ring A. Let $a, a' \in X(\mathfrak{v})$ have the same image in $X(\mathfrak{w})$. If $R \to S$ is the homomorphism corresponding to φ, then strong dominance means that this homomorphism is injective. Since, by assumption, the two homomorphisms $A \to R$ induced respectively by a and a' give rise to the same homomorphism $A \to S$ when composed with the injection $R \subseteq S$, they must already be equal, as we needed to show. □

2.7. Example. Suppose V is the spectrum of an algebraically closed field κ. Zariski closure does not commute with taking κ-rational points, that is to say, the κ-rational points of the Zariski closure of a sieve $\mathfrak{X}$ in X may be bigger than the Zariski closure of $\mathfrak{X}(\kappa)$ (in the usual Zariski topology) in $X(\kappa)$. For instance, the cone $\mathfrak{C}_{\mathbb{L}}(O)$ (defined below after Lemma 2.12) of the origin O on the affine line $\mathbb{L} := \mathbb{A}^1_\kappa$ has Zariski closure equal to $\mathbb{L}$, whereas its κ-rational points consist just of the origin.

The category of sieves on a fixed ambient space is closed under direct limits: if $\mathfrak{X}_i$ for $i \in I$ is a direct system of sieves on a scheme X, then we define a sieve $\mathfrak{X}$ at each point $\mathfrak{z}$, by letting $\mathfrak{X}(\mathfrak{z})$ be the direct limit (as sets) of the subsets $\mathfrak{X}_i(\mathfrak{z})$ of $X(\mathfrak{z})$. It is not hard to see that $\mathfrak{X}$ is again a sieve on X and satisfies the universal property of direct limits in the category of sieves on X.

2.8. Lemma. *A V-scheme X viewed as a (representable) sieve is the direct limit of all its closed subsieves given by zero-dimensional closed subschemes.*

Proof. The collection of all zero-dimensional closed subschemes $Z \subseteq X$ forms a direct system (with the ordering given by closed immersion). Indeed, since zero-dimensional closed subschemes are affine, we may assume $X = \operatorname{Spec} A$ is affine, and $I, I' \subseteq A$ are ideals defining two zero-dimensional closed subschemes Z and Z' respectively. From the short exact sequence

$$0 \to A/(I \cap I') \to A/I \oplus A/I' \to A/(I + I') \to 0$$

it is clear that $I \cap I'$ defines again a zero-dimensional closed subscheme, admitting Z and Z' as closed subschemes, whence showing that the system is directed. Therefore, the corresponding collection of closed subsieves Z°, for Z running over all zero-dimensional closed subschemes of X, is a direct system of sieves on X. Let $\mathfrak{Z}$ be its direct limit. Given a $\mathfrak{z}$-rational point $a\colon \mathfrak{z} \to X$ for some fat point $\mathfrak{z}$, let $\mathfrak{x}$ be its scheme-theoretic image, so that a factors as $\mathfrak{z} \to \mathfrak{x} \subseteq X$, and hence belongs to $\mathfrak{x}(\mathfrak{z})$. As $\mathfrak{x}^\circ$ is a closed subsieve given by a zero-dimensional closed subscheme, a belongs to $\mathfrak{Z}(\mathfrak{z})$, proving that $\mathfrak{Z}(\mathfrak{z}) = X(\mathfrak{z})$. □

2.9. Remark. It is not true, as pointed out to me by Zhixian Zhu, that X is equal to the direct limit $\varinjlim Z$ of the Z as schemes; we merely have a faithfully flat and surjective morphism $\varinjlim Z \to X$ which is trivial on each fat point (in the affine case, the coordinate ring of $\varinjlim Z$ is equal to the product $\prod_x \widehat{\mathcal{O}}_{X,x}$, where the product runs over all closed points x of X).

2.10. Germs of sieves. Strictly speaking, a sieve is a pair $(\mathfrak{X}, X)$ consisting of a sub-pre-sieve $\mathfrak{X}$ of an ambient space X, but we will often treat sieves as abstract objects, that is to say, disregarding their ambient space. This allows us, for instance, to view the same sieve as already defined on a smaller subscheme. To give a more formal treatment, let us call a *germ of a sieve* any equivalence class of pairs $(\mathfrak{Y}, Y)$, where we call two such pairs $(\mathfrak{Y}, Y)$ and $(\mathfrak{Y}', Y')$ equivalent if there exists a pair $(\mathfrak{X}, X)$ and locally closed immersions $\varphi\colon Y \to X$ and $\varphi\colon Y' \to X'$ such that $\varphi_*\mathfrak{Y} = \mathfrak{X} = \varphi'_*\mathfrak{Y}'$. In particular, $\mathfrak{Y}$ is then also a sieve on the intersection $Y \cap Y'$, viewed as a locally closed subscheme of X. Therefore, we may always assume, if necessary, that $\mathfrak{Y}$ is Zariski dense in Y, and then it is also a Zariski dense sieve on any open subsieve of Y containing it. Henceforth, we will often confuse a sieve with the germ it determines. Note that the Zariski closure of a germ is therefore not well-defined.

2.11. Complete sieves. The complement $-\mathfrak{X}$ of a sieve $\mathfrak{X} \subseteq X^\circ$, where we set $-\mathfrak{X}(\mathfrak{z}) := X(\mathfrak{z}) \backslash \mathfrak{X}(\mathfrak{z})$ for any fat point $\mathfrak{z}$, is in general not a sieve, as witnessed by any closed subsieve. The following definition completely characterizes the phenomenon: we call a sieve $\mathfrak{X}$ on X *complete* if

$$\mathfrak{X}(\mathfrak{z}) = X(j)^{-1}(\mathfrak{X}(\tilde{\mathfrak{z}})),$$

for every morphism $j\colon \tilde{\mathfrak{z}} \to \mathfrak{z}$ of fat points, where $X(j)\colon X(\tilde{\mathfrak{z}}) \to X(\mathfrak{z})$ is the map induced functorially by j. More generally, if $\mathfrak{Y} \subseteq \mathfrak{X}$ is an inclusion of sieves on X, then we say that $\mathfrak{Y}$ is *relatively complete in* $\mathfrak{X}$, if

$$\mathfrak{Y}(\mathfrak{z}) = X(j)^{-1}(\mathfrak{Y}(\tilde{\mathfrak{z}})) \cap \mathfrak{X}(\mathfrak{z}),$$

for all morphisms $j\colon \tilde{\mathfrak{z}} \to \mathfrak{z}$.

If V is the spectrum of an algebraically closed field κ, then $\mathfrak{Y}$ is relatively complete in $\mathfrak{X}$ if and only if $\mathfrak{Y}(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z}) \cap \rho_{\mathfrak{z}}^{-1}(\mathfrak{Y}(\kappa))$, for every fat point $\mathfrak{z}$, where $\rho_{\mathfrak{z}}\colon \mathfrak{X}(\mathfrak{z}) \to \mathfrak{X}(\kappa)$ is the residue map sending a rational point to its center (see (5)). In fact, for V arbitrary, we have a similar criterion in that we only have to check the condition for every morphisms of fat points $j\colon \tilde{\mathfrak{z}} \to \mathfrak{z}$ in which $\tilde{\mathfrak{z}}$ has length one (necessarily therefore the spectrum of a field extension of the residue field of $\mathfrak{z}$).

As we shall see in Proposition 7.1 below, open subsieves over an algebraically closed field are complete, and their complements are again sieves. The second property in fact follows from the first, as one easily verifies:

2.12. Lemma. *Given an inclusion of sieves $\mathfrak{Y} \subseteq \mathfrak{X}$, then $\mathfrak{Y}$ is relatively complete in $\mathfrak{X}$ if and only if $\mathfrak{X}\setminus\mathfrak{Y}$ is again a sieve.* □

Assume the base scheme is an algebraically closed field κ. Given a κ-scheme X and a subset $F \subseteq X(\kappa)$, we define the *cone* $\mathfrak{C}_X(F)$ *over* F *on* X to be the sieve given by

$$\mathfrak{C}_X(F)(\mathfrak{z}) := \rho_{\mathfrak{z}}^{-1}(F)$$

for every fat point $\mathfrak{z}$, where as before, $\rho_{\mathfrak{z}}\colon X(\mathfrak{z}) \to X(\kappa)$ is the map induced by the residue map. In other words, a $\mathfrak{z}$-rational point belongs to $\mathfrak{C}_X(F)(\mathfrak{z})$ if and only if its center belongs to F. By our previous discussion on complete sieves over an algebraically closed field, it follows that $\mathfrak{C}_X(F)$ is complete, and its complement is the cone $\mathfrak{C}_X(-F)$. We call a cone $\mathfrak{C}_X(F)$ *constructible*, if $F \subseteq X(\kappa)$ is. More generally, for any sieve $\mathfrak{X}$, the intersection $\mathfrak{X} \cap \mathfrak{C}_X(F)$ is relatively complete in $\mathfrak{X}$, and its complement in $\mathfrak{X}$ is equal to $\mathfrak{X} \cap \mathfrak{C}_X(-F)$. We have the following converse:

2.13. Lemma. *Over an algebraically closed field κ, a subsieve $\mathfrak{Y} \subseteq \mathfrak{X}$ is relatively complete if and only if it is the intersection of $\mathfrak{X}$ and a cone.*

Proof. Let $\mathfrak{Y}$ be relatively complete in $\mathfrak{X}$, and let $F := \mathfrak{Y}(\kappa)$. Then $\mathfrak{Y} = \mathfrak{C}_X(F) \cap \mathfrak{X}$, since both have the same $\mathfrak{z}$-rational points, for any fat point $\mathfrak{z}$. □

Given a sieve $\mathfrak{X}$ on an ambient space X, we define its *completion in* X as the cone $\widehat{\mathfrak{X}}_X := \mathfrak{C}_X(\mathfrak{X}(\kappa))$. By functoriality, $\mathfrak{X}$ is contained in $\widehat{\mathfrak{X}}_X$, and $\widehat{\mathfrak{X}}_X$ is the smallest complete sieve on X containing $\mathfrak{X}$.

2.14. The category of sieves. In the category of pre-sieves, morphisms are just natural transformations between these functors. To obtain a geometrically more relevant notion, we call a natural transformation $s\colon \mathfrak{Y} \to \mathfrak{X}$ between two V-sieves a *morphism of sieves*, if for each morphism of V-schemes $\varphi\colon Z \to Y$ such that

$\mathtt{Im}(\varphi^\circ) \subseteq \mathfrak{Y}$, there exists a morphism of V-schemes $\psi\colon Z \to X$ such that ψ° is equal to the composition

$$\begin{array}{ccccc} Z^\circ & & & & \\ \varphi^\circ \downarrow & & & \overset{\psi^\circ}{\searrow} & \\ \mathfrak{Y} & \xrightarrow{s} & \mathfrak{X} & \hookrightarrow & X^\circ \end{array}$$

where X is some ambient space of $\mathfrak{X}$. That this yields indeed a category, the *category* $\mathbb{Sieve}_V$ *of sieves over* V, follows once we showed that the composition $s \circ t$ of two morphisms $t\colon \mathfrak{Z} \to \mathfrak{Y}$ and $s\colon \mathfrak{Y} \to \mathfrak{X}$ is again a morphism. To see this, let $\varphi\colon T \to Z$ be a morphism of V-schemes such that its image lies in $\mathfrak{Z}$. Hence there exists a morphism of schemes $\psi\colon T \to Y$, for some ambient space Y of $\mathcal{Y}$, such that $\psi^\circ = t \circ \varphi^\circ$. In particular, $\mathtt{Im}(\psi^\circ) \subseteq \mathfrak{Y}$, and hence by definition, there exists a morphism of schemes $\gamma\colon T \to X$, for some ambient space X of $\mathfrak{X}$, such that $\gamma^\circ = s \circ \psi^\circ$. The claim now follows since $\gamma^\circ = s \circ t \circ \varphi^\circ$.

Note that this definition also yields the notion of a morphism between germs of sieves, and we will not make a distinction between the two. In particular, if $\mathfrak{Y}$ is a subsieve on $\mathfrak{X}$, then the inclusion $\mathfrak{Y} \subseteq \mathfrak{X}$ is a morphism of sieves. A natural transformation with domain a representable sieve Y° is a morphism of sieves if and only if it is induced by a genuine morphism of schemes with domain Y: indeed, just take $Z = Y$ and φ the identity morphism in the above definition.

2.15. Corollary. *For any sieve $\mathfrak{X}$, we have an isomorphism of sieves*

$$\mathrm{Mor}_{\mathbb{Sieve}_V}((\cdot)^\circ, \mathfrak{X}) \cong \mathfrak{X}(\cdot).$$

Proof. Let $\mathfrak{z}$ be a fat point and $s\colon \mathfrak{z}^\circ \to \mathfrak{X}$ a morphism of sieves. By our previous observation, this morphism is induced by a morphism $a\colon \mathfrak{z} \to X$ of schemes, where X is some ambient space of $\mathfrak{X}$. Since $s = a^\circ$, we have an inclusion $\mathtt{Im}(a) \subseteq \mathfrak{X}$, and hence $a \in \mathfrak{X}(\mathfrak{z})$ by Lemma 2.4. The converse follows along the same lines. $\square$

2.16. Example. It is not hard to see that the above isomorphism is also continuous in the sense defined below in §3.6, that is to say, a *homeomorphism of sieves*.

Not every natural transformation between representable sieves needs to be a morphism of sieves. The following example was pointed out to me by Zhixian Zhu (it uses some results proven below; see Example 7.10). We will show that the schemic sieve $\mathbb{L}^\circ$ given by the affine line $\mathbb{L} := \mathbb{A}^1_\kappa$ over a field κ is the disjoint union of the open subsieve $\mathbb{L}^\circ_*$ given by the *punctured line*, that is to say, the basic open obtained by removing the origin, and the formal motif $\widehat{\mathbb{L}}^\circ$ given by the formal completion of the affine line at the origin, or, equivalently, the completion in $\mathbb{L}$ of the closed subsieve given by the origin. We may choose now morphisms on each of these two submotives and take their disjoint union, and this will in general no longer be a morphism of sieves. More precisely, global sections will be defined in §3.1 below as morphisms from a sieve into $\mathbb{L}^\circ$, and for the sieves $\mathbb{L}^\circ$, $\mathbb{L}^\circ_*$, and $\widehat{\mathbb{L}}^\circ$ these are in one-one correspondence with respectively the polynomials, the Laurent polynomials, and the power series in one variable x (see (3)). So, taking on $\mathbb{L}^\circ_*$ the morphism induced by the open immersion $\mathbb{L}_* \subseteq \mathbb{L}$, corresponding to x, but on $\widehat{\mathbb{L}}^\circ$ the morphism $\widehat{\mathbb{L}}^\circ \to \mathbb{L}^\circ$ given by x^2, we get a natural transformation (even continuous in the sense given by Proposition 3.9), which cannot be a global section.

Neither is it the case that a bijective morphism of sieves is automatically an isomorphism of sieves. To give an example, we take a closer look at etale morphisms (recall that $\varphi\colon Y \to X$ is called *etale* if it is flat and unramified). We call a morphism of sieves *bijective* (respectively, *injective*, *surjective*, or *with finite fibers*), if it is so at each fat point.

2.17. Theorem. *Over an algebraically closed field κ, a morphism $\varphi\colon Y \to X$ is etale at a closed point $Q \in Y$ if and only if it induces a bijective morphism $\varphi^\circ_Q\colon Y^\circ_Q \to X^\circ_{\varphi(Q)}$ of sieves. In particular, an etale covering $\varphi\colon Y \to X$ induces a surjective morphism $\varphi^\circ\colon Y^\circ \to X^\circ$ of sieves, with finite fibers.*

Proof. Since etale maps are quasi-finite, all (closed) fibers are finite, and hence the second assertion follows from the first. Note that by Zariski's Main Theorem, a surjective etale map is automatically finite. To prove the direct implication, assume $\varphi_Q\colon \mathcal{O}_{X,P} \to \mathcal{O}_{Y,Q}$ is etale, with $P = \varphi(Q)$. Given a fat point with Artinian local coordinate ring $(R, \mathfrak{m})$, an R-rational point on X with center P corresponds to a local homomorphism $a\colon \mathcal{O}_{X,P} \to R$. Since κ is algebraically closed, $\kappa(Q) \cong \kappa \cong R/\mathfrak{m}$, so that we have a natural homomorphism $i\colon \mathcal{O}_{Y,Q} \to R/\mathfrak{m}$ making

$$
\begin{array}{ccc}
\mathcal{O}_{X,P} & \longrightarrow & \mathcal{O}_{Y,Q} \\
\Big\downarrow a & & \Big\downarrow i \\
R & \longrightarrow & R/\mathfrak{m}
\end{array}
\tag{7}
$$

commute. Since $\mathcal{O}_{X,P} \to \mathcal{O}_{Y,Q}$ is formally etale, i lifts to a unique $\mathcal{O}_{X,P}$-algebra homomorphism $b\colon \mathcal{O}_{Y,Q} \to R$ (see [9, §28]). In other words, b is an R-rational point lifting a, showing that $\varphi_Q(R)$ is surjective, and by uniqueness, it is also injective.

Conversely, assume $Y^\circ_Q \to X^\circ_P$ is bijective. To show that $\mathcal{O}_{X,P} \to \mathcal{O}_{Y,Q}$ is etale, it suffices to verify that it is formally etale in the adic topology. So assume we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X,P} & \longrightarrow & \mathcal{O}_{Y,Q} \\ {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ C & \xrightarrow[j]{} & C/\mathfrak{n} \end{array} \tag{8}$$

with $\mathfrak{n} \subseteq C$ an ideal of square zero, containing $\mathfrak{m}_P^n C$ for some n. Hence the image $\bar{B}$ of $\mathcal{O}_{Y,Q}$ in $C/\mathfrak{n}$ is an Artinian local ring, and its pre-image $R := j^{-1}(\bar{B})$ in C is then a fat point. The induced homomorphism $\mathcal{O}_{X,P} \to R$ is an R-rational point, which by assumption lifts to a unique R-rational point $b\colon \mathcal{O}_{Y,Q} \to R$, proving the desired unique factorization in the definition of being formally etale. □

2.18. Remark. The same argument shows that if φ is smooth, then φ° is surjective. Note that the bijection between Y_Q° and X_P° induced by φ_Q° is in fact an isomorphism of sieves: by Corollary 7.3 below, these two sieves are equal to the pro-representable sieves given by the formal completions $\widehat{Y}_Q$ and $\widehat{X}_P$ respectively, and for etale morphisms over an algebraically closed field, these formal schemes are isomorphic via the completion of φ_Q.

However, we can construct an example of a bijective morphism of sieves which is not an isomorphism from this as follows: let $\varphi\colon Y \to X$ be an etale covering. For each closed point $P \in X$, choose a closed point Q on Y lying above P and let $\mathfrak{Y}$ be the union of all Y_Q°. It follows that the restriction of φ° induces a bijection between $\mathfrak{Y}$ and X°, but this cannot be an isomorphism, since $\mathfrak{Y}$ is in general not representable.

2.19. Definition. We call a natural transformation $s\colon \mathfrak{Y} \to \mathfrak{X}$ *rational*, if there exists a morphism of V-schemes $\varphi\colon Y \to X$ such that $\mathfrak{X}$ and $\mathfrak{Y}$ are sieves on respectively X and Y, and such that s is the restriction of $\varphi^\circ\colon Y^\circ \to X^\circ$; we might also express this by saying that *s extends to a morphism of schemes*.

It is easy to see that a rational natural transformation is in fact a morphism of sieves. Moreover, since the definition allows the ambient spaces to be dependent on s, we also defined what it means for a morphism between germs of sieves to be rational. It follows that if $\mathfrak{Y}$ is Zariski dense in Y and $\mathfrak{X}$ is a sieve on X (without any further restriction), then s extends to a morphism $\tilde{Y} \to X$ for some open $\tilde{Y} \subseteq Y$ on which $\mathfrak{Y}$ is also a sieve, that is to say, such that $\mathfrak{Y} \subseteq \tilde{Y}^\circ$. Any inclusion of subsieves is rational, and so is any rational point by Lemma 2.4.

The composition of two rational morphisms is again rational. Indeed, let $s\colon \mathfrak{Z} \to \mathfrak{Y}$ and $t\colon \mathfrak{Y} \to \mathfrak{X}$ be rational, extending respectively to morphisms $\varphi\colon Z \to Y$ and $\psi\colon Y' \to X$. Since Y and Y' are both ambient spaces for $\mathfrak{Y}$, so is their intersection $Y'' := Y \cap Y'$, which is therefore locally closed in either. Hence the restriction of

φ to $\varphi^{-1}(Y'')$ (respectively, the restriction of ψ to Y'') is a morphism extending s (respectively t), and therefore, the composition $\varphi^{-1}(Y'') \to X$ extends $t \circ s$. We can therefore define the *explicit category of sieves*, denoted $\overline{\mathbb{Sieve}}_V$, as the subcategory of all sieves in which the morphisms are only the rational ones.

A note of caution: not every morphism of sieves is rational, and we will discuss some examples later. Moreover, even if it is, one cannot always extend it to any ambient space of the source sieve. An example is in order:

2.20. Example. Let $\mathbf{Hyp} \subseteq \mathbb{L}^2$ be the hyperbola with equation $xy = 1$ over a field κ, and let $\mathbb{L}_*$ be the punctured line. Note that this is again an affine scheme, with coordinate ring $\kappa[x, 1/x]$. The projection $\mathbb{L}^2 \to \mathbb{L}$ onto the first coordinate induces an isomorphism $\mathbf{Hyp} \to \mathbb{L}_*$. Its inverse induces an isomorphism $\mathbb{L}_*^\circ \to \mathbf{Hyp}^\circ$, which is trivially rational, as both sieves are representable. However, although $\mathbb{L}_*^\circ$ is an open subsieve on $\mathbb{L}$, the above isomorphism does not extend (since $1/x$ is not a polynomial).

We may generalize the definitions of pull-back and push-forward along a morphism of sieves as follows. Let $s\colon \mathfrak{Y} \to \mathfrak{X}$ be a morphism of sieves. Given a subsieve $\mathfrak{Y}' \subseteq \mathfrak{Y}$, we define its *push-forward* $s_*\mathfrak{Y}'$ as the sieve defined at each fat point $\mathfrak{z}$ as the image of $\mathfrak{Y}'(\mathfrak{z})$ under $s(\mathfrak{z})$. Similarly, given a subsieve $\mathfrak{X}' \subseteq \mathfrak{X}$, we define its *pull-back* $s^*\mathfrak{X}'$ as the sieve defined at each fat point $\mathfrak{z}$ as the pre-image of $\mathfrak{X}'(\mathfrak{z})$ under $s(\mathfrak{z})$.

3. The topos of sieves

Fix a Noetherian scheme V. Recall that $\mathbb{A}^n_V := \mathbb{A}^n_{\mathbb{Z}} \times V$ is the *affine n-space* over V, and that we will denote it as $\mathbb{L}^n_V = \mathbb{L}^n$.

3.1. Section rings. Given a sieve $\mathfrak{X}$ on a V-scheme X, we define its *global section ring* as

$$H_0(\mathfrak{X}) := \mathrm{Mor}(\mathfrak{X}, \mathbb{L}),$$

where by the latter, we actually mean the collection of morphisms of sieves $\mathfrak{X} \to \mathbb{L}^\circ$, but for notational simplicity, we will identify a scheme with the functor it represents if there is no danger for confusion. For each fat point $\mathfrak{z} = \operatorname{Spec} R$, we have a natural bijection $\Psi_{\mathfrak{z}}\colon \mathbb{L}(\mathfrak{z}) \cong R$ defined as follows. Given a rational point $a\colon \mathfrak{z} \to \mathbb{L}$, it factors through an affine open of the form $\mathbb{L}_\lambda \subseteq \mathbb{L}_V$, for some affine open $\operatorname{Spec}\lambda \subseteq V$, and hence induces a homomorphism $\lambda[y] \to R$, which, again for notational simplicity, we continue to denote by a. Now, set $\Psi_{\mathfrak{z}}(a) := a(y) \in R$. This identification endows $\mathbb{L}$ with a ring structure, and by transfer, then makes $H_0(\mathfrak{X})$ into a ring. Indeed, given morphisms $s, t\colon \mathfrak{X} \to \mathbb{L}$, we define their sum $s+t$ (respectively, their product st) as the morphism which at a fat point $\mathfrak{z}$ maps $a \in \mathfrak{X}(\mathfrak{z})$ to

$$s(\mathfrak{z})(a) + t(\mathfrak{z})(a) = \Psi_{\mathfrak{z}}^{-1}\big(\Psi_{\mathfrak{z}}(s(\mathfrak{z})(a)) + \Psi_{\mathfrak{z}}(t(\mathfrak{z})(a))\big)$$

and a similar formula for $s(\mathfrak{z})(a) \cdot t(\mathfrak{z})(a)$. The functoriality of $s + t$ and st is easy. Moreover, if $\mathfrak{X} = X^\circ$ is representable, so that s and t correspond to elements in $\Gamma(\mathcal{O}_X, X)$, that is to say, are *global sections on* X (see the proof of Proposition 3.3 below), then $s + t$ and st correspond exactly to their sum and product in $\Gamma(\mathcal{O}_X, X)$. Using this, it is now an easy exercise to show that the sum and product of arbitrary global sections are again global sections.

As we shall see below, rational morphisms will play a key role, and so we define the ring of *rational sections* $H_0^{\text{rat}}(\mathfrak{X})$ as the subset of $H_0(\mathfrak{X})$ consisting of all rational morphisms $\mathfrak{X} \to \mathbb{L}$. To see that this is closed under sums and products, let s and s' be two rational sections. By definition, there exist morphisms $X \to \mathbb{L}$ and $X' \to \mathbb{L}$ inducing s and s' respectively, where X and X' are ambient spaces for $\mathfrak{X}$. In particular, the locally closed subscheme $X'' := X \cap X'$ is then also an ambient space for $\mathfrak{X}$. Hence $s + t$ and st extend to the sum and product of the restrictions to X'' of $X \to \mathbb{L}$ and $X' \to \mathbb{L}$, proving that they are also rational.

3.2. Lemma. *Assigning to a sieve $\mathfrak{X}$ its ring of global sections $H_0(\mathfrak{X})$ yields a contravariant functor from $\mathbb{S}\mathrm{ieve}_V$ to the category of $\mathcal{O}_V$-algebras. Similarly, $\mathfrak{X} \mapsto H_0^{rat}(\mathfrak{X})$ is a contravariant functor on the explicit category $\mathbb{S}\overline{\mathrm{ieve}}_V$. In particular, if two sieves are isomorphic, then they have the same global section ring, and if they are rationally isomorphic, then they have the same rational section ring.*

Proof. If $s\colon \mathfrak{X} \to \mathfrak{Y}$ is a morphism of sieves, then pulling-back induces a $\mathcal{O}_V$-algebra homomorphism $H_0(\mathfrak{Y}) \to H_0(\mathfrak{X})$ given by $t \mapsto t \circ s$, for $t\colon \mathfrak{Y} \to \mathbb{L}$. One easily verifies that this constitutes a contravariant functor. Since the pull-back of an rational morphism under a rational morphism is easily seen to be rational again, we get an induced homomorphism $H_0^{\text{rat}}(\mathfrak{Y}) \to H_0^{\text{rat}}(\mathfrak{X})$. □

3.3. Proposition. *The global section ring of a representable functor X° is equal to the ring of global sections $H_0(X) := \Gamma(\mathcal{O}_X, X)$ of the corresponding scheme, and this is also its rational section ring.*

Proof. A global section of X° is induced by a morphism of V-schemes $X \to \mathbb{L}$, and it is well-known that the collection of all these is precisely the ring of global sections on X (see, for instance, [6, II. Exercise 2.4]). □

In particular, if $\mathfrak{X}$ is affine, say, with ambient space $\operatorname{Spec} A$, then $H_0(\mathfrak{X})$ and $H_0^{\text{rat}}(\mathfrak{X})$ are A-algebras.

3.4. Corollary. *The rational section ring $H_0^{rat}(\mathfrak{X})$ of a sieve $\mathfrak{X}$ is the inverse limit of all $H_0(X)$, where X runs over all ambient spaces of $\mathfrak{X}$.*

Proof. If $s\colon \mathfrak{X} \to \mathbb{L}$ is a rational section, then it extends to a morphism $X \to \mathbb{L}$, where X is some ambient space of $\mathfrak{X}$, and hence by the argument in the above proof, it is the image of an element in $H_0(X)$ under the homomorphism $H_0(X) \to H_0^{\text{rat}}(\mathfrak{X})$

induced by the inclusion $\mathfrak{X} \subseteq X^\circ$. If $X' \subseteq X$ is a locally closed subscheme which is also an ambient space for $\mathfrak{X}$, then s extends to a morphism with domain X', and this is therefore necessarily the restriction of the global section in $H_0(X)$ determined by s. This shows that the $H_0(X)$ form an inverse system as X varies over the germ of $\mathfrak{X}$, with limit equal to $H_0^{\text{rat}}(\mathfrak{X})$. □

3.5. Example. If $\mathfrak{X}$ is Zariski dense in X, then the only ambient spaces of $\mathfrak{X}$ inside X are open, so that we may think of the ring of rational sections as a sort of stalk:

$$H_0^{\text{rat}}(\mathfrak{X}) = \varprojlim H_0^{\text{rat}}(U)$$

where U runs over all open subschemes on which $\mathfrak{X}$ is a sieve. In particular, if V is the spectrum of an algebraically closed field κ and $\mathfrak{X}(\kappa) = X(\kappa)$, then there are no proper opens in X on which $\mathfrak{X}$ is a sieve, and hence $H_0^{\text{rat}}(\mathfrak{X}) = H_0(X)$. This holds automatically if X is for instance a fat point.

3.6. The topos of sieves. The Zariski topology on a V-scheme X induces a *topos* on each of its sieves $\mathfrak{X}$. More precisely, the *admissible opens* on $\mathfrak{X}$ are the sieves of the form $\mathfrak{X} \cap U^\circ$, where $U \subseteq X$ runs over all opens of X; and the admissible coverings are all collections of admissible opens $\mathfrak{U}_i \subseteq \mathfrak{X}$ such that their union (as sieves) is equal to $\mathfrak{X}$, that is to say, such that the corresponding opens $U_i \subseteq X$ cover some ambient space of $\mathfrak{X}$. For simplicity, we will simply write $\mathfrak{X} \cap U$ for $\mathfrak{X} \cap U^\circ$. In particular, since X is quasi-compact, any admissible covering contains a finite admissible subcovering. The collection of admissible opens does not depend on the ambient space X, for if $X' \subseteq X$ is a locally closed subscheme on which $\mathfrak{X}$ is also a sieve, then since its topology is induced by that of X, it induces the same admissible opens on $\mathfrak{X}$, and the same admissible coverings. Without going into details, we claim that the collection of admissible opens and admissible coverings yields a Grothendieck topology on $\mathfrak{X}$, turning $\mathbb{Sieve}_V$ into a Grothendieck site. Nonetheless, since for each fat point $\mathfrak{z}$, this induces a topological space on $\mathfrak{X}(\mathfrak{z})$, we will just pretend that we are working in a genuine topological space, and borrow the usual topological jargon. For instance, we call a morphism $s\colon \mathfrak{Y} \to \mathfrak{X}$ *continuous* if the pull-back of any (admissible) open in $\mathfrak{X}$ is an (admissible) open in $\mathfrak{Y}$.

3.7. Lemma. *Suppose V is the spectrum of an algebraically closed field κ. Given a sieve $\mathfrak{X}$ on X and an open $U \subseteq X$ with corresponding open sieve $\mathfrak{U} := \mathfrak{X} \cap U$, we have an equality $\mathfrak{U}(\mathfrak{z}) = \rho_{\mathfrak{z}}^{-1}(\mathfrak{U}(\kappa))$.*

Proof. Let $a\colon \mathfrak{z} \to \mathfrak{X}$ be a $\mathfrak{z}$-rational point with center $x = \rho_{\mathfrak{z}}(a)$ (see (5)). Since the problem is local, we may assume $X = \operatorname{Spec} A$ is affine and $U = \operatorname{Spec} A_f$ is a basic open subset. Hence a corresponds to a κ-homomorphism $a\colon A \to R$, where R is the coordinate ring of $\mathfrak{z}$. Using Lemma 2.4, we see that $a \in \mathfrak{U}(\mathfrak{z})$ if and only if a factors through A_f. The latter is equivalent with $a(f)$ being a unit in R, that is to say, with $x(f) \neq 0$ in κ, which in turn is equivalent with x belonging to $U(\kappa)$ whence to $\mathfrak{U}(\kappa)$. □

3.8. Remark. Over an algebraically closed field κ, unless $\mathfrak{z}$ is the geometric point $\operatorname{Spec}\kappa$ itself, the topological space $\mathfrak{X}(\mathfrak{z})$ is not separated: by Lemma 3.7, two $\mathfrak{z}$-rational points $a, b \in \mathfrak{X}(\mathfrak{z})$ are inseparable if and only if they have the same center, that is to say, if and only if $\rho_{\mathfrak{z}}(a) = \rho_{\mathfrak{z}}(b)$.

3.9. Proposition. *If $s\colon \mathfrak{Y} \to \mathfrak{X}$ is a rational morphism of sieves, then it is continuous.*

Proof. By assumption, s extends to a morphism $\varphi\colon Y \to X$, where $\mathfrak{Y}$ is a sieve on Y and $\mathfrak{X}$ on X. An open $\mathfrak{U} \subseteq \mathfrak{X}$ is of the form $\mathfrak{X} \cap U$, for some open subscheme $U \subseteq X$. Since $\varphi^{-1}(U)$ is open in Y and $\varphi^*\mathfrak{U} = \mathfrak{X} \cap \varphi^{-1}(U)$, the claim follows. □

To make $\mathbb{S}\mathrm{ieve}_V$ into a topos, we need to define structure sheafs for a given sieve $\mathfrak{X}$. We define presheaves $\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{O}_{\mathfrak{X}}^{\mathrm{rat}}$ on $\mathfrak{X}$, by associating to an open $\mathfrak{U} := \mathfrak{X} \cap U$ its $\mathcal{O}_V$-algebra of global sections

$$\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) := H_0(\mathfrak{X} \cap U) \qquad \text{and} \qquad \mathcal{O}_{\mathfrak{X}}^{\mathrm{rat}}(\mathfrak{U}) := H_0^{\mathrm{rat}}(\mathfrak{X} \cap U).$$

The main property is the following acyclicity result:

3.10. Theorem (Acyclicity)**.** *For each sieve $\mathfrak{X}$, the presheaves $\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{O}_{\mathfrak{X}}^{rat}$ are sheaves (in the topos sense).*

Proof. Given a (finite) admissible covering of an admissible open $\mathfrak{U} = \mathfrak{U}_1 \cup \dots \cup \mathfrak{U}_s$ of $\mathfrak{X}$, we have to show the exactness of

$$0 \to H_0(\mathfrak{U}) \to \bigoplus_i H_0(\mathfrak{U}_i) \underset{p_{ji}}{\overset{p_{ij}}{\rightrightarrows}} \bigoplus_{i<j} H_0(\mathfrak{U}_i \cap \mathfrak{U}_j) \tag{9}$$

where $p_{ij}\colon H_0(\mathfrak{U}_i) \to H_0(\mathfrak{U}_i \cap \mathfrak{U}_j)$ is the restriction homomorphism on the global sections induced by the inclusion $\mathfrak{U}_i \cap \mathfrak{U}_j \subseteq \mathfrak{U}_i$; and a similar exact sequence for rational sections

$$0 \to H_0^{\mathrm{rat}}(\mathfrak{U}) \to \bigoplus_i H_0^{\mathrm{rat}}(\mathfrak{U}_i) \underset{p_{ji}}{\overset{p_{ij}}{\rightrightarrows}} \bigoplus_{i<j} H_0^{\mathrm{rat}}(\mathfrak{U}_i \cap \mathfrak{U}_j). \tag{10}$$

By induction, one reduces to $s = 2$, in which case we have to show that

$$\begin{array}{ccc} H_0(\mathfrak{U}) & \xrightarrow{\quad i_2 \quad} & H_0(\mathfrak{U}_2) \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ H_0(\mathfrak{U}_1) & \xrightarrow[\quad p_1 \quad]{} & H_0(\mathfrak{U}_1 \cap \mathfrak{U}_2) \end{array} \tag{11}$$

is a Cartesian square. The construction of the commutative square (11) and the verification that it is commutative, follows easily from Lemma 3.2. Recall that (11), as a commutative square in the category of $\mathcal{O}_V$-algebras, is called *Cartesian* or a *pullback*, if $H_0(\mathfrak{U})$ is universal in this category for making the diagram commute, or, equivalently, if it is the equalizer of the two compositions $p_1 i_1$ and $p_2 i_2$. To verify this property, let $s_j \in H_0(\mathfrak{U}_j)$, for $j = 1, 2$, be such that $s_1|_{\mathfrak{U}_1 \cap \mathfrak{U}_2} = p_1(s_1) = p_2(s_2) = s_2|_{\mathfrak{U}_1 \cap \mathfrak{U}_2}$. We then define $s \in H_0(\mathfrak{U})$ as follows. Given a fat point $\mathfrak{z}$, let $s(\mathfrak{z})$ be the map sending $a \in \mathfrak{U}(\mathfrak{z})$ to $s_j(\mathfrak{z})(a)$ if $a \in \mathfrak{U}_j(\mathfrak{z})$. This is well-defined, since $s_1(\mathfrak{z})$ and $s_2(\mathfrak{z})$ agree on $\mathfrak{U}_1(\mathfrak{z}) \cap \mathfrak{U}_2(\mathfrak{z})$ by assumption. One easily checks that $i_j(s) = s_j$, and, moreover, if the s_j are rational, then so is s. So, remains to verify that this defines a morphism of sieves $s\colon \mathfrak{U} \to \mathbb{L}$, that is to say, a global section of $\mathfrak{U}$. Let X be an ambient space of $\mathfrak{X}$ and let $\varphi\colon Y \to X$ be a morphism such that $\mathtt{Im}(\varphi) \subseteq \mathfrak{U}$. Let $U = U_1 \cup U_2$ be the Zariski opens of X so that $\mathfrak{U} = \mathfrak{X} \cap U$ and $\mathfrak{U}_j = \mathfrak{X} \cap U_j$. In particular, $\varphi^{-1}(U) = Y$. The image of the restriction of φ to $\varphi^{-1}(U_j)$ then lies in $\mathfrak{U}_j$, and hence, by definition of morphism, the composition of this restriction with s_j is induced by a morphism $\psi_j\colon Y \to \mathbb{L}$. It is now easy to see that the ψ_j coincide on their common domain $\varphi^{-1}(U_1) \cap \varphi^{-1}(U_2)$, and hence glue together to a morphism $\psi\colon Y \to \mathbb{L}$, inducing the composition $s \circ \varphi^\circ$, as we needed to prove. □

So we are justified in calling $\mathcal{O}_{\mathfrak{X}}$ the *structure sheaf* of the sieve $\mathfrak{X}$ on a V-scheme X, and $\mathcal{O}^{\text{rat}}_{\mathfrak{X}}$ its *rational structure sheaf.*

3.11. Stalks. Let $\mathfrak{X}$ be a sieve with ambient space X. A closed point $P \in X$ is called a *point* on $\mathfrak{X}$, if the closed immersion $i_P\colon P \subseteq X$, viewed as a P-rational point, belongs to $\mathfrak{X}(P)$, or equivalently, if $P^\circ \subseteq \mathfrak{X}$. We define the stalk at a point $P \in \mathfrak{X}$ as usual as the respective direct limits

$$\mathcal{O}_{\mathfrak{X},P} := \varinjlim \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \qquad \text{and} \qquad \mathcal{O}^{\text{rat}}_{\mathfrak{X},P} := \varinjlim \mathcal{O}^{\text{rat}}_{\mathfrak{X}}(\mathfrak{U})$$

where $\mathfrak{U}$ runs over all admissible opens of $\mathfrak{X}$ such that $P \in \mathfrak{U}$. Clearly, if $\mathfrak{X} = X^\circ$ is representable, then $\mathcal{O}_{X^\circ,P} = \mathcal{O}^{\text{rat}}_{X^\circ,P}$ is just the local ring $\mathcal{O}_{X,P}$ at the closed point $P \in X$ by Proposition 3.3. In fact, we have:

3.12. Proposition. *If $\mathfrak{X}$ is a sieve which is Zariski dense in X, and if P is a point on $\mathfrak{X}$, then $\mathcal{O}^{\text{rat}}_{\mathfrak{X},P} = \mathcal{O}_{X,P}$.*

Proof. One inclusion is immediate, so let $s \in \mathcal{O}^{\text{rat}}_{\mathfrak{X},P}$. Hence there exists an open $U \subseteq X$ containing P such that $s\colon \mathfrak{X} \cap U \to \mathbb{L}$ is a rational section. Since $\mathfrak{X} \cap U$ is then Zariski dense in U, there exists an open ambient space $\tilde{U} \subseteq U$ of $\mathfrak{X} \cap U$ and a morphism $\tilde{U} \to \mathbb{L}$ extending s. This morphism corresponds to a global section of $\tilde{U}$ and hence is an element in $\mathcal{O}_{\tilde{U},P} = \mathcal{O}_{X,P}$, since $\tilde{U}$ is open in X containing P. □

More generally, if $\mathfrak{X}$ is a sieve on X and P a point on $\mathfrak{X}$, then $\mathcal{O}^{\text{rat}}_{\mathfrak{X},P} = \mathcal{O}_{\bar{\mathfrak{X}},P}$, where $\bar{\mathfrak{X}}$ is the Zariski closure (see §2.5) of $\mathfrak{X}$.

3.13. Lemma. *A global section $s\colon \mathfrak{X} \to \mathbb{L}$ of a sieve $\mathfrak{X}$ is a unit if and only if the image of $s(P)$ does not contain zero, for any point $P \in \mathfrak{X}$. If V is the spectrum of an algebraically closed field κ, then this is equivalent with the image of $s(\kappa)$ not containing zero.*

Proof. One direction is clear, so assume that the image of $s(P)$ does not contain zero, for any closed point P. Let $\mathfrak{z}$ be a fat point, and let P be its center. It follows from the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{X}(\mathfrak{z}) & \xrightarrow{\;s(\mathfrak{z})\;} & R \\
\big\downarrow & & \big\downarrow \pi \\
\mathfrak{X}(P) & \xrightarrow[\;s(P)\;]{} & k(P)
\end{array}
\tag{12}
$$

where $k(P)$ is the residue field of P, that the image of $s(\mathfrak{z})$ has empty intersection with the maximal ideal of the coordinate ring R of $\mathfrak{z}$, since π is just the residue map. Hence, for each $a \in \mathfrak{X}(\mathfrak{z})$, its image $s(\mathfrak{z})(a)$ is a unit in R, and hence we can define $t(\mathfrak{z})(a)$ to be its inverse. So remains to check that t is a morphism of sieves $\mathfrak{X} \to \mathbb{L}$, that is to say, a global section, and this is easy. □

3.14. Proposition. *For each point P on a sieve $\mathfrak{X}$, the stalk $\mathcal{O}_{\mathfrak{X},P}$ is a local ring.*

Proof. Let X be an ambient space of $\mathfrak{X}$. We have to show that given two non-units $s, t \in \mathcal{O}_{\mathfrak{X},P}$, their sum is a non-unit as well. Shrinking X if necessary, we may assume that $s, t \in H_0(\mathfrak{X})$. I claim that $s(P)(i_P)$ and $t(P)(i_P)$ are both equal to zero, where $i_P\colon P \subseteq X$ is the closed immersion. Indeed, suppose not, say, $s(P)(i_P) \neq 0$, so that there exists an open $U \subseteq X$ containing P such that $s(Q)$ does not vanish on $U(Q)$ for any closed point $Q \in U$. By Lemma 3.13, this implies that s is a unit in $H_0(\mathfrak{X} \cap U)$ whence in $\mathcal{O}_{\mathfrak{X},P}$, contradiction. Hence $s(P) + t(P)$ also vanishes at i_P and hence cannot be a unit in $\mathcal{O}_{\mathfrak{X},P}$. Note that we in fact proved that the unique maximal ideal consists of all sections $s \in \mathcal{O}_{\mathfrak{X},P}$ such that $s(P)(i_P) = 0$. □

3.15. Theorem. *Any morphism $s\colon \mathfrak{X} \to \mathfrak{Z}$ of sieves with $\mathfrak{X}$ sub-schemic and $\mathfrak{Z}$ affine, is rational. In particular, $H_0(\mathfrak{X}) = H_0^{rat}(\mathfrak{X})$.*

Proof. Let us prove the second assertion first. Let $\varphi\colon Y \to X$ be a morphism of V-schemes, so that $\mathfrak{X} = \mathtt{Im}(\varphi)$. Replacing X by the Zariski closure of $\mathtt{Im}(\varphi)$, we may assume that φ is strongly dominant. Our objective is to show that we have an equality

$$H_0^{\text{rat}}(\mathtt{Im}(\varphi)) = H_0(\mathtt{Im}(\varphi)). \tag{13}$$

Assume first that $Y = \mathfrak{y}$ is a fat point, and hence, since φ is strongly dominant, so is then $X = \mathfrak{x}$. Consider the induced homomorphism of Artinian local rings $R \subseteq S$, which is injective precisely because φ is strongly dominant. Let $s\colon \mathtt{Im}(\varphi) \to \mathbb{L}$ be a global section. By definition of morphism (see §2.14), we can find $q \in S$ which, when viewed as a global section $\mathfrak{y} \to \mathbb{L}$, extends the composition $s \circ \varphi^\circ$. Let $g_1\colon S \to S \otimes_R S$ and $g_2\colon S \to S \otimes_R S$ be given by respectively $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$. Since g_1 and g_2 agree on R, the two corresponding rational points

$\mathfrak{y} \times_{\mathfrak{x}} \mathfrak{y} \to \mathfrak{y}$ have the same image under $s(\mathfrak{y} \times_{\mathfrak{x}} \mathfrak{y})$ and hence, $g_1(q) = g_2(q)$. It follows then from Lemma 3.19 below that $q \in R$.

By Corollary 3.4, this proves (13) whenever the domain of φ is a fat point. If the domain is a zero-dimensional scheme Z, then it is a finite disjoint union $\mathfrak{y}_1 \sqcup \cdots \sqcup \mathfrak{y}_s$ of fat points. By an induction argument, we may assume $s = 2$, so that $\mathtt{Im}(\varphi) = \mathtt{Im}(\varphi|_{\mathfrak{y}_1}) \cup \mathtt{Im}(\varphi|_{\mathfrak{y}_2})$. In particular, $H_0(\mathtt{Im}(\varphi))$ and $H_0^{\text{rat}}(\mathtt{Im}(\varphi))$ satisfy each the same Cartesian square (11) by what we just proved for fat points (instead of induction, we may alternatively use (9) and (10)). By uniqueness, they must therefore be equal, showing that (13) holds whenever the domain is zero-dimensional. For Y arbitrary, we may write the corresponding representable sieve Y° as the direct limit of all its zero-dimensional closed subsieves by Lemma 2.8. Let $\{\mathfrak{Z} := \mathtt{Im}(\varphi|_Z)\}$ be the collection of all sub-schemic motives, where $Z \subseteq Y$ varies over all zero-dimensional closed subschemes of Y. One easily checks, using the universal property of direct limits, that the $\mathfrak{Z}$ form a direct system of sieves with direct limit equal to $\mathtt{Im}(\varphi)$. Therefore, $H_0(\mathtt{Im}(\varphi))$ is the inverse limit of all $H_0(\mathfrak{Z}) = H_0^{\text{rat}}(\mathfrak{Z})$, by Lemma 3.18 below, and by the same argument, this inverse limit is equal to $H_0^{\text{rat}}(\mathtt{Im}(\varphi))$, proving (13).

To prove the first assertion, we may assume, without loss of generality, that $\mathfrak{Z} = Z^\circ$ is representable by an affine scheme Z, and then, since the problem is local, that $X = \operatorname{Spec} A$ is affine. Hence $Z \subseteq \mathbb{L}^n$ is a closed subscheme for some n and some open $\operatorname{Spec} \lambda \subseteq V$. Let s_i be the composition of s with the morphism induced by the projection $\mathbb{L}^n \to \mathbb{L}$ onto the i-th coordinate. Hence $s_i \in H_0(\mathfrak{X}) = H_0^{\text{rat}}(\mathfrak{X})$. Replacing X by an open subscheme if necessary as per Corollary 3.4, we can find $q_i \in A$ such that, viewed as a global section $X \to \mathbb{L}$, it extends s_i. Therefore, the morphism $X \to \mathbb{L}^n$ given by $(q_1, \dots, q_n)$ is an extension of s, as we wanted to show. □

3.16. Remark. By the above proof, we actually showed that if the domain of φ is zero-dimensional, then $H_0(\mathtt{Im}(\varphi))$ is equal to the global section ring of the Zariski closure of $\mathtt{Im}(\varphi)$. However, this is no longer true in the general case, as can be seen from Corollary 3.4. If the target space is not affine, then we have:

3.17. Theorem. *A morphism with source a sub-schemic motif is rational if and only if it is continuous.*

Proof. One direction is just Proposition 3.9, so assume $s\colon \mathfrak{Y} \to \mathfrak{X}$ is continuous and $\mathfrak{Y}$ is sub-schemic. Let Y be the Zariski closure of $\mathfrak{Y}$ and let $X_1, \dots, X_s$ be an affine open covering of X. Hence, the $\mathfrak{X}_i := \mathfrak{X} \cap X_i$ form an open covering of $\mathfrak{X}$. Since s is continuous, each pull-back $\mathfrak{Y}_i := s^*\mathfrak{X}_i$ is open in $\mathfrak{Y}$, so that we can find opens $Y_i \subseteq Y$ such that $\mathfrak{Y}_i = \mathfrak{Y} \cap Y_i$. Let $s_i\colon \mathfrak{Y}_i \to \mathfrak{X}_i$ be the restriction of s to $\mathfrak{Y}_i$. By Theorem 3.15, each s_i is rational, so that, upon shrinking Y and the Y_i if necessary, there is a morphism of schemes $\varphi_i\colon Y_i \to X_i$ inducing s_i. Moreover, from its proof it follows that each φ_i is given by a tuple of global sections of $\mathfrak{Y}_i$, and hence, since s_i and s_j agree on $\mathfrak{Y}_i \cap \mathfrak{Y}_j$, both tuples must agree on their common domain of definition $Y_i \cap Y_j$. Hence the φ_i glue together to a morphism of schemes $Y \to X$ extending s, as we needed to show. □

We conclude with the proof of the two lemmas that were used in the proof of Theorem 3.15.

3.18. Lemma. *Let $\{\mathfrak{X}_i\}$ be a direct system of sieves on some scheme X and let $\mathfrak{X}$ be their direct limit. Then $H_0(\mathfrak{X})$ is the inverse limit of all $H_0(\mathfrak{X}_i)$.*

Proof. Contravariance turns a direct limit into an inverse limit, and the rest is now an easy consequence of the universal property of inverse limits:

$$\begin{aligned} H_0(\mathfrak{X}) &= \operatorname{Mor}(\mathfrak{X}, \mathbb{L}) \\ &= \operatorname{Mor}(\varinjlim_i \mathfrak{X}_i, \mathbb{L}) \\ &= \varprojlim_i \operatorname{Mor}(\mathfrak{X}_i, \mathbb{L}) = \varprojlim_i H_0(\mathfrak{X}_i). \end{aligned}$$

□

3.19. Lemma. *Let $R \subseteq S$ be an injective homomorphism of rings. Then the tensor square*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \otimes_R S \end{array} \tag{14}$$

is Cartesian, that is to say, if $q \otimes 1 = 1 \otimes q$ in $S \otimes_R S$ for some $q \in S$, then in fact $q \in R$.

Proof. Let for simplicity assume that R and S are algebras over some field κ (since we only need the result for R and S Artinian, this already covers any equicharacteristic situation). Let $T := S \otimes_\kappa S$ be the tensor product over κ, and let $\mathfrak{n}$ be the ideal in T generated by all expressions of the form $r \otimes 1 - 1 \otimes r$ for $r \in R$. Hence $S \otimes_R S \cong T/\mathfrak{n}$. If $q \in S$ satisfies $q \otimes 1 = 1 \otimes q$ in $S \otimes_R S$, then viewed as an element in T, the tensor $q \otimes 1 - 1 \otimes q$ lies in $\mathfrak{n}$. The canonical surjection $S \to S/R$ induces a homomorphism of tensor products $T \to (S/R) \otimes_\kappa (S/R)$. Under this homomorphism, $\mathfrak{n}$ is sent to the zero ideal, whence so is in particular $q \otimes 1 - 1 \otimes q$. If the image of q in S/R were non-zero, then we can find a basis of S/R containing q. Hence $q \otimes 1$ and $1 \otimes q$ are two independent basis vectors of $(S/R) \otimes_\kappa (S/R)$, contradicting that they are equal in the latter ring. Hence $q \in R$, as we wanted to show. □

By Lemma 3.2 and properties of tensor products, we have for any two sieves $\mathfrak{X}$ and $\mathfrak{Y}$, a canonical homomorphism

$$H_0(\mathfrak{X}) \otimes_{\mathcal{O}_V} H_0(\mathfrak{Y}) \to H_0(\mathfrak{X} \times \mathfrak{Y}), \tag{15}$$

and a similar formula for rational sections. If $\mathfrak{X}$ and $\mathfrak{Y}$ are both representable, then this is an isomorphism, but not so in general.

3.20. The etale site. We call a natural morphism of sieves $s\colon \mathfrak{Y} \to \mathfrak{X}$ an *et-morphism*, or a *morphism in the etale site*, if there exists a surjective, etale morphism $\varphi\colon Z \to Y$, where Y is some ambient space of $\mathfrak{Y}$, such that the composition $\varphi^*\mathfrak{Y} \to \mathfrak{Y} \to \mathfrak{X}$ is a morphism of sieves (where $\varphi^*\mathfrak{Y}$ is the pull-back and $\varphi^*\mathfrak{Y} \to \mathfrak{Y}$ the morphism induced by φ). Clearly, any (usual) morphism is an et-morphism. Likewise, we call s *algebraic* if the above composition is rational; so to paraphrase, algebraic is rational in the etale sense. By Theorem 3.15 and the fact that a pull-back of a sub-schemic is again sub-schemic, any continuous et-morphism with source a sub-schemic motif is algebraic.

4. Motivic sites

As before, V is a fixed Noetherian scheme. A *motivic site* $\mathbb{M}$ over V is a subcategory of $\mathbb{Sieve}_V$ which is closed under products, and such that for any V-scheme X, the restriction $\mathbb{M}|_X$ (that is to say, the set of all $\mathbb{M}$-sieves on X) forms a lattice. In other words, if $\mathfrak{X}, \mathfrak{Y} \in \mathbb{M}$ are both sieves on a common scheme X, then $\mathfrak{X} \cap \mathfrak{Y}$ and $\mathfrak{X} \cup \mathfrak{Y}$ belong again to $\mathbb{M}$ (including the minimum given by the empty set and the maximum given by X). Sieves in a motivic site $\mathbb{M}$ will often be called *motives*. This nomenclature is to express the fact that a motif represents something geometrical which is not a scheme but ought to be something like a scheme, thus 'motivating' our geometric treatment of it.

We call $\mathbb{M}$ an *explicit* motivic site, if all continuous morphisms are rational. If $\mathbb{M}$ is an arbitrary motivic site, then we let $\overline{\mathbb{M}}$ be the corresponding explicit motivic site, obtained by only taking rational morphisms.

4.1. The Grothendieck ring of a motivic site. Let $\mathbb{M}$ be a motivic site. Given two $\mathbb{M}$-motives $\mathfrak{X}$ and $\mathfrak{Y}$, we say that they are $\mathbb{M}$-*homeomorphic*, if there exists a continuous, bijective $\mathbb{M}$-morphism $\mathfrak{X} \to \mathfrak{Y}$, whose inverse is again a continuous $\mathbb{M}$-morphism. We denote the isomorphism class of a motif $\mathfrak{X}$ by $\langle\mathfrak{X}\rangle$. As the sieves in $\mathbb{M}$ form locally a lattice (on each V-scheme), we can now define its associated Grothendieck ring $\mathbf{Gr}(\mathbb{M})$ as the free Abelian group on isomorphism classes $\langle\mathfrak{X}\rangle$, where $\mathfrak{X}$ runs over all $\mathbb{M}$-motives, modulo the *scissor relations*

$$\langle\mathfrak{X}\rangle + \langle\mathfrak{Y}\rangle - \langle\mathfrak{X} \cup \mathfrak{Y}\rangle - \langle\mathfrak{X} \cap \mathfrak{Y}\rangle$$

for any two $\mathbb{M}$-motives $\mathfrak{X}$ and $\mathfrak{Y}$ on a common V-scheme. We denote the image of an $\mathbb{M}$-motif $\mathfrak{X}$ in $\mathbf{Gr}(\mathbb{M})$ by $[\mathfrak{X}]$. In particular, since each representable functor is in $\mathbb{M}$, we may associate to any V-scheme X its class $[X] := [X^\circ]$ in $\mathbf{Gr}(\mathbb{M})$. We define a multiplication on $\mathbf{Gr}(\mathbb{M})$ by the fiber product (one easily checks that this is well-defined): $[\mathfrak{X}] \cdot [\mathfrak{Y}] := [\mathfrak{X} \times \mathfrak{Y}]$. Since a motivic site has the same objects as its explicit counterpart, we get a canonical surjective homomorphism $\mathbf{Gr}(\overline{\mathbb{M}}) \to \mathbf{Gr}(\mathbb{M})$, which, however, need not be injective, since there are more homeomorphism relations in the latter Grothendieck ring. Note that $V(\mathfrak{z})$ is a singleton, consisting of the structure morphism $\mathfrak{z} \to V$, for every fat V-point $\mathfrak{z}$. So, we set

$$1 := [V], \tag{16}$$

as it is the unit for multiplication.

On occasion, we will encounter variants which are supported only on a subcategory of the category of all V-schemes (that is to say, we only require the restriction of the site to one of the schemes in the subcategory to be a lattice), and we can still associate a Grothendieck ring to it. We will refer to this as a *partial motivic site*. If instead we work on the etale site, the same definitions then yield an *et-motivic* site as a subcategory of the category of sieves on the etale site. The only difference is that morphisms are now et-morphisms. We will not study these in detail here.

Most motivic sites $\mathbb{M}$ will also have additional properties, like for instance being *stable under push-forwards along closed immersions*, meaning that if $i\colon Y \subseteq X$ is a closed subscheme and $\mathfrak{Y}$ a motif on Y in $\mathbb{M}$, then $i_*\mathfrak{Y}$ too is a motif in $\mathbb{M}$. If this is the case, then $\mathbb{M}$ is also closed under disjoint unions: given motives $\mathfrak{X}$ and $\mathfrak{X}'$ on X and X' respectively, then their disjoint union $\mathfrak{X} \sqcup \mathfrak{X}'$ is the union of the push-forwards $i_*\mathfrak{X}$ and $i'_*\mathfrak{X}'$, where $i\colon X \to X \sqcup X'$ and $i'\colon X' \to X \sqcup X'$ are the canonical closed immersions. Moreover, the general scissor relations then are easily seen to be generated by just the disjointness relations

$$\langle \mathfrak{X} \sqcup \mathfrak{Y} \rangle = \langle \mathfrak{X} \rangle + \langle \mathfrak{Y} \rangle \tag{17}$$

for any pair of disjoint motives $\mathfrak{X}$ and $\mathfrak{Y}$ on some common ambient space. Using this, it is now easy to see that any element in $\mathbf{Gr}(\mathbb{M})$ is a difference $[\mathfrak{X}] - [\mathfrak{Y}]$, for some motives $\mathfrak{X}$ and $\mathfrak{Y}$.

4.2. Lemma. *Given a motivic site $\mathbb{M}$ which is stable under push-forwards along closed immersions, two motives $\mathfrak{X}$ and $\mathfrak{Y}$ have the same class in $\mathbf{Gr}(\mathbb{M})$ if and only if there exists a motif $\mathfrak{Z}$ such that $\mathfrak{X} \sqcup \mathfrak{Z}$ and $\mathfrak{Y} \sqcup \mathfrak{Z}$ are $\mathbb{M}$-isomorphic.*

Proof. One direction is immediately, for if $\mathfrak{X} \sqcup \mathfrak{Z}$ and $\mathfrak{Y} \sqcup \mathfrak{Z}$ are isomorphic, then $[\mathfrak{X}] + [\mathfrak{Z}] = [\mathfrak{X} \sqcup \mathfrak{Z}] = [\mathfrak{Y} \sqcup \mathfrak{Z}] = [\mathfrak{Y}] + [\mathfrak{Z}]$ in $\mathbf{Gr}(\mathbb{M})$, from which it follows $[\mathfrak{X}] = [\mathfrak{Y}]$. Conversely, if $[\mathfrak{X}] = [\mathfrak{Y}]$, then by (17), there exist mutually disjoint motives $\mathfrak{A}_i, \mathfrak{B}_i, \mathfrak{C}_i, \mathfrak{D}_i$ such that

$$\langle\mathfrak{X}\rangle + \sum_i \langle\mathfrak{A}_i\rangle + \langle\mathfrak{B}_i\rangle - \langle\mathfrak{A}_i \sqcup \mathfrak{B}_i\rangle = \langle\mathfrak{Y}\rangle + \sum_i \langle\mathfrak{C}_i\rangle + \langle\mathfrak{D}_i\rangle - \langle\mathfrak{C}_i \sqcup \mathfrak{D}_i\rangle$$

in the free Abelian group on isomorphism classes. Bringing the terms with negative signs to the other side, we get an expression in which each term on the left hand side must also occur on the right hand side, that is to say, the collection of all isomorphism classes $\{\langle\mathfrak{X}\rangle, \langle\mathfrak{A}_i\rangle, \langle\mathfrak{B}_i\rangle, \langle\mathfrak{C}_i \sqcup \mathfrak{D}_i\rangle\}$ is the same as the collection of all isomorphism classes $\{\langle\mathfrak{Y}\rangle, \langle\mathfrak{C}_i\rangle, \langle\mathfrak{D}_i\rangle, \langle\mathfrak{A}_i \sqcup \mathfrak{B}_i\rangle\}$. By properties of disjoint union, we therefore get $\langle\mathfrak{X} \sqcup \mathfrak{Z}\rangle = \langle\mathfrak{Y} \sqcup \mathfrak{Z}\rangle$, where $\mathfrak{Z}$ is the disjoint union of all motives $\mathfrak{A}_i$, $\mathfrak{B}_i$, $\mathfrak{C}_i$, and $\mathfrak{D}_i$. □

4.3. Definition (Lefschetz class)**.** The class of the affine line $\mathbb{L}$ plays a pivotal role in what follows; we call it the *Lefschetz class* and keep denoting it by $\mathbb{L}$ (as the context will always make clear whether we mean the scheme, the associated motif, or its class), or $\mathbb{L}_V$ in case we want to make explicit the base scheme V. When dealing with motivic rationality questions (see the forthcoming [11]), we will need to invert this class, and therefore also consider localizations of the form $\mathbf{Gr}(\mathbb{M})_{\mathbb{L}}$.

5. The schemic Grothendieck ring

To connect the theory of motivic sites to the classical construction, we must describe motivic sites whose Grothendieck ring admits a natural homomorphism into the classical Grothendieck ring $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_V)$ (obviously, this fails miserably for the motivic site of all sieves). We will first introduce the various motives of interest in the next few sections, before we settle this issue in Theorem 7.7 below. The smallest motivic site on V is obtained by taking for sieves on a scheme X only the empty sieve and the whole sieve X°. The resulting Grothendieck ring has no non-trivial scissor relations and so we just get the free Abelian ring on isomorphism classes of V-schemes.

To define larger sites, we want to include at least closed subsieves of a scheme X. Any object in the lattice generated by the closed subsieves of X will be called a *schemic motif on* X. Since closed subsieves are already closed under intersection, a schemic motif on X is a sieve of the form

$$\mathfrak{X} = X_1^\circ \cup \cdots \cup X_s^\circ, \tag{18}$$

where the X_i are closed subschemes of X. Let us call a V-scheme X *schemically irreducible* if X° cannot be written as a finite union of proper closed subsieves. In particular, by an easy Noetherian argument, any schemic motif is the union of finitely many schemically irreducible closed subsieves. We call a representation (18) a *schemic decomposition*, if it is *irredundant*, meaning that there are no closed subscheme relations among any two X_i, and the X_i are schemically irreducible. Assume (18) is a schemic decomposition, and let $\mathfrak{X} = Y_1^\circ \cup \cdots \cup Y_t^\circ$ be a second schemic decomposition. Hence, for a fixed i, we have

$$X_i^\circ = X_i^\circ \cap \mathfrak{X} = (X_i \cap Y_1)^\circ \cup \cdots \cup (X_i \cap Y_t)^\circ.$$

Since X_i is schemically irreducible, there is some j such that $X_i = X_i \cap Y_j$, that is to say, $X_i \subseteq Y_j$. Reversing the roles of the two representations, the same argument yields some i' such that $Y_j \subseteq X_{i'}$. Since $X_i \subseteq Y_j \subseteq X_{i'}$, irredundancy implies that these are equalities. Hence, we proved:

5.1. Proposition. *A schemic decomposition is, up to order, unique.* □

We call the X_i in the (unique) schemic decomposition (18) the *schemic irreducible components* of $\mathfrak{X}$. If X has dimension zero, then it is a finite disjoint sum of fat points, its schemic irreducible components. More generally, any schemic motif on X is a disjoint sum of closed subsieves given by fat points, and hence is itself a closed subsieve.

To give a purely scheme-theoretic characterization of being schemically irreducible, recall that a point $x \in X$ is called *associated*, if $\mathcal{O}_{X,x}$ has depth zero, that is to say, if every element in $\mathcal{O}_{X,x}$ is either a unit or a zero-divisor. Any minimal point is associated, and the remaining ones, which are also finite in number, are called *embedded*. The closure of an associated point is called a *primary component*. We say that X is *strongly connected*, if the intersection of all primary components is non-empty, that is to say, if there exists a (closed) point generalizing to each associated point (in the affine case $X = \operatorname{Spec} A$, this means that the associated primes generate a proper ideal). For instance, if X is the union of two parallel lines and one intersecting line, then it is connected but not strongly. For an example of a scheme with embedded points which is not strongly connected, take the affine line with two (embedded) double points given by the ideal $(x^2, xy(y-1))$; a (reduced) example where any two primary, but not all three, components meet, is given by the 'triangle' $xy(x-y-1)$.

5.2. Proposition. *A V-scheme X is schemically irreducible if and only if it is strongly connected.*

Proof. It is easier to work with the contrapositives of these statements, and we will show that their negations are then also equivalent with the existence of finitely many non-zero ideal sheafs $\mathcal{I}_1, \dots, \mathcal{I}_s \subseteq \mathcal{O}_X$ with the property that for each closed point x, there is some n such that $\mathcal{I}_n\mathcal{O}_{X,x} = 0$. To prove the equivalence of this with being schemically reducible, assume $X^\circ = X_1^\circ \cup \cdots \cup X_s^\circ$ for some proper closed subschemes $X_n \subsetneq X$. Let $\mathcal{I}_n$ be the ideal sheaf of X_n and let x be an arbitrary closed point. For each m, the closed immersion $J_x^m X \subseteq X$ is a rational point on X along the m-th co-jet, whence must belong to one of the $X_n(J_x^m X)$, that is to say, $J_x^m X$ is a closed subscheme of X_n by Lemma 2.4. Since there are only finitely many possibilities, there is a single n such that each $J_x^m X$ is a closed subscheme of X_n. By Lemma 2.4, this means that $\mathcal{I}_n \mathcal{O}_{X,x}$ is contained in any power of the maximal ideal $\mathfrak{m}_x$, and hence by Krull's intersection theorem must be zero. Conversely, suppose there are non-zero ideal sheaves $\mathcal{I}_1, \dots, \mathcal{I}_s \subseteq \mathcal{O}_X$ such that at each closed point, at least one vanishes. Let X_n be the closed subscheme defined by $\mathcal{I}_n$, let $\mathfrak{z}$ be a fat point, and let $a\colon \mathfrak{z} \to X$ be a $\mathfrak{z}$-rational point. Let x be the center of $\mathfrak{z}$, a closed point of X, and let $a_x\colon \mathcal{O}_{X,x} \to R$ be the induced local homomorphism on the stalks, where R is the coordinate ring of $\mathfrak{z}$. By assumption, there is some n such that $\mathcal{I}_n \mathcal{O}_{X,x} = 0$, whence so is its image under a_x. By Lemma 2.4, this implies that $a \in X_n(\mathfrak{z})$. Since this holds for any rational point, X° is the union of all X_n°.

We can now prove the equivalence of the above condition with failing to be strongly connected. By (global) primary decomposition (see for instance [5, IV §3.2]), there exist (primary) closed subschemes $Y_n \subseteq X$ and an embedding $\mathcal{O}_X \hookrightarrow \mathcal{O}_{Y_1} \oplus \cdots \oplus \mathcal{O}_{Y_s}$, such that the underlying sets of the Y_n are the primary components of X. Let $\mathcal{J}_n$ be the ideal sheaf of Y_n, so that the above embeddability amounts to $\mathcal{J}_1 \cap \cdots \cap \mathcal{J}_s = 0$. If X is not strongly connected, then the intersection of all Y_n is empty, which means that

$$\mathcal{J}_1 + \cdots + \mathcal{J}_s = \mathcal{O}_X. \tag{19}$$

Let $\mathcal{I}_j$ be the intersection of all $\mathcal{J}_m$ with $m \neq j$, and let X_j be the closed subscheme given by $\mathcal{I}_j$. Let x be any closed point. By (19), we may assume after renumbering that the maximal ideal of x does not contain $\mathcal{J}_1$, that is to say, $\mathcal{J}_1 \mathcal{O}_{X,x} = \mathcal{O}_{X,x}$. Therefore, $\mathcal{I}_1 \mathcal{O}_{X,x} = (\mathcal{I}_1 \cap \mathcal{J}_1)\mathcal{O}_{X,x}$, whence is zero, since $\mathcal{I}_1 \cap \mathcal{J}_1 = 0$. Conversely, assume there are non-zero ideal sheafs $\mathcal{I}_1, \dots, \mathcal{I}_s \subseteq \mathcal{O}_X$ such that $\mathcal{I}_n \mathcal{O}_{X,x} = 0$, for each closed point x and for some n depending on x. This is equivalent with the sum of all $\operatorname{Ann}(\mathcal{I}_n)$ being the unit ideal. Since any annihilator ideal is contained in some associated prime, the sum of all associated primes must also be the unit ideal, and hence the intersection of all primary components is empty. □

From the proof we learn that $\operatorname{Spec} A$ is schemically reducible if and only if there exist finitely many proper ideals whose sum is the unit ideal and whose intersection is the zero ideal. We can even describe an algorithm which calculates its schemic irreducible components. Let $0 = \mathfrak{g}_1 \cap \cdots \cap \mathfrak{g}_n$ be a primary decomposition of the zero ideal in A, and assume $\mathfrak{g}_1 + \cdots + \mathfrak{g}_s = 1$, for some $s \leqslant n$. Then the schemic irreducible components of X are among the schemic irreducible components of the closed subschemes $X_i = \operatorname{Spec}(A/\operatorname{Ann}(\mathfrak{g}_i))$, for $i = 1, \dots, s$. That the X_i can themselves be schemic reducible, whence require further decomposition, is illustrated by the 'square' with equation $x(x-1)y(y-1) = 0$. A sufficient condition for the X_i to be already schemically irreducible is that $s = n$ and no fewer $\mathfrak{g}_i$ generate the unit ideal. By Noetherian induction, this has to happen eventually. In the same vein, we have:

5.3. Proposition. *Let X be a V-scheme, and let $X_1, \dots, X_s \subseteq X$ be closed subschemes with respective ideals of definition $\mathcal{I}_1, \dots, \mathcal{I}_s \subseteq \mathcal{O}_X$. The Zariski closure of the schemic motif $\mathcal{X} := X_1^\circ \cup \dots \cup X_s^\circ$ is the closed subscheme with ideal of definition $\mathcal{I}_1 \cap \dots \cap \mathcal{I}_s$. In particular, $\mathcal{X}$ is equal to its own Zariski closure if and only if $\mathcal{I}_1 + \dots + \mathcal{I}_s = \mathcal{O}_X$.*

Proof. Let Y be the closed subscheme with ideal of definition $\mathcal{I} := \mathcal{I}_1 \cap \dots \cap \mathcal{I}_s$. Since each $X_j \subseteq Y$, the Zariski closure of $\mathfrak{X}$ is contained in Y. To prove the converse, suppose $Z \subseteq X$ is a closed subscheme such that $\mathfrak{X} \subseteq Z^\circ$. We have to show that $Y \subseteq Z$, so suppose not. This mean that $\mathcal{J}\mathcal{O}_Y \neq 0$, where $\mathcal{J}$ is the ideal of definition of Z. Choose a closed point $x \in Y$, with maximal ideal $\mathfrak{m}_x$, such that $\mathcal{J}\mathcal{O}_{Y,x} \neq 0$. By Krull's Intersection Theorem, there is some n such that $\mathcal{J}(\mathcal{O}_{Y,x}/\mathfrak{m}_x^n) \neq 0$. We may write $\mathfrak{m}_x^n = \mathfrak{n}_1 \cap \dots \cap \mathfrak{n}_t$ as a finite intersection of irreducible ideal sheafs,[3] and then for at least one, say for $j = 1$, we must have $\mathcal{J}(\mathcal{O}_{Y,x}/\mathfrak{n}_1) \neq 0$. Let $\mathfrak{z}$ be the fat point with coordinate ring $R := \mathcal{O}_{Y,x}/\mathfrak{n}_1$. By Lemma 2.4, this means that the $\mathfrak{z}$-rational point a given by the inclusion $\mathfrak{z} \subseteq Y$ does not factor through Z. On the other hand, by the same lemma, we have $\mathcal{I}R = 0$. Since the zero ideal is irreducible, at least one of the $\mathcal{I}_j R$ must vanish, showing that a lies in $\mathfrak{X}(\mathfrak{z})$, whence by assumption in $Z(\mathfrak{z})$, contradiction. The last assertion is now immediate from the previous discussion. □

5.4. Corollary. *The global section ring of a schemic motif is equal to that of its Zariski closure.*

Proof. By an induction argument, we may reduce to the case that $\mathfrak{X} = Y^\circ \cup Z^\circ$ where $Y, Z \subseteq X$ are closed subschemes (alternatively, use (9)). In view of the local nature of the problem, we may furthermore reduce to the case that $X = \operatorname{Spec} A$ is affine, so that Y and Z are defined by some ideals $I, J \subseteq A$. In particular, the Cartesian square (11) becomes

$$\begin{array}{ccc} H_0(\mathfrak{X}) & \longrightarrow & A/I \\ \downarrow & & \downarrow \\ A/J & \longrightarrow & A/(I+J). \end{array} \tag{20}$$

However, it is easy to check that putting $A/(I \cap J)$ in the left top corner of this square also yields a Cartesian square, and hence, by uniqueness, we must have $H_0(\mathfrak{X}) =$

[3] An ideal is called *irreducible* if it cannot be written as a finite intersection of strictly larger ideals.

$A/(I \cap J)$. By Proposition 5.3, the Zariski closure of $\mathfrak{X}$ is $\operatorname{Spec}(A/(I \cap J))$, proving the assertion. □

5.5. Definition. We define the *schemic motivic site* over V, denoted $\mathbb{Sch}_V$, as the full subcategory of $\mathbb{Sieve}_V$ consisting of all schemic motives. By our definition of morphism of sieves, the category of V-schemes fully embeds in $\mathbb{Sch}_V$, and the image of this embedding is precisely the full subcategory of representable schemic motives. In fact, by Theorem 3.17, all continuous morphisms in $\mathbb{Sch}_V$ are rational, so that $\mathbb{Sch}_V$ is an explicit motivic site.

5.6. Lemma. *Any element in* $\mathbf{Gr}(\mathbb{Sch}_V)$ *is of the form* $[X] - [Y]$, *for some* V*-schemes* X *and* Y.

Proof. For any two V-schemes X and X', we have $[X] + [X'] = [X \sqcup X']$, where $X \sqcup X'$ denotes their disjoint union. So remains to verify that any element in $\mathbf{Gr}(\mathbb{Sch}_V)$ is a linear combination of classes of schemes. This reduces the problem to the class of a single schemic motif $\mathfrak{X}$ on a V-scheme X. Hence there exist closed subschemes $X_1, \dots, X_n \subseteq X$ such that $\mathfrak{X} = X_1^\circ \cup \dots \cup X_n^\circ$. For each non-empty $I \subseteq \{1, \dots, n\}$, let X_I be the closed subscheme obtained by intersecting all X_i with $i \in I$, and let $|I|$ denote the cardinality of I. A well-known argument deduces from the scissor relations the equality

$$[\mathfrak{X}] = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|}[X_I] \tag{21}$$

in $\mathbf{Gr}(\mathbb{Sch}_V)$, proving the claim. □

5.7. Theorem. *The schemic Grothendieck ring* $\mathbf{Gr}(\mathbb{Sch}_V)$ *is freely generated, as an additive group, by the classes of strongly connected* V*-schemes.*

Proof. Let Γ be the free Abelian group generated by isomorphism classes $\langle X \rangle$ of strongly connected V-schemes X, and let Γ' be the free Abelian group generated by homeomorphism classes $\langle \mathfrak{X} \rangle$ of schemic motives $\mathfrak{X}$. I claim that the composition $\Gamma \subseteq \Gamma' \twoheadrightarrow \mathbf{Gr}(\mathbb{Sch}_V)$ admits an additive inverse $\delta \colon \mathbf{Gr}(\mathbb{Sch}_V) \to \Gamma$.

To construct δ, we will first define an additive morphism $\delta' \colon \Gamma' \to \Gamma$ which is the identity on Γ, and then argue that it vanishes on each scissor relation, inducing therefore a morphism $\delta \colon \mathbf{Gr}(\mathbb{Sch}_V) \to \Gamma$. It suffices, by linearity, to define δ' on an homeomorphism class of a schemic motif

$$\mathfrak{X} = X_1^\circ \cup \dots \cup X_n^\circ, \tag{22}$$

given by a schemic decomposition with $X_i \subseteq X$ strongly connected, closed subschemes. By Noetherian induction, we may furthermore assume that δ' has been defined on the homeomorphism class of any schemic motif on a proper closed subscheme of X. In particular, $\delta'\langle X_I \rangle$ has already been defined, where we borrow the notation from (21). We therefore set

$$\delta'\langle \mathfrak{X} \rangle := \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|} \delta'\langle X_I \rangle \tag{23}$$

By Proposition 5.1, the schemic decomposition is unique, and hence δ' is well-defined. Moreover, if $n = 1$, so that $\mathfrak{X} = X_1^\circ$ is a schemically irreducible closed subsieve, whence strongly connected by Proposition 5.2, then its image under δ' is just $\langle X_1 \rangle$, showing that δ' is the identity on Γ.

Next, let us show that even if (22) is not irredundant, (23) still holds in Γ. We can go from such an arbitrary representation to the schemic decomposition in finitely many steps, by adding or omitting at each step one strongly connected closed subsieve contained in $\mathfrak{X}$. So assume (23) holds for a representation (22), and we now have to show that it also holds for the representation adding a $X_0^\circ \subseteq \mathfrak{X}$, with X_0 strongly connected. Since X_0 is schemically irreducible by Proposition 5.2, it must be a closed subscheme of one of the others, say, of X_1. Let J range over all non-empty subsets of $\{0, \dots, n\}$. To each subset J_+ containing 0 and 1, we associate the subset $J_- := J_+ \setminus \{1\}$. Since $X_0 \subseteq X_1$, we get $X_{J_+} = X_{J_-}$, and as J_- has one element less than J_+, the two terms corresponding to J_+ and J_- in the sum (23) cancel each other out. So, in that sum, only subsets J not containing 0 contribute, which is just the value for the representation without X_0°. We also have to consider the converse case, where instead we omit one, but the argument is the same.

We can now show that (23) is still valid even if the X_i in (22) are not strongly connected. Again we may reduce the problem to adding or omitting a single closed subsieve X_0°. Let $X_0^\circ = Y_1^\circ \cup \dots \cup Y_m^\circ$ be a schemic decomposition for X_0. We have to show that the value of the sum in (23) for the representation $\mathfrak{X} = X_0^\circ \cup \dots \cup X_n^\circ$ is the same as that for the representation

$$\mathfrak{X} = X_1^\circ \cup \dots \cup X_n^\circ \cup Y_1^\circ \cup \dots \cup Y_m^\circ. \tag{24}$$

The first sum is given by

$$\sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|} \delta' \langle X_I \rangle + \sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \delta' \langle X_0 \cap X_I \rangle \tag{25}$$

By Noetherian induction and the fact that $(X_0 \cap X_I)^\circ = (X_I \cap Y_1)^\circ \cup \dots \cup (X_I \cap Y_m)^\circ$, we have an identity

$$\delta' \langle X_0 \cap X_I \rangle = \sum_{\emptyset \neq J \subseteq \{1,\dots,m\}} (-1)^{|J|} \delta' \langle X_I \cap Y_J \rangle$$

for each subset I. Substituting this is in (25) yields the sum corresponding to representation (24) (note that $I \sqcup J$ ranges over all non-empty subsets of $\{1, \dots, n\} \sqcup \{1, \dots, m\}$, as required).

To obtain the induced map δ, we must show next that δ' vanishes on any scissor relation. To this end, let $\mathfrak{Y} = Y_1^\circ \cup \dots \cup Y_t^\circ$ be a second schemic motif on X, again assumed to be given by its schemic decomposition. Put $Z_{ij} := X_i \cap Y_j$, so that $\mathfrak{X} \cup \mathfrak{Y}$ is the union of the X_i° and Y_j°, whereas $\mathfrak{X} \cap \mathfrak{Y}$ is the union of the closed subsieves Z_{ij}°. By our previous argument, we may use these respective representations to calculate

δ' of the scissor relation $\langle \mathfrak{X} \cup \mathfrak{Y}\rangle + \langle \mathfrak{X} \cap \mathfrak{Y}\rangle - \langle \mathfrak{X}\rangle - \langle \mathfrak{Y}\rangle$. Comparing the various sums given by the respective right hand sides of (23), this reduces to the following combinatorial assertion. Given finite subsets I, J, let us call a subset $N \subseteq I \times J$ *dominant*, if its projections onto the first and second coordinates are both surjective; then

$$\sum_{N \text{ dominant}} (-1)^{|N|} = 1.$$

We leave the details to the reader. In conclusion, we have constructed an (additive) map $\delta \colon \mathbf{Gr}(\mathbb{S}\mathbb{c}\mathbb{h}_V) \to \Gamma$ which is the identity on Γ. On the other hand, it follows from (21) that $[\delta[\mathfrak{X}]] = [\mathfrak{X}]$, showing that δ is in fact an isomorphism. □

Immediately from this we get:

5.8. Corollary. *Given a strongly connected V-scheme X, then $[X] = [Y]$ in $\mathbf{Gr}(\mathbb{S}\mathbb{c}\mathbb{h}_V)$, for some V-scheme Y, if and only if $X \cong Y$.* □

5.9. Remark. As in the classical case, the schemic Grothendieck ring is not a domain. In fact, using some examples due to Danielewski, we can show that $\mathbb{L}$ is even a zero-divisor. Indeed, let $Y_n \subseteq \mathbb{L}^3$ be the smooth surface with equation $x^n y + z^2 = 1$. Then all $Y_n \times \mathbb{L}$ are mutually isomorphic, but no two Y_n are isomorphic (see, for instance, [4, 8]). So $[Y_n]\mathbb{L} = [Y_m]\mathbb{L}$, but $[Y_n] \neq [Y_m]$, for $m \neq n$.

6. The sub-schemic Grothendieck ring

To allow for additional relations, we want to include also open subsieves, or more generally, locally closed subsieves (see (2.3.iii)). For applications, it is more appropriate to put this in a larger context. Our point of departure is:

6.1. Lemma. *For any V-scheme, the set of its sub-schemic sieves forms a lattice. Moreover, the product of two sub-schemic sieves is again sub-schemic.*

Proof. Recall that a sub-schemic sieve is just an image sieve. If $\varphi \colon Y \to X$ and $\psi \colon Z \to X$ are morphisms of V-schemes, then $\mathtt{Im}(\varphi) \cap \mathtt{Im}(\psi) = \mathtt{Im}(\varphi \times_X \psi)$, where $\varphi \times_X \psi \colon Y \times_X Z \to X$ is the total morphism in the commutative square

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle \varphi} \\ Z & \xrightarrow[\psi]{} & X \end{array} \tag{26}$$

given by base change. Likewise $\mathtt{Im}(\varphi) \cup \mathtt{Im}(\psi) = \mathtt{Im}(\varphi \sqcup \psi)$, where $\varphi \sqcup \psi \colon Y \sqcup Z \to X$ is the disjoint union of the two morphisms.

As for products, if $\varphi\colon Y \to X$ and $\varphi'\colon Y' \to X'$ are morphisms of V-schemes, then the image of $\varphi \times_V \varphi'\colon Y \times_V Y \to X' \times_V X$ is equal to the product $\mathtt{Im}(\varphi) \times \mathtt{Im}(\varphi')$, showing that the latter is again sub-schemic. □

6.2. Definition. We define the *sub-schemic motivic site* $\mathbb{subSch}_V$ as the full subcategory of $\mathbb{Sieve}_V$ with objects the sub-schemic sieves. By Theorem 3.17, any continuous morphism in this category is rational, so that $\mathbb{subSch}_V$ is again an explicit motivic site.

Instead, we could have opted for a smaller site to take care of open coverings: define the *motivic constructible site* $\mathbb{Con}_V$ by taking on each scheme the lattice generated by locally closed subsieves (note that this is again an explicit motivic site). We have a natural homomorphism of Grothendieck rings $\mathbf{Gr}(\mathbb{Con}_V) \to \mathbf{Gr}(\mathbb{subSch}_V)$, but I do not know whether it is injective and/or surjective.

6.3. Example. The constructible site itself though is strictly smaller than the sub-schemic one, as illustrated by the following example: let $\varphi\colon \mathfrak{l}_4 \to \mathfrak{l}_2$ be the morphism corresponding to the homomorphism $R_2 := \kappa[\xi]/(\xi^2) \to R_4 := \kappa[\xi]/(\xi^4)$ given by $\xi \mapsto \xi^2$. Given a fat point $\mathfrak{z} = \operatorname{Spec} R$, the $\mathfrak{z}$-rational points of $\mathfrak{l}_2$ are in one-one correspondence with the elements in R whose square is zero, whereas $\mathtt{Im}(\varphi)(\mathfrak{z})$ is the subset of all those that are themselves a square, in general a proper subset. As $\mathfrak{l}_2$ is a fat point, it has no non-trivial locally closed subsieves, showing that $\mathtt{Im}(\varphi)$ is sub-schemic but not constructible. Moreover, the Zariski closure of $\mathtt{Im}(\varphi)$ is $\mathfrak{l}_2$.

6.4. Lemma. *If Y is an open in a V-scheme X, then Y° is a complete sieve on X.*

Proof. Let $\mathfrak{v} \to \mathfrak{w}$ be a morphism of fat points. We have to show that any $\mathfrak{w}$-rational point $a\colon \mathfrak{w} \to X$ whose image under $X(\mathfrak{w}) \to X(\mathfrak{v})$ belongs to $Y(\mathfrak{v})$, itself already belongs to $Y(\mathfrak{w})$. The condition that needs to be checked is that if the composition $\mathfrak{v} \to \mathfrak{w} \xrightarrow{a} X$ factors through Y, then so does a. Let $x \in X$ be the center of a. Since x is then also the center of the composition $\mathfrak{v} \to X$, it is a closed point of Y. Therefore, $\mathcal{O}_{X,x} = \mathcal{O}_{Y,x}$. Since a induces a homomorphism $\mathcal{O}_{X,x} \to T$, where T is the coordinate ring of $\mathfrak{w}$, whence a homomorphism $\mathcal{O}_{Y,x} \to T$, we get the desired factorization $\mathfrak{w} \to Y$. □

This will, among other things, allow us often to reduce the calculation of rational points to the affine case. Let $X = X_1 \cup \cdots \cup X_n$ be an open cover. By Lemma 6.4, we get $X^\circ = X_1^\circ \cup \cdots \cup X_n^\circ$. An easy argument on scissor relations, with notation as in (21), yields:[4]

6.5. Lemma. *If $X = X_1 \cup \cdots \cup X_n$ is an open covering of V-schemes, then*

$$[X] = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|}[X_I]$$

[4] By assumption, all V-schemes are separated, and hence the intersection of affines is again affine.

in $\mathbf{Gr}(\mathbb{s}\mathbb{u}\mathbb{b}\mathbb{S}\mathbb{c}\mathbb{h}_V)$. *In particular, the class of a schemic motif lies in the subring generated by classes of affine schemes.* □

6.6. Example. As an example, let us calculate the class of the projective line $\mathbb{P}^1_V$. It admits an open covering $X_1 \cup X_2$ where X_1 and X_2 are obtained by removing respectively the origin and the point at infinity. Since $X_1 \cong X_2 \cong \mathbb{L}$, we have

$$[\mathbb{P}^1_V] = 2\mathbb{L} - \mathbb{L}_*$$

where $\mathbb{L}_*$ denotes the class of the punctured line $X_1 \cap X_2$, the affine line with the origin removed. One would be tempted to think that $\mathbb{L}_*$ is just $\mathbb{L} - 1$, but this is false, as we shall see shortly. However, $X_1 \cap X_2$ is an affine scheme, by Example 2.20, isomorphic to the hyperbola $\mathbf{Hyp} \subseteq \mathbb{L}^2$ with equation $xy - 1 = 0$ under the projection $\mathbb{L}^2 \to \mathbb{L}$ onto the first coordinate. In other words, we have

$$[\mathbb{P}^1_V] = 2\mathbb{L} - [\mathbf{Hyp}]. \tag{27}$$

7. The formal Grothendieck ring

Let $Y \subseteq X$ be a closed subscheme. Recall that the n-th co-jet of X along Y, denoted $J^n_Y X$, is the closed subscheme with ideal sheaf $\mathcal{I}^n_Y$, where $\mathcal{I}_Y \subseteq \mathcal{O}_X$ is the ideal sheaf of Y. The *formal completion* $\widehat{X}_Y$ of X along Y is then the ringed space whose underlying set is equal to the underlying set of Y and whose sheaf of rings is the inverse limit of the sheaves $\mathcal{O}_{J^n_Y X}$. In particular, if $X = \operatorname{Spec} A$ is affine and I the ideal of definition of Y, then the ring of global sections of $\widehat{X}_Y$ is equal to the I-adic completion $\widehat{A}$ of A (see, for instance, [6, II.§9]). We define the *formal completion sieve along* Y to be the sieve $\operatorname{Mor}_V(\cdot, \widehat{X}_Y)$ represented by the formal completion $\widehat{X}_Y$ of X at Y, that is to say, for each fat point $\mathfrak{z}$, it gives the subset of all $\mathfrak{z}$-rational points $\mathfrak{z} \to X$ that factor through $\widehat{X}_Y$. We will simply denote it by $\widehat{X}^\circ_Y$ and call any such sieve again *pro-representable*.[5]

7.1. Proposition. *For a closed subscheme* $Y \subseteq X$*, the formal completion sieve* $\widehat{X}^\circ_Y$ *of* X *along* Y *is equal to the infinite union of all closed subsieves* $(J^n_Y X)^\circ \subseteq X^\circ$*, for* $n \geqslant 1$*. Moreover, we have an identity of sieves*

$$\widehat{X}^\circ_Y = X^\circ \setminus (X \setminus Y)^\circ,$$

showing that $\widehat{X}^\circ_Y$ *is a complete sieve, equal to* $\mathfrak{C}_X(Y)$*.*

Proof. The inclusion $J^n_Y X \subseteq \widehat{X}^\circ_Y$, for any n, is clear since the co-jets of $\widehat{X}_Y$ (with respect to its closed point) are the same as those of the germ (X, Y). Let $\mathfrak{z}$ be a fat point of length l. If $a\colon \mathfrak{z} \to \widehat{X}_Y$ is a $\mathfrak{z}$-rational point, then this must already factor

[5] Note that $\widehat{X}_Y$ is no longer a scheme, but only a locally ringed space with values in the category of $\mathcal{O}_V$-algebras, and so for a (formal) scheme Z, the set $\operatorname{Mor}_V(Z, \widehat{X}_Y)$ is to be understood as the set of morphisms $Z \to \widehat{X}_Y$ of locally ringed spaces with values in the category of $\mathcal{O}_V$-algebras.

through $J^l_Y X$, as any l-th power of a non-invertible section on $\mathfrak{z}$ is zero. Hence $(J^l_Y X)^\circ \subseteq \widehat{X}^\circ_Y$ have the same $\mathfrak{z}$-rational points, proving the first assertion. Since $J^l_Y X$ has the same underlying variety as Y, it lies outside the open $U := X\backslash Y$, and hence $a \notin U(\mathfrak{z})$. To prove the converse inclusion in the second assertion, suppose now that $a\colon \mathfrak{z} \to X$ does not lie in $U(\mathfrak{z})$. In particular, the center of a lies in Y. Let $\operatorname{Spec} A$ be an affine open of X containing the center of a, and let $(R, \mathfrak{m})$ be the (Artinian local) coordinate of $\mathfrak{z}$. Hence a induces a $\mathcal{O}_U$-algebra homomorphism $A \to R$. If I is the ideal locally defining Y in $\operatorname{Spec} A$, then $IR \subseteq \mathfrak{m}$. In particular, $I^l R \subseteq \mathfrak{m}^l = 0$, showing that a lies in $(J^l_Y X)(\mathfrak{z})$ whence in $\widehat{X}_Y(\mathfrak{z})$ by our first inclusion. The completeness of $\widehat{X}^\circ_Y$ now follows from Lemma 2.12, and the last assertion from Lemma 2.13 and the fact that $\widehat{X}_Y(\kappa) = Y(\kappa)$. □

7.2. Remark. Since $\widehat{X}_Y = \mathfrak{C}_X(Y)$, the set $\widehat{X}_Y(\mathfrak{z})$, for each fat point $\mathfrak{z}$, consists exactly of those $\mathfrak{z}$-rational points of X whose center lies in Y. From this characterization, and (2.3.iv) in §2.3, we get:

7.3. Corollary. *For each closed point P on X, we have an equality of sieves $X^\circ_P = \widehat{X}^\circ_P$.* □

Recall that a morphism of formal completions $\widehat{Y}_{\bar{Y}} \to \widehat{X}_{\bar{X}}$ is by definition a scheme-theoretic morphism $\bar{Y} \to \bar{X}$ together with a compatible inverse system of homomorphisms $\mathcal{O}_X/\mathcal{I}^n_{\bar{X}} = \mathcal{O}_{J^n_{\bar{X}}X} \to \mathcal{O}_Y/\mathcal{I}^n_{\bar{Y}} = \mathcal{O}_{J^n_{\bar{Y}}Y}$. The inverse limit of the latter then yields a morphism of sheaves $\mathcal{O}_{\widehat{X}_{\bar{X}}} \to \mathcal{O}_{\widehat{Y}_{\bar{Y}}}$.

7.4. Proposition. *A morphism of formal completions $f\colon \widehat{Y}_{\bar{Y}} \to \widehat{X}_{\bar{X}}$, induces a morphism of sieves $f^\circ\colon \widehat{Y}^\circ_{\bar{Y}} \to \widehat{X}^\circ_{\bar{X}}$.*

Proof. Without loss of generality, we may assume all spaces are affine, say, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, with respective ideals of definition $I \subseteq A$ and $J \subseteq B$ of $\bar{X}$ and $\bar{Y}$. Let $\widehat{A}$ and $\widehat{B}$ denote the respective completions. By assumption, we have a homomorphism $\widehat{A} \to \widehat{B}$ (which might fail to be induced by a homomorphism $A \to B$, so that f° will in general not be rational). To verify that we have a morphism of sieves, assume $B \to C$ induces a morphism with image contained in $\widehat{Y}^\circ_{\bar{Y}}$. In particular, $\operatorname{Spec} C \to Y$ must factor through the underlying variety $\bar{Y}$ of $\widehat{Y}_{\bar{Y}}$, showing that $J^m C = 0$, for some m. Hence, the natural transformation f° is induced by $\operatorname{Spec} C \to X$ given by the composition $A \to \widehat{A} \to \widehat{B} \to B/J^m \to C$. □

The proof of the first assertion of Proposition 7.1 actually gives a stronger statement, which we formalize as follows. A sieve $\mathfrak{X}$ on X is called a *formal motif on* X, if for each fat point $\mathfrak{z}$, there exists a sub-schemic motif $\mathfrak{Y}_\mathfrak{z} \subseteq \mathfrak{X}$ such that $\mathfrak{Y}_\mathfrak{z}(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z})$ (we call the $\mathfrak{Y}_\mathfrak{z}$ the *sub-schemic approximations of* $\mathfrak{X}$, in spite of the fact that they are not unique). A sub-schemic motif is a trivial example of a formal motif; the proof of Proposition 7.1 shows that formal completion sieves are formal too, whose approximations are even schemic. In fact, if follows from the proof that they are strongly formal in the following sense: a formal motif $\mathfrak{X}$ on X is called *strongly formal*, if there exists for each $l \in \mathbb{N}$, a sub-schemic motif $\mathfrak{Y}_l \subseteq \mathfrak{X}$, such that $\mathfrak{Y}_l(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z})$, for all fat points $\mathfrak{z}$ of length at most l.

7.5. Lemma. *If a sieve $\mathfrak{X}$ on a V-scheme X has formal approximations in the sense that for each fat point $\mathfrak{z}$, there exists a formal subsieve $\mathfrak{Y}_\mathfrak{z} \subseteq \mathfrak{X}$ with the same $\mathfrak{z}$-rational points, then $\mathfrak{X}$ itself is formal.*

Proof. By assumption, there exists, for each fat point $\mathfrak{z}$, a sub-schemic approximation $\mathfrak{Z}_\mathfrak{z} \subseteq \mathfrak{Y}_\mathfrak{z}$ with the same $\mathfrak{z}$-rational points, and it is now easy to check that the $\mathfrak{Z}_\mathfrak{z}$ form a sub-schemic approximation of $\mathfrak{X}$. □

7.6. Lemma. *Over an algebraically closed field κ, if a formal motif $\mathfrak{X}$ has no $\mathfrak{z}$-rational points, for some fat point $\mathfrak{z}$, then $\mathfrak{X}$ itself is empty.*

Proof. Suppose first that $\mathfrak{X} = \mathtt{Im}(\varphi)$ is sub-schemic, with $\varphi\colon Y \to X$ some morphism. In order for $\varphi(\mathfrak{z})$ to be empty, $Y(\mathfrak{z})$ has to be empty, whence so must $Y(\kappa)$ be, showing that Y has to be the empty scheme, whence $\mathtt{Im}(\varphi)$ the empty motif. Suppose now that $\mathfrak{X}$ is merely formal, and let $\mathfrak{w}$ be an arbitrary fat point. Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be a sub-schemic approximation with the same $\mathfrak{w}$-rational points. Since $\mathfrak{Y}(\mathfrak{z}) \subseteq \mathfrak{X}(\mathfrak{z}) = \emptyset$, we get $\mathfrak{Y} = \emptyset$, by the sub-schemic case, showing that $\mathfrak{X}(\mathfrak{w}) = \mathfrak{Y}(\mathfrak{w}) = \emptyset$. Since this holds for all fat points $\mathfrak{w}$, the assertion follows. □

Using Lemma 6.1, one easily verifies that the (strongly) formal motives on X form again a lattice, and the product of two (strongly) formal motives is again a (strongly) formal motif, leading to the *formal motivic site* $\mathbb{F}\mathrm{orm}_V$, and its corresponding Grothendieck ring $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_V)$, and similarly, the *strongly formal motivic site* $\mathbb{F}\mathrm{orm}_V^{\mathrm{str}}$, and its corresponding Grothendieck ring $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_V^{\mathrm{str}})$.

7.7. Theorem. *Over an algebraically closed field κ, we have natural ring homomorphisms*

$$\mathbf{Gr}(\mathbb{S}\mathrm{ch}_\kappa) \to \mathbf{Gr}(\mathbb{C}\mathrm{on}_\kappa) \to \mathbf{Gr}(\mathbb{s}\mathrm{ub}\mathbb{S}\mathrm{ch}_\kappa) \to \\ \mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa^{str}) \to \mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa) \to \mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa).$$

Proof. Only the last of these homomorphisms requires an explanation. Given a formal motif $\mathfrak{X}$, we associate to it the class of $\mathfrak{X}(\kappa)$ in the classical Grothendieck ring $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$. Note that by definition, $\mathfrak{X}(\kappa) = \mathtt{Im}(\varphi)(\kappa)$, for some morphism $\varphi\colon Y \to X$ of κ-schemes. In particular, by Chevalley's theorem, $\mathfrak{X}(\kappa)$ is a constructible subset of $X(\kappa)$ and hence its class in $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$ is well-defined. Clearly, this map is compatible with intersections, unions, and products, so that in order for this map to factor through $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)$, we only have to show that it respects homeomorphisms. So assume $s\colon \mathfrak{X} \to \mathfrak{Y}$ is an homeomorphism of formal motives. Let $\mathfrak{Z} \subseteq \mathfrak{X}$ be a sub-schemic approximation of $\mathfrak{X}$ with the same κ-rational points. Its push-forward $s_*\mathfrak{Z}$ is isomorphic with $\mathfrak{Z}$. By Theorem 3.17, the restriction $s|_\mathfrak{Z}$ extends to a morphism $\varphi\colon X \to Y$, where X and Y are some ambient spaces of $\mathfrak{Z}$ and $\mathfrak{Y}$ respectively. Since $\varphi(\kappa)\colon X(\kappa) \to Y(\kappa)$ maps $\mathfrak{Z}(\kappa)$ bijectively onto $s_*\mathfrak{Z}(\kappa)$, these two constructible subsets are isomorphic in the Zariski topology. However, by definition of push-forward, $s_*\mathfrak{Z}(\kappa)$ is the image of $\mathfrak{Z}(\kappa) = \mathfrak{X}(\kappa)$ under $s(\kappa)$, that is to say, is equal to $\mathfrak{Y}(\kappa)$, as we needed to show. □

7.8. Theorem. *Let $s\colon \mathfrak{Y} \to \mathfrak{X}$ be a morphism of V-sieves. If $\mathfrak{Y}$ and $\mathfrak{X}$ are sub-schemic (respectively, strongly formal, formal) motives, then so is the graph of s. Moreover, the pull-back or the push-forward of a sub-schemic (respectively, strongly formal, formal) submotif is again of that form.*

Proof. Let X be an ambient spaces of $\mathfrak{X}$. Since the graph of the composition $\mathfrak{Y} \to \mathfrak{X} \subseteq X^\circ$ is equal to the intersection of the graph $\Gamma(s)$ of s with $\mathfrak{Y} \times \mathfrak{X}$, we may assume from the start that $\mathfrak{X} = X^\circ$. Assume first that $\mathfrak{Y}$ is sub-schemic. By Theorem 3.15, the morphism s extends to a morphism $\varphi\colon Y \to X$ of V-schemes. Let $Z \subseteq Y \times_V X$ be the graph of this morphism, which therefore is a closed subscheme. Since $\Gamma(s)$ is equal to the intersection $Z^\circ \cap (\mathfrak{Y} \times X^\circ)$, it is again sub-schemic.

Suppose next that $\mathfrak{Y}$ is merely a formal motif, and, for each fat point $\mathfrak{z}$, let $\mathfrak{Z}_\mathfrak{z} \subseteq \mathfrak{Y}$ be a sub-schemic approximation. Since $\Gamma(s)$ contains the graph of the restriction of s to $\mathfrak{Z}_\mathfrak{z}$, and since they have the same $\mathfrak{z}$-rational points, the assertion follows from what we just proved for sub-schemic motives, and a similar argument proves the strongly formal case.

Let $\mathfrak{Y}' \subseteq \mathfrak{Y}$ be a sub-schemic or (strongly) formal submotif. Since the push-forward $s_*\mathfrak{Y}$ is the image of the restriction of s to $\mathfrak{Y}'$, we may reduce the problem to showing that $\mathtt{Im}(s)$ is respectively sub-schemic or (strongly) formal. The (strongly) formal case follows easily, as in the previous argument, from the sub-schemic one. So assume once more that $\mathfrak{Y}$ is sub-schemic, say of the form, $\mathtt{Im}(\psi)$ with $\psi\colon Z \to Y$ a morphism of V-schemes. With φ as above, one easily verifies that $\mathtt{Im}(s) = \mathtt{Im}(\varphi \circ \psi)$. To prove the same for the pull-back, simply observe that the pull-back $s^*\mathfrak{X}'$ of a submotif $\mathfrak{X}' \subseteq \mathfrak{X}$ is equal to the image of $\Gamma(s) \cap (\mathfrak{Y} \times \mathfrak{X}')$ under the morphism induced by the projection $Y \times X \to Y$. The result then follows from our previous observations. □

7.9. Corollary. *For each closed germ (X, P), we have isomorphisms of $\mathcal{O}_V$-algebras*

$$H_0^{rat}(X_P^\circ) \cong \mathcal{O}_{X,P} \qquad \text{and} \qquad H_0(X_P^\circ) \cong \hat{\mathcal{O}}_{X,P}.$$

Moreover, the ring of algebraic global sections of X_P° is equal to the Henselization of $\mathcal{O}_{X,P}$.

Proof. Let $\mathfrak{X} := X_P^\circ$, the formal motif determined by the formal completion along P by Corollary 7.3. It follows from Krull's Intersection Theorem, that the Zariski closure of $\mathfrak{X}$ and X have the same stalk, and so we may assume that $\mathfrak{X}$ is Zariski dense. An open subscheme $U \subseteq X$ is an ambient space of $\mathfrak{X}$ if and only if $P \in U$ by 2.3(2.3.iv). By Corollary 3.4, the ring of rational sections $H_0^{\mathrm{rat}}(\mathfrak{X})$ is therefore the direct limit of all $H_0(U)$, where $U \subseteq X$ runs over all opens containing P, that is to say, to $\mathcal{O}_{X,P}$. On the other hand, since the co-jets $J_P^n X$ are approximations of $\mathfrak{X}$, the inverse limit of the $H_0(J_P^n X)$ is equal to $H_0(\mathfrak{X})$ by Lemma 3.18. Since

$H_0(J^n_P X) = \mathcal{O}_{X,P}/\mathfrak{m}^n_P$, this proves the second isomorphism. The last one follows along the same lines, observing that an algebraic global section factors through a morphism $Y \to X$ which is etale at P, and the direct limit of all local etale extensions of $\mathcal{O}_{X,P}$ is precisely its Henselization. □

7.10. Example. Let $\widehat{\mathbb{L}}$ be the class of the formal completion of the affine line along the origin O (recall that this is also the class of the localization $\mathbb{L}_O$). It follows from Proposition 7.1 that $\widehat{\mathbb{L}} = \mathbb{L} - \mathbb{L}_*$ (see Example 6.6). In particular, (27) becomes

$$[\mathbb{P}^1_V] = \mathbb{L} + \widehat{\mathbb{L}}.$$

We will shortly generalize this in Proposition 7.11 below, but let us first construct from this an example of a non-rational morphism. Let $f(x)$ be a power series in a single indeterminate which is not a polynomial. The homomorphism $\kappa[t] \to \kappa[[x]]$ given by $t \mapsto f$ induces a morphism of sieves $s_f \colon \widehat{\mathbb{L}}^\circ \to \mathbb{L}^\circ$. Since f is not a polynomial, it cannot extend to a morphism of schemes, that is to say, it is not rational. Its graph, in accordance with Theorem 7.8, is the formal motif with approximations the graphs of the rational morphisms given by the various truncations of f.

We can also use this to give a counterexample to Proposition 3.9 for global sections. In general, given a global section $s \colon \mathfrak{X} \to \mathbb{L}$ of a formal motif $\mathfrak{X}$, define a sieve $\mathfrak{X}_s$ by letting $\mathfrak{X}_s(\mathfrak{z})$ consist of all $\mathfrak{z}$-rational points $a \in \mathfrak{X}(\mathfrak{z})$ such that $s(\mathfrak{z})(a)$ is a unit, for each fat point $\mathfrak{z}$. Since $\mathfrak{X}_s = s^*\mathbb{L}^\circ_*$, it is a strongly formal motif by Theorem 7.8. Applied to the global section s_f above, $\widehat{\mathbb{L}}^\circ_{s_f}$ is the intersection of the open subsieves given by the truncations of f, whence not an admissible open in $\mathfrak{X}$ for the Zariski topos. In particular, s_f is not continuous. Put differently, the submotives of the form $\mathfrak{X}_s$ form in general a basis for a Grothendieck topology which is stronger than the Zariski one.

7.11. Proposition. *For each n, the class of projective n-space is given by*

$$[\mathbb{P}^n_V] = \sum_{m=0}^{n} \mathbb{L}^m \cdot \widehat{\mathbb{L}}^{n-m}$$

in $\mathbf{Gr}(\mathbb{F}\mathrm{orm}^{str}_V)$.

Proof. Let $(x_0 : \dots : x_n)$ be the homogeneous coordinates of $\mathbb{P}^n_V$, and let X_i be the basic open given as the complement of the x_i-hyperplane. Hence every X_i is isomorphic with $\mathbb{L}^n$ and their union is equal to $\mathbb{P}^n_V$. Therefore, by Lemma 6.5, we have

$$[\mathbb{P}^n_V] = \sum_{\emptyset \neq I \subseteq \{0,\dots,n\}} (-1)^{|I|}[X_I] \tag{28}$$

in $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_V)$. So we need to calculate the class of each X_I. One easily verifies that, for $m \geqslant 0$, any intersection of m different opens X_i is isomorphic to the open $\mathbb{L}^{n-m} \times (\mathbb{L}_*)^m$, where $\mathbb{L}_*$ is the affine line minus a point. Since $\mathbb{L}_* = \mathbb{L} - \widehat{\mathbb{L}}$ by Proposition 7.1, the class of such an intersection is equal to the product $\mathbb{L}^{n-m}(\mathbb{L} -$

$\widehat{\mathbb{L}})^m$. Since there are $\binom{n+1}{m}$ terms with $|I| = m$ in (28), the class of $\mathbb{P}^n_V$ is equal to $g(\mathbb{L}, \widehat{\mathbb{L}})$, where

$$g(t,u) := \sum_{m=0}^{n} (-1)^m \binom{n+1}{m} t^{n-m}(t-u)^m.$$

By the binomial theorem, $t^{n+1} - (t-u)g(t,u) = (t-(t-u))^{n+1} = u^{n+1}$, and hence

$$g(t,u) = \frac{t^{n+1} - u^{n+1}}{t-u} = \sum_{m=0}^{n} t^m u^{n-m},$$

as we wanted to show. □

7.12. Example. To see that the homomorphisms in Theorem 7.7 are not injective, we construct a non-zero element in the kernel of $\mathbf{Gr}(\mathbb{S}\mathrm{ch}_\kappa) \to \mathbf{Gr}(\mathbb{F}\mathrm{orm}^{\mathrm{str}}_\kappa)$. Let $C \subseteq \mathbb{L}^2$ be the nodal curve with equation $y^2 = x^3 + x^2$. Since C is rational, as a variety, $C\backslash O$ is isomorphic to $\mathbb{L}_*$, and hence $[C] = \mathbb{L}$ in $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$. This equality no longer holds in the formal Grothendieck ring, but we have nonetheless a non-trivial relation which collapses to this in the classical Grothendieck ring: let $X \subseteq \mathbb{L}^2$ be the variety given as the union of two distinct lines, that is to say, given by $xy = 0$. Since $\widehat{X}_O \cong \widehat{C}_O$, they have the same class in $\mathbf{Gr}(\mathbb{F}\mathrm{orm}^{\mathrm{str}}_\kappa)$ by Proposition 7.4. Let $U := \mathbb{L}^2\backslash O$, given as the union of the two basic open sets D_x and D_y (obtained by respectively inverting x and y). Since $X \cap D_x \cap D_y = \varnothing$, we have $[X \cap U] = [X \cap D_x] + [X \cap D_y]$, and the latter two are both equal to $\mathbb{L}_*$. On the other hand, $C \cap D_x$ is the punctured parabola with equation $(y/x)^2 = x + 1$, and hence its class is also equal to $\mathbb{L}_*$ (via projection). Since $X \cap U = X\backslash O$ and $C \cap D_x = C\backslash O$, two applications of Proposition 7.1 yield $[X] = [X \cap U] + [\widehat{X}_O]$ and $[C] = [C \cap D_x] + [\widehat{C}_O]$. Putting everything together, we get

$$[X] = [C] + \mathbb{L}_* \tag{29}$$

(note that the image of $[X] - \mathbb{L}_*$ in $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$ is indeed $\mathbb{L}$). However, since X is not isomorphic with $C \sqcup \mathbb{L}_*$, this equality does not yet hold in $\mathbf{Gr}(\mathbb{S}\mathrm{ch}_\kappa)$ by Theorem 5.7. I suspect that it even fails in $\mathbf{Gr}(\mathrm{sub}\mathbb{S}\mathrm{ch}_\kappa)$.

It is not yet clear to me, what the kernels of the homomorphisms in Theorem 7.7 are, not even into the classical Grothendieck ring. Clearly, the ideal $\mathcal{I}$ generated by all differences $[X] - [X^{\mathrm{red}}]$, with X a κ-scheme, lies in the kernel of $\mathbf{Gr}(\mathbb{S}\mathrm{ch}_\kappa) \to \mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$, but I do not know whether it constitutes the whole kernel (this seems quite unlikely for the schemic Grothendieck ring but it could hold for the formal one). A possibly smaller ideal is the ideal $\mathcal{I}_0$ generated by the differences $1 - [\mathfrak{z}]$, with $\mathfrak{z}$ a fat point.

7.13. Restriction of scalars. Let $f\colon W \to V$ be a morphism of schemes. We want to compare the respective Grothendieck rings over these base schemes. In [11, §3], we will study this problem in more detail through adjunction. Here we only discuss a simpler solution, to wit, *restriction of scalars*, producing an additive map $f_*\colon \mathbf{Gr}(\mathbb{S}\mathrm{ch}_W) \to \mathbf{Gr}(\mathbb{S}\mathrm{ch}_V)$, which extends to the sub-schemic and the strongly formal Grothendieck rings as well. Given a scheme Y over W, we may also view it as a scheme over V via f, and to denote the latter, we will write f_*Y. Now, given a fat V-point $\mathfrak{z}$, any $\mathfrak{z}$-rational point $a\colon \mathfrak{z} \to f_*Y$, induces a W-scheme structure on $\mathfrak{z}$ via the W-structure on Y, and we denote the corresponding fat W-point by $\tilde{\mathfrak{z}}_a$. Hence, a now becomes a morphism $\tilde{\mathfrak{z}}_a \to Y$ of W-schemes, and as such, will be denoted $\tilde{a}$. This allows us to define, for any sieve $\mathfrak{Y}$ on Y, its *restriction of scalars* $f_*\mathfrak{Y}$ as the sieve on f_*Y containing, for every fat V-point $\mathfrak{z}$, exactly those $\mathfrak{z}$-rational points $a\colon \mathfrak{z} \to f_*Y$ for which $\tilde{a}\colon \tilde{\mathfrak{z}}_a \to Y$ belongs to $\mathfrak{Y}(\tilde{\mathfrak{z}}_a)$. The reader easily verifies that $f_*\mathfrak{Y}$ is indeed a contravariant functor, that is to say, a sieve on f_*Y. From this, it is already clear that f_* preserves unions and intersections of sieves. By definition, $f_*(Y^\circ) = (f_*Y)^\circ$, so it also preserves schemic motives. By functoriality, one easily verifies that given a morphism $\varphi\colon Y' \to Y$ of W-schemes, we have an equality

$$f_*\mathtt{Im}(\varphi) = \mathtt{Im}(f_*\varphi), \tag{30}$$

where $f_*\varphi\colon f_*Y' \to f_*Y$ is the corresponding morphism over V, showing that f_* preserves sub-schemic motives as well. So, remains to show that if $\mathfrak{Y}$ is strongly formal, then so is its restriction $f_*\mathfrak{Y}$. Let $\mathfrak{Z}_l \subseteq \mathfrak{Y}$ be the sub-schemic approximations of $\mathfrak{Y}$, with the same $\mathfrak{y}$-rational points for any fat W-point $\mathfrak{y}$ of length at most l. Given a fat V-point $\mathfrak{z}$ of length l, consider the sub-schemic V-motif $\mathfrak{X}_l := f_*(\mathfrak{Z}_l)$. Clearly, $\mathfrak{X}_l \subseteq f_*\mathfrak{Y}$, and we will be done if we can show that they have the same $\mathfrak{z}$-rational points. So, let $a\colon \mathfrak{z} \to f_*Y$ be a $\mathfrak{z}$-rational point in $f_*\mathfrak{Y}$. By definition, $\tilde{a}\colon \tilde{\mathfrak{z}}_a \to Y$ is a $\tilde{\mathfrak{z}}_a$-rational point in $\mathfrak{Y}(\tilde{\mathfrak{z}}_a)$, whence in $\mathfrak{Z}_l(\tilde{\mathfrak{z}}_a)$, since $\tilde{\mathfrak{z}}_a$ has also length l. By definition of f_* once more, this means that $a \in \mathfrak{X}_l(\mathfrak{z})$, as we wanted to show. I do not know whether the restriction of scalars of a formal motif is again formal. In any case, we proved:

7.14. Theorem. *A morphism $f\colon W \to V$ of schemes induces additive maps $f_*\colon \mathbf{Gr}(\mathbb{S}\mathrm{ch}_W) \to \mathbf{Gr}(\mathbb{S}\mathrm{ch}_V)$, $f_*\colon \mathbf{Gr}(\mathrm{sub}\mathbb{S}\mathrm{ch}_W) \to \mathbf{Gr}(\mathrm{sub}\mathbb{S}\mathrm{ch}_V)$, and $f_*\colon \mathbf{Gr}(\mathbb{F}\mathrm{orm}_W^{str}) \to \mathbf{Gr}(\mathbb{F}\mathrm{orm}_V^{str})$ by restriction of scalars.* □

Note that f_* does not preserve products, as $f_*(Y_1 \times_W Y_2)$ is in general only a closed subscheme of $f_*Y_1 \times_V f_*Y_2$. However, we have one important case of semi-linearity (we leave the verification to the reader; see [11, Lemma 3.9] for a more general form):

$$f_*(\mathbb{L}_W^n \cdot \alpha) = \mathbb{L}_V^n \cdot f_*\alpha \tag{31}$$

for every $\alpha \in \mathbf{Gr}(\mathbb{F}\mathrm{orm}^{\mathrm{str}}_W)$. In the notation of (16), we have $f_*1 = [W](\neq 1)$.

7.15. Lemma. *Given a morphism of schemes $f\colon W \to V$, a strongly formal V-motif $\mathfrak{X}$ is in the image of $f_*\colon \mathbb{F}\mathrm{orm}^{str}_W \to \mathbb{F}\mathrm{orm}^{str}_V$ if and only if it admits an ambient space of the form f_*Y, for some W-scheme Y.*

Proof. One direction is clear by definition, so assume $\mathfrak{X}$ is a strongly formal motif on f_*Y, for Y some W-scheme. Define a sieve $\mathfrak{Y}$ on Y as follows: given a fat W-point $\mathfrak{w}$, let $\mathfrak{Y}(\mathfrak{w})$ consists of all $\mathfrak{w}$-rational points $a\colon \mathfrak{w} \to Y$ such that their restriction $f_*a\colon f_*\mathfrak{w} \to f_*Y$ lies in $\mathfrak{X}(f_*\mathfrak{w})$. I claim that $\mathfrak{Y}$ is strongly formal and $f_*\mathfrak{Y} = \mathfrak{X}$. To prove the latter, given a fat V-point $\mathfrak{z}$, a $\mathfrak{z}$-rational point $a\colon \mathfrak{z} \to f_*Y$ belongs by definition to $f_*\mathfrak{Y}$ if and only if $\tilde{a}\colon \tilde{\mathfrak{z}}_a \to Y$ belongs to $\mathfrak{Y}(\tilde{\mathfrak{z}}_a)$ if and only if $f_*\tilde{a}\colon f_*\tilde{\mathfrak{z}}_a \to f_*Y$ belongs to $\mathfrak{X}(f_*\tilde{\mathfrak{z}}_a)$. The assertion then follows since $f_*\tilde{\mathfrak{z}}_a = \mathfrak{z}$ and $a = f_*\tilde{a}$.

To prove that $\mathfrak{Y}$ is strongly formal, by the usual approximation argument (on the length of a fat point), we may reduce to the case that $\mathfrak{X} = \mathtt{Im}(\varphi)$ for some morphism $\varphi\colon Z \to f_*Y$. Considering Z as a W-scheme $\tilde{Z}$ via the composition $Z \to Y \to W$, the V-morphism φ induces a W-morphism $\tilde{\varphi}\colon \tilde{Z} \to Y$, and it remains to show that $\mathfrak{Y} = \mathtt{Im}(\tilde{\varphi})$. However, this is immediate from the fact that $f_*\tilde{\varphi} = \varphi$ and a similar argument as the one proving (30). □

7.16. Corollary. *A morphism $f\colon W \to V$ is split if and only if the restriction of scalars functor $f_*\colon \mathbb{F}\mathrm{orm}^{str}_W \to \mathbb{F}\mathrm{orm}^{str}_V$ is surjective.*

Proof. Recall that f being *split* means that there exists a morphism $s\colon V \to W$, called a *section*, or also, a *V-rational point*, such that $f \circ s$ is the identity morphism on V. In particular, for any strongly formal V-motif $\mathfrak{X}$, we have $\mathfrak{X} = f_*(s_*\mathfrak{X})$, showing that f_* is surjective.

Conversely, if f_* is surjective, then in particular, V° is obtained by restriction of scalars, and hence hence admits an ambient space of the form f_*Y for some W-scheme Y, by Lemma 7.15. The induced morphism $V \to Y$ composed with the structure morphism $Y \to W$ is then the desired section. □

8. Complements

As noted in the introduction, a complement of a sieve is in general no longer a sieve, as functoriality fails (this phenomenon is not present when formulating the theory in the model-theoretic framework of [10]). By Lemma 2.12, only complete sieves have a complement which is again a sieve. So we start with a characterization of the complete sieves among the formal motives. Throughout this section, we work over an algebraically closed field κ.

8.1. Theorem. *Over an algebraically closed field, a sieve is a complete formal motif if and only if it is a constructible cone. Moreover, any such motif is then strongly formal.*

Proof. Recall that a cone is constructible, if it is of the form $\mathfrak{C}_X(F)$ with $F \subseteq X(\kappa)$ constructible. Suppose $\mathfrak{X}$ is complete and formal. By Lemma 2.13, it is a cone, say, of the form $\mathfrak{C}_X(F)$ for $F \subseteq X(\kappa)$. In fact, $F = \mathfrak{X}(\kappa)$, and hence, by definition, of the form $\mathtt{Im}(\varphi)(\kappa)$ for some morphism $\varphi\colon Y \to X$. By Chevalley's theorem, F is constructible. To prove the converse, since cones are easily seen to commute with union and intersection, and since any constructible subset of $\mathfrak{X}(\kappa)$ is an intersection and union of closed and open subsets, it suffices to prove that $\mathfrak{C}_X(F)$ is formal, whenever F is a closed or an open subset. The open case follows immediately from Lemma 6.4, and in the closed case, we have $\mathfrak{C}_X(F) = \widehat{X}^{\circ}_F$ by Proposition 7.1 (note that the completion of a scheme along a subscheme only depends on the underlying variety of the subscheme, so that $\widehat{X}_F$ is well-defined). This then also proves the last assertion. □

8.2. Corollary. *Over an algebraically closed field, a Boolean combination of (strongly) formal motives on an ambient space X is a (strongly) formal motive if and only if it is a sieve on X (that is to say, is functorial).*

Proof. Before we start, observe that if a disjoint union $\mathfrak{X} \sqcup \mathfrak{Y}$ is a sieve on a scheme X, then so must both $\mathfrak{X}$ and $\mathfrak{Y}$ be: indeed, given a morphism $j\colon \tilde{\mathfrak{z}} \to \mathfrak{z}$ of fat points, it maps a $\mathfrak{z}$-rational point a of $\mathfrak{X}$ to a $\tilde{\mathfrak{z}}$-rational point $\tilde{a}$ of $\mathfrak{X} \sqcup \mathfrak{Y}$. However, since a and $\tilde{a}$ have the same center, and since $\mathfrak{X}(\kappa)$ and $\mathfrak{Y}(\kappa)$ are disjoint, we must have $\tilde{a} \in \mathfrak{X}(\tilde{\mathfrak{z}})$.

Now, let $\mathfrak{Z}$ be a sieve, equal to a Boolean combination of formal motives on X. We can write it as a disjoint union with disjuncts of the form $\mathfrak{X}\backslash\mathfrak{Y}$ (see the discussion in the beginning of the proof of Proposition 8.6), and by our previous observation, each of these disjuncts is again a sieve. Hence, by induction on the number of disjuncts, we may assume $\mathfrak{Z} = \mathfrak{X}\backslash\mathfrak{Y}$ with $\mathfrak{Y} \subseteq \mathfrak{X}$ formal motives on X. By Lemmas 2.12 and 2.13 combined, $\mathfrak{Y} = \mathfrak{X} \cap \mathfrak{C}_X(F)$, where $F = \mathfrak{Y}(\kappa) \subseteq X(\kappa)$ is constructible. It is now easy to see that $\mathfrak{Z} = \mathfrak{X} \cap \mathfrak{C}_X(\mathfrak{X}(\kappa)\backslash F)$ and so we are done by Theorem 8.1 and the fact that $\mathfrak{X}(\kappa)$ is also constructible. □

In [11], we will introduce motivic integration, but even to integrate step functions, one needs to be able to partition a motif (which inevitably requires complementation). The remainder of this section is to describe a more restricted motivic site than $\mathbb{F}\mathrm{orm}_\kappa$ which allows for complements but which still yields the same formal Grothendieck ring.

8.3. Pure and split points. Let $\mathbb{F}\mathrm{at}^{\mathrm{pure}}_\kappa$ and $\mathbb{F}\mathrm{at}^{\mathrm{split}}_\kappa$ be the respective categories of *pure points* and *split points* over κ, whose objects are fat points over κ and whose morphisms are respectively pure and split epimorphisms. Recall that a morphism $\varphi\colon Y \to X$ is called *pure* if $\mathcal{O}_X \to \mathcal{O}_Y$ is injective, and remains so after tensoring with any $\mathcal{O}_X$-module; and that it is called *split*, if there exists a morphism, also called a *section*, $\sigma\colon X \to Y$ such that $\varphi\sigma$ is the identity on X. Faithfully flat morphisms are pure, and so are split morphisms, whence $\mathbb{F}\mathtt{at}_\kappa^{\text{split}}$ is a subcategory of $\mathbb{F}\mathtt{at}_\kappa^{\text{pure}}$, and, of course, both are (non-full) subcategories of $\mathbb{F}\mathtt{at}_\kappa$. Each structure morphism $\mathfrak{z} \to \operatorname{Spec}\kappa$ is a split epimorphism, and by base change, so is each projection map $\mathfrak{z}\mathfrak{w} \to \mathfrak{w}$.

We will call a contravariant functor $\mathfrak{X}$ from $\mathbb{F}\mathtt{at}_\kappa^{\text{pure}}$ (respectively, from $\mathbb{F}\mathtt{at}_\kappa^{\text{split}}$) to the category of sets, a *pure* (respectively, a *split*) pre-sieve. If, moreover, we have an inclusion morphism $\mathfrak{X} \subseteq X^\circ$, where X° is the restriction of the representable functor of a κ-scheme X, then we call $\mathfrak{X}$ a *pure* (respectively, a *split*) sieve. In particular, ordinary pre-sieves or sieves (that is to say, defined on $\mathbb{F}\mathtt{at}_\kappa$) when restricted to $\mathbb{F}\mathtt{at}_\kappa^{\text{split}}$ are split—and to emphasize this, we may call them *full* sieves—, but as the next result shows, in view of Theorem 8.1, not every split sieve is the restriction of a full sieve:

8.4. Proposition. *The complement of a schemic motif $\mathfrak{X} \subseteq X^\circ$ is a pure sieve. The complement $-\mathfrak{X}$ of a formal motif $\mathfrak{X} \subseteq X^\circ$ is a split sieve.*

Proof. Any (full) schemic motif is the union of closed subsieves, and hence its complement is the intersection of complements of closed subsieves. Since the intersection of (pure or split) sieves is again a sieve, we only need to verify that the complement of a single closed subsieve $Y^\circ \subseteq X^\circ$ is a pure sieve. The only thing to show is functoriality, so let $\mathfrak{w} \to \mathfrak{z}$ be a pure morphism of fat points. We have to show that under the induced map $X(\mathfrak{z}) \to X(\mathfrak{w})$ any $\mathfrak{z}$-rational point a not in $Y(\mathfrak{z})$ is mapped to a point not in $Y(\mathfrak{w})$. Since fat points are affine, we may replace X by an affine open, and so assume from the start that it is affine with coordinate ring A. Let I be the ideal defining Y, and let R and S be the coordinate rings of $\mathfrak{z}$ and $\mathfrak{w}$ respectively. The $\mathfrak{z}$-rational point a corresponds to a morphism $A \to R$; it does not belong to $Y(\mathfrak{z})$ if and only if the image IR of I under $A \to R$ is non-zero. Suppose towards a contradiction that $a \in Y(\mathfrak{w})$, so that $IS = 0$. Since $R \to S$ is by assumption pure, we must have $IR = IS \cap R = 0$, contradiction.

Assume next that $\mathfrak{X}$ is a sub-schemic motif, that is to say, $\mathfrak{X} = \mathtt{Im}(\varphi)$ for some morphism $\varphi\colon Y \to X$. Let $\lambda\colon \mathfrak{w} \to \mathfrak{z}$ be a split epimorphism of fat points and let $a\colon \mathfrak{z} \to X$ be a $\mathfrak{z}$-rational point outside $\mathtt{Im}(\varphi)(\mathfrak{z})$. We have to show that the image $a \circ \lambda$ of a in $X(\mathfrak{w})$ does not lie in the image of $\varphi(\mathfrak{w})$. Towards a contradiction, assume the opposite, so that $a \circ \lambda$ factors through Y, giving rise to a commutative diagram

$$\begin{array}{ccc} \mathfrak{w} & \xrightarrow{\quad b \quad} & Y \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathfrak{z} & \xrightarrow[\quad a \quad]{} & X. \end{array} \tag{32}$$

By assumption, there exists a section $\sigma\colon \mathfrak{z} \to \mathfrak{w}$ so that $\lambda\sigma$ is the identity on $\mathfrak{z}$. Let $\tilde{b}$ be the image of b under $Y(\sigma)\colon Y(\mathfrak{w}) \to Y(\mathfrak{z})$, that is to say, $\tilde{b} = b \circ \sigma$. The image of $\tilde{b}$ under $\varphi(\mathfrak{z})\colon Y(\mathfrak{z}) \to X(\mathfrak{z})$ is by (32) equal to

$$\varphi(\mathfrak{z})(\tilde{b}) = \varphi \circ \tilde{b} = \varphi \circ b \circ \sigma = a \circ \lambda \circ \sigma = a$$

showing that a lies in the image of $\varphi(\mathfrak{z})$, contradiction.

Lastly, assume that $\mathfrak{X}$ is formal, so that there exists for each fat point $\mathfrak{z}$ a sub-schemic motif $\mathfrak{Y}_{\mathfrak{z}} \subseteq \mathfrak{X}$ such that $\mathfrak{Y}_{\mathfrak{z}}(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z})$. Let $\lambda\colon \mathfrak{w} \to \mathfrak{z}$ be a split epimorphism. Since $\mathfrak{Y}_{\mathfrak{w}} \subseteq \mathfrak{X}$, we have $-\mathfrak{X}(\mathfrak{z}) \subseteq -\mathfrak{Y}_{\mathfrak{w}}(\mathfrak{z})$. By what we just proved, $-\mathfrak{Y}_{\mathfrak{w}}(\mathfrak{z})$, is sent under $X(\lambda)\colon X(\mathfrak{z}) \to X(\mathfrak{w})$ inside $-\mathfrak{Y}_{\mathfrak{w}}(\mathfrak{w})$, and by construction, the latter is equal to $-\mathfrak{X}(\mathfrak{w})$. A fortiori, $-\mathfrak{X}(\mathfrak{z})$ is then sent inside $-\mathfrak{X}(\mathfrak{w})$, proving the assertion. □

8.5. Remark. It is important to note that we may not apply this argument to an arbitrary split sieve, since a section of a split morphism is not split and hence does not induce a morphism on the rational points of the split sieve. The point in the above argument is that formal motives are pre-sieves on the full category of fat points, and hence any section does induce a map between their rational points.

The argument of the proof shows that we could replace pure points by the slightly larger category of cyclically pure points, since that is the only property needed: recall that $R \to S$ is *cyclically pure* if $I = IS \cap R$ for any ideal $I \subseteq R$.

We call any Boolean combination of closed subsieves a *pure-schemic* motif. By what we just proved, any such motif is indeed a pure sieve. Our aim is to define motivic sites of schemic, sub-schemic and formal motives with respect to pure and/or split points, but without changing the corresponding Grothendieck ring. To not obtain too many morphisms, we only allow morphisms that extend to true motives.[6] More precisely, we define the *pure-schemic* motivic site $\mathbb{Sch}_\kappa^{\mathrm{pure}}$, as the category with objects all pure-schemic motives, and with morphisms all natural transformations $s\colon \mathfrak{Y} \to \mathfrak{X}$ of pure schemic motives which extend to a morphism of schemic motives in the sense that there are schemic motives $\mathfrak{X}' \supseteq \mathfrak{X}$ and $\mathfrak{Y}' \supseteq \mathfrak{Y}$ and a morphism of schemic motives $s'\colon \mathfrak{Y}' \to \mathfrak{X}'$ whose restriction to $\mathfrak{Y}$ is s. To not introduce unwanted isomorphisms, we moreover require that if s is injective and continuous, then so

[6] I do not know whether the analogous notion of morphism as for full sieves is sufficiently strong.

must its extension s' be. Likewise, we call any Boolean combination of sub-schemic (respectively, strongly formal, formal) motives a *split-sub-schemic* (respectively, a *strongly split-formal*, *split-formal*) motif, and we define the *split-sub-schemic* motivic site $\mathbb{subSch}_\kappa^{\text{split}}$ (respectively, the *strongly split-formal* motivic site $\mathbb{Form}_\kappa^{\text{sspl}}$, the *split-formal* motivic site $\mathbb{Form}_\kappa^{\text{split}}$), as the category with objects all split-sub-schemic (respectively, strongly split-formal, split-formal) motives, and as morphisms all natural transformations which extend to a morphism of sub-schemic (respectively, strongly formal, formal) motives, with continuous, injective morphisms extending to continuous, injective ones. By Corollary 8.2, a (strongly) split-formal motif is a full (strongly) formal motif if and only if it is a (full) pre-sieve. All these sites satisfy the same properties as ordinary motivic sites, apart from being defined only over a restricted category, but have the additional property that their restriction to any scheme is a Boolean lattice. At any rate, we can define their corresponding Grothendieck rings.

8.6. Proposition. *We have equalities of Grothendieck rings*

$$\begin{aligned}
\mathbf{Gr}(\mathbb{Sch}_\kappa^{pure}) &= \mathbf{Gr}(\mathbb{Sch}_\kappa),\\
\mathbf{Gr}(\mathbb{subSch}_\kappa^{split}) &= \mathbf{Gr}(\mathbb{subSch}_\kappa),\\
\mathbf{Gr}(\mathbb{Form}_\kappa^{sspl}) &= \mathbf{Gr}(\mathbb{Form}_\kappa^{str}),\\
\mathbf{Gr}(\mathbb{Form}_\kappa^{split}) &= \mathbf{Gr}(\mathbb{Form}_\kappa).
\end{aligned}$$

Proof. I will only give the argument for the case of most interest to us, the formal motives, and leave the remaining cases, with analogous proof, to the reader. Before we do this, let us first discuss briefly Boolean lattices. Let $\mathcal{B}$ be a Boolean lattice. Given a finite collection of subsets $X_1, \dots, X_n \in \mathcal{B}$, and an n-tuple $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ with entries ± 1, let X_ϵ be the subset given by the intersection of all X_i with $\varepsilon_i = 1$ and all $-X_i$ with $\varepsilon_i = -1$. Then any element in the Boolean sublattice $\mathcal{B}(X_1, \dots, X_n)$ of $\mathcal{B}$ generated by $X_1, \dots, X_n$ is a disjoint union of the X_ε. In particular, if all X_i belong to a sublattice $\mathcal{L} \subseteq \mathcal{B}$, then any element in $\mathcal{B}(X_1, \dots, X_n)$ is a disjoint union of sets of the form $C \setminus D$ with $D \subseteq C$ in $\mathcal{L}$.

We now define a map γ from the free Abelian group $\mathbb{Z}[\mathbb{Form}_\kappa^{\text{split}}]$ to $\mathbf{Gr}(\mathbb{Form}_\kappa)$ as follows. By the above argument, a typical element in $\mathbb{Form}_\kappa^{\text{split}}$ is a disjoint union of split formal motives of the form $\mathfrak{X} \setminus \mathfrak{Y}$ with $\mathfrak{Y} \subseteq \mathfrak{X}$ (full) formal motives. We define its γ-value to be the element $[\mathfrak{X}] - [\mathfrak{Y}]$. This is well-defined, for if it is also equal to a difference of motives $\tilde{\mathfrak{X}} \setminus \tilde{\mathfrak{Y}}$ then one easily checks that $\mathfrak{X} \cup \tilde{\mathfrak{Y}} = \tilde{\mathfrak{X}} \cup \mathfrak{Y}$ and $\mathfrak{X} \cap \tilde{\mathfrak{Y}} = \tilde{\mathfrak{X}} \cap \mathfrak{Y}$, so that

$$[\mathfrak{X}] + [\tilde{\mathfrak{Y}}] = [\tilde{\mathfrak{X}}] + [\mathfrak{Y}].$$

We extend this to disjoint sums by taking the sum of the disjoint components, and then extend by linearity, to the entire free Abelian group. It is not hard to verify that γ preserves all scissor relations. So we next check that it preserves also homeomorphism relations. We may again reduce to an homeomorphism of the form $s\colon \mathfrak{X}\backslash\mathfrak{Y} \to \tilde{\mathfrak{X}}\backslash\tilde{\mathfrak{Y}}$ with $\mathfrak{Y} \subseteq \mathfrak{X}$ and $\tilde{\mathfrak{Y}} \subseteq \tilde{\mathfrak{X}}$ formal motives. By assumption, s extends to an injective morphism $s'\colon \mathfrak{X}' \to \tilde{\mathfrak{X}}'$ with $\mathfrak{X}'$ and $\tilde{\mathfrak{X}}'$ formal motives. Upon replacing $\mathfrak{X}$ and $\mathfrak{X}'$ with their common intersection, we may assume that they are equal. Since s' is injective, it induces an homeomorphism between $\mathfrak{X}$ and its image, as well between $\mathfrak{Y}$ and its image. Hence $\mathfrak{X}\backslash\mathfrak{Y} \cong s'(\mathfrak{X})\backslash s'(\mathfrak{Y})$ and, since s' extends s, the latter must be equal to $\tilde{\mathfrak{X}}\backslash\tilde{\mathfrak{Y}}$, yielding

$$\gamma(\mathfrak{X}\backslash\mathfrak{Y}) = [\mathfrak{X}] - [\mathfrak{Y}] = [s'(\mathfrak{X})] - [s'(\mathfrak{Y})] = \gamma(\tilde{\mathfrak{X}}\backslash\tilde{\mathfrak{Y}})$$

as we wanted to show. Hence, γ induces a map $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa^{\mathrm{split}}) \to \mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)$. By construction, it is surjective and the identity on $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)$ (when viewing a full motif as a split motif), showing that it is a bijection. By construction it is also additive, and the reader readily verifies that it preserves products, thus showing that it is an isomorphism. □

8.7. Remark. Consider the pure-schemic motif $\mathfrak{l}_3^\circ\backslash\mathfrak{l}_2^\circ$. It has no κ-rational points, but is does have an $\mathfrak{l}_3$-rational point, namely the identity morphism on $\mathfrak{l}_3$. This example shows that the analogue of Lemma 7.6 does not hold for split formal motives.

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