SINGULARITIES FOR THE LAY

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In this paper, we will take a closer look at a very important geometric phenomenon: a singularity. From the dawn of human reasoning, mathematics has played a crucial role in our understanding of the surrounding world. The need to understand and manipulate numbers led to the development of **arithmetic**, and later, in medieval times, to the emergence of **algebra** (Al-Khwarizmi); the need to understand, describe and predict the physical world gave birth to **geometry** (Euclid), which then led to the emergence of **calculus** (Newton). Even in our days, a lot of fundamental research is done in algebra as well as in geometry, and in this paper we'll see one of the many ways in which they meet. In spite of all this research, often by the greatest minds of their times, many questions in these disciplines remain unanswered, while new ones are raised as old ones get answered. Who raises these questions and why? Well, firstly, mathematicians themselves pose questions, in an effort to deepen their understanding of the mathematical world. They often invent some new mathematical object (one that might not even exist in our physical world), and then ask several questions about it. However, questions often also originate from within other sciences, which use mathematics in their description of the physical world. Physical theories, such as relativity theory, quantum physics and string theory, are based on very sophisticated mathematics, which is far from well understood, either physically or mathematically. For instance, what are black holes? And how can we describe them mathematically? The second question is as relevant as the first, because once we have a satisfactory mathematical description, properties can be derived theoretically, which then can be verified experimentally. Another great unsolved puzzle in modern cosmology lies at the very inception of our universe, the so-called Big Bang. Black holes and Big Bangs are often referred to as **singularities**. The sci-fi literature also freely uses this notion, to the extent that "singularity" has become part of our popular vocabulary. So what is a singularity? As this is a concept from geometry, we will have to look first at this field in more detail.

Geometry

We are familiar with geometric objects such as lines and curves, planes and surfaces, cubes and solids, etc. I grouped these objects according to their **dimension**, loosely speaking, the number of distinct directions needed to walk on these objects. For instance on the plane given by the floor you can walk in a north/south direction and then in an east/west direction to reach any point on the floor you want. This shows that the floor is a two-dimensional object, and for the same reason so is any surface. Since our physical world is three-dimensional (north/south, east/west and up/down), we have a hard time to visualize any geometric object of dimension higher than three, like, say, a **hyper-cube**. However, mathematically, there is no difficulty in introducing higher dimensional objects. Modern physics relies heavily on higher dimensional geometry: relativity theory views our time-space continuum as a four-dimensional object; string theory even pictures our world as having ten dimensions, and possibly even eleven!

Singularities

When asked to visualize a geometric object, we may think of a circle or the surface of a ball. These are very smooth objects, and only when we try to come up with some less obvious examples, we might try to picture something more contorted. For instance, let us look at the curve that looks like the number **8**. Everything seems very smooth, except in the one point in the middle of our **8**, where the upper and the lower part meet. What exactly is different at that point? To understand this better, let us first look at any other point on the **8**, say the point on top. Near that point, there is clearly some curvature: the curve is pointing downwards on either side. Let us zoom in at that point. This can be done by enlarging the picture, or by putting it under a microscope. After we zoom in, we see that near the point on the top, there is still downwards curvature, but since we zoomed in, the curvature has diminished: the curve looks flatter than before. Each time we zoom in further, the curvature gets less and less, so that after a couple of zooms at the top of the **8**, the picture becomes really a flat (horizontal) line. We say that the top of the **8** is **smooth** since zooming in flattens the curve near that point. The same is true for any other point other than the midpoint, the only difference being that after repeated zooming in the flattened picture is not necessarily horizontal, but it still is a flat line. In fact the line one gets in this way is precisely the **tangent line** as studied in calculus. Does the same thing happen in the middle of the **8**? When we zoom in the curvature gets less, but we still see an intersection point, looking a little like a curly ***x***. Further zooming in will make the two intersecting curves look straighter, but they keep intersecting each other so that the final picture is like a straight **x**. Zooming in did not result in a complete flattening into the shape of a line and so the midpoint is not a smooth point (geometers call this type of point a **node**).

Let us look at another example, the curve that looks like a **3**. The point on top, for instance, is again smooth, because repeated zooming in will make it look like a horizontal line. But there is one extraordinary point, the midpoint, where the two semicircles of the **3** meet. If we now zoom in at that point, the curvature gets less, but we still see the sudden nick given by the midpoint. The picture is something like a curly ***v*** on its side. Further zooming in results in some straightening, but the nick never goes away, and in fact, gets sharper (this type of point is called a **cusp**). A non-smooth point, like the node in **8** or the cusp in **3**, is called a **singularity**, because zooming in does not flatten the picture. The definition is the same in higher dimensions. For instance, a point on a surface will be smooth, if after repeated zooming in, the picture looks like a flat plane. Whereas for curves we only have a finite number of singularities, in dimensions two and up, there can be infinitely many singularities. In fact, the collection of all singularities is again a geometric object, and hence as a geometric object on its own, can have singularities or not. Here are some further examples. A cone has one singular point, its vertex (the reader should convince himself that this is the only point where zooming in will not flatten the picture). Consider a piece of paper folded in half and then opened up slightly. Every point on the fold is a singularity (the sharp nick caused by the fold will not go away by zooming in), so the singularities form a whole line here. This line itself has no singularities, but if we would have folded the piece of paper instead along the shape of a **3**, then again all points on this fold would be singularities, with the cusp in the middle of the fold being a singularity inside these singularities. It is clear that as the dimension gets higher, the potential complexity of the singularities will increase.

We have been a little careless in the above description when we were talking about the curve that looks like a **3**. This figure has two endpoints, so that it is not a closed curve. As a rule, we should not look at such non-closed curves. For instance, we should view the **3**-curve as part of a larger, closed curve: if we rotate the **3** counter-clockwise over 90°, it can be viewed as the top part of the heart-shaped curve ♥, which is indeed a closed curve. Of course the original midpoint, which is now on top, is still a singularity. In the sequel, we will always close off any geometric object by adding all its endpoints, including those *at infinity* (I will not make this more explicit, appealing to the reader's intuition of what happens at infinity). The technical term for a closed geometric object is a **complete variety**. Moreover, since we want to apply tools from algebra to study these geometric objects and their singularities, we will also assume that they are algebraic, in the sense that they are given by polynomial equations (more on this later).

<u>Resolving singularities</u>

Although many geometric objects do not have singularities (like an ellipse, a sphere, the surface of a vase), many others do. These singularities, however, make it harder to study the geometric object in question: at a singularity, there is no longer a tangent line (or tangent space in higher dimensions), which is an indication that the standard tools from calculus are no longer available. Curiously enough, the singularities of an object are often its most important features and so we cannot dismiss them. For instance, in behavioral sciences, one studies the learning curve of an individual. The most interesting points on these curves are the sharp bents in this curve (i.e., the singularities) that occur when the person has a sudden insight, a moment of revelation. We could paraphrase this by saying that a *flash of genius is a singularity*.

If we need to study singularities in general yet the usual tools from calculus are unavailable, can we perhaps not rid ourselves of them without changing too much the original object? Of course, we cannot just cut them out because we insisted that our objects are closed, i.e., without endpoints, and cutting would just produce such an endpoint. This is also clear from the example of the learning curve: we definitely do not want to erase the flash of genius, but we want to *dissect* it in order to study its peculiarities. So we need a more refined technique, called **resolution of singularities**. The goal is to find a second geometric object, this time without any singularities, that is very similar to our original object. The kind of similarity we envision here is called by geometers **birational equivalence**: we are allowed to cut up the original object in finitely many pieces and then glue them together in an algebraic way to get a smooth object. I am a bit vague here with the term *in an algebraic way*, but it excludes for instance another trivial way of taking out a singularity, namely by knocking it out with a sledge-hammer. Such a mutilation is not algebraic and therefore not allowed. For instance, we cannot just unfold the piece of paper and 'iron' the fold out of it. A very common technique to obtain resolution of singularities is by a process called a **blowing up**. Let's look once more at the **8**-curve. Imagine a rope laid down on the floor in the shape of an **8**. The rope crosses itself precisely in the midpoint. If we now lift one branch of the rope and leave the other on the floor, we get a curve in three-dimensional space (looking a bit like a roller coaster) and this new curve has no longer singularities. In other words, we viewed the **8** inside a three-dimensional space (instead of the original two-dimensional surface of the floor), where we have now more room to maneuver, so that we can pull apart (or blow up) the branches of the curve. Can we always do this blowing up trick to get rid of all singularities? The answer is yes, in all dimensions. This famous result was proven by one of the leading mathematicians of our time, Heisuke Hironaka[2]. Although a blowing up produces a very similar object, we still might loose some valuable information about the original object (the lost information is hidden in what geometers call **cohomology**). Sometimes we are lucky and we hardly loose anything by blowing up: points with this property are called **rational singularities**. The definition is too technical for this survey, as it involves cohomology. It should however be clear that there is a real interest in identifying a rational singularity, for once we know that this is the case, we can blow up and work without singularities, being sure that we have not lost much in the process. The definition of a rational singularity is unfortunately rather abstract and thus hard to verify. Therefore, we will now embark on a quest for a practical characterization of rational singularities. We will need to visit various disciplines of mathematics and learn a little about them, before we can resolve our problem.

Algebraic geometry

There is a very strong link between geometry and algebra, which I will briefly describe since we will need it to conduct our study of rational singularities. The link is made through the concept of **coordinates.** We all have heard about the x/y-plane or **Cartesian** plane, named after the 17th century philosopher and scientist René Descartes: each point in the plane is given by its x and y coordinates. For instance, when giving directions to a cab driver in Manhattan, you describe your destination by two coordinates: *street* and *avenue*. So, when we tell him “twenty-fourth and third”, we are referring to the point on Manhattan’s grid system given by the intersection of twenty-fourth street and third avenue. A geometric object is now represented by all points whose coordinates satisfy certain equations. For instance in the Cartesian plane the equation of the circle with radius one is $x^2+y^2=1$. We said that we would only study geometric objects of an algebraic nature. This refers to the fact that their defining equations must be polynomial (like the above equation of the circle), that is to say, formulated in terms of addition and multiplication only, but for instance not in terms of sine or cosine. This part of geometry is accordingly called **algebraic geometry**. In conclusion, we can study geometric objects via their defining equations, to which all the tools from algebra can be applied.

Power rule

The tool from algebra that will aid us to characterize rational singularities is given by the *power rule*. Since this will play a pivotal role, we need to spend some time on it. Most of us learned in high school about the **double product formula** for squares: $(a+b)^2=a^2+2ab+b^2$. In fact, we were emphatically told that the square of a sum is not the sum of the squares, but that there is an error term given by the double product ($2ab$). Things get even more complicated when we take higher powers: the formula for third powers is $(a+b)^3=a^3+3a^2b+3ab^2+b^3$. Again a very common error is to ignore the inner terms or *error* terms (i.e., the terms $3a^2b$ and $3ab^2$) and pretend that we have a *simplified power rule*, namely to think erroneously that the third power of a sum is the sum of the third powers. Let us take a closer look at these error terms, or more precisely at the numbers in front of them, their so-called **coefficients**. For the second power, the error term has coefficient 2; for the third power, both error terms have coefficient 3. Is this a general pattern? Unfortunately not, as we can already see from the formula for fourth powers: $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$. The error terms are now $4a^3b, 6a^2b^2$ and $4ab^3$, of which the middle one has a coefficient, 6, different from 4. And as we take even higher powers, things probably only can get worse. Well, not entirely: something nice happens for certain powers, namely for those given by a prime number. Recall that a **prime number** is a non-divisible number (more precisely, a number divisible only by one and by itself), like 3 or 17, but not like 4, because 4 is also divisible by 2. Before we look at the behavior of prime powers, let me mention that the general power rule is due to the 11th century Persian mathematician and poet Omar Khayyam (western sources erroneously accredit the rule to Newton). There is also a very nice and easy way to calculate the coefficients of the error terms via the so-called triangle of Pascal, named after the 17th century mathematician and philosopher Blaise Pascal. For instance, if we list all the coefficients in the fourth power rule, we get the sequence 1, 4, 6, 4, 1 (note that the coefficients of the two end terms a^4 and b^4 are both equal to 1). Now, to find the coefficients in the fifth power rule, we just take the consecutive sums 1+4=5, 4+6=10, 6+4=10, 4+1=5, yielding the formula $(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$. From this we can then calculate the coefficients in the sixth power rule (namely, 6, 15, 20, 15 and 6), and so on. Returning to the promised properties of prime powers, let us take the prime number 5 as our example. The coefficients of the error terms are the numbers 5, 10, 10 and 5 we just calculated via Pascal's triangle. Although they are not all equal to 5, each is at least a **multiple** of 5. Note that this is not true for the fourth power rule, since 6 is a coefficient that is not divisible by 4. One can prove that whenever the power is a prime number p, each error term will have a coefficient that is a multiple of p. We express this by saying that the *simplified power rule holds up to multiples of p*. This was first observed by the 17th century lawyer and amateur mathematician Pierre de Fermat (whose famous ‘Last Theorem’ has been a long outstanding problem in mathematics, which only recently was solved by Andre Wiles).

Prime characteristic world

Let us now come back to our quest for a characterization of a rational singularity. In this section, we will exploit the simplified power rule to provide such a characterization, but only *up to multiples of a prime number*. In the next section we will encounter an even stronger power rule with the aid of which we can then remove the modifier ‘up to multiples of a prime number’. Nonetheless, even results ‘up to multiples of a prime number’ are quite interesting and there exists a whole branch of geometry that deals just with these types of problems: this is so-called **prime characteristic** geometry. It has the advantage that in the prime characteristic world the simplified power rule holds. However, the simplified power rule is an algebraic property, whereas a rational singularity is a geometric concept. To turn this algebraic rule into a geometric property, we must view the process of taking powers as an operation performed on geometric objects (again the transition between algebra and geometry is made via coordinates). The ensuing geometric power map is called the **Frobenius map**, after the 19^{th} century mathematician Ferdinand Frobenius. The next result shows that there is a strong relationship in the prime characteristic world between the Frobenius map and the notion of a singularity; the statement is quite reminiscent of our previous definition using zooming in, although the words have a slightly different technical meaning, which I will not further explain. Kunz’s Theorem states[3]: *A point on a geometric object in a prime characteristic world is a singularity precisely when the Frobenius map at that point is not* *flat*. So what about *rational* singularities? The answer is provided by a result of Karen Smith[6]; she gives a characterization of rational singularities in the prime characteristic world in terms of the Frobenius map. Although the statement is too technical for this article (it has to do with the absence of Frobenius-invariant subgroups in cohomology), it is a workable one. How can we use this now to solve the problem in the 'real' world?

Infinite power rule

The answer will involve yet another field from mathematics: logic. *Logic* should not be understood here as *the art of formal reasoning*, as it was studied in antiquity by philosophers, but as its modern development into a mathematical discipline. In this new discipline one tries to cope with foundational issues raised by various mathematical paradoxes (leading to the ultimate question: is math correct?) and with issues about infinity (how many infinities are there?). Another branch of modern logic, called **model-theory**, systematically analyzes mathematical structures (**models**) and their relationships. A typical question that one seeks to answer then is to which extent a true statement about one mathematical structure carries over to another. Using model-theory, I showed that since the Frobenius map exists in the prime characteristic world, it must also exist in the real world. In fact, another tool from model-theory, called an **ultraproduct**, even gives a construction of such a Frobenius map as an infinite power. To be more precise, the power ρ is the ultraproduct of all prime numbers (we can think of ρ as a limit of prime numbers), yielding a truly simplified power rule $(a+b)^\rho=a^\rho+b^\rho$. We finally have reached the end of her journey: in analogy with Smith's result, I formulated a new criterion for a singularity to be rational in terms of the 'real' Frobenius[4]. Let me conclude with a couple of benefits of this description.

Quotient singularities and log-terminal singularities

We will look at two important branches in algebraic geometry and discuss the types of singularities that are encountered in these fields. First, there is a great deal of research done on getting a complete list, or *classification*, of all smooth geometric objects. Put differently, one seeks to answer the question how two geometric objects without singularities can be told apart. This is an extremely difficult question, and answers are known only in small dimensions. Nonetheless, geometers have made some estimated guesses for arbitrary dimensions; this is the so-called **Mori program** (named after the Japanese mathematician Shigefumi Mori). Strangely enough, it appears that in order to describe all the geometric objects without singularities, one has to include in one's study also geometric objects with certain mild singularities, called **log-terminal singularities**. These are rational singularities with even nicer properties and are perhaps the "least harmful" of all singularities. Again a definition is outside the scope of this article, but it involves detailed knowledge of the blowing up that resolves the singularity.

Another branch of geometry tries to understand not just one geometric object, but a whole collection of related geometric objects. For instance, what can we say about the collection of all smooth curves? By a clever encoding, one can often turn such a collection into a single geometric object (the technical term is a **moduli space**). To understand this sort of self-reference—which is quite common in mathematics–, think of a catalog of books, which itself is a book. For instance, in the case of curves, there is a number associated to the curve, called its **genus** g, and all curves of a fixed genus g (at least two) form a geometric object of dimension $3(g-1)$. Unfortunately, even if the collection consists of smooth geometric objects, the geometric object associated to the entire collection will in general have singularities. The type of singularity that thus occurs is called a **quotient singularity**. We also find quotient singularities in orbit spaces and symmetry spaces, and it is fair to say that they are ubiquitous in geometry. So how bad a singularity is a quotient singularity? It was proven by Jean-Francois Boutot[1], using deep theorems, that they are rational (recall that this implies that one does not loose much information after blowing up). The new theory of the Frobenius map that I described in the previous section, not only gives a new and substantially easier proof of this important result, but gives the following stronger result[5], which provides a link between the two important types of singularities that we encountered in this section: *every quotient singularity is log-terminal*.

Conclusion

We started our trip with an informal discussion of an intriguing geometric phenomenon, a **singularity**. We saw that although singularities are not so easy to analyze, it is imperative that we understand them better, for they often carry very valuable information. There is a process that resolves these singularities, namely **blowing up**, but inevitably results in some loss of information. We identified a class of singularities for which this loss is minimal, the **rational singularities**. In order to find a characterization of the latter, we had to turn to a problem in algebra about powers. There we found that prime numbers play a distinguished role: they allow for a simplified power rule if we are willing to work in a **prime characteristic** world. This imaginary world, where everything is only taken up to multiples of a prime number, does connect to our real world via **ultraproducts** (a notion from logic). This led to the existence of a privileged map, called the **Frobenius map**, which we then used to characterize among all singularities, the rational ones. We concluded our trip with an application to a class of frequently encountered singularities, the so-called **quotient singularities**, and showed that they are the 'nicest' among all, namely that they are **log-terminal**.

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