J. ALGEBRAIC GEOMETRY
**00** (XXXX) 000–000
S 1056-3911(XX)0000-0

# SCHEMIC GROTHENDIECK RINGS II: JET SCHEMES AND MOTIVIC INTEGRATION

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**Abstract**

We generalize the notion of a jet scheme (truncated arc space) to arbitrary fat points via adjunction, and show that this yields for each fat point, an endomorphism on each schemic Grothendieck ring as defined in [17]. We prove that some of the analogues for linear jets still hold true, like locally trivial fibration over the smooth locus. In this formalism, we can define several generating zeta series, *motivic series*, the rationality of which can now be investigated. We use the theory of jet schemes to define a local motivic integration with values in the formal Grothendieck ring.

## 1. Introduction

Modeled on $p$-adic integration, Kontsevich [9] formulated a general integration technique for smooth varieties over an algebraically closed field $\kappa$, called *motivic integration*. This was extended by Denef and Loeser [1, 2, 3] to arbitrary varieties to achieve *motivic rationality*, by which they mean the fact that the rationality of a certain generating series from geometry or number-theory, like the Igusa-zeta series, is "motivated" by the rationality of its motivic counterpart. Here, the motivic counterpart is supposed to specialize to the given classical series via some multiplicative function, like a counting function or Euler characteristic. The two main ingredients of this construction are the Grothendieck ring of varieties over $\kappa$, in which the integration takes its values, and the *truncated arc space* $L(X)$ of a variety $X$, that is to say, the reduced Hilbert scheme classifying all jets $\operatorname{Spec}\kappa[[\xi]] \to X$.

In [17], we generalized the concept of a Grothendieck ring to include also schemes with nilpotent structure. The idea is to view a scheme as a contravariant functor, not on all $\kappa$-algebras, but only on Artinian local $\kappa$-algebras, the so-called *fat points*. We defined a *formal motif* as a certain subfunctor of a representable functor which can be approximated by images of scheme-theoretic maps (more details are given in §2), and build from these the *formal Grothendieck ring* $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)$. In the present paper, we turn to the second ingredient and define jet schemes via adjunction given by base change over a fat point $\mathfrak{z}$. The resulting jet motif $\nabla_{\mathfrak{z}}\mathfrak{X}$ is again formal and classifies

Received September 19, 2011.

all maps from $\mathfrak{z}$ to $\mathfrak{X}$. Applied to a scheme $X$, with $\mathfrak{z}$ equal to the $n$-th co-jet $\mathfrak{l}_n := \operatorname{Spec}(\kappa[t]/t^n\kappa[t])$, we recover the classical truncated arc/jet scheme. The major advantage over the classical construction is that each jet map $\nabla_{\mathfrak{z}}$ now operates on the formal Grothendieck ring $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)$. Jets behave well over smooth varieties, as in the classical case (Theorem 4.14): the canonical morphism $\nabla_{\mathfrak{z}}X \to X$ is a locally trivial fibration over the non-singular locus of $X$, with general fiber some affine space. Hence, in the smooth case, $[\nabla_{\mathfrak{z}}X] = [X]\mathbb{L}^{l(d-1)}$, where $d$ and $l$ are respectively the dimension of $X$ and the length of (the coordinate ring of) $\mathfrak{z}$, and where $\mathbb{L} := [\mathbb{A}^1_\kappa]$ is the *Lefschetz class*.

Using the formalism of adjunction, we discuss some variants of jets: *deformed jets* in §6, and *extendable jets* in §8. For the definition of the latter, we discuss in §7 a compactification of the category of fat points, the category of *limit points*, given as direct limits of fat points (e.g., the formal completion $\widehat{Y}_P$). Although we can extend the notion of jets to any limit point, the corresponding scheme is no longer of finite type, and will be called an *arc scheme*.

In §9, we discuss some of the motivic series that can now be defined using this formalism. Since they or their classical variants specialize to generating series that are known to be rational, we ask whether they are already rational over the formal Grothendieck ring, or rather, over its localization $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)_{\mathbb{L}}$; this is what is meant by *motivic rationality*.

The final section, §10 is devoted to motivic integration. We only develop the finitistic theory, that is to say, over a fixed fat point, leaving the case of a limit point to a future paper. One of the great disadvantages of the categorical approach is that fibers are in general not functorial (after all, a fiber is the complement of the remaining fibers). We can overcome this, without changing the resulting Grothendieck ring, by restricting to the category of *split fat points*, as discussed in [17, §9]. Our motivic integration will take values in the localization $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)_{\mathbb{L}}$. A functor $s$, viewed on the category of split fat points, from a formal motif $\mathfrak{X}$ on $X$ to the constant sheaf with values in this localization $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)_{\mathbb{L}}$ is called a *formal invariant* if all its fibers are formal motives, with only finitely many non-empty. We then define

$$\int s\, d_{\mathfrak{z}}X := \mathbb{L}^{-dl} \sum_{g \in \mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)_{\mathbb{L}}} g \cdot [\nabla_{\mathfrak{z}}(s^{-1}(g))],$$

where $d$ is the dimension of $X$ and $l$ the length of $\mathfrak{z}$. This motivic integral can be calculated locally (Theorem 10.4).

**Notation and terminology.** Varieties are assumed to be reduced, but not necessarily irreducible. Given a scheme $X$, we let $X^{\mathrm{red}}$ denote its *underlying variety* or *reduction*. We often denote a morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ by the same letter as the corresponding ring homomorphism $A \to B$, whenever this causes no confusion. By a *germ* $(X, Y)$ we mean a scheme $X$ together with a closed subscheme $Y \subseteq X$. Most of the time $Y$ is an irreducible subvariety, that is to say, the closure of a point $y \in X$, and we simply write $(X, y)$ for this germ. If $Y$ is a closed point, we call the germ *closed*. The *$n$-th co-jet* $J^n_Y X$ of a germ $(X, Y)$ is the closed subscheme defined by $\mathcal{I}^n_Y$, where $\mathcal{I}_Y$ is the ideal of definition of $Y$.[1] The *formal completion* $\widehat{X}_Y$ of the germ $(X, Y)$ is the locally ringed space obtained as the direct limit of the $J^n_Y X$ (see [7, II.§9]). For instance, if $Y = P$ is a closed point with maximal ideal $\mathfrak{m}_P$, then the ring of global sections of $\widehat{X}_P$ is the $\mathfrak{m}_P$-adic completion $\widehat{\mathcal{O}}_{X,P}$ of $\mathcal{O}_{X,P}$.

We denote the affine line $\mathbb{A}^1_V := \mathbb{A}^1_{\mathbb{Z}} \times V$ over a base scheme $V$ by $\mathbb{L}$, or $\mathbb{L}_V$ if we want to emphasize the base scheme, and also use this notation for its class in a Grothendieck ring. The formal completion of the germ $(\mathbb{L}, O)$, where $O$ is the origin, is denoted $\widehat{\mathbb{L}}$, and the *punctured line* $\mathbb{L}\backslash O$, that is to say, the open subscheme obtained by removing the origin, is denoted $\mathbb{L}_*$. Recall the formula $\mathbb{L} = \mathbb{L}_* + \widehat{\mathbb{L}}$ in $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)$ from [17, Proposition 7.1]. The $n$-th co-jet of $(\mathbb{L}, O)$ will be denoted $\mathfrak{l}_n = \operatorname{Spec} R_n$, where $R_n := (\kappa[x]/(x^n))$.

## 2. The formal Grothendieck ring

In this section, I give a brief overview of the results, as we need them, from [17]. Fix a Noetherian, separated, Jacobson scheme $V$ as a base scheme, which often is just the spectrum of an algebraically closed field $\kappa$. By a *scheme* $X$, we mean a separated scheme of finite type over $V$, and we let $\mathbb{S}\mathrm{ch}_V$ denote the category of schemes over $V$. Viewing a scheme $X$ as a contravariant functor on the subcategory $\mathbb{F}\mathrm{at}_V \subseteq \mathbb{S}\mathrm{ch}_V$ of all fat points, we denote it for emphasis by $X^\circ$. Recall that a *fat ($V$-)point* $\mathfrak{z}$ is a scheme of the form $\operatorname{Spec} R$ with $R$ a finite, local $V$-algebra (that is to say, a one-point scheme). More precisely, we let $X(\mathfrak{z})$ denote the collection of all *$\mathfrak{z}$-rational points*, by which we mean $V$-morphisms $\mathfrak{z} \to X$. In other words, $X^\circ = \operatorname{Mor}_V(\cdot, X)$. By a *sieve* on $X$, we mean a subfunctor $\mathfrak{X}$ of $X^\circ$, and we denote the category of sieves over $V$ by $\mathbb{S}\mathrm{ieve}_V$. Morphisms are a bit more tricky, as we cannot allow just any natural transformation (see [17, Example 2.16]). One first makes the category of sieves into a topos via the Zariski topology of the ambient space ([17, §3]), and requires that all morphisms be continuous, although this is still not sufficient. Without going into details, the most important class of morphisms are the rational ones, where a morphism $s\colon \mathfrak{Y} \to \mathfrak{X}$ is called *rational*, if it is induced by a scheme-theoretic morphism $\varphi\colon Y \to X$ of some ambient spaces, meaning that $s = \mathfrak{Y}|_{\varphi^\circ}$ (see [17, §2.14]). In particular, we have

$$\operatorname{Mor}_{\mathbb{S}\mathrm{ieve}_V}(\mathfrak{z}^\circ, \mathfrak{X}) \cong \mathfrak{X}(\mathfrak{z}). \tag{1}$$

[1] Note that many authors take instead the $n + 1$-th power.

for every fat point $\mathfrak{z}$.

By applying set-theoretic constructions (such as inclusion, intersection, complement,...) point-wise, that is to say, on each set $\mathfrak{X}(\mathfrak{z})$, we can extend them to any sieve. Now, the key notion is that of a *motivic site* $\mathbb{M}$, which is a category of sieves closed under Cartesian products, restricting on each scheme $X$ to a lattice (with respect to intersection and union); an $\mathbb{M}$-sieve on $X$ is called an $\mathbb{M}$-*motif with ambient space* $X$. To a motivic site $\mathbb{M}$, we associate its Grothendieck ring $\mathbf{Gr}(\mathbb{M})$ as the free Abelian group on $\mathbb{M}$-homeomorphism classes $\langle \mathfrak{X} \rangle$ of motives $\mathfrak{X}$ modulo the *scissor relations*

$$\langle \mathfrak{X} \rangle + \langle \mathfrak{Y} \rangle - \langle \mathfrak{X} \cup \mathfrak{Y} \rangle - \langle \mathfrak{X} \cap \mathfrak{Y} \rangle$$

for any two motives $\mathfrak{X}$ and $\mathfrak{Y}$ with common ambient space (so that $\mathfrak{X} \cup \mathfrak{Y}$ and $\mathfrak{X} \cap \mathfrak{Y}$ are again motives in $\mathbb{M}$). In [17], we give three motivic sites, each extending the previous one, with the property that each of their Grothendieck rings admits a homomorphism into the classical Grothendieck ring $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_V)$ of varieties. Namely, given an arbitrary scheme $X$, we call a sieve $\mathfrak{X}$

- (2.i) a *schemic* motif when it is a finite union of (functors represented by) closed subschemes, yielding the Grothendieck ring $\mathbf{Gr}(\mathbb{S}\mathrm{ch}_V)$;
- (2.ii) to close the latter site under homorphic images, we call $\mathfrak{X}$ *sub-schemic* if its equals the sieve given by the image of some (scheme-theoretic) morphism $Y \to X$ (that is to say, on each fat point $\mathfrak{z}$, it consists of the $\mathfrak{z}$-rational points on $X$ that factor through $Y$), yielding the *sub-schemic* Grothendieck ring $\mathbf{Gr}(\mathrm{sub}\mathbb{S}\mathrm{ch}_V)$;
- (2.iii) to include certain complements of sub-schemic motives, we say $\mathfrak{X}$ is *formal*, if it can be approximated by sub-schemic motives, yielding the *formal Grothendieck ring* $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_V)$.

For our purposes, we will mainly work with the latter class, and so let me give a more detailed definition: $\mathfrak{X}$ is a formal motif on $X$, if there exists, for each fat point $\mathfrak{z}$, a morphism $\varphi_\mathfrak{z} \colon Y_\mathfrak{z} \to X$ whose image lies inside $\mathfrak{X}$ (meaning that $\varphi_\mathfrak{z}(\mathfrak{x})\left(Y_{(}\mathfrak{x})\right) \subseteq \mathfrak{X}(\mathfrak{x})$ for all fat points $\mathfrak{x}$) and is equal to it at the fat point $\mathfrak{x} = \mathfrak{z}$ itself. If, moreover, we can choose the morphism $\varphi_\mathfrak{z}$ only depending on the length of $\mathfrak{z}$, then we call $\mathfrak{X}$ *strongly formal*, and we denote the collection of all strongly formal motives by $\mathbb{F}\mathrm{orm}_V^{\mathrm{str}}$. Any morphism with sub-schemic source is rational ([17, Theorem 3.17]), and hence any morphism with formal source can be approximated by rational ones. The most important example of a sub-schemic motif on a scheme $X$ is the functor $U^\circ$ represented by a Zariski open $U \subseteq X$, and that of strongly formal motif, is its complement $X^\circ \setminus U^\circ$, which is represented by the formal completion $\widehat{X}$ of $X$ along the complement $X \setminus U$. We proved in [17, Theorem 7.6], that if $V$ is the spectrum of an algebraically closed field $\kappa$, then there exists a canonical homomorphism from $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)$ to the classical Grothendieck ring $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$, sending the class of a motif $\mathfrak{X}$ to the class of its $\kappa$-rational points $\mathfrak{X}(\kappa)$. For the purposes in this paper, we mainly need strongly formal motives. As we also need to invert the *Lefschetz class* $\mathbb{L}$, we therefore work, especially in the latter part of this paper, mostly in $\mathbf{G}_V := \mathbf{Gr}(\mathbb{F}\mathrm{orm}_V^{\mathrm{str}})_{\mathbb{L}}$. Moreover, the base field is more often than not just an algebraically closed field $\kappa$, and so we write $\mathbf{G}$ for $\mathbf{G}_\kappa$.

## 3. Adjunction

Let $V$ and $W$ be two Noetherian schemes. By a *schemic adjunction* over $(V, W)$, we mean a pair of functors $\eta\colon \mathbb{F}\mathrm{at}_W \to \mathbb{F}\mathrm{at}_V$ and $\nabla\colon \mathbb{S}\mathrm{ch}_V \to \mathbb{S}\mathrm{ch}_W$, called respectively the *left* and *right adjoint*, such that, for each fat $W$-point $\mathfrak{z}$ and each $V$-scheme $X$, we have an adjunction isomorphism

$$\Theta_{\mathfrak{z},X}\colon X(\eta(\mathfrak{z})) = \mathrm{Mor}_V(\eta(\mathfrak{z}), X) \cong \mathrm{Mor}_W(\mathfrak{z}, \nabla X) = \nabla X(\mathfrak{z}), \tag{2}$$

which is functorial in both arguments. Whenever $\mathfrak{z}$ and $X$ are clear from the context, we may just denote this isomorphism by $\Theta$, or even omit it altogether, thus identifying $X(\eta(\mathfrak{z}))$ with $\nabla X(\mathfrak{z})$. More generally, by an (arbitrary) *adjunction* we mean the same as above, except that the right adjoint now only takes values in the category of sieves, that is to say, is a functor $\nabla\colon \mathbb{S}\mathrm{ch}_V \to \mathbb{S}\mathrm{ieve}_W$, where we identify the category of $V$-schemes with its image as the full subcategory of representable sieves. Of course, the morphisms on the right hand side of (2) are now to be taken in $\mathbb{S}\mathrm{ieve}_W$, where the last equality is then given by (1) (note that all morphisms are in fact rational). If each $\nabla X$ is sub-schemic or formal, then we call the adjunction respectively *sub-schemic* or *formal*.

We can formulate the adjunction property as a representability question: given a functor $\eta\colon \mathbb{F}\mathrm{at}_W \to \mathbb{F}\mathrm{at}_V$ and a $V$-scheme $X$, let $\nabla_\eta X$ be the functor over $W$ associating to a fat $W$-point $\mathfrak{z}$, the set of rational points $X(\eta(\mathfrak{z}))$. We have adjunction when each functor $\nabla_\eta X$ is a sieve as $X$ varies over all $V$-schemes; the adjunction is then (sub-)schemic or formal, if each $\nabla_\eta X$ is respectively a (sub-)schemic or formal motif. From this perspective, $\nabla_\eta$ is the right adjoint of $\eta$, and we simply call $\nabla_\eta$ the adjunction. To extend this to a functor $\nabla_\eta\colon \mathbb{S}\mathrm{ieve}_V \to \mathbb{S}\mathrm{ieve}_W$, let $\mathfrak{X}$ be a sieve on a $V$-scheme $X$, and define its *adjoint* $\nabla_\eta \mathfrak{X}$ as the functor over $W$ given by

$$\nabla_\eta \mathfrak{X}(\mathfrak{z}) := \Theta_{\mathfrak{z},X}\big(\mathfrak{X}(\eta(\mathfrak{z}))\big)$$

for any $W$-point $\mathfrak{z}$. It follows immediately from (2) that $\nabla_\eta X = \nabla X$, and hence $\nabla_\eta \mathfrak{X}$ is a subsieve of $\nabla X$. The adjunction isomorphism (2) then becomes

$$\mathrm{Mor}_{\mathbb{S}\mathrm{ieve}_V}(\eta(\mathfrak{z}), \mathfrak{X}) \cong \mathrm{Mor}_{\mathbb{S}\mathrm{ieve}_W}(\mathfrak{z}, \nabla_\eta \mathfrak{X}). \tag{3}$$

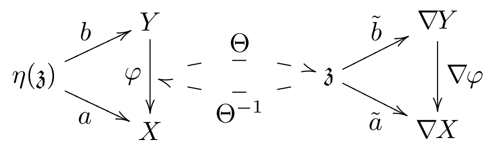
**3.1. Lemma.** *If $\varphi\colon Y \to X$ is a morphism of $V$-schemes, then, with $\nabla\varphi\colon \nabla Y \to \nabla X$ the induced morphism of $W$-sieves, we have an equality of sieves*

$$\nabla_\eta \mathtt{Im}(\varphi) = \mathtt{Im}(\nabla\varphi). \tag{4}$$

*In particular, sub-schemic adjunctions preserve sub-schemic as well as formal motives, whereas formal adjunctions preserve formal motives.*

*Proof.* We verify (4) on a fat $W$-point $\mathfrak{z}$. Functoriality of adjunction implies that we have a one-one correspondence of diagrams

(5)




where the right triangle is in $\mathbb{Sieve}_W$. So, if $\tilde a \in \mathtt{Im}(\nabla\varphi)(\mathfrak{z})$, then by (1), we can find $\tilde b$ making the right triangle in (5) commute. Taking the image under $\Theta^{-1}_{\mathfrak{z},X}$ yields the commutative triangle on the left, showing that $\Theta^{-1}(\tilde a) \in \mathtt{Im}(\varphi)(\eta(\mathfrak{z}))$, and hence that $\tilde a \in (\nabla_\eta \mathtt{Im}(\varphi))(\mathfrak{z})$. The converse holds for the same reason, by going this time from left to right.

It then follows from [17, Theorem 7.8] that the adjoint of a sub-schemic motif is again sub-schemic, in case $\eta$ is sub-schemic itself. Suppose next that $\mathfrak{X}$ is formal, and, for each fat $V$-point $\mathfrak{w}$, let $\mathfrak{Y}_\mathfrak{w} \subseteq \mathfrak{X}$ be a sub-schemic approximation with the same $\mathfrak{w}$-rational points. For each fat $W$-point $\mathfrak{z}$, let $\tilde{\mathfrak{Y}}_\mathfrak{z}$ be defined as $\nabla_\eta(\mathfrak{Y}_{\eta(\mathfrak{z})})$. By what we just proved, $\tilde{\mathfrak{Y}}_\mathfrak{z} \subseteq \nabla_\eta\mathfrak{X}$ is a sub-schemic submotif, and one easily verifies that both sieves have the same $\mathfrak{z}$-rational points, proving the last assertion for sub-schemic adjunctions. The case of a formal adjunction then follows from previously cited theorem and [17, Lemma 7.5]. □

**3.2. Remark.** The proof as it stands, does not work for strongly formal motives. However, with an additional assumption, met in every single application, we can also deal with this case. Namely, let us call $\eta\colon \mathbb{Fat}_W \to \mathbb{Fat}_V$ *bounded*, if $\ell(\eta(\mathfrak{z})) \leqslant \ell(\mathfrak{z})$ for all fat $W$-points $\mathfrak{z}$ (in fact, all we need is that the length of $\eta(\mathfrak{z})$ is bounded by a function only depending on $\ell(\mathfrak{z})$). With this additional assumption, modify the above proof by letting $\tilde{\mathfrak{Y}}_l$ be $\nabla_\eta(\mathfrak{Y}_l)$, where $\mathfrak{Y}_l \subseteq \mathfrak{X}$ is now a strong approximation of $\mathfrak{X}$ by sub-schemic motives.

**3.3. Proposition.** *A formal adjunction $\nabla_\eta$ induces a homomorphism of Grothendieck rings $\nabla_\eta\colon \mathbf{Gr}(\mathbb{Form}_V) \to \mathbf{Gr}(\mathbb{Form}_W)$. If $\nabla_\eta$ is strongly formal and $\eta$ is bounded, we get a homomorphism $\nabla_\eta\colon \mathbf{Gr}(\mathbb{Form}^{str}_V) \to \mathbf{Gr}(\mathbb{Form}^{str}_W)$. If $\nabla_\eta$ is (sub-)schemic, we get a homomorphism of the corresponding (sub-)schemic Grothendieck rings.*

*Proof.* By Lemma 3.1 and Remark 3.2, adjunction preserves motivic sites of the same respective type, (sub-)schemic or formal. As it is compatible with unions and intersections, it preserves scissor relations, and as it is functorial, it preserves isomorphisms as well as products. □

Before we describe some important instances in which we have adjunction, with applications discussed in §§4 and 6, we give an example of a formal adjunction.

**3.4. Example.** Given a fat point $\mathfrak{z}$ over an algebraically closed field $\kappa$, and $r \geqslant 2$, let $\Upsilon(\mathfrak{z}) := \Upsilon_r(\mathfrak{z})$ be the fat point with coordinate ring $\kappa + \mathfrak{m}^r \subseteq R$, where $(R, \mathfrak{m})$ is the Artinian local ring corresponding to $\mathfrak{z}$. Note that $\Upsilon$ is bounded, and we have a strongly dominant morphism $\mathfrak{z} \to \Upsilon(\mathfrak{z})$. For simplicity, let us take $r = 2$. For fixed $n$, let $\mathfrak{l} := \mathfrak{l}_n$ be the $n$-th co-jet of a point on a line, with coordinate ring $S := \kappa[\xi]/(\xi^n)$. For each $l$, let $\mathfrak{w}_l$ be the fat point in $\mathbb{L}^{2l}$ with ideal of definition generated by all $\xi_i^l$ and $Q^n$, where

$$Q := \xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{2l-1}\xi_{2l}.$$

Let $\mathfrak{Y}_l$ be the image sieve of the morphism $\varphi_l \colon \mathfrak{w}_l \to \mathfrak{l}$ induced by $\xi \mapsto Q$. I claim that $\nabla_\Upsilon \mathfrak{l}$ is approximated by the $\mathfrak{Y}_l$, from which it then follows that it is strongly formal. To this end, fix a fat point $\mathfrak{z}$ with coordinate ring $(R, \mathfrak{m})$ and let $l$ be its length. An $\Upsilon(\mathfrak{z})$-rational point $a \in \mathfrak{l}(\Upsilon(\mathfrak{z}))$ is completely determined by the image, denoted again $a$, of $\xi$ in $\kappa + \mathfrak{m}^2$. Since $a^n = 0$, we must in particular have $a \in \mathfrak{m}^2$ (note that $\mathfrak{m}^2$ is the maximal ideal of $\Upsilon(\mathfrak{z})$), and hence can be written as $a = b_1b_2 + \cdots + b_{2l-1}b_{2l}$, for some $b_i \in \mathfrak{m}$. Since $b_i^l = 0$ and $a = Q(b_1, \ldots, b_{2l})$, the assignment $\xi_i \mapsto b_i$ induces a morphism $\mathfrak{z} \to \mathfrak{w}_l$ which factors through $\varphi_l$. In other words, $a \in \mathfrak{Y}_l(\mathfrak{z})$. Conversely, since $Q$ is quadratic, any $\mathfrak{z}$-rational point factoring through $\varphi_l$ must extend to $\Upsilon(\mathfrak{z})$.

Presumably, this argument should extend to any fat point other than $\mathfrak{l}$ and any power $r \geqslant 2$. To extend this to higher dimensional schemes, we face the problem that a rational point can be given by non-units. This forces us to be able to single out the field elements inside an Artinian local ring $R$. In characteristic $p$, this can be done: the elements of $\kappa \subseteq R$ are precisely the $p^l$-th powers. Using this, a slight modification of the above argument then yields $\nabla_\Upsilon \mathbb{L}$ as a strongly formal motif: in the above, replace $\mathfrak{w}_l$ by $\mathbb{L}_{\mathfrak{w}_l}$ and $\mathfrak{Y}_l$ by the image of the morphism $\mathbb{L}_{\mathfrak{w}_l} \to \mathbb{L}_\kappa$ given by $\xi \mapsto \xi_0^{p^l} + Q$. It seems likely that we can again extend this argument to arbitrary schemes and arbitrary $r \geqslant 2$.

**Augmentation.** Fix a morphism of Noetherian schemes $f \colon W \to V$. Via $f$, any $W$-scheme $Y$ becomes a $V$-scheme, and to make a notational distinction between these two scheme structures, we denote the latter by $f_*Y$. We will show that $f_*$ constitutes a left adjoint, where the corresponding right adjoint is given by base change: given a $V$-scheme $X$, we set

$$f^*X := W \times_V X.$$

**3.5. Theorem.** *If $f\colon W \to V$ is a morphism of finite type of Noetherian Jacobson schemes, then $f_*$ is a bounded functor from $\mathbb{Fat}_W$ to $\mathbb{Fat}_V$, and as such, it is the left adjoint of $f^*$. The corresponding adjunction associates to a $V$-sieve $\mathfrak{X}$ on a $V$-scheme $X$, the $W$-sieve $\nabla_{f_*}\mathfrak{X}$ on $f^*X$, inducing ring homomorphisms*

$$\begin{aligned}
\nabla_{f_*}&\colon \mathbf{Gr}(\mathbb{Sch}_V) \to \mathbf{Gr}(\mathbb{Sch}_W)\\
\nabla_{f_*}&\colon \mathbf{Gr}(\mathbb{subSch}_V) \to \mathbf{Gr}(\mathbb{subSch}_W)\\
\nabla_{f_*}&\colon \mathbf{Gr}(\mathbb{Form}_V^{str}) \to \mathbf{Gr}(\mathbb{Form}_W^{str})\\
\nabla_{f_*}&\colon \mathbf{Gr}(\mathbb{Form}_V) \to \mathbf{Gr}(\mathbb{Form}_W).
\end{aligned}$$

*Proof.* Let $\mathfrak{z}$ be a fat $W$-point with coordinate ring $R$ and let $y$ be its center, that is to say, the closed point on $W$ given as the image under the structure morphism $\mathfrak{z} \to W$. By the generalized Nullstellensatz ([6, Theorem 4.19]), the image $x := f(y)$ is a closed point on $V$, and the residue field extension $\kappa(x) \subseteq \kappa(y)$ is finite. As $\kappa(y) \subseteq R/\mathfrak{m}$ is also finite, $f_*\mathfrak{z}$ is a fat $V$-point (note that $R$ is also the coordinate ring of $f_*\mathfrak{z}$, so that $f_*$ is bounded). The adjunction of $f_*$ and $f^*$ is well-known (and, in any case, easily checked; see, for instance [7, Chapter II.5], but note that left and right are switched there since they are formulated in the dual category of sheaves), proving that $\nabla_{f_*}X = f^*X$. The last statement follows from Proposition 3.3. □

If we drop the condition that the schemes are Jacobson, then we must require $f$ to be proper as well.

**3.6. Remark.** Although $f_*\colon \mathbb{Fat}_W \to \mathbb{Fat}_V$ is an embedding of categories, it is, however, not full: so are the closed subschemes in $\mathbb{L}^2$ defined by the ideals $(x^2, y^3)$ and $(x^3, y^2)$ isomorphic as fat $\kappa$-points, but not as fat $\kappa[x]$-points. Nonetheless, $\mathbb{Fat}_W$ is cofinal in $\mathbb{Fat}_V$, or, in the terminology of §7 below, both have the same universal point.

**Diminution.** Let $f\colon W \to V$ be a finite and faithfully flat morphism of Noetherian schemes. As opposed to the previous section, we will now consider $f^*$ as a left adjoint. For technical reasons (see Remark 3.9 below for how to circumvent these), we make the following additional assumptions:

(†) $V$ is of finite type over an algebraically closed field $\kappa$ and $f$ induces an isomorphism on the underlying varieties.

The second condition implies that for any closed point $x \in V$ there is a unique closed point $y \in W$ lying above it, and hence the closed fiber $f^{-1}(x)$ is a local scheme. Under these assumptions, the base change $f^*\mathfrak{z}$ of a fat $V$-point $\mathfrak{z}$ is a fat $W$-point. Indeed, since the problem is local, we may assume $V = \operatorname{Spec}\lambda$ and $W = \operatorname{Spec}\mu$ are affine. Let $(R, \mathfrak{m})$ be the coordinate ring of $\mathfrak{z}$, and let $\mathfrak{p} := \lambda \cap \mathfrak{m}$ be the induced maximal ideal of $\lambda$, defining its center. The coordinate ring of $f^*\mathfrak{z}$ is then $S := R \otimes_\lambda \mu$. By base change, $S$ is finite (and flat) over $R$, whence in particular Artinian. By base change, $S$ is also finite as a $\mu$-module, and $\ell(S) \leqslant \ell(R)$. Since $\mathfrak{m}$ is nilpotent, any maximal ideal of $S$ must contain $\mathfrak{m}S$. Since $S/\mathfrak{m}S = R/\mathfrak{m}\otimes_{\lambda/\mathfrak{p}}\mu/\mathfrak{p}\mu$, since $\lambda/\mathfrak{p} = \kappa$ by the Nullstellensatz, and since $R/\mathfrak{m}$ is a finite extension of the latter, whence trivial, $S/\mathfrak{m}S = \mu/\mathfrak{p}\mu$ is local by assumption (†), showing that $S$ itself is an Artinian local ring, thus proving the claim. Moreover, $f^*$ is bounded.

**3.7. Theorem.** *If $f\colon W \to V$ is a finite and faithfully flat morphism satisfying (†), then $f^*$ is the left adjoint of a schemic adjunction, inducing natural homomorphisms*

$$\nabla_{f*}\colon \mathbf{Gr}(\mathbb{S}\mathbb{c}\mathbb{h}_W) \to \mathbf{Gr}(\mathbb{S}\mathbb{c}\mathbb{h}_V)$$
$$\nabla_{f*}\colon \mathbf{Gr}(\mathbb{s}\mathbb{u}\mathbb{b}\mathbb{S}\mathbb{c}\mathbb{h}_W) \to \mathbf{Gr}(\mathbb{s}\mathbb{u}\mathbb{b}\mathbb{S}\mathbb{c}\mathbb{h}_V)$$
$$\nabla_{f*}\colon \mathbf{Gr}(\mathbb{F}\mathbb{o}\mathbb{r}\mathbb{m}_W^{str}) \to \mathbf{Gr}(\mathbb{F}\mathbb{o}\mathbb{r}\mathbb{m}_V^{str})$$
$$\nabla_{f*}\colon \mathbf{Gr}(\mathbb{F}\mathbb{o}\mathbb{r}\mathbb{m}_W) \to \mathbf{Gr}(\mathbb{F}\mathbb{o}\mathbb{r}\mathbb{m}_V).$$

*More precisely, for any $W$-scheme $Y$, there exists a $V$-scheme $\nabla_{f*}Y$ and a canonical morphism $\rho_Y\colon f^*(\nabla_{f*}Y) \to Y$ of $W$-schemes, such that, for any fat $V$-point $\mathfrak{z}$, the map sending a $\mathfrak{z}$-rational point $a\colon \mathfrak{z} \to \nabla_{f*}Y$ to the $f^*\mathfrak{z}$-rational point $\rho_Y \circ f^*a\colon f^*\mathfrak{z} \to Y$, induces an isomorphism $(\nabla_{f*}Y)(\mathfrak{z}) = Y(f^*\mathfrak{z})$.*

*If $Z \subseteq Y$ is a closed immersion, then so is $\nabla_{f*}Z \to \nabla_{f*}Y$.*

*Proof.* Since $f$ is finite and flat, $W$ is locally free over $V$. Since we may construct each $\nabla_{f*}Y$ locally and then, by the uniqueness of the universal property of adjoints, glue the pieces together, we may assume that $Y = \operatorname{Spec} B$, $V = \operatorname{Spec}\lambda$, and $W = \operatorname{Spec}\mu$ are affine, and that $\mu$ is free over $\lambda$ (in all applications, we will already have global freeness anyway). Let $\alpha_1, \dots, \alpha_l$ be a basis of $\mu$ over $\lambda$. Write $B := \mu[x]/(h_1, \dots, h_s)$, for some polynomials $h_i$ over $\mu$, and $x$ a $n$-tuple of variables. Let $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_l)$ be a row of $l$ many $n$-tuples of variables $\tilde{x}_i$, for $i = 1, \dots, l$, and define a generic $n$-tuple of jets

$$\hat{x} := \alpha_1\tilde{x}_1 + \dots + \alpha_l\tilde{x}_l \tag{6}$$

in $(\mu[\tilde{x}])^n$. Given any $g \in \mu[x]$, let $\nabla_j g \in \lambda[\tilde{x}]$ be defined by the expansion

$$g(\hat{x}) = \sum_{j=1}^{l} \alpha_j \nabla_j g. \tag{7}$$

Applying (7) with $g = h_i$, for $i = 1, \dots, s$, we get polynomials $\nabla_j h_i$ in $\lambda[\tilde{x}]$ and we let $A$ be the residue ring of $\lambda[\tilde{x}]$ modulo the ideal generated by all these $\nabla_j h_i$, with $i = 1, \dots, s$ and $j = 1, \dots, l$. I claim that $X := \operatorname{Spec} A$ represents $\nabla_{f*}Y$. It follows from (7) that the map $x \mapsto \hat{x}$ yields a $\mu$-algebra homomorphism $B \to f^*A$, where $f^*A := A \otimes_\lambda \mu$ is the base change, and hence a $\mu$-morphism $\rho_Y\colon f^*X \to Y$. Fix a fat $\lambda$-point $\mathfrak{z}$, and a $\mathfrak{z}$-rational point $a\colon \mathfrak{z} \to X$. By base change, we get a $\mu$-algebra homomorphism $f^*\mathfrak{z} \to f^*X$ which composed with $\rho_Y$ induces a $f^*\mathfrak{z}$-point $\Theta(a)\colon f^*\mathfrak{z} \to Y$. To prove that the map $a \mapsto \Theta(a)$ establishes an adjunction isomorphism, we construct its converse. Given an $f^*\mathfrak{z}$-rational point $b\colon f^*\mathfrak{z} \to Y$, let $B \to R \otimes_\lambda \mu$ be the corresponding $\mu$-algebra homomorphism, where $R$ is the coordinate ring of $\mathfrak{z}$. The latter homomorphism is uniquely determined by a tuple $\mathbf{u}$ in $R \otimes_\lambda \mu$ such that all $h_i(\mathbf{u}) = 0$. Expanding this tuple as

$$\mathbf{u} = \alpha_1 \tilde{\mathbf{u}}_1 + \cdots + \alpha_l \tilde{\mathbf{u}}_l \tag{8}$$

yields a (unique) tuple $\tilde{\mathbf{u}} := (\tilde{\mathbf{u}}_1, \ldots, \tilde{\mathbf{u}}_l)$ over $R$ such that all $\nabla_j h_i(\tilde{\mathbf{u}}) = 0$, determining, therefore, a $\lambda$-algebra homomorphism $A \to R$, whence a $\lambda$-morphism $\Lambda(b)\colon \mathfrak{z} \to X$. So remains to verify that $\Lambda$ and $\Theta$ are mutual inverses. Starting with the $f^*\mathfrak{z}$-rational point $b$, we get the $\mathfrak{z}$-rational point $\Lambda(b)$, which in turn induces the $f^*\mathfrak{z}$-rational point $\Theta(\Lambda(b))$, given as the composition $\rho_Y \circ f^*\Lambda(b)$. The latter corresponds by (8) to the $\mu$-algebra homomorphism $B \to f^*A \to f^*R$ given by $x \mapsto \hat{x} \mapsto \mathbf{u}$, showing that $\Theta(\Lambda(b)) = b$. If, on the other hand, we start with the $\mathfrak{z}$-rational point $a$, given by $\tilde{x} \mapsto \tilde{\mathbf{u}}$, we get the $f^*\mathfrak{z}$-rational point $\Theta(a)$, given by $x \mapsto \mathbf{u}$, where $\mathbf{u}$ is as in (8). Hence $\Lambda(\Theta(a))$ is given by $\tilde{x} \mapsto \tilde{\mathbf{u}}$, that is to say, is equal to $a$, as we needed to show.

To prove the last assertion, assume that $Z$ is a closed subscheme of $Y$, so that its coordinate ring is of the form $B/(h_{s+1}, \ldots, h_t)B$ for some additional polynomials $h_i \in \mu[x]$. Hence $\nabla_{f^*} Z$ is the closed subscheme of $\nabla_{f^*} Y$ given by the $\nabla_j h_i$ for $s < i \leqslant t$. □

Immediately from the above proof, by taking $Y = f^*X$, we have the following result, which we will use in the next section:

**3.8. Corollary.** *If $f\colon W \to V$ is a finite and faithfully flat morphism satisfying (†), then we have for each $V$-scheme $X$, a canonical $V$-morphism $\rho_X\colon \nabla_{f^*} f^*X \to X$. If $Z \subseteq X$ is a closed immersion, then so is $\nabla_{f^*} f^*Z \to \nabla_{f^*} f^*X$.* □

**3.9. Remark.** Without assumption (†), the pull-back of a fat $V$-point $\mathfrak{z}$ is only a zero-dimensional $W$-scheme, and hence a disjoint sum of fat $W$-points $f^*\mathfrak{z} = \mathfrak{w}_1 \sqcup \cdots \sqcup \mathfrak{w}_s$. We can then still make sense of $Y(f^*\mathfrak{z})$, as the disjoint union $Y(\mathfrak{w}_1) \sqcup \cdots \sqcup Y(\mathfrak{w}_s)$, and the adjunction condition then becomes that this must be equal to $(\nabla_{f^*} Y)(\mathfrak{z})$. Since nowhere in the above proof we used that $f^*R$ is local, we therefore can omit condition (†) from the statements of Theorem 3.7 and Corollary 3.8.

Caveat: do not confuse diminution with the restriction of scalars operation $f_*$ defined in [17, §7.13], which is not an adjunction. In particular, $f_*$ is not multiplicative. Using the present notation, we can now generalize [17, (29)] as follows:

**3.10. Lemma.** *Given a morphism $f\colon W \to V$ of finite type of Noetherian Jacobson schemes, for every $\alpha \in \mathbf{Gr}(\mathbb{F}\mathrm{orm}_V^{str})$ and every $\beta \in \mathbf{Gr}(\mathbb{F}\mathrm{orm}_W^{str})$, we have an identity*

$$\alpha \cdot f_*\beta = f_*(\nabla_{f_*}\alpha \cdot \beta) \tag{9}$$

*in $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_V^{str})$. In particular, the image of $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_W^{str})$ under $f_*$ is an ideal $\mathfrak{I}_W$ in $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_V^{str})$, called the* ideal of $W$-motives.

*Proof.* By linearity, we may reduce to the case that $\alpha = [\mathfrak{X}]$ and $\beta = [\mathfrak{Y}]$, with $\mathfrak{X}$ and $\mathfrak{Y}$ strongly formal motives on some $V$-scheme $X$ and some $W$-scheme $Y$ respectively. Given a fat $V$-point $\mathfrak{z}$ and a $\mathfrak{z}$-rational map $a\colon \mathfrak{z} \to f_*(\nabla_{f_*} X \times_W Y) = X \times_V f_*Y$, let $\pi_1$ and $\pi_2$ be the respective projections of $X \times_V f_*Y$ to $X$ and $f_*Y$. Now, $\tilde{a}\colon \tilde{\mathfrak{z}}_a \to f^*X \times_W Y$ belongs to $\left(\nabla_{f_*}\mathfrak{X} \times \mathfrak{Y}\right)(\tilde{\mathfrak{z}}_a)$ if and only if $\pi_1 \circ \tilde{a}$ and $\pi_2 \circ \tilde{a}$ belong respectively to

$$\nabla_{f_*}\mathfrak{X}(\tilde{\mathfrak{z}}_a) = \mathfrak{X}(f_*\tilde{\mathfrak{z}}_a) = \mathfrak{X}(\mathfrak{z})$$

and $\mathfrak{Y}(\tilde{\mathfrak{z}}_a)$. But this just means that $a \in \mathfrak{X}(\mathfrak{z}) \times f_*\mathfrak{Y}(\mathfrak{z})$, proving (9). The second assertion now easily follows from this. □

Taking $\beta = 1$ and using that $f_*1 = [W]$, we see that the image of the composite map $f_* \circ \nabla_{f_*}\colon \mathbf{Gr}(\mathbb{F}\mathrm{orm}^{\mathrm{str}}_V) \to \mathbf{Gr}(\mathbb{F}\mathrm{orm}^{\mathrm{str}}_W) \to \mathbf{Gr}(\mathbb{F}\mathrm{orm}^{\mathrm{str}}_V)$ is the ideal generated by $[W]$. In view of [17, Corollary 7.16], we ask whether $f$ is split if and only if $\mathfrak{I}_W$ is the unit ideal. In this generality, we do not know the answer, but by restricting to schemic motives, we get the following criterion. Let $\mathbf{M}_V(W)$ be the residue ring $\mathbf{Gr}(\mathbb{S}\mathrm{ch}_V)/\mathfrak{I}_W$, then we have:

**3.11. Corollary.** *Over a Noetherian, Jacobson $V$, a $V$-scheme $W$ has a $V$-rational point if and only if $\mathbf{M}_V(W) = 0$.*

*Proof.* Saying that $f\colon W \to V$ has a $V$-rational point just means that $f$ is split, and by [17, Corollary 7.16], this is equivalent with the surjectivity of $f_*$ (on schemic motives). This proves already one direction. For the converse, if $\mathfrak{I}_W$ is the unit ideal, it contains $1 = [V]$ and hence $[V] = [f_*Y] - [f_*Z]$ in $\mathbf{Gr}(\mathbb{S}\mathrm{ch}_V)$, for some $W$-schemes $Y$ and $Z$. Since any schemic irreducible component of $f_*Y$ is also obtained by restriction of scalars (use for instance [17, Lemma 7.15]), we may assume $f_*Y$ is schemic irreducible, and hence $V \sqcup f_*Z \cong f_*Y$ by [17, Corollary 5.8], whence $V = f_*Y$ by the irreducibility assumption. It follows that $f_*$ is surjective, whence $f$ is split. □

Thus, we may rephrase the Faltings-Mordell theorem that over a number field $\kappa$, any curve $C$ of genus at least two, has a (non-empty, whence dense) open subset $U \subseteq C$ such that $\mathbf{M}_\kappa(U) \neq 0$. The Bombieri-Lang conjecture can then be stated in the same vain: if $X$ is a smooth variety of general type over $\kappa$, does there exist an open $U \subseteq X$ such that $\mathbf{M}_\kappa(U) \neq 0$. For adjunctions, (9) corresponds to the following commutation rule in a Cartesian square:

**3.12. Theorem** (Projection Formula)**.** *Let $f\colon W \to V$ be a finite and faithfully flat morphism of Noetherian schemes satisfying (†), let $u\colon \tilde{V} \to V$ be a morphism of finite type, with either $V$ Jacobson or $u$ proper, and let*

(10)
$$\begin{array}{ccc} \tilde W & \xrightarrow{\ \tilde f\ } & \tilde V \\ {\scriptstyle \tilde u}\downarrow & & \downarrow{\scriptstyle u} \\ W & \xrightarrow[\ f\ ]{} & V \end{array}$$

*be the base change diagram, where* $\tilde W := W \times_V \tilde V$*. We have an identity of adjunctions*

$$\nabla_{\tilde f *} \nabla_{\tilde u *} = \nabla_{u *} \nabla_{f *}$$

*from* $W$*-sieves to* $\tilde V$*-sieves.*

*Proof.* Note that $\tilde f$ is again finite and faithfully flat, satisfying (†), so that the diminution $\nabla_{\tilde f *}$ makes sense. To prove the identity we have to check it on each $W$-sieve $\mathfrak{Y}$ and each fat $\tilde V$-point $\mathfrak{z}$, becoming

$$(\nabla_{\tilde f *} \nabla_{\tilde u *} \mathfrak{Y})(\mathfrak{z}) = \mathfrak{Y}(\tilde u_* \tilde f^* \mathfrak{z}) \stackrel{?}{=} \mathfrak{Y}(f^* u_* \mathfrak{z}) = (\nabla_{u *} \nabla_{f *} \mathfrak{Y})(\mathfrak{z}).$$

But one easily verifies that we have an equality of fat $W$-points

$$\tilde u_* \tilde f^* \mathfrak{z} = f^* u_* \mathfrak{z}$$

concluding the proof of the theorem. □

**Frobenius transform.** Assume for the remainder of this section that the base ring is a field $\kappa$ of characteristic $p > 0$. Let us denote the Frobenius homomorphism $a \mapsto a^p$ on a $\kappa$-algebra $A$ by $\mathbf{F}$, or in case we need to specify the ring by $\mathbf{F}_A$, so that we have in particular a commutative diagram

(11)
$$\begin{array}{ccc} \kappa & \xrightarrow{\ \mathbf{F}_\kappa\ } & \kappa \\ \downarrow & & \downarrow \\ A & \xrightarrow[\ \mathbf{F}_A\ ]{} & A. \end{array}$$

Due to the functorial nature, we can glue these together and hence obtain on any $\kappa$-scheme $X$ a corresponding endomorphism $\mathbf{F}_X$.

Diagram (11) implies that $\mathbf{F}_A$ is not a $\kappa$-algebra homomorphism. To overcome this difficulty, we assume $\kappa$ is perfect, so that $\mathbf{F}$ is an isomorphism on $\kappa$. To make (11) into a $\kappa$-algebra homomorphism, we must view the second copy of $A$ with a different $\kappa$-algebra structure, namely, the one inherited from the composite homomorphism $\kappa \xrightarrow{\mathbf{F}} \kappa \to A$. Several notational devices have been proposed (see for instance [7, Chapter IV, Remark 2.4.1] or [18, Chapter 8.1.c]), but we will use the one already introduced in the previous section: the push-forward of $A$ along $\mathbf{F}$ will be denoted $\mathbf{F}_*A$. In other words, $\mathbf{F}_*A$ is $A$ with its $\kappa$-action given by $u \cdot a = u^p a$. Since $\kappa$ is perfect, $A \cong \mathbf{F}_*A$ as rings, and in many instances, even as $\kappa$-algebras. In particular, (11) yields a $\kappa$-algebra homomorphism $A \xrightarrow{\mathbf{F}} \mathbf{F}_*A$, called the *$\kappa$-linear Frobenius*. The image of the $\kappa$-linear Frobenius homomorphism $A \xrightarrow{\mathbf{F}} \mathbf{F}_*A$ is the subring of $A$ consisting of all $p$-th powers, and we will simply denote it by $\mathbf{F}A$ (rather than the more common $A^p$, which might lead to confusions with Cartesian powers). Hence, pushing forward the inclusion homomorphism $\mathbf{F}A \subseteq A$ gives a factorization of the $\kappa$-algebra homomorphism $\mathbf{F}$ as $A \twoheadrightarrow \mathbf{F}_*\mathbf{F}A \subseteq \mathbf{F}_*A$, where the first homomorphism is an isomorphism if and only if $A$ is reduced. For instance, if $A = \kappa[x]$, then $\mathbf{F}A = \kappa[x^p]$, so that this factorization is given by the sequence of $\kappa$-algebra homomorphisms

$$
\begin{array}{ccccccc}
\kappa[x] & \xrightarrow{\cong} & \mathbf{F}_*\kappa[x^p] & \subseteq & \mathbf{F}_*\kappa[x] & \xrightarrow{\cong} & \kappa[x] \\
 & & & & h & \xmapsto{\sigma} & \tilde{h} \\
g & \longmapsto & g^p & & \longmapsto & & g(x^p),
\end{array}
\tag{12}
$$

where $\tilde{h}$ is obtained from $h$ by replacing each coefficient with its (unique) $p$-th root. So, from this we can calculate $\mathbf{F}_*A$ for $A$ of the form $\kappa[x]/(f_1, \dots, f_s)$ as

$$\mathbf{F}_*A \cong \kappa[x]/(\tilde{f}_1, \dots, \tilde{f}_s),$$

with $\tilde{f}_i = \sigma(f_i)$ as in (12). The $\kappa$-linear Frobenius $A \to \mathbf{F}_*A$ is then the induced homomorphism by the composite map $g \mapsto g(x^p)$ from (12).

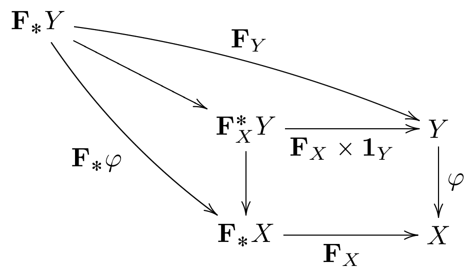
Similarly, viewing $X$ as a $\kappa$-scheme via the composition $X \to \operatorname{Spec}\kappa \xrightarrow{\mathbf{F}} \operatorname{Spec}\kappa$, it will be denoted by $\mathbf{F}_*X$, yielding a morphism $\mathbf{F}_X \colon \mathbf{F}_*X \to X$ of $\kappa$-schemes, called the *$\kappa$-linear Frobenius*. Its scheme-theoretic image will be denoted by $\mathbf{F}X$, so that we have a strongly dominant morphism $X \to \mathbf{F}X$, yielding a factorization

$$\mathbf{F}_X \colon \mathbf{F}_*X \to \mathbf{F}_*\mathbf{F}X \subseteq X \tag{13}$$

of $\mathbf{F}_X$, where the closed immersion $\mathbf{F}_*\mathbf{F}X \subseteq X$ is the identity if and only if $X$ is a variety. In particular, $\mathbf{F}X$ is the Zariski closure of $\mathtt{Im}(\mathbf{F}_X)$ in $X$.

We could view $\mathbf{F}_\kappa$ as an automorphism of the base to get by Theorem 3.5 an adjunction pair $(\mathbf{F}_\kappa^*, \mathbf{F}_{\kappa*})$. However, since $X$ and $\mathbf{F}_{\kappa*}\mathbf{F}_\kappa^*X$ are isomorphic as $\kappa$-schemes, this merely induces an action of the Frobenius. For the same reason, diminution does not induce any interesting endomorphism on the Grothendieck ring. Instead we take a relative point of view. To a morphism $\varphi \colon Y \to X$ of $\kappa$-schemes, we can associate two commutative squares; the base change and the Frobenius square.

Combined into a single commutative diagram of $\kappa$-morphisms, we have




where $\mathbf{F}^*_X Y := \mathbf{F}_*X \times_X Y$ is the pull-back of $Y$ along $\mathbf{F}_X$, called the *Frobenius transform of* $Y$ *in* $X$, and where the canonical projection $\mathbf{F}_X \times \mathbf{1}_Y \colon \mathbf{F}^*_X Y = \mathbf{F}_*X \times_X Y \to Y$ is called the *relative Frobenius on* $Y$ *over* $X$. In case $\varphi$ is a closed immersion, the natural morphism $\mathbf{F}_*Y \to \mathbf{F}^*_X Y$ is then also a closed immersion. We can calculate it explicitly in case $X = \operatorname{Spec} A$ is affine and $Y$ is defined by the ideal $I \subseteq A$. Traditionally, one denotes the ideal generated by the image of $I$ under the Frobenius $\mathbf{F}_A$ by $I^{[p]}$; it is the ideal generated by all $f^p$ with $f \in I$. With this notation, we have

$$\mathbf{F}^*_X Y = \mathbf{F}_*(\operatorname{Spec} A/I^{[p]}).$$

In particular, applying $\sigma$ from (12) to the previous isomorphism in case $X$ is affine space, we get:

**3.13. Corollary.** *If* $Y$ *is the closed subscheme of* $\mathbb{L}^n$ *with ideal of definition* $(f_1, \dots, f_s)$*, then* $\mathbf{F}^*_{\mathbb{L}^n} Y$ *is the closed subscheme of* $\mathbb{L}^n$ *with ideal of definition* $(f_1(x^p), \dots, f_s(x^p))$*, and the relative Frobenius* $\mathbf{F}_{\mathbb{L}^n} \times \mathbf{1}_Y$ *is the map induced by* $x \mapsto x^p$. □

The assignment $\mathfrak{z} \mapsto \mathbf{F}\mathfrak{z}$ constitutes a bounded functor on $\mathbb{F}\mathtt{at}_\kappa$, which will play the role of left adjoint. However, in this case, the adjunction will only be subschemic, via the following right adjoint. For each $\kappa$-scheme $Y$, we define a subschemic motif $\mathfrak{F}_Y$, called its *Frobenius motif*. In order to do this, we will work locally: show that it is a right adjoint locally, and then deduce its uniqueness and existence, as well as right adjointness, globally. So let $Y$ be affine, say, a closed subscheme of $\mathbb{L}^n$, and let $\mathbf{F}_{\mathbb{L}^n} \times \mathbf{1}_Y \colon \mathbf{F}^*_{\mathbb{L}^n} Y \to Y$ be the corresponding relative Frobenius. Set $\mathfrak{F}_Y := \mathtt{Im}(\mathbf{F}_{\mathbb{L}^n} \times \mathbf{1}_Y)$, so that it is a sub-schemic motif on $Y$. To see that this is independent from the choice of closed immersion, we prove the adjunction formula

$$Y(\mathbf{F}\mathfrak{z}) = \mathfrak{F}_Y(\mathfrak{z}) \tag{14}$$

for any fat point $\mathfrak{z}$. More precisely, the canonical (strongly dominant) morphism $\mathfrak{z} \to \mathbf{F}\mathfrak{z}$ induces a map $Y(\mathbf{F}\mathfrak{z}) \to Y(\mathfrak{z})$. By [17, Lemma 2.6] it is injective, and we want to show that its image is $\mathfrak{F}_Y(\mathfrak{z})$. Let $(f_1, \dots, f_s)$ be the ideal defining $Y$. By Corollary 3.13, the Frobenius transform $\mathbf{F}^*_{\mathbb{L}^n} Y$ is given by the ideal

$(f_1(x^p),\dots,f_s(x^p))$. An $\mathbf{F}\mathfrak{z}$-rational point $a$ in $Y$ corresponds to a $\kappa$-algebra homomorphism $A\to \mathbf{F}R$, where $R$ is the coordinate ring of $\mathfrak{z}$, and hence to a solution of $f_1=\dots=f_s=0$ in $R$ of the form $\mathbf{r}^p$. The image $a'$ of $a$ in $Y(\mathfrak{z})$ corresponds to the composition $A\to\mathbf{F}R\subseteq R$. Since $\mathbf{r}$ is a solution in $R$ of the equations defining $\mathbf{F}^*_{\mathbb{L}^n}Y$, it induces a $\mathfrak{z}$-rational point $b\colon\mathfrak{z}\to\mathbf{F}^*_{\mathbb{L}^n}Y$ such that $a'=(\mathbf{F}_{\mathbb{L}^n}\times\mathbf{1}_Y)(\mathfrak{z})(b)$, proving that $a'\in\mathfrak{F}_Y(\mathfrak{z})$. Conversely, by reversing these arguments, we see that any such $\mathfrak{z}$-rational point is induced by a $p$-th power in $R$, and hence comes from a $\mathbf{F}\mathfrak{z}$-rational point. This concludes the proof of (14) when $Y$ is affine, and proves in particular that $\mathfrak{F}_Y$ does not depend on the choice of closed immersion. For arbitrary $Y$, let $Y_1,\dots,Y_m$ be an open affine covering. For each $Y_i$ and each intersection $Y_i\cap Y_j$, we have an equality (14). Hence we may glue all pieces together to obtain a sub-schemic motif $\mathfrak{F}_Y$ satisfying (14). In particular, in view of Proposition 3.3, we proved:

**3.14. Theorem.** *The functors $\mathfrak{z}\mapsto\mathbf{F}\mathfrak{z}$ and $Y\mapsto\mathfrak{F}_Y$ constitute a sub-schemic adjunction. In particular, we get induced endomorphisms $\nabla_{\mathbf{F}}$ on $\mathbf{Gr}(\mathbb{subSch}_\kappa)$, $\mathbf{Gr}(\mathbb{Form}^{str}_\kappa)$, and $\mathbf{Gr}(\mathbb{Form}_\kappa)$.* □

Unraveling the definitions, the action of this adjunction on a motif $\mathfrak{Y}$ on a scheme $Y$ is given by

$$\nabla_{\mathbf{F}}\mathfrak{Y}=\mathfrak{Y}\cap\mathfrak{F}_Y.$$

Moreover, if $Y=\mathbb{L}^n$, then $\mathfrak{F}_Y$ is just $\mathtt{Im}(\mathbf{F}_{\mathbb{L}^n})$, the image of the $\kappa$-linear Frobenius. Therefore, if a motif $\mathfrak{Y}$ has an ambient space which is affine, we may take it to be an affine space $\mathbb{L}^n$, so that

$$\nabla_{\mathbf{F}}\mathfrak{Y}=\mathfrak{Y}\cap\mathtt{Im}(\mathbf{F}_{\mathbb{L}^n}).$$

**3.15. Families of motives.** Let $s\colon\mathfrak{Y}\to\mathfrak{X}$ be a rational morphism of $V$-sieves (see Remark 3.17 below for the non-rational case). Hence, we can find ambient spaces $Y$ and $X$ of $\mathfrak{Y}$ and $\mathfrak{X}$ respectively, and a morphism $\varphi\colon Y\to X$ of $V$-schemes extending $s$. We explain now how we may view $s$ as a family of $V$-sieves, by associating to each *$V$-rational point* $a$, that is to say, any $V$-morphism $a\colon V\to X$, a $V$-sieve $\mathfrak{Y}_a$ as follows. We may view $\mathfrak{Y}$ as an $X$-sieve via $\varphi$ by restriction of scalars, denoted $\varphi_*\mathfrak{Y}$ (see [17, §7.13]). Using $a$ as augmentation map, we define

$$\mathfrak{Y}_a:=\nabla_{a_*}\varphi_*\mathfrak{Y},$$

called the *specialization of $\mathfrak{Y}$ at $a$*. By Theorem 3.5, this is a sieve on the base change $\nabla_{a_*}Y=a^*Y=V\times_X Y$. To see that the specialization $\mathfrak{Y}_a$ is independent from the choice of ambient space $Y$, we simply observe that

$$\mathfrak{Y}_a(\mathfrak{z})=V(\mathfrak{z})\times_{\mathfrak{X}(\mathfrak{z})}\mathfrak{Y}(\mathfrak{z})=\{(r,b)\in V(\mathfrak{z})\times\mathfrak{Y}(\mathfrak{z})\mid a(\mathfrak{z})(r)=s(\mathfrak{z})(b)\}\tag{15}$$

as a subset of $V(\mathfrak{z})\times_{X(\mathfrak{z})}Y(\mathfrak{z})$, for any fat $V$-point $\mathfrak{z}$. Immediately from Theorem 3.5, we have:

**3.16. Proposition.** *The specialization of a schemic, sub-schemic, or strongly formal motif is again of the same type.* □

**3.17. Remark.** We can even apply this theory to a non-rational morphism $s\colon \mathfrak{Y} \to \mathfrak{X}$ of strongly formal motives. Indeed, let $\mathfrak{Z}_l \subseteq \mathfrak{Y}$ be sub-schemic approximations of the stronlgy formal motif $\mathfrak{Y}$. By [17, Theorem 3.15], the restriction $s|_{\mathfrak{Z}_l}$ is rational. Hence, given $a\colon V \to X$, the specialization $(\mathfrak{Z}_l)_a$ is sub-schemic by Proposition 3.16. Using (15), it is not hard to show that these specializations $(\mathfrak{Z}_l)_a$ are approximations of $\mathfrak{Y}_a$ (as defined by the right hand side of (15)), showing that the latter is strongly formal too.

## 4. Jet schemes

From now on, our base scheme will (almost always) be a field $\kappa$, often even assumed to be algebraically closed. Fix a fat point $\mathfrak{z}$ and let $j\colon \mathfrak{z} \to \operatorname{Spec}\kappa$ be its structure morphism. Clearly, it is flat and finite and satisfies condition (†) when $\kappa$ is algebraically closed, and so both augmentation and diminution with respect to $j$ are well-defined (in the non-algebraically closed case, we apply Remark 3.9). We define the *jet functor* of $\mathfrak{z}$, as a double adjunction[2]

$$\nabla_{\mathfrak{z}} := \nabla_{j^*} \circ \nabla_{j_*}.$$

In other words, given a motif $\mathfrak{X}$ on a $\kappa$-scheme $X$, and a fat point $\mathfrak{w}$, we have

$$(\nabla_{\mathfrak{z}}\mathfrak{X})(\mathfrak{w}) = \mathfrak{X}(j_* j^* \mathfrak{w})$$

where $j_* j^* \mathfrak{w}$ is the product $\mathfrak{z} \times_\kappa \mathfrak{w}$ viewed as a fat point over $\kappa$, denoted henceforth simply by $\mathfrak{z}\mathfrak{w}$. Applied to a $\kappa$-scheme $X$, we get the so-called *jet scheme* $\nabla_{\mathfrak{z}} X$, whose $\mathfrak{w}$-rational points are in one-one correspondence with the $\mathfrak{z}\mathfrak{w}$-rational points of $X$, and so we will identify

$$X(\mathfrak{z}\mathfrak{w}) = (\nabla_{\mathfrak{z}} X)(\mathfrak{w}).$$

Moreover, we have by Corollary 3.8 a canonical morphism

$$\rho_X\colon \nabla_{\mathfrak{z}} X \to X. \tag{16}$$

**4.1. Remark.** In other words, $\nabla_{\mathfrak{z}} X$ is the Hilbert scheme classifying all maps from $\mathfrak{z}$ to $X$. When $\mathfrak{l}_n = \operatorname{Spec}(\kappa[\xi]/(\xi^n))$, the resulting jet scheme is also known in the literature as a *truncated arc scheme*.[3]

[2]See §6 below for the corresponding single adjunction.

[3]This was also the terminology in an earlier version of this paper posted on ArXiv.

**4.2. Remark.** By the argument in the proof of Theorem 3.7, for any fat point $\mathfrak{z}$, we may choose a basis $\Delta = \{\alpha_0, \ldots, \alpha_{l-1}\}$ of its coordinate ring $(R, \mathfrak{m})$ with some additional properties. In particular, unless noted explicitly, we will always assume that the first base element is $1$ and that the remaining ones belong to $\mathfrak{m}$. Moreover, once the basis is fixed, we let $\tilde{x}$ be the $l$-tuple of jet variables $(\tilde{x}_0, \ldots, \tilde{x}_{l-1})$, so that $\widehat{x} = \tilde{x}_0 + \alpha_1\tilde{x}_1 + \cdots + \alpha_{l-1}\tilde{x}_{l-1}$ is the corresponding generic jet. It follows from (7) that $\nabla_0 f = f(\tilde{x}_0)$, for any $f \in \kappa[x]$. By [14, §2.1], we may choose $\Delta$ so that, with $\mathfrak{a}_i := (\alpha_i, \ldots, \alpha_{l-1})R$, we have a Jordan-Holder composition series

$$\mathfrak{a}_l = 0 \subsetneq \mathfrak{a}_{l-1} \subsetneq \mathfrak{a}_{l-2} \subsetneq \cdots \subsetneq \mathfrak{a}_1 = \mathfrak{m} \subsetneq \mathfrak{a}_0 = R.$$

Without giving the details, we may construct $\Delta$ as follows: write $R$ as a homomorphic image of $\kappa[y]$ so that $y := (y_1, \ldots, y_e)$ generates $\mathfrak{m}$, and let $\mathfrak{a}(\alpha)$, for $\alpha \in \mathbb{Z}^n_{\geqslant 0}$, be the ideal in $R$ generated by all $y^\beta$ with $\beta$ lexicographically larger than $\alpha$. Then we may take $\Delta$ to be all monomials $y^\alpha$ such that $y^\alpha \notin \mathfrak{a}(\alpha)$.

Given $r \in R$, we expand it as in (8) in the basis as $r = r_0 + \alpha_1 r_1 + \cdots + \alpha_{l-1} r_{l-1}$, with $r_i \in \kappa$. I claim that $r_j = 0$ for $j < i$ whenever $r \in \mathfrak{a}_i$. Indeed, if not, let $j < i$ be minimal so that there exists a counterexample with $r_j \neq 0$. By minimality, $r = \alpha_j r_j + \alpha_{j+1} r_{j+1} + \cdots \in \mathfrak{a}_i$ showing that $\alpha_j \in \mathfrak{a}_{j+1}$, since $r_j$ is invertible. However, this implies that $\mathfrak{a}_j = \mathfrak{a}_{j+1}$, contradiction. From this, it is now easy to see that the first $m$ basis elements of $\Delta$ form a basis of $R_m := R/\mathfrak{a}_{m+1}$. Therefore, calculating $\nabla_m f$ in (7) for $f \in \kappa[x]$ does not depend on whether we work over $R$ or over $R_m$, and hence, in particular, $\nabla_m f \in \kappa[\tilde{x}_0, \ldots, \tilde{x}_m]$ for every $m < l$.

**4.3. Remark.** In case $X = \operatorname{Spec} A$ is affine, with $A = \kappa[x]/I$, then the proof of Theorem 3.7 also provides a recipe for calculating rational points. Namely, let $\tilde{A}$ be the coordinate ring of the jet scheme $\nabla_{\mathfrak{z}} X$, where $\mathfrak{z}$ is a fat point with coordinate ring $R$. Given any fat point $\operatorname{Spec} S$, an $S$-rational point of the jet scheme, that is to say, a homomorphism $a\colon \tilde{A} \to S$ defines an $R \otimes S$-rational point on $X$ as follows. For each variable $x_i$, let $\widehat{x}_i$ be the corresponding generic jet (see (6)), then $A \to R \otimes S$ is given by sending $x_i$ to $a(\widehat{x}_i)$, the value of the generic jet at the $S$-rational point $a$.

With these observations, we can now prove the following important openness property of jets:

**4.4. Theorem.** *Given a $\kappa$-scheme $X$, a fat point $\mathfrak{z}$, and an open $U \subseteq X$, we have isomorphisms*

$$\nabla_{\mathfrak{z}} U \cong \rho_X^{-1}(U) \cong \nabla_{\mathfrak{z}} X \times_X U. \tag{17}$$

*Proof.* By the universal property of adjunction, whence of jets, it suffices to verify (17) in case $X = \operatorname{Spec} B \subseteq \mathbb{L}^n$ is affine and $U = \operatorname{Spec}(B_f)$ is a basic open subset. Let $A$ be the coordinate ring of $\nabla_{\mathfrak{z}} X$. Since $U$ is the closed subscheme of $\mathbb{L}_B$ given by $g := fy - 1 = 0$, the corresponding jet scheme $\nabla_{\mathfrak{z}} U$ is the closed subscheme of $\mathbb{L}^n_A$ with coordinate ring $A' := A[\tilde{y}]/(\nabla_0 g, \ldots, \nabla_{l-1} g)$, where $l$ is the length of $\mathfrak{z} =$

$\operatorname{Spec} R$ and the $\nabla_i g$ are given by (7), with $\tilde{y}$ a tuple of $l$ variables. By Remark 4.2, we may calculate the $\nabla_i g$ using any basis $\alpha_0 = 1, \dots, \alpha_{l-1}$ of $R$, and so we may assume it has the properties discussed in that remark. In particular, by the last observation in that remark, each $\nabla_i g$ only depends on $\tilde{y}_0, \dots, \tilde{y}_i$. Clearly, $\nabla_0 g = (\nabla_0 f)\tilde{y}_0 - 1$. In particular, the $A$-subalgebra of $A'$ generated by $\tilde{y}_0$ is just the localization $A_{\nabla_0 f}$. We will prove by induction, that each $\tilde{y}_i$ belongs to this subalgebra, and hence $A' = A_{\nabla_0 f}$, as we needed to prove.

To verify the claim, we may assume by induction that $\tilde{y}_0, \dots \tilde{y}_{i-1}$ belong to $A_{\nabla_0 f}$. The coefficient of $\alpha_i$ in the expansion of the product

$$(\nabla_0 f + \alpha_1 \nabla_1 f + \dots + \alpha_{l-1}\nabla_{l-1} f)(\tilde{y}_0 + \alpha_1 \tilde{y}_1 + \dots + \alpha_{l-1}\tilde{y}_{l-1}) \tag{18}$$

is equal to $\nabla_i g$, whence zero in $A'$. As observed in Remark 4.2, the choice of basis allows us to ignore all terms with $\alpha_j$ for $j > i$. Put differently, upon replacing $R$ by $R/(\alpha_{i+1}, \dots, \alpha_{l-1})R$, which does not effect the calculation of $\nabla_i g$, we may assume that they are zero in (18). Hence,

$$\nabla_i g = (\nabla_0 f)\tilde{y}_i + \text{terms involving only } \tilde{y}_0, \dots, \tilde{y}_{i-1}$$

proving the claim, since $\nabla_0 f$ is clearly invertible in $A_{\nabla_0 f}$. □

Before we proceed, some simple examples are in order. Jet spaces are sensitive to singularities, as the next examples show:

**4.5. Example.** Let us calculate the jet scheme of the cusp $C$ given by the equation $x^2 - y^3 = 0$ along the fat point $\mathfrak{z}$ with coordinate ring the four dimensional algebra $R := \kappa[\xi, \zeta]/(\xi^2, \zeta^2)$, using the basis $\Delta := \{1, \xi, \zeta, \xi\zeta\}$ (in the order listed), and corresponding jet variables $\tilde{x} = (\tilde{x}_{00}, \tilde{x}_{10}, \tilde{x}_{01}, \tilde{x}_{11})$ and $\tilde{y} = (\tilde{y}_{00}, \tilde{y}_{10}, \tilde{y}_{01}, \tilde{y}_{11})$. One easily calculates that $\nabla_{\mathfrak{z}} C$ is given by the equations

$$\begin{aligned}
\tilde{x}_{00}^2 &= \tilde{y}_{00}^3 \\
2\tilde{x}_{00}\tilde{x}_{10} &= 3\tilde{y}_{00}^2\tilde{y}_{10} \\
2\tilde{x}_{00}\tilde{x}_{01} &= 3\tilde{y}_{00}^2\tilde{y}_{01} \\
2\tilde{x}_{00}\tilde{x}_{11} + 2\tilde{x}_{10}\tilde{x}_{01} &= 3\tilde{y}_{00}^2\tilde{y}_{11} + 6\tilde{y}_{00}\tilde{y}_{10}\tilde{y}_{01}.
\end{aligned}$$

Note that above the singular point $\tilde{x}_{00} = 0 = \tilde{y}_{00}$, the fiber consist of two 4-dimensional hyperplanes, whereas above any regular point, it is a 3-dimensional affine space, the expected value by Theorem 4.14 below.

**4.6. Example.** Another example is classical: let $R_2 = \kappa[\xi]/(\xi^2)$ be the ring of dual numbers and $\mathfrak{l}_2 := \operatorname{Spec}(R_2)$ the corresponding fat point. Then one verifies that a $\kappa$-rational point on $\nabla_{\mathfrak{l}_2} X$ is given by a $\kappa$-rational point $P$ on $X$, and a tangent vector $\mathbf{v}$ to $X$ at $P$, that is to say, an element in the kernel of the Jacobian matrix $\operatorname{Jac}_X(P)$.

**4.7. Example.** As a last example, we calculate $\nabla_{\mathfrak{l}_n} \mathfrak{l}_m$, where $\mathfrak{l}_n$ is the $n$-th co-jet of the origin on the affine line, that is to say, $\operatorname{Spec}(\kappa[\xi]/(\xi^n))$. With $\hat{x} = \tilde{x}_0 + \xi\tilde{x}_1 +$

$\cdots + \xi^{n-1}\tilde{x}_{n-1}$, we will expand $\tilde{x}^m$ in the basis $\{1, \xi, \dots, \xi^{n-1}\}$ of $\kappa[\xi]/(\xi^n)$; the coefficients of this expansion then generate the ideal of definition of $\nabla_{\mathfrak{l}_n}\mathfrak{l}_m$. A quick calculation shows that these generators are the polynomials

$$g_s(\tilde{x}_0, \dots, \tilde{x}_{n-1}) := \sum_{i_1+\cdots+i_m=s} \tilde{x}_{i_1}\tilde{x}_{i_2}\cdots\tilde{x}_{i_m}$$

for $s = 0, \dots, n-1$, where the $i_j$ run over $\{0, \dots, n-1\}$. Note that $g_0 = \tilde{x}_0^m$. One shows by induction that $(\tilde{x}_0, \dots, \tilde{x}_s)$ is the unique minimal prime ideal of $\nabla_{\mathfrak{l}_n}\mathfrak{l}_m$, where $s = \lceil \frac{n}{m} \rceil$ is the *round-up* of $n/m$, that is to say, the least integer greater than or equal to $n/m$. In particular, $\nabla_{\mathfrak{l}_n}\mathfrak{l}_m$ is irreducible (but not reduced) of dimension $n - \lceil \frac{n}{m} \rceil$.

Immediately from Theorems 3.5 and 3.7, we get:

**4.8. Theorem.** *For each fat point $\mathfrak{z}$, the jet functor $\nabla_{\mathfrak{z}}$ induces a ring endomorphism on each of the motivic Grothendieck rings $\mathbf{Gr}(\mathbb{Sch}_\kappa)$, $\mathbf{Gr}(\mathbb{subSch}_\kappa)$, $\mathbf{Gr}(\mathbb{Form}_\kappa^{str})$, and $\mathbf{Gr}(\mathbb{Form}_\kappa)$.* □

Applied to the affine line, we get the following simple formula

$$\nabla_{\mathfrak{z}}\mathbb{L} = \mathbb{L}^{\ell(\mathfrak{z})} \tag{19}$$

either as an identity of schemes or as a relation in the Grothendieck ring. In case of complete formal motives, we can calculate the jet scheme by base change:

**4.9. Lemma.** *For any closed immersion $Y \subseteq X$ of $\kappa$-schemes, and any fat point $\mathfrak{z}$, we have isomorphisms*

$$\nabla_{\mathfrak{z}}(\widehat{X}_Y) \cong \nabla_{\mathfrak{z}}X \times_X \widehat{X}_Y \cong (\widehat{\nabla_{\mathfrak{z}}X})_{\rho^{-1}(Y)},$$

*where $\rho\colon \nabla_{\mathfrak{z}}X \to X$ is the canonical map from (16).*

*Proof.* Let $U := X \setminus Y$. By [17, Proposition 7.1], we have an equality of sieves

$$-\widehat{X}_Y^\circ = U^\circ \tag{20}$$

on $X$. By [17, Theorem 7.7], we may pull back (20) under the map $\rho\colon \nabla_{\mathfrak{z}}X \to X$, to get a relation

$$-(\nabla_{\mathfrak{z}}X \times_X \widehat{X}_Y)^\circ = \rho^*(-\widehat{X}_Y^\circ) = \rho^*U^\circ = (\nabla_{\mathfrak{z}}X \times_X U)^\circ = (\nabla_{\mathfrak{z}}U)^\circ$$

where we used the openness of jets (Theorem 4.4) for the last equality. On the other hand, taking jets in identity (20), yields

$$-\nabla_{\mathfrak{z}}\widehat{X}_Y^\circ = \nabla_{\mathfrak{z}}(-\widehat{X}_Y^\circ) = \nabla_{\mathfrak{z}}U^\circ = (\nabla_{\mathfrak{z}}U)^\circ$$

where one easily checks that jet functors commute with complements of complete sieves. Combining both identities and taking complements then proves the first isomorphism.

To see the second isomorphism, we may assume, in view of the local nature of jets, that $X = \operatorname{Spec} A$ is affine. Let $I \subseteq A$ be the ideal of definition of $Y$, so that the global sections of $\widehat{X}_Y$ is the completion $\widehat{A}_I$ of $A$ with respect to $I$. Let $A[y]/J$

be the coordinate ring of the jet scheme $\nabla_{\mathfrak{z}} X$, for some $J \subseteq A[y]$ and some tuple of variables $y$. By the first isomorphism, the global section ring of $\nabla_{\mathfrak{z}} \widehat{X}_Y$ is equal to the base change $\widehat{A}_I[y]/J\widehat{A}_I[y]$. The ideal defining $\rho^{-1}(Y)$ in $\nabla_{\mathfrak{z}} X$ is $I(A[y]/J)$, and the completion of $A[y]/J$ with respect to this ideal is $\widehat{A}_I[y]/J\widehat{A}_I[y]$, proving the second isomorphism. □

In view of [17, Corollary 7.3], jets commute with localization in the following sense, where $X_P$ is the local scheme $\operatorname{Spec}(\mathcal{O}_{X,P})$:

**4.10. Corollary.** *For any closed germ $(X, P)$ and any fat point $\mathfrak{z}$, we have an isomorphism $\nabla_{\mathfrak{z}} X_P \cong (\nabla_{\mathfrak{z}} X) \times_X X_P$.* □

**4.11. Corollary.** *The jet of a complete formal motif is again complete. More precisely, if $F \subseteq X$ is a constructible subset, then*

$$\nabla_{\mathfrak{z}}(\mathfrak{C}_X(F)) = \mathfrak{C}_{\nabla_{\mathfrak{z}} X}(\rho^{-1}(F)), \tag{21}$$

*for any fat point $\mathfrak{z}$, where $\rho\colon \nabla_{\mathfrak{z}} X \to X$ is the canonical map.*

*Proof.* By [17, Proposition 7.1], the cone of a closed subset $Y$ is $\widehat{X}_Y$, and by [17, Lemma 6.4], that of an open subset $U$ is just $U$. Identity 21 holds in either case by Lemma 4.9 and Theorem 4.4 respectively. Since cones and arcs pass both through unions and intersections, we proved (21) in general. The first assertion now follows from [17, Theorem 8.1]. □

A similar result holds for etale maps (compare with [5, Lemma 2.9]):

**4.12. Theorem.** *For any etale morphism $\varphi\colon Y \to X$ and any fat point $\mathfrak{z}$, we have an isomorphism of schemes $\nabla_{\mathfrak{z}} Y \cong \nabla_{\mathfrak{z}} X \times_X Y$.*

*Proof.* By functoriality, we have a commutative diagram

$$\begin{array}{ccc} \nabla_{\mathfrak{z}} Y & \xrightarrow{\nabla_{\mathfrak{z}}\varphi} & \nabla_{\mathfrak{z}} X \\ \downarrow{\scriptstyle \rho_Y} & & \downarrow{\scriptstyle \rho_X} \\ Y & \xrightarrow[\varphi]{} & X \end{array} \tag{22}$$

whence a morphism $\nabla_{\mathfrak{z}} Y \to \nabla_{\mathfrak{z}} X \times_X Y$. To construct its inverse, we may work over a fixed fat point $\mathfrak{v}$. Hence let $b \in \nabla_{\mathfrak{z}} X(\mathfrak{v})$ and $\bar{a} \in Y(\mathfrak{v})$ have the same image in $X(\mathfrak{v})$. Let $\tilde{b}$ be the $\mathfrak{z}\mathfrak{v}$-rational point of $X$ induced by $b$. Hence, if $Q$ is the center of $\bar{a}$, then $\varphi(Q)$ is the center of $\tilde{b}$. Since $\tilde{b}$ factors through $X_{\varphi(Q)}$, there is a unique $\mathfrak{z}\mathfrak{v}$-rational point $\mathfrak{z}\mathfrak{v} \to Y_Q$ lifting the latter by [17, Theorem 2.17]. Let $\tilde{a}$ be its composition with the natural morphism $Y_Q \to Y$, and let $a\colon \mathfrak{v} \to \nabla_{\mathfrak{z}} Y$ be the induced $\mathfrak{v}$-rational point of the jet scheme. We leave it to the reader to verify that the assignment $(b, \bar{a}) \mapsto a$ is the desired inverse morphism. □

It is easy to check that

$$\nabla_{\mathfrak{v}}\nabla_{\mathfrak{w}} = \nabla_{\mathfrak{v}\mathfrak{w}} = \nabla_{\mathfrak{w}}\nabla_{\mathfrak{v}}, \tag{23}$$

so that all jet functors commute with one another. If $\kappa$ has positive characteristic, we also have a Frobenius adjoint acting on the sub-schemic and formal Grothendieck rings, and we have the following commutation relation

$$\nabla_{\mathfrak{z}}\nabla_{\mathbf{F}} = \nabla_{\mathbf{F}}\nabla_{\mathbf{F}\mathfrak{z}} \tag{24}$$

for any fat point $\mathfrak{z}$. Indeed, we verify this on an arbitrary motif $\mathfrak{X}$ and a fat point $\mathfrak{w}$. The left hand side of (24) becomes

$$\begin{aligned}(\nabla_{\mathfrak{z}}\nabla_{\mathbf{F}})(\mathfrak{X})(\mathfrak{w}) &= \nabla_{\mathfrak{z}}(\nabla_{\mathbf{F}}\mathfrak{X})(\mathfrak{w})\\ &= (\nabla_{\mathbf{F}}\mathfrak{X})(\mathfrak{z}\mathfrak{w})\\ &= \mathfrak{X}(\mathbf{F}(\mathfrak{z}\mathfrak{w})),\end{aligned}$$

whereas the right hand side becomes

$$\begin{aligned}(\nabla_{\mathbf{F}}\nabla_{\mathbf{F}\mathfrak{z}})(\mathfrak{X})(\mathfrak{w}) &= \nabla_{\mathbf{F}}(\nabla_{\mathbf{F}\mathfrak{z}}\mathfrak{X})(\mathfrak{w})\\ &= (\nabla_{\mathbf{F}\mathfrak{z}}\mathfrak{X})(\mathbf{F}\mathfrak{w})\\ &= \mathfrak{X}((\mathbf{F}\mathfrak{z})(\mathbf{F}\mathfrak{w})),\end{aligned}$$

and these are both equal since an easy calculation shows that $\mathbf{F}(\mathfrak{z}\mathfrak{w}) = (\mathbf{F}\mathfrak{z})(\mathbf{F}\mathfrak{w})$.

**Jets and locally trivial fibrations.** By adjunction, any morphism $\bar{\mathfrak{z}} \to \mathfrak{z}$ of fat points induces a natural transformation of jet functors $\nabla_{\mathfrak{z}} \to \nabla_{\bar{\mathfrak{z}}}$. In particular, taking $\bar{\mathfrak{z}}$ to be the geometric point given by $\kappa$ itself, we get a canonical morphism $\nabla_{\mathfrak{z}}\mathfrak{X} \to \mathfrak{X}$, for any motif $\mathfrak{X}$, since $\nabla_\kappa$ is the identity functor. In case $\mathfrak{X} = X^\circ$ is representable, this is none other than the canonical morphism $\rho_X\colon \nabla_{\mathfrak{z}}X \to X$ from (16). To formulate the key property of this morphism, we need a definition.

We call a morphism $Y \to X$ of $\kappa$-schemes a *locally trivial fibration with fiber* $Z$ if for each (closed) point $P \in X$, we can find an open $U \subseteq X$ containing $P$ such that the restriction of $Y \to X$ to $U$ is isomorphic with the projection $U \times_\kappa Z \to U$.

**4.13. Lemma.** *If $f\colon Y \to X$ is a locally trivial fibration of $\kappa$-schemes with fiber $Z$, then $[Y] = [X]\cdot[Z]$ in $\mathbf{Gr}(\mathbb{sub}\mathbb{Sch}_\kappa)$.*

*Proof.* By definition and compactness, there exists a finite open covering $X = X_1 \cup \dots \cup X_n$, so that

$$f^{-1}(X_i) \cong X_i \times_\kappa Z,$$

for $i = 1, \dots, n$. In fact, for any non-empty subset $I \subseteq \{1, \dots, n\}$, we have an isomorphism $f^{-1}(X_I) \cong X_I \times_\kappa Z$, and hence, after taking classes in $\mathbf{Gr}(\mathbb{sub}\mathbb{Sch}_\kappa)$, we get $[f^{-1}(X_I)] = [X_I]\cdot[Z]$. Since the $f^{-1}(X_i)$ form an open affine covering of

$Y$ and pre-images commute with intersection, a double application of [17, Lemma 6.5] yields

$$[Y] = \sum_{\varnothing \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|}[f^{-1}(X_I)] = \sum_{\varnothing \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|}[X_I]\cdot[Z] = [X]\cdot[Z]$$

in $\mathbf{Gr}(\mathbb{s}\mathbb{u}\mathbb{b}\mathbb{S}\mathbb{c}\mathbb{h}_\kappa)$. □

**4.14. Theorem.** *If $X$ is a $d$-dimensional smooth $\kappa$-scheme and $\bar{\mathfrak{z}} \subseteq \mathfrak{z}$ a closed immersion of fat points, then the canonical map $\nabla_{\mathfrak{z}} X \to \nabla_{\bar{\mathfrak{z}}} X$ is a locally trivial fibration with fiber $\mathbb{L}^{d(l-\bar{l})}$, where $l$ and $\bar{l}$ are the respective lengths of $\mathfrak{z}$ and $\bar{\mathfrak{z}}$. In particular,*

$$[\nabla_{\mathfrak{z}} X] = [X] \cdot \mathbb{L}^{d(l-1)}$$

*in* $\mathbf{Gr}(\mathbb{s}\mathbb{u}\mathbb{b}\mathbb{S}\mathbb{c}\mathbb{h}_\kappa)$.

*Proof.* Let $R$ and $\bar{R}$ be the Artinian local coordinate rings of $\mathfrak{z}$ and $\bar{\mathfrak{z}}$ respectively. Since jets can be calculated locally, we may assume $X$ is the (affine) closed subscheme of $\mathbb{L}^m$ with ideal of definition $(f_1, \dots, f_s)$. Since the composition of locally trivial fibrations is again a locally trivial fibration, with general fiber the product of the fibers, we may reduce to the case that $\bar{R} = R/\alpha R$ with $\alpha$ an element in the socle of $R$, that is to say, such that $\alpha\mathfrak{m} = 0$, where $\mathfrak{m}$ is the maximal ideal of $R$. Let $\Delta$ be a basis of $R$ as in Remark 4.2, with $\alpha_{l-1} = \alpha$ (since $\alpha$ is a socle element, such a basis always exists). In particular, $\Delta - \{\alpha\}$ is a basis of $\bar{R}$. We will use these bases to calculate both jet maps.

To calculate a general fiber of the map $s\colon \nabla_{\mathfrak{z}} X \to \nabla_{\bar{\mathfrak{z}}} X$, fix a fat point $\mathfrak{w}$ with coordinate ring $S$, and a $\mathfrak{w}$-rational point $\bar{b}\colon \mathfrak{w} \to \nabla_{\bar{\mathfrak{z}}} X$, given by a tuple $\tilde{\mathbf{u}}$ over $S$. The fiber $s(\mathfrak{w})^{-1}(\bar{b})$, is equal to the fiber of $X(\mathfrak{z}\mathfrak{w}) \to X(\bar{\mathfrak{z}}\mathfrak{w})$ above $\bar{a}$, where $\bar{a}\colon \bar{\mathfrak{z}}\mathfrak{w} \to X$ is the $\bar{\mathfrak{z}}\mathfrak{w}$-rational point corresponding to $\bar{b}$, that is to say, the composition $\bar{\mathfrak{z}}\mathfrak{w} \to \bar{\mathfrak{z}} \times \nabla_{\bar{\mathfrak{z}}} X \to X$ given by Theorem 3.7 (see Corollary 3.8). Being a rational point, $\bar{a}$ corresponds therefore to a solution $\mathbf{u}$ in $\bar{R} \otimes_\kappa S$ of the equations $f_1 = \dots = f_s = 0$, where the relation with the tuple $\tilde{\mathbf{u}}$ is given by equation (8). Let $x$ be the center of $\bar{a}$, that is to say, the closed point given as the image of $\bar{a}$ under the canonical map $X(\bar{\mathfrak{z}}\mathfrak{w}) \to X(\kappa)$. Since $X$ is smooth at $x$, the Jacobian $(s \times n)$-matrix $\mathrm{Jac}_X := (\partial f_i/\partial x_j)$ has rank $m - d$ at $x$. Replacing $X$ by an affine local neighborhood of $x$ and rearranging the variables if necessary, we may assume that the first $(m-d) \times (m-d)$-minor in $\mathrm{Jac}_X$ is invertible on $X$.

The surjection $R \to \bar{R}$ induces a surjection $R \otimes_\kappa S \to \bar{R} \otimes_\kappa S$. The fiber above $\bar{a}$ is therefore defined by the equations $f_j(\mathbf{u} + \tilde{x}_{l-1}\alpha) = 0$, for $j = 1, \dots, s$. By Taylor expansion, this becomes

$$0 = f_j(\mathbf{u} + \tilde{x}_{l-1}\alpha) = \Big(\sum_{i=1}^{m} \frac{\partial f_j}{\partial x_i}(\mathbf{u})\tilde{x}_{l-1,i}\Big)\alpha \tag{25}$$

since $f_j(\mathbf{u}) = 0$ and $\alpha^2 = 0$ in $R\otimes_\kappa S$. In fact, since $\mathbf{u} \equiv \tilde{\mathbf{u}}_0 \mod \mathfrak{m}(R\otimes_\kappa S)$ and $\alpha\mathfrak{m} = 0$, we may replace each $\partial f_j/\partial x_i(\mathbf{u})$ in (25) by $\partial f_j/\partial x_i(\tilde{\mathbf{u}}_0)$. Hence, the fiber above $\bar{a}$ is is the linear subspace of $(R \otimes S)^m$ defined as the kernel of the Jacobian $\mathrm{Jac}_X(\tilde{\mathbf{u}}_0)$. In view of the shape of the Jacobian of $X$, we can find $g_{ij} \in \kappa[x]$ such that

$$\tilde{x}_{l-1,i} = \sum_{j>m-d} g_{ij}(\tilde{\mathbf{u}}_0)\tilde{x}_{l-1,j}$$

for all $i \leqslant m-d$, by Kramer's rule. Therefore, viewing the parameter $\tilde{\mathbf{u}}_0$ as varying over $X(\mathfrak{w})$, the fiber of $s(\mathfrak{w})$ is the constant space $\mathbb{L}^d$, as we needed to show. Applying this to $\nabla_{\mathfrak{z}} X \to X$, (note that $X = \nabla_\kappa X$) we get a locally trivial fibration with fiber equal to $\mathbb{L}^{d(l-1)}$, so that the last assertion follows from Lemma 4.13. □

Calculations, like for instance Example 4.7, suggest that even for certain non-reduced schemes, there might be an underlying locally trivial fibration (here we write $X^{\mathrm{red}}$ for the underlying reduced variety of a scheme $X$):

**4.15. Question.** Let $\mathfrak{z}$ be a fat point of length $l$ and $X$ a $d$-dimensional $\kappa$-scheme. If the reduction of $X$ is smooth, when is the induced reduction map $(\nabla_{\mathfrak{z}} X)^{\mathrm{red}} \to X^{\mathrm{red}}$ a locally trivial fibration with fiber $\mathbb{L}^m$, for some $m$?

**4.16. Remark.** As we shall see in Example 5.4 below, $m$ can be different from $d(l-1)$, the value that we get in the reduced case. In many cases, the answer seems to be affirmative, but there are exceptions, see Example 4.17 below.

Moreover, as can be seen from Table (1) below, taking jets does not commute with reduction, that is to say, $(\nabla_{\mathfrak{z}} X)^{\mathrm{red}}$ is in general not equal to the jet space $\nabla_{\mathfrak{z}}(X^{\mathrm{red}})$ of the reduction of $X$, nor even to the reduction of the latter jet space.

**4.17. Example.** The simplest instance to which Question 4.15 applies is when $X$ itself is a fat point $\mathfrak{x}$. The expectation then is that

$$(\nabla_{\mathfrak{z}}\mathfrak{x})^{\mathrm{red}} \cong \mathbb{L}^m \tag{26}$$

for some $m$ (for expected values, see Example 5.4 below). Example 4.7 provides instances in which (26) holds. However, the following is a counterexample: let $\mathfrak{z} := J^4_O C$, where $C$ is the cuspidal curve with equation $\xi^2 - \zeta^3 = 0$ and $O$ the origin, its unique singularity. Let us calculate its *auto-arcs* $\nabla_{\mathfrak{z}}\mathfrak{z}$. As the monomials in $\xi$ and $\zeta$ of degree at most two together with $\xi\zeta^2$ form a basis of the coordinate ring $R$ of $\mathfrak{z}$, its length is 7 and the generic jets are

$$\tilde{x} = \tilde{x}_0 + \tilde{x}_1\xi + \cdots + \tilde{x}_6\xi\zeta^2 \quad \text{and} \quad \tilde{y} = \tilde{y}_0 + \tilde{y}_1\xi + \cdots + \tilde{y}_6\xi\zeta^2.$$

Since the jet scheme $\nabla_{\mathfrak{z}}\mathfrak{z}$ lies above the origin, its reduction lies in the subvariety of $\mathbb{L}^{14}$ defined by $\tilde{x}_0 = \tilde{y}_0 = 0$, and hence, we may put these two to zero in the generic jets and work inside the affine space $\mathbb{L}^{12}$ given by the remaining jet variables. From the fact that $\xi^3 = 0$ in $R$, the jet scheme is contained in the closed subscheme of $\mathbb{L}^{12}$

by the coefficients of the expansion of

$$\tilde{x}^3 = 3\tilde{x}_1\tilde{x}_2^2\xi\zeta^2 + \tilde{x}_2^3\xi^2 + \dots$$

In particular, since $\tilde{x}_2^3$ vanishes, the reduction lies in the subvariety given by $\tilde{x}_2 = 0$, and so we may again put this variable equal to zero and work in the corresponding 11-dimensional affine space. The remaining equations come from the expansion of

$$\begin{aligned}\tilde{x}^2 - \tilde{y}^3 &= (\tilde{x}_1\xi + \tilde{x}_3\xi^2 + \tilde{x}_4\xi\zeta + \tilde{x}_5\zeta^2 + \dots)^2 - (\tilde{y}_1\xi + \tilde{y}_2\zeta + \dots)^3 \\ &= (\tilde{x}_1^2 - \tilde{y}_2^3)\xi^2 + (2\tilde{x}_1\tilde{x}_5 - 3\tilde{y}_1\tilde{y}_2^2)\xi\zeta^2\end{aligned}$$

showing that the reduced jet space is the singular variety with equations $\tilde{x}_1^2 - \tilde{y}_2^3 = 2\tilde{x}_1\tilde{x}_5 - 3\tilde{y}_1\tilde{y}_2^2 = 0$. Note that the latter can be viewed as the tangent bundle of the cusp. More precisely, instead of the anticipated (26), we obtain the following modified form of the auto-arc variety

$$(\nabla_{J_O^4 C}(J_O^4 C))^{\text{red}} \cong \nabla_{\mathfrak{l}_2} C \times \mathbb{L}^7,$$

a singular 9-dimensional variety. However, I could not find such a form for values higher than 4.

**Locally constructible sieves.** We say that a sieve $\mathfrak{X}$ on a $\kappa$-scheme $X$ is *locally constructible*, if $\mathfrak{X}(\mathfrak{z})$ is constructible in $X(\mathfrak{z})$, for each fat point $\mathfrak{z}$, by which we mean that $\nabla_{\mathfrak{z}}\mathfrak{X}(\kappa)$ is constructible in the Zariski topology on the variety $\nabla_{\mathfrak{z}}X(\kappa)$ viewed as the space of closed points of $\nabla_{\mathfrak{z}}X$.

**4.18. Proposition.** *Any formal motif is locally constructible.*

*Proof.* This follows from Chevalley's theorem and Theorem 4.8 in case $\mathfrak{X}$ is subschemic, since, for a morphism $\varphi\colon Y \to X$ of $\kappa$-schemes, $\mathtt{Im}(\varphi)(\mathfrak{z})$, as a subset of $\nabla_{\mathfrak{z}}X(\kappa)$, is the image of the map $\nabla_{\mathfrak{z}}Y(\kappa) \to \nabla_{\mathfrak{z}}X(\kappa)$. The formal case then follows from this, since there exists a sub-schemic motif $\mathfrak{Y} \subseteq \mathfrak{X}$ such that $\mathfrak{Y}(\mathfrak{z}) = \mathfrak{X}(\mathfrak{z})$. □

**4.19. Remark.** Let $Z$ be a zero-dimensional $\kappa$-scheme, so that it is a disjoint union of fat points $Z = \mathfrak{z}_1 \sqcup \dots \sqcup \mathfrak{z}_s$. Although the structure morphism $j\colon Z \to \operatorname{Spec}\kappa$ no longer satisfies (†), we can still define, for each scheme $X$, its jet scheme $\nabla_Z X$ along $Z$ as the double adjunction $\nabla_{j*} \circ \nabla_{j_*}$ in view of the discussion in Remark 3.9. An easy calculation then shows that

$$\nabla_Z X = \prod_{i=1}^{s} \nabla_{\mathfrak{z}_i} X. \tag{27}$$

In particular, (19) generalizes to $\nabla_Z \mathbb{L} = \mathbb{L}^{\ell(Z)}$.

## 5. Dimension

In this section, we assume $\kappa$ is an algebraically closed field. The dimension of an jet scheme $\nabla_{\mathfrak{z}}X$ is a subtle invariant depending on $\mathfrak{z}$ and $X$, and not just on their respective length $l$ and dimension $d$, as Table (1) shows. The underlying cause for this phenomenon is the fact that taking reduction does not commute with taking jets. To exemplify this behavior, we list, for small lengths, some defining equations of jets and their reductions for three different closed subschemes $X$ with the same underlying one-dimensional variety, the union of two lines in the plane. Here $\mathfrak{l}_e$ denotes the closed point with coordinate ring $\kappa[\xi]/(\xi^e)$, that is to say, the $e$-th co-jet of the origin on the affine line.

TABLE 1. Jet equations, their reductions and dimension $\delta$.

| $X$ | $e$ | $xy = 0$ | $\delta$ | $x^2y = 0$ | $\delta$ | $x^2y^3 = 0$ | $\delta$ |
|---|---|---|---|---|---|---|---|
| $\nabla_{\mathfrak{l}_e}$ | 1 | $\tilde{x}_0\tilde{y}_0,$ | | $\tilde{x}_0^2\tilde{y}_0,$ | | $\tilde{x}_0^2\tilde{y}_0^3,$ | |
| | 2 | $\tilde{x}_0\tilde{y}_1 + \tilde{x}_1\tilde{y}_0,$ | | $2\tilde{x}_0\tilde{x}_1\tilde{y}_0 + \tilde{x}_0^2\tilde{y}_1,$ | | $2\tilde{x}_0\tilde{x}_1\tilde{y}_0^3 + 3\tilde{x}_0^2\tilde{y}_0^2\tilde{y}_1,$ | |
| | 3 | $\tilde{x}_0\tilde{y}_2 + \tilde{x}_1\tilde{y}_1 + \tilde{x}_2\tilde{y}_0$ | | $\tilde{x}_0^2\tilde{y}_2 + 2\tilde{x}_0\tilde{x}_1\tilde{y}_1 + (2\tilde{x}_0\tilde{x}_2 + \tilde{x}_1^2)\tilde{y}_0$ | | $3\tilde{x}_0^2(\tilde{y}_0\tilde{y}_1^2 + \tilde{y}_0^2\tilde{y}_2) + 6\tilde{x}_0\tilde{x}_1\tilde{y}_0^2\tilde{y}_1 + (\tilde{x}_1^2 + 2\tilde{x}_0\tilde{x}_2)\tilde{y}_0^3$ | |
| $\nabla^{\mathrm{red}}_{\mathfrak{l}_e}$ | 1 | $\tilde{x}_0\tilde{y}_0,$ | 1 | $\tilde{x}_0\tilde{y}_0,$ | 1 | $\tilde{x}_0\tilde{y}_0,$ | 1 |
| | 2 | $\tilde{x}_0\tilde{y}_1, \tilde{x}_1\tilde{y}_0,$ | 2 | $\tilde{x}_0\tilde{y}_1,$ | 3 | [no new equation] | 3 |
| | 3 | $\tilde{x}_0\tilde{y}_2, \tilde{x}_1\tilde{y}_1, \tilde{x}_2\tilde{y}_0$ | 3 | $\tilde{x}_0\tilde{y}_2, \tilde{x}_1\tilde{y}_0$ | 4 | $\tilde{x}_1\tilde{y}_0$ | 5 |

As substantiated by the data in this table, we have the following general estimate:

**5.1. Lemma.** *The dimension of $\nabla_{\mathfrak{z}}X$ is at least $dl$, where $d$ is the dimension of $X$ and $l$ the length of $\mathfrak{z}$, with equality if $X$ is smooth.*

*Proof.* Assume first that $X$ is reduced so that it contains a non-empty open subset $U$ which is non-singular. By Theorem 4.4, the pull-back $\nabla_{\mathfrak{z}}U = U \times_X \nabla_{\mathfrak{z}}X$ is a open subset of $\nabla_{\mathfrak{z}}X$. Moreover, by Theorem 4.14 the dimension of $\nabla_{\mathfrak{z}}U$ is equal to $dl$.

For $X$ arbitrary, let $V := X^{\mathrm{red}}$ be the variety underlying $X$. The closed immersion $V \subseteq X$ yields a closed immersion $\nabla_{\mathfrak{z}}V \subseteq \nabla_{\mathfrak{z}}X$ by Corollary 3.8. The result now follows from the reduced case applied to $V$. □

We will call the difference $\dim(\nabla_{\mathfrak{z}}X) - dl$ the *defect of $X$ at $\mathfrak{z}$*. Smooth varieties therefore have no defect. An interesting question is which varieties do not have defect. In the linear case, locally closed intersections with rational singuarities have this property by [13, Theorem 0.1]. The bound given by Lemma 5.1 is far from optimal, as can be seen by taking the jet scheme of a fat point (see, for instance, Example 4.7). Calculations lead me to believe that the dimension of $\nabla_{\mathfrak{l}_n}\mathfrak{z}$ is equal to $n + 1$, for $n \geqslant 1$, where $\mathfrak{z}$ is the fat point in $\mathbb{L}^2$ with equations $x^3 = xy = y^3 = 0$ (a case not covered by Lemma 9.7 below). To obtain motivic rationality of the Igusa-zeta series and other motivic generating series, to be discussed shortly in §9, we must understand the growth of these defects. Given a closed germ $(Y, P)$ and a scheme $X$, let us define the *asymptotic defect* of $X$ along $(Y, P)$ as

$$\delta_{(Y,P)}(X) := \limsup_n \frac{\dim(\nabla_{J^n_P Y} X)}{\ell(J^n_P Y)} - \dim(X) \tag{28}$$

In other words, the asymptotic defect is the limsup of the $J^n_P Y$-defect over the length $j^n_P(Y)$ of $J^n_P Y$, where the latter grows as a polynomial in $n$ of degree $\dim(Y, P)$ by Hilbert-Samuel theory. It is not true that the asymptotic defect is always zero, as the example below shows, but we expect:

**5.2. Conjecture.** *The asymptotic defect is attained, that is to say, the sequence on the right hand side of* (28) *becomes eventually stationary.*

**5.3. Example.** Suppose $Y$ is of the form $Z \times \mathfrak{z}$ for some fat point $\mathfrak{z}$ for which $X$ has a defect $a$, that is to say, $\dim(\nabla_{\mathfrak{z}} X) = dl + a$, where $d = \dim(X)$ and $l = \ell(\mathfrak{z})$. Let $Q$ be the projection of $P$ on $Z$, let $j^n_P(Y)$ and $j^n_Q(Z)$ be the length of $J^n_P Y$ and $J^n_Q Z$ respectively. In particular, $J^n_P Y \cong J^n_Q Z \times \mathfrak{z}$ for $n \geqslant l$. Hence, $j^n_P(Y) = l j^n_Q(Z)$, and

$$\nabla_{J^n_P Y} X = \nabla_{J^n_Q Z}(\nabla_{\mathfrak{z}} X)$$

by (23). From this, one easily calculates that

$$l\delta_{(Y,P)}(X) = \delta_{(Z,Q)}(\nabla_{\mathfrak{z}} X) + a$$

showing that both defects cannot be zero, if the $\mathfrak{z}$-defect $a$ of $X$ is non-zero. In fact, since defects are always non-negative, we must have $0 < a/l \leqslant \delta_{(Y,P)}(X)$.

**Auto-arcs.** The growth of the dimension of auto-arcs (see Example 4.17), that is to say, the function

$$\delta(\mathfrak{z}) := \dim(\nabla_{\mathfrak{z}} \mathfrak{z})$$

for $\mathfrak{z}$ a fat point, is still quite puzzling. By Example 4.7, we have $\delta(\mathfrak{l}_n) = n - 1$. However, the next example shows that $\delta(\mathfrak{z})$ can be bigger than $\ell(\mathfrak{z})$.

**5.4. Example.** Let $\mathfrak{o}_n := J^n_O \mathbb{L}^2$ be the $n$-th co-jet of the origin in the plane with ideal of definition $(\xi, \zeta)^n$. Its length is equal to $o_n := \binom{n+1}{2}$, with a basis consisting of all monomials in $\xi$ and $\zeta$ of degree strictly less than $n$. Let

$$\widehat{x} := \sum_{i+j<n} \tilde{x}_{ij} \xi^i \zeta^j \quad \text{and} \quad \widehat{y} := \sum_{i+j<n} \tilde{y}_{ij} \xi^i \zeta^j \tag{29}$$

be the generic jets, so that $\nabla_{\mathfrak{o}_n} \mathfrak{o}_n$ is the closed subscheme of $\mathbb{L}^{o_n}$ given by the coefficients of the monomials $\widehat{x}^i \widehat{y}^{n-i}$, for $i = 0, \dots, n$. Since the jet scheme $\nabla_{\mathfrak{o}_n} \mathfrak{o}_n$ lies above the origin, its defining equations contain the ideal $(\tilde{x}_{00}, \tilde{y}_{00})^n$. To calculate its dimension, we may take its reduction, which means that we may put $\tilde{x}_{00}$ and $\tilde{y}_{00}$ equal to zero in (29). However, any monomial of degree $n$ in the generic jets is then identical zero, showing that the reduction of the jet scheme is given by $\tilde{x}_{00} = \tilde{y}_{00} = 0$, and hence, its dimension is equal to

$$\delta(\mathfrak{o}_n) = 2\binom{n+1}{2} - 2 = n^2 + n - 2.$$

One might be tempted to propose therefore that $\delta(\mathfrak{z})$ is equal to the embedding dimension of $\mathfrak{z}$ times its length minus one, but the next example disproves this. Namely, without proof, we state that $\delta(\mathfrak{z}) = 7$ for $\mathfrak{z}$ the fat point in the plane with equations $\xi^2 = \xi\zeta^2 = \zeta^3 = 0$ (note that $\mathfrak{z}$ has length 5 and embedding dimension 2, so that the expected value would be $2 \times 4 = 8$). Note that the auto-arc space $\nabla_{\mathfrak{z}}\mathfrak{z}$ is often, but not always an affine space (see Question 4.15 and the example following it).

It seems plausible that $\delta(J^n_P Y)$ grows as a polynomial in $n$ of degree $d$, for any $d$-dimensional closed germ $(Y, P)$. In particular, we expect the limit

$$e(Y, P) := \lim_{n\to\infty} \frac{\delta(J^n_P Y)}{\ell(J^n_P Y)}$$

to exist. For instance, an easy extension of the above examples yields the equality $e(\mathbb{L}^m, O) = m$. In view of Question 4.15, we would even expect that the *auto-Igusa-zeta* series

$$\zeta_{\mathfrak{z}}(t) := \sum_{n=1}^{\infty} \mathbb{L}^{-d\ell(J^n_P Y)}[\nabla_{J^n_P Y}(J^n_P Y)]t^n$$

is rational over the localization of the classical Grothendieck ring with respect to $\mathbb{L}$, for any $d$-dimensional closed germ $(Y, P)$, a phenomenon that we will study in §9 below under the name of *motivic rationality* (and where we also explain the choice of power of $\mathbb{L}$). What about its motivic rationality over the localization $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)_{\mathbb{L}}$ of the formal Grothendieck ring?

**Dimension of a motif.** Given a formal motif $\mathfrak{X}$ on a $\kappa$-scheme $X$, we define its *dimension* as the dimension of $\mathfrak{X}(\kappa)$. This is well-defined since $\mathfrak{X}(\kappa)$ is a constructible subset of $X(\kappa)$ by Proposition 4.18. If $\mathfrak{X} = X^\circ$ is representable, then its dimension is precisely the dimension of the scheme $X$. On the other hand, if $\mathfrak{X}$ is the formal completion of $X$ at a closed point, then $\mathfrak{X}$ has dimension zero, whereas its global section ring (see [17, §3.1]) has dimension equal to that of $X$ at $P$ by [17, Corollary 7.8].

**5.5. Proposition.** *If two formal motives have the same class in the formal Grothendieck ring* $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa)$*, then they have the same dimension.*

*Proof.* Since dimension is determined by the $\kappa$-rational points, we may take, using [17, Theorem 7.6], the image of this common class in $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$, where the result is known to hold. □

As we will work over $\mathbf{G} := \mathbf{Gr}(\mathbb{F}\mathrm{orm}^{\mathrm{str}}_\kappa)_{\mathbb{L}}$ below, we extend the notion of dimension into an integer valued invariant on this localized Grothendieck ring by defining the dimension of $[\mathfrak{X}]\cdot\mathbb{L}^{-i}$ to be $\dim(\mathfrak{X})-i$, for any strongly formal motif $\mathfrak{X}$ and any $i\in\mathbb{Z}$. In particular, if $X$ has dimension $d$ and $\mathfrak{z}$ length $l$, then $[\nabla_{\mathfrak{z}}X]\cdot\mathbb{L}^{-dl}$ has positive dimension, which is the reason behind the introduction of this power of the Lefschetz class in the formulas below. This also gives us the Kontsevich filtration by dimension on $\mathbf{G}$. Namely, for each $m\in\mathbb{N}$, let $\Gamma_m(\mathbf{G})$ be the subgroup generated by all classes $[\mathfrak{X}]\cdot\mathbb{L}^{-i}$ of dimension at most $-m$. This is a descending filtration and the completion of $\mathbf{G}$ with respect to this filtration will be denoted $\widehat{\mathbf{G}}$. However, since we define motivic filtration locally (see §10 below), we will not make use of it. For a motif $\mathfrak{X}$, we can, in view of Proposition 5.5, define its *weightless class* in $\mathbf{G}$ as

$$[[\mathfrak{X}]] := \frac{[\mathfrak{X}]}{\mathbb{L}^{\dim(\mathfrak{X})}}, \tag{30}$$

a notation that will be handy at times.

## 6. Deformed jets

We continue with the setup from §4: let $j_{\mathfrak{z}}\colon \mathfrak{z}\to\operatorname{Spec}\kappa$ be the structure morphism of a fat point $\mathfrak{z}$ over an algebraically closed field $\kappa$. Instead of looking at the double adjunction giving rise to the jet functor $\nabla_{\mathfrak{z}}$, we consider here the diminution part only, that is to say, the right adjoint $\nabla_{j_{\mathfrak{z}}^*}$ satisfying for each $\mathfrak{z}$-sieve $\mathfrak{Y}$ on a $\mathfrak{z}$-scheme $Y$ and each fat $\kappa$-point $\mathfrak{w}$, an isomorphism

$$(\nabla_{j_{\mathfrak{z}}^*}\mathfrak{Y})(\mathfrak{w})\cong\mathfrak{Y}(j_{\mathfrak{z}}^*\mathfrak{w})$$

where this time, we have to view $j_{\mathfrak{z}}^*\mathfrak{w}=\mathfrak{z}\mathfrak{w}$ as a fat $\mathfrak{z}$-point. By Theorem 3.7, we associate in particular to any $\mathfrak{z}$-scheme $Y$, a $\kappa$-scheme $\nabla_{j_{\mathfrak{z}}^*}Y$. In particular, if $Y=j_{\mathfrak{z}}^*X$ is obtained from a $\kappa$-scheme $X$ by base change, then

$$\nabla_{j_{\mathfrak{z}}^*}Y=\nabla_{\mathfrak{z}}X \tag{31}$$

by Corollary 3.8.

Apart from $j_{\mathfrak{z}}$, we also have the residue field morphism $\pi_{\mathfrak{z}}\colon\operatorname{Spec}\kappa\to\mathfrak{z}$. To a $\mathfrak{z}$-scheme $Y$, we can therefore also associate the base change $\bar{Y}:=\pi_{\mathfrak{z}}^*Y$, called the *closed fiber* of $Y$. We can think of $Y$ as a *fat deformation* of $j_{\mathfrak{z}}^*\bar{Y}$. Indeed, since $\kappa\times_{\mathfrak{z}}\kappa=\kappa$, any $\kappa$-rational point of $Y$ is also a $\kappa$-rational point on $\bar{Y}$, that is to say, $Y(\kappa)=\bar{Y}(\kappa)=j_{\mathfrak{z}}^*\bar{Y}(\kappa)$, showing that $Y$ and $j_{\mathfrak{z}}^*\bar{Y}$ have the same underlying variety.

**6.1. Example.** For instance, if $C$ is the curve $x^2-y^3$, then the $\mathfrak{l}_3$-scheme $X:=\operatorname{Spec}R_3[x,y]/(x^2-y^3-\xi^2)$ has closed fiber $C$, and $X(\kappa)=C(\kappa)$. Note however that $X(\mathfrak{l}_3)\neq C(\mathfrak{l}_3)$. In fact, truncation yields a map $X(\mathfrak{l}_3)\to C(\mathfrak{l}_2)$.

Hence, by (31), we may likewise think of $\nabla_{j_{\mathfrak{z}}^*}Y$ as a fat deformation of the jet space $\nabla_{\mathfrak{z}}\bar{Y}$ of its closed fiber, justifying the term *deformed jet space* for $\nabla_{j_{\mathfrak{z}}^*}Y$. This construction is compatible then with specializations in the following sense. Fix a $\kappa$-scheme $Z$. The base change $j_Z\colon Z \times \mathfrak{z} \to Z$ is again a finite, flat homomorphism satisfying condition (†), thus allowing us to consider the diminution $\nabla_{j_Z^*}$, associating to any $Z \times \mathfrak{z}$-scheme $Y$, a $Z$-scheme $\nabla_{j_Z^*} Y$, called the *relative jet scheme* of $Y$. The deformed jet space is then given by the special case when $Z = \operatorname{Spec} \kappa$.

**6.2. Proposition.** *Let $\mathfrak{z}$ be a fat $\kappa$-point and $Z$ a $\kappa$-scheme. For every $Z \times \mathfrak{z}$-scheme $Y$, viewed as a family over $Z$ in the sense of §3.15, and for any $\kappa$-rational point $a$ on $Z$, we have an isomorphism*

$$\nabla_{j_{\mathfrak{z}}^*}(Y_{\tilde a}) = (\nabla_{j_Z^*} Y)_a$$

*of $\kappa$-schemes, where $\tilde a\colon \mathfrak{z} \to Z \times \mathfrak{z}$ is the base change of $a$.*

*Proof.* Immediately from Theorem 3.12 applied to the base change diagram

$$\begin{array}{ccc} \mathfrak{z} & \xrightarrow{\ j_{\mathfrak{z}}\ } & \operatorname{Spec}\kappa \\ {\scriptstyle \tilde a}\downarrow & & \downarrow{\scriptstyle a} \\ Z \times \mathfrak{z} & \xrightarrow[\ j_Z\ ]{} & Z. \end{array} \tag{32}$$

□

So, returning to Example 6.1, let $X \subseteq \mathbb{L}^3_{\mathfrak{l}_3}$ be the hypersurface with equation $x^2 - y^3 - z\xi^2$. As a family over $\mathbb{L}_{\mathfrak{l}_3}$ via projection onto the last coordinate, its specializations $X_a$ are all isomorphic if $a \neq 0$, whereas the special fiber is $X_0 = C \times \mathfrak{l}_3$. The corresponding relative jet scheme $\nabla_{j^*_{\mathbb{L},\mathfrak{l}_3}} X$ is given by

$$\tilde x_0^2 - \tilde y_0^3 = 2\tilde x_0 \tilde y_1 - 3\tilde y_0^2 \tilde y_1 = 2\tilde x_0 \tilde x_2 + \tilde x_1^2 - 3\tilde y_0 \tilde y_1^2 - 3\tilde y_0^2 \tilde y_2 - \tilde z_0 = 0$$

Its specializations are again all isomorphic (to the third order Milnor fiber; see below) whereas the special fiber is isomorphic to $\nabla_{\mathfrak{l}_3} C$.

## 7. Limit points

The closed subscheme relation defines a partial order relation on $\mathbb{F}\mathtt{at}_\kappa$, that is to say, we say that $\bar{\mathfrak{z}} \leqslant \mathfrak{z}$ if and only if $\bar{\mathfrak{z}}$ is a closed subscheme of $\mathfrak{z}$ (and not just isomorphic to one). Consider a direct system of fat points with a least element, viewed as schemes. It follows that all fat points in the system must have the same center (to wit, the center of the least element). In other words, any fat point in the directed system has the same underlying closed point, and so we will call such a system a *point system*.

We want to adjoin to the category of fat points its direct limits, but the problem is that the category of schemes is not closed under direct limits either. However, the category of locally ringed spaces is: if $(X_i, \mathcal{O}_{X_i})$ form a direct system, then their direct limit is the topological space $X := \varinjlim X_i$ endowed with the structure sheaf $\mathcal{O}_X := \varprojlim \mathcal{O}_{X_i}$. Since we will assume that all fat points have the same underlying topological space, namely a single point, the construction simplifies: the direct limit of a point system is simply the one-point space with its unique stalk given as the inverse limit of all the coordinate rings of the fat points in the system. A *morphism* in this setup will mean a morphism of locally ringed spaces with values in the category of $\kappa$-algebras. For example, if $R$ is any $\kappa$-algebra and $\mathfrak{o}$ the locally ringed space with underlying set the origin and (unique) stalk $R$, and if $X = \operatorname{Spec} A$ is any affine scheme, then $\operatorname{Mor}_\kappa(\mathfrak{o}, X)$ is in one-one correspondence with the set of $\kappa$-algebra homomorphisms $\operatorname{Hom}_{\kappa-\text{alg}}(A, R)$.

Let $\mathbb{X} \subseteq \mathbb{F}\mathtt{at}_\kappa$ be a point system. Its direct limit $\varinjlim \mathbb{X}$, as a one-point locally ringed space, is called a *limit point*. Some examples of limit points are:

(7.i) If $\mathbb{X}$ is finite, the direct limit is just its maximum, whence a fat point.

(7.ii) Given a closed germ $(Y, P)$, its formal completion $\widehat{Y}_P$ is the direct limit of the co-jets $J^n_P Y$, whence a limit point.

(7.iii) The direct limit of all fat points with the same center is called the *universal point* and is denoted $\mathfrak{u}_\kappa$, or just $\mathfrak{u}$. Any limit point admits a closed immersion into $\mathfrak{u}$. In particular, up to isomorphism, $\mathfrak{u}$ does not depend on the underlying point.

**7.1. Lemma.** *The stalk of the universal point $\mathfrak{u}_\kappa$ is isomorphic to the power series ring over $\kappa$ in countably many indeterminates.*

*Proof.* Any fat point is a closed subscheme of some formal scheme $\widehat{(\mathbb{L}^n)}$. Hence suffices to show that the inverse limit of the power series rings $S_n := \kappa[[x_1, \dots, x_n]]$ under the canonical projections $S_m \to S_n$ given by modding out the variables $x_i$ for $n < i \leqslant m$ is isomorphic to the power series ring $\kappa[[x]]$ in countably many indeterminates $x = (x_1, x_2, \dots)$. To this end, let $f_n \in S_n$ be a compatible sequence in the inverse system. For each exponent $\nu = (\nu_1, \nu_2, \dots)$ in the direct sum $\mathbb{N}^{(\mathbb{N})}$ of countably many copies of $\mathbb{N}$, and each $n$, let $a_{n,\nu} \in \kappa$ be the coefficient of $x^\nu := x_1^{\nu_1} \cdots x_{i(\nu)}^{\nu_{i(\nu)}}$ in $f_n$, where $i(\nu)$ is the largest index for which $\nu_i$ is non-zero. Compatibility means that there exists for each $\nu$ an element $a_\nu \in \kappa$ such that $a_\nu = a_{n,\nu}$ for all $n > i(\nu)$. Hence $f := \sum_\nu a_\nu x^\nu \in \kappa[[x]]$ is the limit of the sequence $f_n$, proving the claim. □

To make the limit points into a category, denoted $\widehat{\mathbb{F}\mathtt{at}_\kappa}$, take morphisms to be direct limits of morphisms of fat points. More precisely, given point systems $\mathbb{X}, \mathbb{Y} \subseteq \mathbb{F}\mathtt{at}_\kappa$ with respective direct limits $\mathfrak{x}$ and $\mathfrak{y}$, then a morphism (of locally ringed spaces) $\varphi\colon \mathfrak{x} \to \mathfrak{y}$ is a *morphism of limit points* if for each $\mathfrak{z} \in \mathbb{X}$ there exists a $\mathfrak{v} \in \mathbb{Y}$

such that $\varphi(\mathfrak{z}) \subseteq \mathfrak{v}$, or, dually, if the induced morphism $\varprojlim \mathcal{O}_{\mathfrak{v}} \to \varprojlim \mathcal{O}_{\mathfrak{z}}$ has the property that for each $\mathfrak{z} \in \mathbb{X}$, we can find a $\mathfrak{v} \in \mathbb{Y}$ such that this morphism factors through $\mathcal{O}_{\mathfrak{v}} \to \mathcal{O}_{\mathfrak{z}}$. In this way, the category $\widehat{\mathbb{F}\mathtt{at}}_\kappa$ of limit points is an extension of the category $\mathbb{F}\mathtt{at}_\kappa$ of fat points, which in a sense acts as its *compactification*. In particular, any limit point $\mathfrak{x}$ admits a canonical structure morphism $j_{\mathfrak{x}} \colon \mathfrak{x} \to \operatorname{Spec} \kappa$. We also extend the partial order relation on $\mathbb{F}\mathtt{at}_\kappa$ to one on $\widehat{\mathbb{F}\mathtt{at}}_\kappa$ as follows. Firstly, we say that $\mathfrak{z} \leqslant \mathfrak{x}$ for $\mathfrak{z}$ a fat point and $\mathfrak{x} = \varinjlim \mathbb{X}$ a limit point, if $\mathfrak{z} \leqslant \mathfrak{v}$ for some fat point $\mathfrak{v} \in \mathbb{X}$. It follows that there is a canonical embedding $\mathfrak{z} \subseteq \mathfrak{x}$ which induces a surjection on the stalks, and which we therefore call a *closed embedding* in analogy with the scheme-theoretic concept. We then say for a limit point $\mathfrak{y} = \varinjlim \mathbb{Y}$ that $\mathfrak{y} \leqslant \mathfrak{x}$ if for every $\mathfrak{z} \in \mathbb{Y}$ we have $\mathfrak{z} \leqslant \mathfrak{x}$. It follows that we have a canonical morphism of limit points $\mathfrak{y} \to \mathfrak{x}$ which is again surjective on their stalks, and hence can rightly be called once more a *closed embedding*. One checks that this defines indeed a partial order on limit points extending the one on fat points.

We call a limit point $\mathfrak{x}$ *bounded* if it is the direct limit of fat points of embedding dimension at most $n$, for some $n$. Formal completions of closed germs are examples of bounded limit points, whereas $\mathfrak{u}$ clearly is not. In fact, any bounded limit point arises in a similar, analytical way:

**7.2. Proposition.** *The bounded limit points are in one-one correspondence with analytic germs. More precisely, the stalks of bounded limit points are precisely the complete Noetherian local rings with residue field $\kappa$.*

*Proof.* Let $\mathfrak{x}$ be a bounded limit point, say, realized as the direct limit of fat points $\mathfrak{z}_i \subseteq \mathbb{L}^n$ centered at the origin, for some fixed $n$. Let $\kappa[x]/\mathfrak{a}_i$ be the coordinate ring of $\mathfrak{z}_i$, so that $\mathfrak{a}_i \subseteq \kappa[x]$ is $\mathfrak{m}$-primary, where $\mathfrak{m}$ is the maximal ideal generated by the variables. Let $I$ be the intersection of all $\mathfrak{a}_i\kappa[[x]]$. I claim that $\mathfrak{x}$ has stalk equal to $S := \kappa[[x]]/I$. Indeed, by a theorem of Chevalley ([12, Exercise 8.7]), there exists for each $l$, some $i$ such that $\mathfrak{a}_i S \subseteq \mathfrak{m}^l S$. In particular, the inverse limit is simply the $\mathfrak{m}$-adic completion of $S$, which is of course $S$ itself.

The converse is also obvious: given a complete Noetherian ring $(S, \mathfrak{m})$ with residue field $\kappa$, then by Cohen's structure theorem, it is of the form $\kappa[[x]]/I$ for some ideal $I$. One easily checks that it is the coordinate ring of the direct limit of the corresponding co-jets $\operatorname{Spec}(S/\mathfrak{m}^l)$. □

Any limit point $\mathfrak{x} = \varinjlim \mathbb{X}$ defines a functor $\mathfrak{x}^\circ$ by assigning to a fat point $\mathfrak{z}$ the set of morphisms $\operatorname{Mor}_{\widehat{\mathbb{F}\mathtt{at}}_\kappa}(\mathfrak{z}, \mathfrak{x})$.

**7.3. Corollary.** *The functor $\mathfrak{x}^\circ$ defined by a limit point $\mathfrak{x} = \varinjlim \mathbb{X}$ is the inverse limit of the representable functors $\mathfrak{v}^\circ$ for $\mathfrak{v} \in \mathbb{X}$. If $\mathfrak{x}$ is moreover bounded, then $\mathfrak{x}^\circ$ is a strongly formal motif.*

*Proof.* Given a fat point $\mathfrak{z} = \operatorname{Spec} R$, we have to show that $\mathfrak{v}(\mathfrak{z})$ for $\mathfrak{v} \in \mathbb{X}$ forms an inverse system with inverse limit equal to $\operatorname{Mor}_{\widehat{\mathbb{F}\mathtt{at}}_\kappa}(\mathfrak{z}, \mathfrak{x})$. Let $(S, \mathfrak{m})$ be the stalk of $\mathfrak{x}$, that is to say, the inverse limit of the coordinate rings of the fat points belonging to $\mathbb{X}$. The first statement is immediate by functoriality, and for the second, note that since $R$ has finite length,

$$\operatorname{Mor}_{\widehat{\mathbb{F}\mathtt{at}_\kappa}}(\mathfrak{z},\mathfrak{x}) \cong \operatorname{Hom}_{\kappa-\mathrm{alg}}(S,R). \tag{33}$$

More precisely, any $\kappa$-algebra homomorphism $a\colon S \to R$ factors through $S/\mathfrak{m}^l \to R$, for $l = \ell(R)$. Moreover, if $n$ is the embedding dimension of $R$, then there exists a complete, Noetherian residue ring $\bar S$ of $S$ of embedding dimension at most $n$ such that $a$ factors as $S \to \bar S/\mathfrak{m}^l\bar S \to R$. By the same argument as in the proof of Proposition 7.2, there is some $\mathfrak{v} = \operatorname{Spec} T \in \mathbb{X}$ such that $T \to \bar S/\mathfrak{m}^l\bar S$, showing that $a$ is already induced by the morphism $\mathfrak{z} \to \mathfrak{v}$. In fact, if $\mathfrak{x}$ is bounded, then we may choose $\mathfrak{v}$ independent from $a$, showing that $\mathfrak{x}^\circ(\mathfrak{z}) = \mathfrak{v}(\mathfrak{z})$. Since all $\mathfrak{v} \in \mathbb{X}$ embed in the same affine space, $\mathfrak{x}^\circ$ is a locally schemic sieve on this space, whence a strongly formal motif. □

Let $\mathfrak{x} = \varinjlim \mathbb{X}$ be a limit point. Given a contravariant functor $\mathfrak{X}$ on $\mathbb{F}\mathtt{at}_\kappa$, the collection of all $\mathfrak{X}(\mathfrak{v})$ for $\mathfrak{v} \in \mathbb{X}$ is an inverse system of sets given by the maps $i_{\mathfrak{v},\mathfrak{w}}\colon \mathfrak{X}(\mathfrak{w}) \to \mathfrak{X}(\mathfrak{v})$ if $\mathfrak{v} \leqslant \mathfrak{w}$ in $\mathbb{X}$, where $i_{\mathfrak{v},\mathfrak{w}}$ is induced by the embedding $\mathfrak{v} \subseteq \mathfrak{w}$. We denote the inverse limit of this system simply by $\mathfrak{X}(\mathfrak{x})$. It follows from the definition of morphisms of limit points that $\mathfrak{X}$ becomes a functor on the category of limit points. In other words, any contravariant functor on $\mathbb{F}\mathtt{at}_\kappa$ extends to one on $\widehat{\mathbb{F}\mathtt{at}_\kappa}$; this principle will simply be called *continuity*. Since inverse limits commute with functors, one easily verifies that if $s\colon \mathfrak{X} \to \mathfrak{Y}$ is a natural transformation, then for any limit point $\mathfrak{x}$, this induces a map $\mathfrak{X}(\mathfrak{x}) \to \mathfrak{Y}(\mathfrak{x})$, showing that extension by continuity is functorial.

If $\mathbb{X}$ is a point system in $\mathbb{F}\mathtt{at}_\kappa$ with limit $\mathfrak{x}$, and if $\mathfrak{z}$ is any fat point with canonical morphism $j_\mathfrak{z}\colon \mathfrak{z} \to \operatorname{Spec}\kappa$, then the base change $j_\mathfrak{z}^*\mathbb{X}$ consisting of all $\mathfrak{z}\mathfrak{v}$ for $\mathfrak{v} \in \mathbb{X}$ is again a point system, whose limit we simply denote by $\mathfrak{z}\mathfrak{x}$ (the reader can check that this defines a product in the category $\widehat{\mathbb{F}\mathtt{at}_\kappa}$). Repeating this argument on the first factor then shows that we may even multiply any two limit points. However, this multiplication does no longer behave as well as before. For instance, since the base change $j_\mathfrak{z}^*(\mathbb{F}\mathtt{at}_\kappa)$ by any fat point $\mathfrak{z}$ is equal to the whole category $\mathbb{F}\mathtt{at}_\kappa$, we get $\mathfrak{z}\mathfrak{u} = \mathfrak{u}$.

Let $j_\mathfrak{x}\colon \mathfrak{x} \to \operatorname{Spec}\kappa$ be the structure morphism of the limit point $\mathfrak{x}$. Strictly speaking, as this is only a direct limit of finite, flat morphisms, the theory of diminution does not apply, and neither that of augmentation. Nonetheless, without going into details, one could develop the theory under this weaker condition, although we will only give an ad hoc argument in the case we need it. So, given a sieve $\mathfrak{X}$ on $\widehat{\mathbb{F}\mathtt{at}_\kappa}$, we define $\nabla_\mathfrak{x}\mathfrak{X} := \nabla_{j_\mathfrak{x}^*}\nabla_{(j_\mathfrak{x})_*}\mathfrak{X}$ at a limit point $\mathfrak{y}$ as the set $\mathfrak{X}(\mathfrak{x}\mathfrak{y})$, where we view $\mathfrak{x}\mathfrak{y}$ again as a limit point (over $\kappa$).

**7.4. Lemma.** *For any limit point $\mathfrak{x}$ and any $\kappa$-scheme $X$, the adjoint $\nabla_\mathfrak{x}(X^\circ)$ is representable, by the so-called* arc scheme $\nabla_\mathfrak{x}X$ along $\mathfrak{x}$.

*Proof.* Let $\mathfrak{x}$ be the direct limit of the directed subset $\mathbb{X} \subseteq \mathbb{F}\mathtt{at}_\kappa$. Suppose first that $X$ is affine. Since the $\nabla_{\mathfrak{w}} X$ for $\mathfrak{w} \in \mathbb{X}$ form an inverse system of affine schemes, their inverse limit is a well-defined affine scheme $\tilde{X}$ with coordinate ring the direct limit of the coordinate rings of the jet schemes along fat points in $\mathbb{X}$. By continuity, it suffices to verify that $\nabla_{\mathfrak{x}}(X^\circ) = \tilde{X}^\circ$ on $\mathbb{F}\mathtt{at}_\kappa$. To this end, fix a fat point $\mathfrak{z}$. From

$$\begin{aligned}(\nabla_{\mathfrak{x}}(X^\circ))(\mathfrak{z}) = X(\mathfrak{x}\mathfrak{z}) &= \varprojlim_{\mathfrak{w}\in\mathbb{X}} X(\mathfrak{w}\mathfrak{z}) \\ &= \varprojlim_{\mathfrak{w}\in\mathbb{X}} \mathrm{Mor}_\kappa(\mathfrak{z}, \nabla_{\mathfrak{w}} X) \\ &= \mathrm{Mor}_\kappa(\mathfrak{z}, \varprojlim_{\mathfrak{w}\in\mathbb{X}} \nabla_{\mathfrak{w}} X) \\ &= \mathrm{Mor}_\kappa(\mathfrak{z}, \tilde{X}) = \tilde{X}(\mathfrak{z}),\end{aligned}$$

where we used the universal property of inverse limits in the third line, the claim now follows. The general case follows from this by the open nature of jet schemes (Theorem 4.4) and the fact that if $X$ admits an open affine covering of cardinality $N$, then so does any jet scheme $\nabla_{\mathfrak{z}} X$ by base change. □

**7.5. Remark.** We can take this still one step further by using Remark 4.19 and the fact proven in [17, Lemma 2.8] that any scheme $Y$ is the direct limit, as a sieve, of its zero-dimensional closed subsieves. Thus we define $\nabla_Y X$ as the inverse limit of the $\nabla_Z X$, where $Z \subseteq Y$ runs over all zero-dimensional closed subschemes, and call it the *scheme of $Y$-arcs on $X$*.

## 8. Extendable jets

Let $\widehat{Y}$ be a formal completion of a closed germ $(Y, O)$, viewed as the limit point of the co-jets $J^n_O Y$, and let $X$ be a $\kappa$-scheme. By Lemma 7.4, we have an associated arc scheme $\nabla_{\widehat{Y}} X$. For each $n$, we have a canonical map $\nabla_{\widehat{Y}} X \to \nabla_{J^n_O Y} X$, which in general is not surjective (it is so, by Theorem 4.14, when $X$ is smooth). To study this image, we make the following definitions.

Given a closed embedding $\mathfrak{v} \subseteq \mathfrak{w}$, the image sieve given by the canonical map $\nabla_{\mathfrak{w}} X \to \nabla_{\mathfrak{v}} X$ is called the *motif of $\mathfrak{w}$-extendable jets on $X$ along $\mathfrak{v}$*, and will be denoted $\nabla_{\mathfrak{w}/\mathfrak{v}} X$. By construction, it is sub-schemic. Let $\nabla_n X := \nabla_{J^n_O Y} X$, and $\nabla_{m/n} X := \nabla_{J^m_O Y / J^n_O Y} X$, for $m \geqslant n$. Since the map $\nabla_{\widehat{Y}} X \to \nabla_n X$ is not of finite type, the corresponding image sieve, denoted $\nabla_{\widehat{Y}/n} X$ and called the *$n$-th order $\widehat{Y}$-extendable jets on $X$*, may fail to be sub-schemic. We do have:

**8.1. Theorem.** *For each $\kappa$-scheme $X$, for each formal completion $\widehat{Y}$ of a closed germ, and for each $n$, the $n$-th order extendable jets on $X$ along this formal completion, $\nabla_{\widehat{Y}/n} X$, is a formal motif.*

*Proof.* Without loss of generality, we may assume that $\widehat{Y}$ is the completion of $Y$ at the origin. It is clear that $\nabla_{\hat{Y}/n}X$ is the intersection of all $\nabla_{m/n}X$, for $m \geqslant n$. To show that it is a formal motif, it suffices to show that its complement can be approximated by sub-schemic motives. Since each $\nabla_{m/n}X$ is sub-schemic, this will follow if we can show that for each fat point $\mathfrak{z}$, there is some $m_{\mathfrak{z}}$ such that $\nabla_{\hat{Y}/n}X(\mathfrak{z}) = \nabla_{m_{\mathfrak{z}}/n}X(\mathfrak{z})$.

Recall that for $(R, \mathfrak{m})$ a quotient of a power series ring $\kappa[[\xi]]$ modulo an ideal generated by polynomials, we have uniform strong Artin Approximation, in the sense that for any polynomial system of equations $f_1 = \cdots = f_s = 0$ and every $n$, there exists some $N$, such that any solution of this system in $R/\mathfrak{m}^N$ is congruent modulo $\mathfrak{m}^n$ to a solution in $R$: see for instance [15, Theorem 7.1.10], where the proof is only given for the power series ring itself, but immediately generalizes to any quotient by a polynomial ideal, whence in particular to the stalk of the formal completion $\widehat{Y}$. This means that $\nabla_{\hat{Y}/n}X(\kappa) = \nabla_{m/n}X(\kappa)$, for some $m \geqslant n$. To obtain a similar identity over an arbitrary fat point $\mathfrak{z}$, we apply the same result but replacing $X$ by the jet scheme $\nabla_{\mathfrak{z}}X$, yielding the existence of a $m_{\mathfrak{z}} \geqslant n$ such that

$$\nabla_{\hat{Y}/n}X(\mathfrak{z}) = \nabla_{\hat{Y}/n}(\nabla_{\mathfrak{z}}X)(\kappa) = \nabla_{m_{\mathfrak{z}}/n}(\nabla_{\mathfrak{z}}X)(\kappa) = \nabla_{m_{\mathfrak{z}}/n}X(\mathfrak{z}),$$

as required. □

**8.2. Remark.** I believe that with some greater care on how the bound $m_{\mathfrak{z}}$ depends on $\mathfrak{z}$, one should be able to show that in fact $\nabla_{\hat{Y}/n}X$ is strongly formal. Since we may no longer have the required strong Artin Approximation estimate, I do not know whether this result generalizes to arbitrary limit points, that is to say, is $\nabla_{\mathfrak{y}/\mathfrak{x}}X$ a formal motif, for limit points $\mathfrak{x} \leqslant \mathfrak{y}$. The first case to look at is when $\mathfrak{x}$ is a fat point and $\mathfrak{y}$ is bounded (but not a formal completion).

## 9. Motivic generating series

Although we can work in greater generality, we will assume once more that our base scheme is an algebraically closed field $\kappa$, and, unless noted, we will work over the localized Grothendieck ring $\mathbf{G} := \mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa^{\mathrm{str}})_{\mathbb{L}}$ of strongly formal motives. We can similarly extend the theory to arbitrary formal motives, but except possibly for Poincaré series (see below), this does not seem to be necessary.

**Motivic Igusa-zeta series.** For any $\kappa$-scheme $X$ and any closed germ $(Y, P)$, we define the *motivic Igusa-zeta series of $X$ along the germ* $(Y, P)$ as the formal power series

$$\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(X) := \sum_{n=1}^{\infty} \mathbb{L}^{-d \cdot j^n_P(Y)} [\nabla_{J^n_P Y} X]\, t^n$$

in $\mathbf{G}[[t]]$, where $d$ is the dimension of $X$ and $j^n_P(Y)$ the length of the $n$-th co-jet $J^n_P Y$ (which is also equal to the Hilbert-Samuel polynomial of $\mathcal{O}_{Y,P}$ for large $n$). This definition generalizes the one in [1] or [3, §4], where $Y$ is just the germ of a point on a line (and the classes are taken inside the classical Grothendieck ring).

**9.1. Theorem.** *If $X$ is a smooth $d$-dimensional variety and $(Y, P)$ a closed germ, then*

$$\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(X) = \frac{[[X]]\, t}{1-t},$$

*over* $\mathbf{G}$.

*Proof.* Recall that $[[X]] = [X]/\mathbb{L}^d$ is the weightless class of $X$ defined in (30). Since $X$ is smooth, we have

$$[\nabla_{J^n_P Y} X] = [X]\mathbb{L}^{d(j^n_P(Y)-1)} \tag{34}$$

by Theorem 4.14, from which the assertion follows easily. □

With aid of (34) applied to affine space (or (19)), we can write down a more suggestive formula for the motivic Igusa-zeta series

$$\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(X) := \sum_{n=1}^{\infty} \nabla_{J^n_P Y}[[X]]\, t^n. \tag{35}$$

To discuss motivic rationality, we have to keep in mind that $\mathbf{G}$ is most likely not a domain, nor even reduced (as neither is $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$). Let $\Sigma$ be the multiplicative subset consisting of products of units of the form $u - v\mathbb{L}^a t^b$ with $u, v \in \kappa$ and $u \neq 0$, and with $a, b \in \mathbb{Z}$ and $b \geqslant 1$. We call $f \in \mathbf{G}[[t]]$ *strongly rational*, if it is of the form $P/s$ with $P \in \mathbf{G}[t]$ and $s \in \Sigma$. In particular, by the previous result, the motivic Igusa-zeta series of a smooth variety is strongly rational.

**9.2. Question.** When is the motivic Igusa-zeta series $\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(X)$ of a $\kappa$-scheme $X$ along an arbitrary closed germ $(Y, P)$ strongly rational over $\mathbf{G}$?

More generally, given any strongly formal motif $\mathfrak{X}$ on a $d$-dimensional $\kappa$-scheme $X$, we define its Igusa-zeta series along the germ $(Y, P)$ as the formal power series

$$\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(\mathfrak{X}) := \sum_{n=1}^{\infty} \nabla_{J^n_P Y}\left(\frac{[\mathfrak{X}]}{\mathbb{L}^d}\right) t^n$$

and ask about its strong rationality.

**9.3. Corollary.** *Given an irreducible scheme $X$ and a closed germ $(Y, P)$, its motivic Igusa zeta series is strongly rational if and only if $\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(\widehat{X}_{sing})$ is strongly rational, where $\widehat{X}_{sing}$ is the formal completion of $X$ along its singular locus.*

*Proof.* Let $U$ be the regular locus of $X$, so that $U$ is either empty, in which case the statement trivially holds, or otherwise, it has the same dimension $d$ as $X$. By [17, Proposition 7.1], we have an equality $[X] = [U] + [\widehat{X}_{\mathrm{sing}}]$ in $\mathbf{G}$. Diving by $\mathbb{L}^d$ and taking the jet operator $\nabla_n := \nabla_{J^n_P Y}$ yields

$$\nabla_n[[X]] = \nabla_n[[U]] + \nabla_n[\frac{\widehat{X}_{\text{sing}}}{\mathbb{L}^d}]$$

Multiplying with $t^n$ and summing over all $n$ then yields an identity of motivic series

$$\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(X) = \mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(U) + \mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(\widehat{X}_{\text{sing}}). \tag{36}$$

By Theorem 9.1 the first series on the right hand side is strongly rational, proving the claim. □

Building upon this phenomenon, let us define an equivalence relation on $\mathbf{G}[[t]]$ by calling $f \approx g$, if there exists $s, t \in \Sigma$ such that $sf - tg$ is strongly rational. In particular, if $f \approx g$ and one is strongly rational, then so is the other. We can now weaken the requirement in Question 9.2, by asking when two schemes have $\approx$-equivalent motivic Igusa-zeta series. An identity like [17, (28)], namely $[X] = [C] + \mathbb{L}_*$ in $\mathbf{G}$, where $X$ is the union of two lines given by $xy = 0$ and $C$ is the nodal curve with equation $y^2 = x^3 + x^2$, shows that both curves have $\approx$-equivalent motivic Igusa-zeta series by applying the jet operator to this identity and using Theorem 9.1.

**9.4. Weighted motifs.** Example 5.3 implies that not every motivic Igusa-zeta series can be rational: with the notation of loc. cit., the Igusa-zeta series along $(Y, P)$ requires us to use the factor $\mathbb{L}^{-dj^n_P(Y)}$ whereas along $(Z, Q)$, we need $\mathbb{L}^{-(dl+a)j^n_Q(Z)}$. In case $a > 0$ and $(Y, P)$ has dimension at least two, the rationality of the first precludes that of the second, in view of the extra factor $\mathbb{L}^{aj^n_Z(Q)}$, since these exponents grow at least quadratically in $n$, and hence can never appear in a rational function (see also (38) below). To circumvent this, we introduce the *jet of a weighted motif* as follows. Given a fat point $\mathfrak{z}$ of length $l$, let $\nabla_{\mathfrak{z}}$ act on $\mathbf{G} \oplus \mathbb{Q}$ as

$$\nabla_{\mathfrak{z}}[\mathfrak{X}, q] := [\nabla_{\mathfrak{z}}\mathfrak{X}] \cdot \mathbb{L}^{[ql]} \tag{37}$$

for any strongly formal motif $\mathfrak{X}$ and any rational number $q$. Thus, we can rewite (35) also as

$$\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(X) := \sum_{n=1}^{\infty} \nabla_{J^n_P Y}[X, -d]t^n$$

With this notation, we define a *weighted motivic Igusa-zeta series* for any weighted strongly formal motif $(\mathfrak{X}, q)$ as

$$\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(\mathfrak{X}, q) := \sum_{n=1}^{\infty} \nabla_{J^n_P Y}[\mathfrak{X}, q - d]t^n$$

and ask for which values of $q$ is the series strongly rational. We postulate, in view of Conjecture 5.2, that $-q$ should be equal to $\delta_{(Y,P)}(X)$, the asymptotic defect, at least when $(Y, P)$ has dimension two or higher. Continuing with Example 5.3, we have, for any $q \in \mathbb{Q}$, an equality of weighted motivic Igusa-zeta series

$$\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(X, q) = \mathrm{Igu}^{\mathrm{mot}}_{(Z,Q)}(\nabla_{\mathfrak{z}}X, ql + a). \tag{38}$$

**9.5. Remark.** There is another possible solution to avoid quadratic or higher powers of $\mathbb{L}$ to appear in the general term of the motivic Igusa-zeta series, by making each term weightless (in the sense of dimension as given by (30)). Namely, define the *weightless motivic Igusa-zeta series*

$$\mathrm{Igu}^{\mathrm{w}}_{(Y,P)}(X) := \sum_{n=1}^{\infty} [[\nabla_{J^n_P Y} X]]\, t^n \tag{39}$$

for any $\kappa$-scheme $X$ and any closed germ $(Y, P)$. Conjecture 5.2 is then equivalent with the existence of some $q$ for which the difference $\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(X, q) - \mathrm{Igu}^{\mathrm{w}}_{(Y,P)}(X)$ is a polynomial (it is zero for $X$ smooth and $q = 0$). When is (39) strongly rational?

**9.6. Example.** The present point of view even gives interesting new results over the classical Grothendieck ring, since we may take the image of the motivic Igusa zeta series in $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$. Continuing with the calculations made in Example 4.7, let $m$ be a positive integer and consider the image of $\mathrm{Igu}(\mathfrak{l}_m) := \mathrm{Igu}^{\mathrm{mot}}_{(\mathbb{L},O)}(\mathfrak{l}_m)$ over the localized classical Grothendieck ring $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)_{\mathbb{L}}$. This amounts to taking the reduced scheme underlying each jet scheme $\nabla_{\mathfrak{l}_n}\mathfrak{l}_m$, and as shown above, this reduction is just $\mathbb{L}^{n-\lceil \frac{n}{m}\rceil}$. Write $n = sm - r$ for some unique $s \geqslant 1$ and $0 \leqslant r < m$, so that $\lceil \frac{n}{m}\rceil = s$. Over $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)_{\mathbb{L}}$, we have

$$\mathrm{Igu}(\mathfrak{l}_m) = \sum_{r=0}^{m-1}\sum_{s=1}^{\infty} \mathbb{L}^{sm-r-s} t^{sm-r} = \frac{\sum_{r=0}^{m-1}(\mathbb{L}t)^{-r}}{(1-\mathbb{L}^{m-1}t^m)}.$$

In particular, whenever Question 4.15 holds affirmatively, the image of the motivic Igusa zeta series of the fat point would be strongly rational over the classical Grothendieck ring. Skipping the easy calculations, where one uses that jets commute with products, we have for instance that

$$\mathrm{Igu}(\mathfrak{z}) = \frac{t + \mathbb{L}^2 t^2}{(1-\mathbb{L}^2 t^2)}$$

where $\mathfrak{z}$ is the fat point $\mathfrak{l}_2^2 (= \mathfrak{l}_2 \times \mathfrak{l}_2)$ with coordinate ring $\kappa[x, y]/(x^2, y^2)$. Similarly, one calculates that $\mathrm{Igu}(\mathfrak{l}_a \times \mathfrak{l}_b)$ is strongly rational with denominator dividing $1 - \mathbb{L}^{2ab-a-b}t^{ab}$ (the exact denominator is given by diving both exponents by the greatest common divisor of $a$ and $b$); and $\mathrm{Igu}(\mathfrak{l}^d_m)$ has denominator $(1-\mathbb{L}^{d(m-1)}t^m)$. However, the reduced Igusa zeta-series does not characterize the fat point uniquely, as can be seen from the next result:

**9.7. Lemma.** *Let $m \geqslant 2$ and suppose $\mathfrak{z}$ is a closed subscheme of $\mathfrak{l}^d_m$ given by equations of order at least $m$. Then $\nabla_{\mathfrak{l}_n}\mathfrak{z}$ and $\nabla_{\mathfrak{l}_n}\mathfrak{l}^d_m$ have the same underlying variety, whence the same dimension, for all $n$. In particular,* $\mathrm{Igu}(\mathfrak{z}) = \mathrm{Igu}(\mathfrak{l}^d_m)$.

*Proof.* Let $x, y, \dots$ be the $d$ variables defining $\mathfrak{z}$ and let $1, \dots, \xi^{n-1}$ be the basis of $\mathfrak{l}_n$ with respect to which we calculate jets. By assumption, its ideal of definition contains all powers $x^m, y^m, \dots$ and some additional polynomials of order at least $m$. Since jets commute with products, the minimal prime ideal of $\nabla_{\mathfrak{l}_n}\mathfrak{l}^d_m = (\nabla_{\mathfrak{l}_n}\mathfrak{l}_m)^d$

is equal to $(\tilde{x}_0, \tilde{y}_0, \dots, \tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_s, \tilde{y}_s, \dots)$ with $s = \lceil \frac{n}{m} \rceil$ by Example 4.7. Let $\bar{A}$ be the coordinate ring of the reduction of $\nabla_{\mathfrak{l}_n} \mathfrak{l}^d_m$. Hence the generic jets $\widehat{x}, \widehat{y}, \dots$ all have order at least $s$ in $\xi$ over $\bar{A}$. If $f(x, y, \dots)$ is some additional equation of $\mathfrak{z}$, by assumption of order at least $m$ in the variables $x, y, \dots$, then $f(\widehat{x}, \widehat{y}, \dots)$ has order at least $sm \geqslant n$ in $\xi$ over $\bar{A}$, and hence does not contribute in the calculation of $\nabla_{\mathfrak{l}_n} \mathfrak{z}$, showing that $\bar{A}$ is also the coordinate ring of the reduction of $\nabla_{\mathfrak{l}_n} \mathfrak{z}$. □

We can take this construction even a step further: let $X$ be a $d$-dimensional $\kappa$-scheme, $\mathfrak{X}$ a strongly formal motif on $X$, and $\mathfrak{y}$ a limit point. In order to study $\nabla_{\mathfrak{y}}[[X]]$, we need to understand the asymptotics of the direct system $\nabla_{\mathfrak{z}} X$ with $\mathfrak{z}$ running over all fat points in a point system $\mathbb{Y}$ with limit $\mathfrak{y}$. To which extent is this independent from the choice of point system $\mathbb{Y}$? Suppose $\mathbb{Y}$ is a countable chain $\{\mathfrak{z}_1 \subseteq \mathfrak{z}_2 \subseteq \dots\}$, so that we can build the *motivic Igusa-zeta series of* $\mathfrak{X}$ *along* $\mathbb{Y}$, as the power series over $\mathbf{G}$ given by

$$\mathrm{Igu}^{\mathrm{mot}}_{\mathbb{Y}}(\mathfrak{X}) := \sum_{n=1}^{\infty} \nabla_{\mathfrak{z}_n}[\mathfrak{X}, -d] t^n. \tag{40}$$

We again may ask about its strong rationality (which, of course, will require a certain 'naturality' of the chain) , and how these series for two different point systems $\mathbb{Y}$ and $\mathbb{Y}'$ with the same limit, are related to each other.

**Motivic Hilbert series.** Given a motivic site $\mathbb{M}$, we let $\mathbb{M}_0$ be its restriction to the subcategory of zero-dimensional schemes, that is to say, the union of all $\mathbb{M}|_Z$, where $Z$ runs over all zero-dimensional $\kappa$-schemes. As the product of two zero-dimensional schemes is again zero-dimensional, $\mathbb{M}_0$ is a partial motivic site, and hence has an associated Grothendieck ring $\mathbf{Gr}_0(\mathbb{M}) := \mathbf{Gr}(\mathbb{M}_0)$, called the *Grothendieck ring of* $\mathbb{M}$ *in dimension zero*. There is a natural homomorphism $\mathbf{Gr}_0(\mathbb{M}) \to \mathbf{Gr}(\mathbb{M})$, which in general will fail to be injective, as there are a priori more relations in the latter Grothendieck ring. In particular, applied to (sub-)schemic or (strongly) formal motives, we get the corresponding Grothendieck rings in dimension zero $\mathbf{Gr}_0(\mathbb{Sch}_\kappa)$, $\mathbf{Gr}_0(\mathbb{subSch}_\kappa)$, $\mathbf{Gr}_0(\mathbb{Form}^{\mathrm{str}}_\kappa)$, and $\mathbf{Gr}_0(\mathbb{Form}_\kappa)$.

**9.8. Proposition.** *The schemic Grothendieck ring in dimension zero,* $\mathbf{Gr}_0(\mathbb{Sch}_\kappa)$*, is freely generated, as an additive group, by the isomorphism classes of fat points.*

*In particular, there is a canonical homomorphism* $\ell\colon \mathbf{Gr}_0(\mathbb{Sch}_\kappa) \to \mathbb{Z}$ *extending the length function.*

*Proof.* A zero-dimensional scheme $Z$ is a disjoint union $\mathfrak{z}_1 \sqcup \dots \sqcup \mathfrak{z}_s$ of fat points (in a unique way). Moreover, since any fat point is strongly connected, this unique decomposition in fat points is its schemic decomposition. Hence, by the proof of [17, Theorem 5.7], the image of $[Z]$ under the composition $\mathbf{Gr}_0(\mathbb{Sch}_\kappa) \to \mathbf{Gr}(\mathbb{Sch}_\kappa) \xrightarrow{\delta} \Gamma$ is $\langle \mathfrak{z}_1 \rangle + \dots + \langle \mathfrak{z}_s \rangle$, where $\Gamma$ is the free Abelian group on isomorphism classes of strongly indecomposable $\kappa$-schemes. Since $[Z] = [\mathfrak{z}_1] + \dots + [\mathfrak{z}_s]$

in $\mathbf{Gr}_0(\mathbb{S}\mathrm{ch}_\kappa)$, this composition is an isomorphism. The last assertion is now immediate. □

Let $(X,P)$ be a closed germ over $\kappa$. For $t$ a single variable, we define the *motivic Hilbert series* as the series

$$\mathrm{Hilb}^{\mathrm{mot}}(X,P) := \sum_{n=1}^{\infty} [J_P^n X]\, t^n$$

in $\mathbf{Gr}_0(\mathbb{S}\mathrm{ch}_\kappa)[[t]]$. Extending the homomorphism $\ell$ from Proposition 9.8 to the power series ring $\mathbf{Gr}_0(\mathbb{S}\mathrm{ch}_\kappa)[[t]]$ by letting it act on the coefficients of a power series, $\ell(\mathrm{Hilb}_P(X))$ is a rational function in $\mathbb{Z}[[t]]$ by Hilbert-Samuel theory (it is the first difference of the classical Hilbert series of $X$ at $P$). However, $\mathrm{Hilb}^{\mathrm{mot}}(X,P)$ will in general not be (strongly) rational, even over $\mathbf{G}$, as is already clear from taking $(X,P)=(\mathbb{L},O)$. Although no longer specializing to a classical series, we may also consider the more general series

$$\mathrm{Hilb}^{\mathrm{mot}}(X,x) := \sum_{n=0}^{\infty} [J_x^n X]\, t^n$$

where $x$ is any point on $X$ (not necessarily closed).

**9.9. Theorem.** *Let $\kappa$ be an algebraically closed field of cardinality $2^\gamma$ for some infinite cardinal $\gamma$ (under the Generalized Continuum hypothesis this means any uncountable algebraically closed field). The assignment $\widehat{X}_P \mapsto \mathrm{Hilb}^{mot}(X,P)$ is a complete invariant in the sense that for closed germs $(X,P)$ and $(Y,Q)$ over $\kappa$, their completions $\widehat{X}_P$ and $\widehat{Y}_Q$ are abstractly isomorphic (that is to say, over $\mathbb{Z}$) if and only if they have the same motivic Hilbert series in $\mathbf{Gr}_0(\mathbb{S}\mathrm{ch}_\kappa)$.*

*Proof.* Immediate from Proposition 9.8 and the classification results [16, Theorem 1 and §8.3]. □

**Motivic Hilbert-Kunz series.** Assume that $\kappa$ has characteristic $p$. Recall that for a given closed subscheme $Y \subseteq X$, we defined in §3 its Frobenius transform in $X$ as the pull-back $\mathbf{F}_X^* Y := \mathbf{F}_* X \times_X Y$ of $Y$ along $\mathbf{F}_X$. We may also take the pull-back with respect to the powers $\mathbf{F}_X^n$ of the Frobenius, yielding the *$n$-th Frobenius transform* $\mathbf{F}_X^{n*} Y$. If $Y$ has dimension zero, then so does any of its Frobenius transforms, and so the following series, called the *motivic Hilbert-Kunz series*,

$$\mathrm{HK}_Y^{\mathrm{mot}}(X) := \sum_{n=1}^{\infty} [\mathbf{F}_X^{n*} Y]\, t^n$$

is a well-defined series in $\mathbf{Gr}_0(\mathbb{S}\mathrm{ch}_\kappa)[[t]]$. Taking the length function $\ell$ yields the classical Hilbert-Kunz series, of which not too much is known (one expects it to be rational). Of course, we could also take $Y$ to be of higher dimension, and get the corresponding motivic Hilbert-Kunz series in $\mathbf{Gr}(\mathbb{S}\mathrm{ch}_\kappa)[[t]]$.

Instead of transforms we could take iterated Frobenius motives $\mathfrak{F}_Y^n := \nabla_{\mathbf{F}^n} Y$ given as the image sieve of the $n$-th relative Frobenius $\mathbf{F}_{\mathbb{L}^s}^n \times \mathbf{1}_Y$, for some closed immersion $Y \subseteq \mathbb{L}^s$, in case $Y$ is affine, and by glueing for the general case, yielding a series

$$\mathrm{Fr}^{\mathrm{mot}}(Y) := \sum_{n=1}^{\infty} [\nabla_{\mathbf{F}^n} Y]\, t^n$$

in $\mathbf{Gr}_0(\mathbb{S}\mathrm{ch}_\kappa)[[t]]$. Note that $\mathfrak{F}^n_Y(\kappa) = Y(\mathbf{F}^n\kappa) = Y(\kappa)$ by Theorem 3.14, so that this series becomes the rational function $[Y]/(1-t)$ in $\mathbf{Gr}(\mathbb{V}\mathrm{ar}_\kappa)$.

**Motivic Milnor series.** Let $(Y, P)$ be a closed germ with formal completion $\widehat{Y}_P$, so that $\widehat{Y}^\circ_P$ is a strongly formal motif approximated by its co-jets. However, this is not the only way to locally approximate it with schemic submotives. Given a system of parameters $\xi_1, \dots, \xi_e$ in $\mathcal{O}_{Y,P}$ (that is to say, a tuple of length $e = \dim(\mathcal{O}_{Y,P})$ generating an ideal primary to the maximal ideal), let $\mathfrak{y}_n$ be the fat point with coordinate ring $B_n := \mathcal{O}_{Y,P}/(\xi_1^n, \dots, \xi_e^n)\mathcal{O}_{Y,P}$, and $j_{\mathfrak{y}_n} \colon \mathfrak{y}_n \to \operatorname{Spec}\kappa$ the canonical morphism. The reader can check that given a fat point $\mathfrak{z}$, there exists some $n$ such that $\mathfrak{y}_n(\mathfrak{z}) = \widehat{Y}_P(\mathfrak{z})$, that is to say, $\widehat{Y}_P$ is the limit point corresponding to the point system $\{\mathfrak{y}_n\}_n$ (see §7), called the *deformations of $Y$ with respect to the system of parameters $\xi_1, \dots, \xi_e$*. Recall that by the Monomial Theorem, the element $(\xi_1 \cdots \xi_e)^{n-1}$ is a non-zero element in the socle of $B_n$ (meaning that the ideal it generates has length one).

Let $X \subseteq \mathbb{L}^{d+1}$ be the ($d$-dimensional) hypersurface with equation $f(x) = 0$ and let $X_n \subseteq \mathbb{L}^{d+1}_{\mathfrak{y}_n}$ be the deformed hypersurface with equation $f(x) - (\xi_1 \cdots \xi_e)^{n-1} = 0$. In other words, it is the general fiber in the family $W_n \subseteq \mathbb{L}^{d+2}_{\mathfrak{y}_n}$ over the last coordinate $z$, given by the equation $f(x) - z(\xi_1 \cdots \xi_e)^{n-1} = 0$, whereas the special fiber is just the base change $j^*_{\mathfrak{y}_n} X$. We define the *$n$-th order Milnor fiber of $X$ along the germ* $(Y, P)$ as the deformed jet space

$$M_n(X) := \nabla_{j^*_{\mathfrak{y}_n}} X_n.$$

Hence, by Proposition 6.2, with $j_{d,n} \colon \mathbb{L}^{d+2}_{\mathfrak{y}_n} \to \mathbb{L}^{d+2}$ the base change of $j_{\mathfrak{y}_n}$, the specializations of the relative jet scheme are

$$(41) \qquad \begin{aligned} (\nabla_{j^*_{d,n}} W_n)_a &= \nabla_{\mathfrak{y}_n} X && \text{if } a \text{ is the zero section;} \\ &= M_n(X) && \text{otherwise.} \end{aligned}$$

We define the associated Milnor series

$$\mathrm{Mil}^{\mathrm{mot}}_{(Y,P)}(X) := \sum_{n=1}^{\infty} \mathbb{L}^{-d\ell(\mathfrak{y}_n)} [M_n(X)] t^n$$

as a power series in $t$ over $\mathbf{G}$. By (40) and (41), this series can be viewed as a deformation of the motivic Igusa-zeta series along the deformations $\mathfrak{y}_n$. When $(Y, P)$ is the germ of a point on a line, we get the schemic variant of the series introduced by Denef-Loeser et al. Therefore, in view of Question 9.2, we ask when the motivic Milnor series is also strongly rational, and in fact, as a rational function, to have degree zero. Assuming this to be true, we can calculate the limit of this series when $t \to \infty$, and this conjectural limit, presumably in $\mathbf{G}$, will be called the *motivic Milnor fiber* of $X$ along the closed germ $(Y, P)$.

**9.10. Proposition.** *The motivic Milnor series of a smooth hypersurface $X$ along a closed germ is strongly rational and its motivic Milnor fiber is equal to $-[[X]]$.*

*Proof.* With notation as above, the defining equations of $M_n(X)$ are the same as those of $\nabla_{\mathfrak{y}_n} X$, except for the last equation, which has an additional term $-1$ (here we choose a basis for the coordinate ring of $\mathfrak{y}_n$ with last element $(\xi_1 \cdots \xi_e)^{n-1}$, which is possible by the Monomial Theorem). Therefore, the argument in Theorem 4.14 shows that $[M_n(X)] = [X]\mathbb{L}^{d(\ell(\mathfrak{y}_n)-1)}$, where $d$ is the dimension of $X$. It follows that $\mathrm{Mil}^{\mathrm{mot}}_{(Y,P)}(X) = \frac{[[X]]t}{1-t}$, from which the last claim is now also clear. □

For $Y$ of higher dimension, we may again need weighted (see §9.4) or weightless (see Remark 9.5) variants for rationality to happen, and so, setting $[M_n(X, q)] := [M_n(X)] \cdot \mathbb{L}^{\lceil -q\ell(\mathfrak{y}_n)\rceil}$, we define the *weighted motivic Milnor series* as

$$\mathrm{Mil}^{\mathrm{mot}}_{(Y,P)}(X, q) := \sum_{n=1}^{\infty} [M_n(X, q - d)]t^n,$$

and, with the notation from (30), the *weightless motivic Milnor series* as

$$\mathrm{Mil}^{\mathrm{w}}_{(Y,P)}(X) := \sum_{n=1}^{\infty} [[M_n(X)]]t^n.$$

We may again ask whether there is a $q$ such that $\mathrm{Mil}^{\mathrm{w}}_{(Y,P)}(X) - \mathrm{Mil}^{\mathrm{mot}}_{(Y,P)}(X, q)$ is a polynomial.

**Motivic Hasse-Weil series.** Another important generating series in algebraic geometry whose rationality—proven by Dwork in [4]—is postulated to be motivic, is the Hasse-Weil series of a scheme over a finite field $\mathbb{F}_q$: its general coefficient is the number of rational points over the finite extensions $\mathbb{F}_{q^n}$. To turn this into an abstract counting principle, we use the inversion formula relating the number of degree $n$ effective zero cycles on $X$ to the number of rational points in an extension of degree $n$, and observe that the former cycles are in one-one correspondence with the rational points on the $n$-fold symmetric product $X^{(n)}$ of $X$ (given as the quotient of $X^n$ modulo the action of the symmetric group on $n$-tuples). Therefore, following Kapranov [8], we propose the following motivic variant, the *Motivic Hasse-Weil series*:

$$\mathrm{HW}^{\mathrm{mot}}_X := \sum_{n=0}^{\infty} [X^{(n)}]\, t^n,$$

as a power series over $\mathbf{G}$. Kapranov himself proved rationality of the image of this series over $\mathbf{Gr}(\mathbb{V}\mathsf{ar}_{\mathbb{C}})_{\mathbb{L}}$, as well as a functional equation, for certain smooth, projective irreducible curves, but the general case is still open. We know from work of Larsen and Lunts on smooth surfaces ([10]), that, in general, this cannot hold over the Grothendieck ring itself: in [11], they show that rationality over the Grothendieck ring is equivalent with the complex surface having negative Kodaira dimension. It is therefore natural to conjecture the same properties for our motivic variant $\mathrm{HW}_X^{\mathrm{mot}}$.

**Motivic Poincare series.** Given a closed germ $(Y,P)$ with formal completion $\widehat{Y}$, viewed as a limit point, and a $\kappa$-scheme $X$, by Theorem 8.1, we can now define the *motivic Poincaré series of $X$ along* $(Y,P)$ as the formal series

$$\mathrm{Poin}^{\mathrm{mot}}_{(Y,P)}(X) := \sum_{n=1}^{\infty} \mathbb{L}^{-dj^n_P(Y)}[\nabla_{\widehat{Y}/n} X]t^n$$

over $\mathbf{G}$, where $d$ is the dimension of $X$, where $\nabla_{\widehat{Y}/n}X$ denotes the $n$-th order $\widehat{Y}$-extendable jets on $X$ (see §8), and where, as before, $j^n_P(Y)$ is the length of the $n$-th co-jet $J^n_P Y$ (strictly speaking, we do not yet know whether extendable jets are strongly formal, and so we should really work in $\mathbf{Gr}(\mathbb{Form}_\kappa)_{\mathbb{L}}$, but see Remark 8.2). Denef and Loeser proved in [2] that along the germ of a point on the line, the image of this series in the localized classical Grothendieck ring is rational, provided $\kappa$ has characteristic zero. Extending the notation to weighted motifs (see §9.4), by putting $\nabla_{\widehat{Y}/n}[X,q] := [\nabla_{\widehat{Y}/n}X]\mathbb{L}^{\lceil qj^n_P(Y)\rceil}$, and defining the *weighted motivic Poincaré series* as

$$\mathrm{Poin}^{\mathrm{mot}}_{(Y,P)}(X,q) := \sum_{n=1}^{\infty} [\nabla_{\widehat{Y}/n}X, q-d]t^n$$

it is therefore natural to ask:

**9.11. Question.** For any closed germ $(Y,P)$ and any $\kappa$-scheme $X$, for which $q$ is the associated motivic Poincaré series $\mathrm{Poin}^{\mathrm{mot}}_{(Y,P)}(X,q)$ strongly rational over $\mathbf{G}$?

The question is answered in the affirmative by Theorem 9.1 for smooth $X$ with $q=0$, since then $\nabla_{\widehat{Y}/n}X = \nabla_{J^n_P Y}X$ by Theorem 4.14, so that the Poincare series and Igusa-zeta series agree. Similarly, we define the *weightless motivic Poincaré series* as

$$\mathrm{Poin}^{\mathrm{w}}_{(Y,P)}(X) := \sum_{n=1}^{\infty} [[\nabla_{\widehat{Y}/n}X]]t^n.$$

Given $X$ and a formal completion $\widehat{Y}$, we may ask for each $n$, which are the fat points $\mathfrak{z}$ containing the $n$-th co-jet $\mathfrak{j}_n := J^n_O Y$ such that $\nabla_{\widehat{Y}/\mathfrak{j}_n}X \subseteq \nabla_{\mathfrak{z}/\mathfrak{j}_n}X$, that is to say, when are $\widehat{Y}$-extendable jets also $\mathfrak{z}$-extendable? For instance, if $\widehat{Y} = \widehat{\mathbb{L}}$ is the completion of the affine line, then by Theorem 4.14, we can extend along any co-jet of a non-singular germ $(W,O)$, since there exist closed immersions $\mathfrak{j}_n \subseteq J^n_O W \subseteq \mathfrak{j}^d_n$, where $d$ is the dimension of $(W,O)$. However, I do not know whether we can extend along the fat point given by, say, $x^4=y^4=x^3-y^2=0$. For which schemes $X$ can every $\widehat{Y}$-extendable jet be extended along any fat point? This is true if $X$ is smooth, but are there any other cases?

## 10. Motivic integration

Unlike the Kontsevitch-Denef-Loeser motivic integration, we will only define integration on the jet schemes, and not on the arc schemes. We continue to work over the localized Grothendieck ring $\mathbf{G} := \mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa^{\mathrm{str}})_{\mathbb{L}}$, with $\kappa$ an algebraically closed field (we leave the general formal case as an exercise). Before we develop the theory, we discuss a naive approach.

**Motivic measure.** We fix a fat point $\mathfrak{z}$. Our goal is to define a motivic measure $\mu_{\mathfrak{z}}$ on strongly formal motives. To this end, we define

$$\mu_{\mathfrak{z}}(\mathfrak{X}) := [\nabla_{\mathfrak{z}}\mathfrak{X}]$$

in $\mathbf{G}$. In particular, this measure does not depend on the ambient space of $\mathfrak{X}$, only on its germ. Using Theorem 4.8, we can extend the motivic measure to an endomorphism on $\mathbf{G}$. We would like to normalize this measure, with the ultimate goal—which, however, we do not discuss in this paper—to make the comparison between different fat points and take limits. One way to normalize is to make the value weightless (in the sense of dimension; see (30)), by

$$\bar{\mu}_{\mathfrak{z}}(\mathfrak{X}) := [[\nabla_{\mathfrak{z}}\mathfrak{X}]]$$

This, of course, is no longer additive, and the corresponding integral will no longer satisfy Proposition 10.3 below. For this reason, we will normalize differently below, by fixing an ambient space. Following integration theory practice, we would like to say that

$$\int \mathbf{1}_{\mathfrak{X}}\, d_{\mathfrak{z}}x := \mu_{\mathfrak{z}}(\mathfrak{X}) := [\nabla_{\mathfrak{z}}\mathfrak{X}]$$

and extend this to arbitrary step functions. Here, a *step function* would be a formal, finite sum $s = \sum g_i \mathbf{1}_{\mathfrak{X}_i}$ with $g_i \in \mathbf{G}$ and $\mathfrak{X}_i$ strongly formal motives. However, how to interpret this as a function? As usual, we should do this at each fat point $\mathfrak{w}$, and interpret $\mathbf{1}_{\mathfrak{X}}(\mathfrak{w})$ as the characteristic function on $X(\mathfrak{w})$ of $\mathfrak{X}(\mathfrak{w})$, where $X$ is an ambient space of $\mathfrak{X}$. Likewise, provided $X$ is an ambient space for all $\mathfrak{X}_i$, we let $s(\mathfrak{w})$ be the function $X(\mathfrak{w}) \to \mathbf{G}$ associating to a $\mathfrak{w}$-rational point $a \in X(\mathfrak{w})$ the sum of all $g_i$ for which $a \in \mathfrak{X}_i(\mathfrak{w})$. However, the main obstruction is that this point-wise defined function is in general not functorial. The reason is the non-functorial nature of fibers, which in turn stems from the lack of complements in categories—note that the complement of a fiber is the union of the other fibers. To recover functoriality, we work over a subcategory of fat points as in [17, §8], the material of which we quickly review.

**Split points.** Let $\mathbb{F}\mathrm{at}_\kappa^{\mathrm{split}}$ be the category of *split points* over $\kappa$, whose objects are fat points over $\kappa$ and whose morphisms are split epimorphisms $\varphi\colon \mathfrak{z} \to \mathfrak{w}$, that is to say, admitting a *section*, $\sigma\colon \mathfrak{w} \to \mathfrak{z}$ such that $\varphi\sigma$ is the identity on $\mathfrak{w}$. Each structure morphism $\mathfrak{z} \to \operatorname{Spec}\kappa$ is a split epimorphism, and by base change, so is each projection map $\mathfrak{z}\mathfrak{w} \to \mathfrak{w}$. We call a Boolean combination of strongly formal motives a *strongly split-formal motif*. The key result proven in [17, Proposition 8.4]–explaining also the name–, is that a split-formal motif is a functor on $\mathbb{F}\mathrm{at}_\kappa^{\text{split}}$ (it is in general no longer a functor on $\mathbb{F}\mathrm{at}_\kappa$). To make this into a motivic site, the *strongly split-formal* motivic site $\mathbb{F}\mathrm{orm}_\kappa^{\text{sspl}}$, we take as morphisms those natural transformations between strongly split-formal motives that extend to a morphism of (full) strongly formal motives (see [17, §8] for more details). We show in [17, Proposition 8.6] that the Grothendieck ring of $\mathbb{F}\mathrm{orm}_\kappa^{\text{sspl}}$ has not changed: it is equal to the formal Grothendieck ring $\mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa^{\text{str}})$. For strongly formal motives $\mathfrak{X} \subseteq \mathfrak{Y}$ and a fat point $\mathfrak{z}$, we therefore set

$$\nabla_{\mathfrak{z}}\,(\mathfrak{Y}\backslash\mathfrak{X}) := \nabla_{\mathfrak{z}}\mathfrak{Y}\backslash\nabla_{\mathfrak{z}}\mathfrak{X}. \tag{42}$$

Since jets commute with unions, this is well-defined, and extends the jet operator to any strongly split-formal motif. Moreover, after taking classes, (42) gives the correct value in the Grothendieck ring.

**10.1. Lemma.** *If $\mathfrak{X}$ is a strongly split-formal motif and $\mathfrak{z}$ a fat point such that $\nabla_{\mathfrak{z}}\mathfrak{X}$ is empty, then $\mathfrak{X}$ too is empty. In particular, all jet maps are injective on each ambient space.*

*Proof.* By the argument in the proof of [17, Proposition 8.6], we may reduce to the case that $\mathfrak{X}$ is of the form $\mathfrak{Y}\backslash\mathfrak{Z}$ with $\mathfrak{Z} \subseteq \mathfrak{Y}$ (full) strongly formal motives. Let $\mathfrak{w}$ be an arbitrary fat point. The closed immersion $\mathfrak{w} \subseteq \mathfrak{z}\mathfrak{w}$ induces maps $\mathfrak{Z}(\mathfrak{z}\mathfrak{w}) \to \mathfrak{Z}(\mathfrak{w})$ and $\mathfrak{Y}(\mathfrak{z}\mathfrak{w}) \to \mathfrak{Y}(\mathfrak{w})$. Since composing the closed immersion with the (split) projection $\mathfrak{z}\mathfrak{w} \to \mathfrak{w}$ is the identity, the two above maps are surjective. Since $\nabla_{\mathfrak{z}}\mathfrak{X}$ is the empty motif, it has no $\mathfrak{w}$-rational points, that is to say, $\mathfrak{Z}(\mathfrak{z}\mathfrak{w}) = \mathfrak{Y}(\mathfrak{z}\mathfrak{w})$ by (42). Surjectivity then yields that $\mathfrak{Z}(\mathfrak{w}) = \mathfrak{Y}(\mathfrak{w})$, whence $\mathfrak{X}(\mathfrak{w}) = \emptyset$. Since this holds for any fat point $\mathfrak{w}$, we see that $\mathfrak{X}$ is the empty motif.

To prove the last assertion, assume $\nabla_{\mathfrak{z}}\mathfrak{X} = \nabla_{\mathfrak{z}}\mathfrak{Y}$ for $\mathfrak{X}, \mathfrak{Y}$ strongly split-formal motives on a scheme $X$. By what we just proved, $\mathfrak{X}\backslash(\mathfrak{X} \cap \mathfrak{Y})$ and $\mathfrak{Y}\backslash(\mathfrak{X} \cap \mathfrak{Y})$ are both empty, from which the claim now follows. $\square$

From now on, we will work in the category of strongly split-formal motives $\mathbb{F}\mathrm{orm}_\kappa^{\text{sspl}}$, and we view the class of any such motif as an element in the localized Grothendieck ring $\mathbf{G} := \mathbf{Gr}(\mathbb{F}\mathrm{orm}_\kappa^{\text{str}})_{\mathbb{L}}$. Let $\underline{\mathbf{G}}$ be the *constant pre-sieve with values in* $\mathbf{G}$, that is to say, the contravariant functor on the category of split points which associates to any fat point the set $\mathbf{G}$ and to any split epimorphism of fat points the identity on $\mathbf{G}$. Given a morphism, that is to say, a natural transformation, $s\colon \mathfrak{X} \to \underline{\mathbf{G}}$, we define, for each $g \in \mathbf{G}$, the *fiber* $s^{-1}(g)$ as the subfunctor of $\mathfrak{X}$ given at each fat point $\mathfrak{z}$ by the fiber $s(\mathfrak{z})^{-1}(g)$ of $s(\mathfrak{z})\colon \mathfrak{X}(\mathfrak{z}) \to \mathbf{G}$ at $g$. If both $\mathfrak{X}$ and all fibers are strongly split-formal motives, and $s$ has only finitely many non-empty fibers, then we call $s$ a *formal invariant*.[4]

**10.2. Corollary.** *The formal invariants on a strongly split-formal motif $\mathfrak{X}$ form an algebra over* $\mathbf{G}$.

*Proof.* Clearly, any multiple of a formal invariant by an element in $\mathbf{G}$ is again a formal invariant. Let $s, t\colon \mathfrak{X} \to \underline{\mathbf{G}}$ be formal invariants. We have to show that $s+t$ and $st$ are also formal invariants. Functoriality is easily verified, so we only need to show that the fibers are again strongly split-formal motives. Fix a fat point $\mathfrak{z}$, and an element $g \in \mathbf{G}$. A $\mathfrak{z}$-rational point $a \in \mathfrak{X}(\mathfrak{z})$ lies in $(s+t)^{-1}(g)(\mathfrak{z})$ (respectively, in $(st)^{-1}(g)(\mathfrak{z})$), if $s(\mathfrak{z})(a) + t(\mathfrak{z})(a) = g$ (respectively, if $s(\mathfrak{z})(a) \cdot t(\mathfrak{z})(a) = g$). Since $s(\mathfrak{z})$ and $t(\mathfrak{z})$ have finite image, their are only finitely many ways that $g$ can be written as a sum $p+q$ (respectively, a product $pq$), with $p$ in the image of $s(\mathfrak{z})$ and $q$ in the image of $t(\mathfrak{z})$. Hence, the rational point $a$ lies in the intersection $\big(s(\mathfrak{z})^{-1}(p)\big) \cap \big(t(\mathfrak{z})^{-1}(q)\big)$, for one of these finitely many choices of $p$ and $q$. Since a finite union of intersections of strongly split-formal motives is again strongly split-formal, the result follows. □

**Motivic integrals.** Let $X$ be a $\kappa$-scheme, $\mathfrak{z}$ a fat point, and $s\colon \mathfrak{X} \to \underline{\mathbf{G}}$ a formal invariant with $\mathfrak{X}$ a strongly split-formal motif on $X$. We define the *motivic integral* of $s$ on $X$ along $\mathfrak{z}$ as

$$\int s\, d_{\mathfrak{z}}X := \mathbb{L}^{-dl} \sum_{g \in \mathbf{G}} g \cdot [\nabla_{\mathfrak{z}}(s^{-1}(g))], \tag{43}$$

where $d$ is the dimension of $X$ and $l$ the length of $\mathfrak{z}$. Note that the sum on the right hand side of (43) is finite by definition, so that $\int s\, d_{\mathfrak{z}}X$ is a well-defined element in $\mathbf{G}$. At the reduced fat point, $\operatorname{Spec}\kappa$, we drop the subscript in the measure, and so this integral becomes

$$\int s\, dX := \mathbb{L}^{-d} \sum_{g \in \mathbf{G}} g \cdot [s^{-1}(g)].$$

To a strongly formal motif $\mathfrak{Y}$ on $X$, we can associate two invariants. Firstly, the constant map, denoted again $\mathfrak{Y}$, which at each fat point is the constant map sending every rational point to $[\mathfrak{Y}]$. One easily calculates that

$$\int \mathfrak{Y}\, d_{\mathfrak{z}}X = [\mathfrak{Y}] \int d_{\mathfrak{z}}X = [\mathfrak{Y}] \cdot \mathbb{L}^{-dl} \cdot [\nabla_{\mathfrak{z}}X].$$

In particular, $X \mapsto \int dX = [[X]]$ is the weightless class map. It follows from Theorem 4.8 and Proposition 5.5 that the integral $\int \mathfrak{Y}\, d_{\mathfrak{z}}X$ only depends on the classes of $\mathfrak{Y}$ and $X$. Moreover, by our previous discussion $\int d_{\mathfrak{z}}X$ has positive dimension.

[4] I do not know whether the finitude of the non-empty fibers does not already follow from the other assumptions.

Secondly, we define the *characteristic function* $\mathbf{1}_{\mathfrak{Y}}$ of $\mathfrak{Y}$ by the rule that $\mathbf{1}_{\mathfrak{Y}}(\mathfrak{z})$ is the characteristic function of $\mathfrak{Y}(\mathfrak{z})$, that is to say, the formal invariant sending a rational point $a \in X(\mathfrak{z})$ to $1$, if $a \in \mathfrak{Y}(\mathfrak{z})$, and to zero otherwise, for any fat point $\mathfrak{z}$. Any formal invariant can be written as a $\mathbf{G}$-linear combination of characteristic functions, and, in fact, the decomposition

$$s = \sum_{i=1}^{n} g_i \mathbf{1}_{\mathfrak{Y}_i} \tag{44}$$

is unique if the non-empty strongly formal submotives $\mathfrak{Y}_i$ are mutually disjoint (note that then necessarily $\mathfrak{Y}_i = s^{-1}(g_i)$). Therefore, (44) is called the *fiber decomposition* of $s$.

Since $\mathbf{1}_{\mathfrak{Y}}^{-1}(1) = \mathfrak{Y}$, in the notation of weighted motifs (see §9.4), we get

$$\int_{\mathfrak{Y}} d_{\mathfrak{z}}X = \int \mathbf{1}_{\mathfrak{Y}}\, d_{\mathfrak{z}}X = \nabla_{\mathfrak{z}}[\mathfrak{Y}, -d] \tag{45}$$

where we followed the common practice of writing in general

$$\int_{\mathfrak{Y}} s\, d_{\mathfrak{z}}X := \int s \cdot \mathbf{1}_{\mathfrak{Y}}\, d_{\mathfrak{z}}X.$$

In this notation, we have

$$\int s\, d_{\mathfrak{z}}X = \sum_{g \in \mathbf{G}} g \int_{s^{-1}(g)} d_{\mathfrak{z}}X$$

The weighted variant is simply given by scaling as

$$\int s\, d_{\mathfrak{z}}(X, q) := \sum_{g \in \mathbf{G}} g \cdot \nabla_{\mathfrak{z}}[s^{-1}(g), q - d] = \mathbb{L}^{[ql]} \int s\, d_{\mathfrak{z}}X.$$

**10.3. Proposition.** *For each $\kappa$-scheme $X$ and each fat point $\mathfrak{z}$, the motivic integral on $X$ along $\mathfrak{z}$ is a $\mathbf{G}$-linear functional on the $\mathbf{G}$-algebra of formal invariants.*

*Proof.* Motivic integration is clearly preserved under multiplication by a constant $g \in \mathbf{G}$. To prove additivity, we may induct on the number of characteristic functions, and reduce to the case of a sum $s + h\mathbf{1}_{\mathfrak{Z}}$, that is to say, we have to prove

$$\int s + h\mathbf{1}_{\mathfrak{Z}}\, d_{\mathfrak{z}}X = \int s\, d_{\mathfrak{z}}X + \int h\mathbf{1}_{\mathfrak{Z}}\, d_{\mathfrak{z}}X. \tag{46}$$

Let (44) be the fiber decomposition of $s$. Since the fiber decomposition of $s + h\mathbf{1}_{\mathfrak{Z}}$ is then

$$\sum_{i=1}^{n} g_i \mathbf{1}_{\mathfrak{Y}_i - \mathfrak{Z}} + \sum_{i=1}^{n} (g_i + h)\mathbf{1}_{\mathfrak{Y}_i \cap \mathfrak{Z}} + h\mathbf{1}_{\mathfrak{Z} - \mathfrak{Y}}$$

where $\mathfrak{Y}$ is the union of the $\mathfrak{Y}_i$, the left hand side of (46) is

$$\mathbb{L}^{-dl}\Big(\sum_{i=1}^{n} g_i[\nabla_{\mathfrak{z}}(\mathfrak{Y}_i \backslash \mathfrak{Z})] + \sum_{i=1}^{n} (g_i + h)[\nabla_{\mathfrak{z}}(\mathfrak{Y}_i \cap \mathfrak{Z})] + h[\nabla_{\mathfrak{z}}(\mathfrak{Z} \backslash \mathfrak{Y})]\Big),$$

where $d$ and $l$ are respectively the dimension of $X$ and the length of $\mathfrak{z}$. Grouping together the $n+1$ terms with coefficient $h$, and for each $i$, the two terms with coefficient $g_i$, this sum becomes

$$\mathbb{L}^{-dl}(\sum_{i=1}^{n} g_i[\nabla_{\mathfrak{z}}\mathfrak{Y}_i] + h[\nabla_{\mathfrak{z}}\mathfrak{Z}]),$$

since $\nabla_{\mathfrak{z}}$ acts on the Grothendieck ring by Theorem 4.8, and since both $\mathfrak{Y}_i\backslash\mathfrak{Z}$ and $\mathfrak{Z}\backslash\mathfrak{Y}$ are disjoint from $\mathfrak{Y}_i \cap \mathfrak{Z}$. However, this is just the right hand side of (46), and so we are done. □

Let $s\colon \mathfrak{X} \to \underline{\mathbf{G}}$ be a formal invariant on a $\kappa$-scheme $X$. Given an open $U \subseteq X$, let $s|_U$ denote the restriction of $s$ to $\mathfrak{X} \cap U^\circ$. It is easy to see that $s|_U$ is a formal invariant on $U$. Let $U_1, \dots, U_n$ be an open covering of $X$. For each non-empty subset $I \subseteq \{1, \dots, n\}$, let $U_I$ be the intersection of all $U_i$ with $i \in I$. We have the following local formula for the motivic integral (here we call a scheme *equidimensional* if every non-empty open has the same dimension as the scheme):

**10.4. Theorem.** *Let $s\colon \mathfrak{X} \to \underline{\mathbf{G}}$ be a formal invariant on an equidimensional $\kappa$-scheme $X$, let $\mathfrak{z}$ be a fat point, and let $U_1, \dots, U_n$ be an open covering of $X$. Then we have an equality*

$$\int s\, d_{\mathfrak{z}}X = \sum_{\emptyset\neq I\subseteq\{1,\dots,n\}} (-1)^{|I|} \int s|_{U_I}\ d_{\mathfrak{z}}U_I. \tag{47}$$

*Proof.* Given $g \in \mathbf{G}$, one easily verifies that we have an equality of motives

$$(s|_{U_I})^{-1}(g) = s^{-1}(g) \cap U_I^\circ,$$

for each $I \subseteq \{1, \dots, n\}$. Applying the scissor relations to this, we get an identity

$$[s^{-1}(g)] = [\bigcup_{i=1}^{n} (s|_{U_i})^{-1}(g)] = \sum_{\emptyset\neq I\subseteq\{1,\dots,n\}} (-1)^{|I|}[(s|_{U_I})^{-1}(g)]$$

in $\mathbf{G}$. Applying the jet morphism $\nabla_{\mathfrak{z}}$ as per Theorem 4.8, we get

$$[\nabla_{\mathfrak{z}}(s^{-1}(g))] = \sum_{\emptyset\neq I\subseteq\{1,\dots,n\}} (-1)^{|I|}[\nabla_{\mathfrak{z}}((s|_{U_I})^{-1}(g))].$$

Since by assumption all non-empty $U_I$ have the same dimension as $X$ (and, of course, the empty ones do not contribute), the result follows from (43). □

**Relations among motivic series.** Let $\mathbf{G}_0 := \mathbf{Gr}_0(\mathbb{S}\mathrm{ch}_\kappa)$. We define, for any element $\alpha \in \mathbf{G}_0$, the integral

$$\int s\, d_\alpha X := \sum_{i=1}^{s} n_i \int s\, d_{\mathfrak{z}_i} X$$

where $\alpha = n_1[\mathfrak{z}_1] + \cdots + n_s[\mathfrak{z}_s]$ is the unique decomposition in classes of fat points given by Proposition 9.8. We then formally extend this over $\mathbf{G}_0[[t]]$, by treating $t$ as a constant. In this sense, we get, for a closed germ $(Y, P)$, and a $\kappa$-scheme $X$, the following identity of power series:

$$\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(X) = \int d_{\mathrm{Hilb}^{\mathrm{mot}}(Y,P)} X.$$

As before, let us call a power series over $\mathbf{G}_0$ *strongly rational*, if it is of the form $p/s$ with $p \in \mathbf{G}_0[t]$ a polynomial and $s \in \Sigma_0$, where $\Sigma_0$ consist of all non-zero polynomials over $\kappa$. Likewise, given $f, f' \in \mathbf{G}_0[[t]]$, we say $f \approx f'$, if there exist $s, s' \in \Sigma_0$ such that $sf - s'f'$ is strongly rational.

**10.5. Theorem.** *Let* $(Y, P)$ *and* $(Y', P')$ *be two closed germs with the property that* $\mathrm{Hilb}^{mot}(Y, P) \approx \mathrm{Hilb}^{mot}(Y', P')$ *over* $\mathbf{G}_0$*, then for any scheme* $X$*, we have* $\mathrm{Igu}^{\mathrm{mot}}_{(Y,P)}(X) \approx \mathrm{Igu}^{\mathrm{mot}}_{(Y',P')}(X)$ *over* $\mathbf{G}$.

*Proof.* Let $H$ and $H'$ denote the respective motivic Hilbert series $\mathrm{Hilb}^{\mathrm{mot}}(Y, P)$ and $\mathrm{Hilb}^{\mathrm{mot}}(Y', P')$. By assumption

$$sH = s'H' + \frac{p}{t}, \tag{48}$$

with $s, s', t \in \kappa[t]$ non-zero and $p \in \mathbf{G}_0[t]$. In general, if $q \in \kappa(t)$ and $h \in \mathbf{G}_0[[t]]$, then an easy calculation shows that $d_{qh} = q d_h$, meaning that for each scheme $X$ and each formal invariant $s$, we have $\int s\, d_{qh}X = q \int s\, d_h X$, and the claim now follows by additivity applied to (48). ☐

## 11. Appendix: lattice rings

Let $\mathbb{M}$ be a motivic site over an algebraically closed field $\kappa$ and let $X$ be a $\kappa$-scheme. By assumption, $\mathbb{M}|_X$ is a lattice, and so we can define its *lattice group* $\Lambda^X(\mathbb{M})$ as the free Abelian group on $\mathbb{M}$-motives on $X$ modulo the scissor relations

$$\langle \mathfrak{X} \rangle + \langle \mathfrak{Y} \rangle - \langle \mathfrak{X} \cup \mathfrak{Y} \rangle - \langle \mathfrak{X} \cap \mathfrak{Y} \rangle$$

for any two $\mathbb{M}$-motives $\mathfrak{X}$ and $\mathfrak{Y}$ on $X$. In other words, same definition as for the Grothendieck ring, but without the homeomorphism relations. In particular, there is a natural linear map $\Lambda^X(\mathbb{M}) \to \mathbf{Gr}(\mathbb{M})$. We will denote the class of a motif $\mathfrak{X}$ again by $[\mathfrak{X}]$. For each $n$, consider the embedding $\mathbb{M}|_{X^n} \to \mathbb{M}|_{X^{n+1}}$ via the rule $\mathfrak{X} \mapsto \mathfrak{X} \times X^\circ$. One verifies that this induces a well-defined linear map $\Lambda_n := \Lambda^{X^n}(\mathbb{M}) \to \Lambda_{n+1} := \Lambda^{X^{n+1}}(\mathbb{M})$, where $X^n$ is the $n$-fold Cartesian power of $X$. Moreover, the Cartesian product defines a multiplication $\Lambda_m \times \Lambda_n \to \Lambda_{m+n}$, for all $m, n$. Hence $\oplus_n \Lambda_n$ is a graded ring, called the *graded lattice ring* of $\mathbb{M}$ on $X$, and denoted $\Lambda^X_\bullet(\mathbb{M})$. The linear maps $\Lambda_n \to \mathbf{Gr}(\mathbb{M})$ combine to form a ring homomorphism $\Lambda^X_\bullet(\mathbb{M}) \to \mathbf{Gr}(\mathbb{M})$.

We can now state a combinatorial property of the motivic integral:

**11.1. Proposition.** *Over a $\kappa$-scheme $X$ and a fat point $\mathfrak{z}$, we can define for each formal invariant $s\colon \mathfrak{X} \to \underline{\mathbf{G}}$ on $X$ and each $g \in \Lambda^X(\mathbb{F}\mathrm{orm}_\kappa^{str})$, an integral $\int_g s\, d_{\mathfrak{z}} X$, such that if $g$ is the class in $\Lambda^X(\mathbb{F}\mathrm{orm}_\kappa^{str})$ of a strongly formal motif $\mathfrak{Y}$ on $X$, then*

$$\int_g s\, d_{\mathfrak{z}} X = \int_{\mathfrak{Y}} s\, d_{\mathfrak{z}} X.$$

*Proof.* By definition, $g$ is a $\mathbb{Z}$-linear combination of classes of strongly formal motives on $X$, say, of the form $g = n_1[\mathfrak{Y}_1] + \cdots + n_s[\mathfrak{Y}_s]$. Define

$$\int_g s\, d_{\mathfrak{z}} X := \sum_{i=1}^{s} n_i \int_{\mathfrak{Y}_i} s\, d_{\mathfrak{z}} X.$$

To show that this is well-defined, we have to verify this only for scissor relations, that is to say, we have to show that

$$\int_{\mathfrak{Y}} s\, d_{\mathfrak{z}} X + \int_{\mathfrak{Y}'} s\, d_{\mathfrak{z}} X = \int_{\mathfrak{Y}\cup\mathfrak{Y}'} s\, d_{\mathfrak{z}} X + \int_{\mathfrak{Y}\cap\mathfrak{Y}'} s\, d_{\mathfrak{z}} X$$

for $\mathfrak{Y}, \mathfrak{Y}'$ strongly formal motives on $X$. This is immediate from the easily proven identity of characteristic functions

$$\mathbf{1}_{\mathfrak{Y}} + \mathbf{1}_{\mathfrak{Y}'} = \mathbf{1}_{\mathfrak{Y}\cup\mathfrak{Y}'} + \mathbf{1}_{\mathfrak{Y}\cap\mathfrak{Y}'}.$$

□

Using this, we can now show that the lattice rings are not very interesting invariants (and hence only by also taking homeomorphism relations, do we get something significant):

**11.2. Corollary.** *The natural map sending a strongly formal motif on some Cartesian power of $X$ to its class in $\Lambda_\bullet^X(\mathbb{F}\mathrm{orm}_\kappa^{str})$ is injective.*

*Proof.* Note that there are no non-trivial relations among classes of motives on different Cartesian powers of $X$, so after replacing $X$ by one of its Cartesian powers, we may reduce to the case that $\mathfrak{X}$ and $\mathfrak{Y}$ are strongly formal motives on $X$ having the same class in $\Lambda^X(\mathbb{F}\mathrm{orm}_\kappa^{\mathrm{str}})$. By Proposition 11.1, we have

$$\int_{\mathfrak{X}} s\, d_{\mathfrak{z}} X = \int_{\mathfrak{Y}} s\, d_{\mathfrak{z}} X \tag{49}$$

for any formal invariant $s$ on $X$ and any fat point $\mathfrak{z}$. Take $s := \mathbf{1}_{\mathfrak{X}}$. The left hand side of (49) is equal to $\mathbb{L}^{-dl}[\nabla_{\mathfrak{z}}\mathfrak{X}]$ (as an element in $\mathbf{G}$), whereas the right hand side is equal to $\mathbb{L}^{-dl}[\nabla_{\mathfrak{z}}(\mathfrak{X} \cap \mathfrak{Y})]$, where $d$ and $l$ are respectively the dimension of $X$ and the length of $\mathfrak{z}$. Using that $\nabla_{\mathfrak{z}}$ preserves scissor relations, we get $\nabla_{\mathfrak{z}}(\mathfrak{X}\backslash\mathfrak{Y}) = 0$. Hence $\mathfrak{X}\backslash\mathfrak{Y} = \varnothing$ by Lemma 10.1, showing that $\mathfrak{Y} \subseteq \mathfrak{X}$. Replacing the role of $\mathfrak{X}$ and $\mathfrak{Y}$ then proves the other inclusion. □

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