LEFSCHETZ EXTENSIONS, TIGHT CLOSURE, AND BIG COHEN-MACAULAY ALGEBRAS

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ABSTRACT. We associate to every equicharacteristic zero Noetherian local ring R a faithfully flat ring extension which is an ultraproduct of rings of various prime characteristics, in a weakly functorial way. Since such ultraproducts carry naturally a non-standard Frobenius, we can define a new tight closure operation on R by mimicking the positive characteristic functional definition of tight closure. This approach avoids the use of generalized Néron Desingularization and only relies on Rotthaus' result on Artin Approximation in characteristic zero. If R is moreover equidimensional and universally catenary, then we can also associate to it in a canonical, weakly functorial way a balanced big Cohen-Macaulay algebra.

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Introduction

In this paper, we investigate when a ring of characteristic zero can be embedded in an ultraproduct of rings of positive characteristic. Recall that an ultraproduct of a family of rings is a sort of 'average' of its members; see §1 for more details. To facilitate the discussion, let us call a ring of characteristic zero a *Lefschetz ring* if it is realized as an ultraproduct of rings of prime characteristic. The designation alludes to an old heuristic principle in algebraic geometry regarding transfer between positive and zero characteristic, which Weil [62] attributes to Lefschetz. A *Lefschetz field* is a Lefschetz ring which happens to be a field. To model-theorists it is well-known that the field $\mathbb{C}$ of complex numbers is Lefschetz. Moreover, any field of characteristic zero embeds into a Lefschetz field. It follows that *any* domain of characteristic zero embeds into a Lefschetz ring, but in doing so, we loose the entire ideal theory of the domain. It is therefore natural to impose that the embedding preserves enough of the ideal structure, leading to:

Question. Given a Noetherian ring R of characteristic zero, can we find a *faithfully flat* ring extension of R which is Lefschetz?

Suppose that R is a ring of characteristic zero which admits a faithfully flat Lefschetz extension D. Hence D is an ultraproduct of a family (D_w) of rings D_w of prime characteristic; infinitely many different prime characteristics must occur. Each D_w can be viewed as a kind of 'reduction modulo p', or *approximation*, of R. Faithful flatness guarantees that the D_w retain enough properties of the original ring. (See §5 below.) For an easy example consider the following criterion for ideal membership in R: given $f_0, \ldots, f_s \in R$ and given $f_{iw} \in D_w$ whose ultraproduct is equal to the image of f_i in D, we have $f_0 \in (f_1, \ldots, f_s)R$ if and only if $f_{0w} \in (f_{1w}, \ldots, f_{sw})D_w$ for almost all w.

The main motivation for posing the above question stems from the following observations. Any ring of prime characteristic p admits an endomorphism which is at the same time algebraic and canonical, to wit, the Frobenius $\mathbf{F}_p \colon x \mapsto x^p$. This has an immense impact on the homological algebra of a prime characteristic ring, as is witnessed by a myriad of papers exploiting this fact. To mention just a few: Peskine-Szpiro [42] on homological conjectures, Hochster-Roberts [32] on the Cohen-Macaulay property of rings of invariants, Hochster [24] on big Cohen-Macaulay algebras and Mehta-Ramanathan [41] on Frobenius splitting of Schubert varieties. This approach has found its culmination in the *tight closure theory* of Hochster-Huneke [27, 28, 35]. (For a more extensive history of the subject, see [35, Chapter 0]; the same book is also an excellent introduction to tight closure theory.)

Hochster and Huneke also developed tight closure in characteristic zero (see [31] or [35, Appendix 1]), but without any appeal to an endomorphism and relying on deep theorems about Artin Approximation and Néron desingularization. Any Lefschetz ring D, however, is endowed with a *non-standard Frobenius* $\mathbf{F}_\infty$, obtained by taking the ultraproduct of the Frobenii on the D_w. The endomorphism $\mathbf{F}_\infty$ acts on the subring R of D, and although it will in general not leave R invariant, its presence makes it possible to generalize the characteristic p functional definition of tight closure to any Noetherian ring R admitting a faithfully flat Lefschetz extension. This was carried out in [55] for the case where R is an algebra of finite type over $\mathbb{C}$. Here we had a canonical choice for a faithfully flat Lefschetz extension, called the *non-standard hull* of R. The resulting closure operation was termed *non-standard tight closure*. Variants and further results can be found in [50, 51, 57, 58, 53].

Let us briefly recall the construction of the non-standard hull of a finitely generated algebra A over a Lefschetz field K, and at the same time indicate the problem in the non-affine case. For ease of exposition assume that K is an ultraproduct of fields K_p of

characteristic p, with p ranging over the set of prime numbers. (See also Proposition 1.4 below.) If A is of the form $K[X]/I$, where I is an ideal of $K[X] = K[X_1, \ldots, X_n]$, and we have already constructed a faithfully flat Lefschetz extension D of $K[X]$, then D/ID is a faithfully flat Lefschetz extension of A. So we may assume $A = K[X]$. There is an obvious candidate for a Lefschetz ring, namely the ultraproduct $K[X]_\infty$ of the $K_p[X]$. In a natural way $K[X]_\infty$ is a K-algebra. Taking the ultraproduct of the constant sequence X_i in $K_p[X]$ yields an element in $K[X]_\infty$, which we continue to write as X_i. By Łos' Theorem (see Theorem 1.1 below), the elements $X_1, \ldots, X_n \in K[X]_\infty$ are algebraically independent over K and hence can be viewed as indeterminates over K. This yields a canonical embedding of $K[X]$ into $K[X]_\infty$. Van den Dries observed that this embedding is faithfully flat [16, 18], thus giving a positive answer to the question above for finitely generated K-algebras.

In [52, §3.3], the Artin-Rotthaus Theorem [3] was used to extend results from the finitely generated case to the complete case. This ad hoc application will be replaced in this paper by constructing a faithfully flat Lefschetz extension for *every* Noetherian local ring of equal characteristic zero. However, for the proof, a stronger form of Artin Approximation is needed, to wit [47]. By the Cohen Structure Theorem, any equicharacteristic zero Noetherian local ring has a faithfully flat extension which is a homomorphic image of a power series ring $K[[X]]$ (where K is as before), so the problem is essentially reduced to $K[[X]]$. There is again a natural candidate for a faithfully flat Lefschetz extension, namely the ultraproduct $K[[X]]_\infty$ of the $K_p[[X]]$. Since $K[X]_\infty$ is a subring of $K[[X]]_\infty$, so is $K[X]$. Moreover, one easily verifies that $K[[X]]_\infty$ with the X-adic topology is complete, that is to say, each Cauchy sequence in $K[[X]]_\infty$ has a limit in $K[[X]]_\infty$. The obstruction in extending the above argument from $K[X]$ to $K[[X]]$ is that the X-adic topology on $K[[X]]_\infty$ is not Hausdorff, and hence these limits are not unique. Therefore, to send $f \in K[[X]]$ to an element in $K[[X]]_\infty$, we must pick a limit in $K[[X]]_\infty$ of the Cauchy sequence (f_n), where $f_n \in K[X]$ is the truncation of f at degree n. It is not at all obvious how to do this systematically in order to get a ring homomorphism $h\colon K[[X]] \to K[[X]]_\infty$. (It is not hard to prove that such an h, once defined, must be faithfully flat.) An example exhibits some of the subtleties encountered: Let us say that a power series $f \in L[[X]]$, where L is a field, *does not involve the variable* X_i if $f \in L[[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]]$. Similarly we say that an element of $K[[X]]_\infty$ *does not involve the variable* X_i if it is the ultraproduct of power series in $K_p[[X]]$ not involving X_i. Using an example of from [46] we explain in §4.33 why there can be no homomorphism $h\colon K[[X]] \to K[[X]]_\infty$ with the property that for arbitrary i, if $f \in K[[X]]$ does not involve the variable X_i, then neither does $h(f)$. (Of course there is always a limit of the f_n in $K[[X]]_\infty$ which has this property.) To circumvent these kinds of problems, we use Artin Approximation to derive the following positive answer to the question posed at the beginning:

Theorem. *For each equicharacteristic zero Noetherian local ring R, we can construct a local Lefschetz ring $\mathfrak{D}(R)$ and a faithfully flat embedding $\eta_R\colon R \to \mathfrak{D}(R)$.*

In fact, the result also holds for semi-local rings. More importantly, $\mathfrak{D}$ can be made *functorial* in a certain way, which is crucial for applications. See Theorem 4.3 for the precise statement.

Organization of the paper. Sections 1–4 of Part 1 contain a proof of the theorem above. The proof will be self-contained except for the use of Rotthaus' result [47]. We also discuss further connections with Artin Approximation and cylindrical approximation. In Section 5

we investigate which algebraic properties are carried over from R to the rings whose ultra-product is $\mathfrak{D}(R)$. The reader who is mostly interested in the applications of the theorem might skip this section at first reading and immediately proceed to Part 2 (referring back to Section 5 whenever necessary).

We apply our main theorem in two ways. First, in Section 6 we define (*non-standard*) tight closure in any equicharacteristic zero Noetherian local ring and prove the basic facts (such as its triviality on regular rings, Colon Capturing and Briançon-Skoda). In contrast with the Hochster-Huneke version from [31] or [35, Appendix 1] we do not have to invoke generalized Néron desingularization. In order for this paper not to become too long, issues such as the existence of test elements, persistence of tight closure, detailed comparison with other tight closure operations, F-rationality and F-regularity will be postponed to a future publication.

Our second application is a direct construction of a balanced big Cohen-Macaulay algebra for each equicharacteristic zero Noetherian local ring, simpler than the one given in [30]. This construction is weakly functorial on the subcategory of equidimensional and universally catenary rings of bounded cardinality. Using non-standard hulls, the second author gave a similar construction for finitely generated algebras over a field [57]. The method, which itself relies on a result of [29], easily extends to the present situation, at least for complete domains with algebraically closed residue field.

Conventions. Throughout, m and n range over the set $\mathbb{N} := \{0, 1, 2, \dots\}$ of natural numbers. By 'ring' we always mean 'commutative ring with multiplicative identity 1'.

Part 1. **Faithfully Flat Lefschetz Extensions**

After some preliminaries on ultraproducts in §1 and on nested rings in §2, in §4 we prove the theorem from the introduction (in the form of Theorem 4.3). The construction of the desired Lefschetz extensions is achieved via cylindrical approximation in equicharac-teristic zero, which is a corollary of Rotthaus' theorem [47], as we explain in §3. In §5 we then discuss the relationship between R and the components of $\mathfrak{D}(R)$.

1. Ultraproducts

Let $\mathcal{W}$ be an infinite set. A *non-principal ultrafilter* on $\mathcal{W}$ is a collection of infinite subsets of $\mathcal{W}$ which is closed under finite intersections and has the property that for any $W \subseteq \mathcal{W}$, either W or its complement $\mathcal{W} \setminus W$ belongs to the collection. (One should think of the subsets W which are in the ultrafilter as 'big' and those not in it as 'small'.) Given an infinite set $\mathcal{W}$, any collection of infinite subsets of $\mathcal{W}$ which is closed under finite intersections can be enlarged to a non-principal ultrafilter on $\mathcal{W}$. (See for instance [33, Theorem 6.2.1].) Applying this to the collection of co-finite subsets of $\mathcal{W}$ implies that on every infinite set there exists at least one non-principal ultrafilter. With a few exceptions we will always consider a fixed ultrafilter on a given infinite set, so there is no need to name the ultrafilter. Henceforth we call a set $\mathcal{W}$ endowed with some non-principal ultrafilter an *ultraset*.

In the remainder of this section we let $\mathcal{W}$ be an ultraset, and we let w range over $\mathcal{W}$. For each w let A_w be a ring. The *ultraproduct* of the family (A_w) (with respect to $\mathcal{W}$) is by definition the quotient of the product $\prod_w A_w$ modulo the ideal $\mathcal{I}_{\text{null}}$ consisting of the sequences almost all of whose entries are zero. Here and elsewhere, a property is said to hold *for almost all indices* if the subset of all w for which it holds lies in the ultrafilter. We

will often denote the ultraproduct of the family (A_w) by

$$(1.0.1) \qquad \operatorname*{ulim}_{w \in \mathcal{W}} A_w := \prod_{w \in \mathcal{W}} A_w \bigg/ \mathcal{I}_{\text{null}}.$$

Sometimes we denote such an ultraproduct simply by A_∞, and we also speak, somewhat imprecisely, of 'the ultraproduct of the A_w' (with respect to $\mathcal{W}$). Given a sequence $a = (a_w)$ in $\prod_w A_w$ we call its canonical image in A_∞ the *ultraproduct* of the a_w and denote it by

$$a_\infty := \operatorname*{ulim}_{w \in \mathcal{W}} a_w.$$

Similarly if $\mathbf{a}_w = (a_{1w}, \ldots, a_{nw}) \in (A_w)^n$ for each $w \in \mathcal{W}$ and $a_{i\infty}$ is the ultraproduct of the a_{iw} for $i = 1, \ldots, n$, then $\mathbf{a}_\infty := (a_{1\infty}, \ldots, a_{n\infty}) \in (A_\infty)^n$ is called the *ultraproduct* of the n-tuples $\mathbf{a}_w$. If all A_w are the same, say equal to the ring A, then the resulting ultraproduct is called an *ultrapower* of A (with respect to $\mathcal{W}$), denoted by

$$A^{\mathcal{W}} := \operatorname*{ulim}_{w \in \mathcal{W}} A.$$

The map $\delta_A \colon A \to A^{\mathcal{W}}$ which sends $a \in A$ to the ultraproduct of the constant sequence with value a is a ring embedding, called the *diagonal embedding* of A into $A^{\mathcal{W}}$. We will always view $A^{\mathcal{W}}$ as an A-algebra via δ_A. Hence if A is an S-algebra (for some ring S), then so is $A^{\mathcal{W}}$ in a natural way.

Let A_∞ and B_∞ be ultraproducts, with respect to the same ultraset $\mathcal{W}$, of rings A_w and B_w respectively. If for each w we have a map $\varphi_w \colon A_w \to B_w$, then we obtain a map $\varphi_\infty \colon A_\infty \to B_\infty$, called the *ultraproduct* of the φ_w (with respect to $\mathcal{W}$), by the rule

$$a = \operatorname*{ulim}_{w} a_w \mapsto \varphi_\infty(a) := \operatorname*{ulim}_{w} \varphi_w(a_w).$$

(The right-hand side is independent of the choice of the a_w such that $a = \operatorname{ulim}_w a_w$.) Almost all φ_w are homomorphisms if and only if φ_∞ is a homomorphism, and the φ_w are injective (surjective) if and only if φ_∞ is injective (surjective, respectively).

These definitions apply in particular to ultrapowers, that is to say, the case where all A_w and B_w are equal to respectively A and B. In fact, we then can extend them to arbitrary S-algebras, for some base ring S. For instance, let A and B be S-algebras, and let $\varphi \colon A \to B$ be an S-algebra homomorphism. The *ultrapower* of φ (with respect to $\mathcal{W}$), denoted $\varphi^{\mathcal{W}}$, is the ultraproduct of the $\varphi_w := \varphi$. One easily verifies that $\varphi^{\mathcal{W}} \colon A^{\mathcal{W}} \to B^{\mathcal{W}}$ is again an S-algebra homomorphism.

The main model-theoretic fact about ultraproducts is called Łoś' Theorem. For most of our purposes the following equational version suffices.

1.1. Theorem (Equational Łoś' Theorem). *Given a system $\mathcal{S}$ of equations and inequalities*

$$f_1 = f_2 = \cdots = f_s = 0, \; g_1 \neq 0, \; g_2 \neq 0, \ldots, g_t \neq 0$$

with $f_i, g_j \in \mathbb{Z}[X_1, \ldots, X_n]$, the tuple $\mathbf{a}_\infty$ is a solution of $\mathcal{S}$ in A_∞ if and only if almost all tuples $\mathbf{a}_w$ are solutions of $\mathcal{S}$ in A_w.

In particular it follows that any ring-theoretic property that can be expressed "equationally" holds for A_∞ if and only if it holds for almost all the rings A_w. For example, the ring A_∞ is reduced (a domain, a field) if and only if almost all the rings A_w are reduced (domains, fields, respectively). All these statements can deduced from Łoś' Theorem using appropriately chosen systems $\mathcal{S}$. For instance, a ring B is reduced if and only if the system $X^2 = 0, X \neq 0$ (in the single indeterminate X) has no solution in B. We leave the details of these and future routine applications of Łoś' Theorem to the reader. An example of a

property which *cannot* be transferred between A_∞ and the A_w in this way is Noetherianity. (However, A_∞ is Artinian of length $\leq l$ if and only if almost all A_w are Artinian of length $\leq l$, see [36, Proposition 9.1].) Also note that if almost all A_w are algebraically closed fields, then A_∞ is an algebraically closed field; the converse is false in general, as [36, Example 2.16] shows.

We refer to [13], [19] or [33] for in-depth discussions of ultraproducts. A brief review by the second author, adequate for our present needs, can be found in [55, §2]. Using induction on the quantifier complexity of a formula, Theorem 1.1 readily implies the "usual" version of Łos' Theorem, stating that in A_∞, the tuple $\mathbf{a}_\infty$ satisfies a given (first-order) formula in the language of rings if and only if almost all $\mathbf{a}_w$ satisfy the same formula (in A_w). In particular, a sentence in the language of rings holds in A_∞ if and only if it holds in almost all A_w. Similarly, if for each w we are given an endomorphism $\varphi_w \colon A_w \to A_w$ of A_w, then its ultraproduct φ_∞ is an endomorphism of A_∞, and a formula in the language of difference rings (= rings with a distinguished endomorphism) holds for the tuple $\mathbf{a}_\infty$ in $(A_\infty, \varphi_\infty)$ if and only if it holds for almost all $\mathbf{a}_w$ in (A_w, φ_w). On occasion, we invoke these stronger forms of Łos' Theorem. (See for instance, [33, Theorem 9.5.1] for a very general formulation.)

The ultraproduct construction also extends to more general algebraic structures than rings. For example, if for each w we are given an A_w-module M_w, we may define

$$M_\infty := \operatorname*{ulim}_{w \in \mathcal{W}} M_w := \prod_{w \in \mathcal{W}} M_w \Big/ \mathcal{M}_{\mathrm{null}}$$

where $\mathcal{M}_{\mathrm{null}}$ is the submodule of $\prod_w M_w$ consisting of the sequences almost all of whose entries are zero. Then M_∞ is a module over A_∞ in a natural way. If the A_∞-module M_∞ is generated by $m_{1\infty}, \ldots, m_{s\infty}$, then the A_w-module M_w is generated by $m_{1w}, \ldots, m_{sw}$, for almost all w. It is possible to formulate a version of Łos' Theorem for modules. Since this will not be needed in the present paper, let us instead illustrate the functoriality inherent in the ultraproduct construction by establishing a fact which will be useful in §4. Suppose that for each $w \in \mathcal{W}$ we are given an A_w-algebra B_w and an A_w-module M_w.

1.2. Proposition. *If M_∞ has a resolution*

$$\cdots \to (A_\infty)^{n_{i+1}} \xrightarrow{\varphi_i} (A_\infty)^{n_i} \xrightarrow{\varphi_{i-1}} (A_\infty)^{n_{i-1}} \to \cdots \xrightarrow{\varphi_0} (A_\infty)^{n_0} \to M_\infty \to 0$$

by finitely generated free A_∞-modules $(A_\infty)^{n_i}$ and B_∞ is coherent, then as B_∞-modules

$$(1.2.1) \qquad \operatorname{Tor}_i^{A_\infty}(B_\infty, M_\infty) \cong \big(\operatorname{Tor}_i^{A_w}(B_w, M_w) \big)_\infty$$

for every $i \in \mathbb{N}$.

Here the module on the right-hand side of (1.2.1) is the ultraproduct of the B_w-modules $\operatorname{Tor}_i^{A_w}(B_w, M_w)$. Before we begin the proof, first note that we may identify the free A_∞-module $(A_\infty)^n$ with the ultraproduct $(A_w^n)_\infty$ of the free A_w-modules A_w^n in a canonical way. Under this identification, if $\mathbf{a}_{1w}, \ldots, \mathbf{a}_{mw}$ are elements of A_w^n, then the A_∞-submodule of $(A_\infty)^n$ generated by the ultraproducts $\mathbf{a}_{1\infty}, \ldots, \mathbf{a}_{m\infty} \in (A_\infty)^n$ of the $\mathbf{a}_{1w}, \ldots, \mathbf{a}_{mw}$, respectively, corresponds to the ultraproduct N_∞ of the A_w-submodules $N_w := A_w \mathbf{a}_{1w} + \cdots + A_w \mathbf{a}_{mw}$ of A_w^n (an A_∞-submodule of $(A_w^n)_\infty$). The canonical surjections $\pi_w \colon A_w^n \to A_w^n/N_w$ induce a surjection $\pi_\infty \colon (A_\infty)^n = (A_w^n)_\infty \to (A_w^n/N_w)_\infty$ whose kernel is N_∞. Hence we may identify $(A_\infty)^n/N_\infty$ and $(A_w^n/N_w)_\infty$.

Proof (Proposition 1.2). The A_∞-linear maps φ_i are given by certain $n_{i-1} \times n_i$-matrices with entries in A_∞. Hence each φ_i is an ultraproduct $\varphi_i = \operatorname{ulim}_w \varphi_{i,w}$ of A_w-linear maps

$\varphi_{i,w} \colon A_w^{n_{i+1}} \to A_w^{n_i}$ with $\ker \varphi_i = (\ker \varphi_{i,w})_\infty$ and $\operatorname{im} \varphi_i = (\operatorname{im} \varphi_{i,w})_\infty$. Hence for given $i > 0$ the complex

$$A_w^{n_{i+1}} \xrightarrow{\varphi_{i,w}} A_w^{n_i} \xrightarrow{\varphi_{i-1,w}} A_w^{n_{i-1}} \to \cdots \xrightarrow{\varphi_{0,w}} A_w^{n_0} \to M_w \to 0$$

is exact for almost all w, by Łos' Theorem. On the other hand, tensoring the free resolution of M_∞ from above with B_∞ we obtain the complex

$$\cdots \to (B_\infty)^{n_{i+1}} \xrightarrow{\psi_i} (B_\infty)^{n_i} \to \cdots \xrightarrow{\psi_0} (B_\infty)^{n_0} \to B_\infty \otimes_{A_\infty} M_\infty \to 0$$

where $\psi_i := 1 \otimes \varphi_i$. (We identify $(B_\infty)^{n_i}$ and $B_\infty \otimes_{A_\infty} (A_\infty)^{n_i}$ as usual, for each i.) Writing each ψ_i as an ultraproduct $\psi_i = \operatorname{ulim}_w \psi_{i,w}$ of B_w-linear maps $\psi_{i,w} \colon B_w^{n_{i+1}} \to B_w^{n_i}$ yields, for given $i > 0$, that $\psi_{i,w} = 1 \otimes \varphi_{i,w}$ for almost all w, hence $\operatorname{Tor}_i^{A_w}(B_w, M_w) \cong \ker \psi_{i-1,w}/\operatorname{im}\psi_{i,w}$ for almost all w. Since B_∞ is coherent, the B_∞-module $\ker \psi_{i-1}$ is finitely generated, and we get

$$\operatorname{Tor}_i^{A_\infty}(B_\infty, M_\infty) = \ker \psi_{i-1}/\operatorname{im}\psi_i \cong \big(\ker \psi_{i-1,w}/\operatorname{im}\psi_{i,w}\big)_\infty.$$

This proves the case $i > 0$ of the proposition. Using the remarks preceding the proof it is easy to show that $B_\infty \otimes_{A_\infty} M_\infty \cong \big(B_w \otimes_{A_w} M_w\big)_\infty$, proving the case $i = 0$. $\qquad\square$

1.3. **Lefschetz rings.** An ultraproduct A_∞ of rings A_w with respect to an ultraset $\mathcal{W}$ will be called *Lefschetz* (with respect to $\mathcal{W}$) if almost all of the A_w are of prime characteristic and A_∞ is of characteristic zero. (The condition on $\operatorname{char}(A_\infty)$ holds precisely if for each prime number p, the set $\{w : \operatorname{char}(A_w) = p\}$ does not belong to the ultrafilter of $\mathcal{W}$.) A Lefschetz field is a Lefschetz ring that happens to be a field; in this case almost all A_w are fields. The following proposition is a well-known consequence of Łos' Theorem. We let p range over the set of prime numbers. As usual $\mathbb{F}_p$ denotes the field with p elements and $\mathbb{F}_p^{\mathrm{alg}}$ its algebraic closure.

1.4**. Proposition.** *There is a* (*non-canonical*) *isomorphism between the field of complex numbers* $\mathbb{C}$ *and an ultraproduct of the* $\mathbb{F}_p^{\mathrm{alg}}$.

Proof. Equip the set $\mathcal{P}$ of prime numbers with a non-principal ultrafilter and let F_∞ be the ultraproduct of the $\mathbb{F}_p^{\mathrm{alg}}$ with respect to the ultraset $\mathcal{P}$. By the remarks following Theorem 1.1, we see that F_∞ is an algebraically closed field. Since l is a unit in F_p, for every prime l distinct from p, it is a unit in F_∞, by Łos' Theorem. Consequently, F_∞ has characteristic zero. The cardinality of F_∞ is that of the continuum; see [13, Proposition 4.3.7]. Any two algebraically closed fields of characteristic zero, of the same uncountable cardinality, are isomorphic, since they have the same transcendence degree over $\mathbb{Q}$. Hence $F \cong F_\infty$. $\qquad\square$

1.5. *Remark.* Note that that the particular choice of non-principal ultrafilter on $\mathcal{P}$ used in the proof above is irrelevant. The same argument may also be employed to show, more generally: every algebraically closed field of characteristic zero of uncountable cardinality 2^λ (for some infinite cardinal λ) is isomorphic to a Lefschetz field F_∞ with respect to $\mathcal{P}$ all of whose components F_p are algebraically closed fields of characteristic p. It follows that every field of characteristic zero can be embedded into a Lefschetz field all of whose components are algebraically closed fields. Moreover, under the assumption of the Generalized Continuum Hypothesis ($2^\lambda = \lambda^+$ for all infinite cardinals λ) every uncountable algebraically closed field of characteristic zero is Lefschetz.

The following class of Lefschetz rings will be of special interest to us:

1.6. Definition. A Lefschetz ring A_∞ (with respect to the ultraset $\mathcal{W}$) will be called an *analytic Lefschetz ring* (with respect to $\mathcal{W}$) if almost all of the A_w are complete Noetherian local rings of prime equicharacteristic with algebraically closed residue field. Let A_∞ and B_∞ be analytic Lefschetz rings. An ultraproduct $\varphi_\infty \colon A_\infty \to B_\infty$ of local ring homomorphisms $\varphi_w \colon A_w \to B_w$ will be called a *homomorphism of analytic Lefschetz rings* (with respect to $\mathcal{W}$).

By Łos' Theorem every analytic Lefschetz ring is a local ring, and every homomorphism of analytic Lefschetz rings is a local homomorphism of local rings. If $A = A_\infty$ is a Lefschetz ring (an analytic Lefschetz ring) with respect to $\mathcal{W}$ and I a finitely generated proper ideal of A, then A/I is isomorphic to a Lefschetz ring (an analytic Lefschetz ring, respectively) with respect to the same ultraset $\mathcal{W}$. Hence if the maximal ideal of the analytic Lefschetz ring A_∞ is finitely generated, then the residue field of A_∞ may be identified with the ultraproduct K_∞ of the residue fields K_w of A_w in a natural way. In particular K_∞ is itself Lefschetz and algebraically closed.

1.7. Example. For fixed n let $A_w := K_w[[X_1, \ldots, X_n]]$ be the ring of formal power series in indeterminates $X_1, \ldots, X_n$ over an algebraically closed field K_w of characteristic $p(w) > 0$. If for every integer $p > 0$, almost all $p(w)$ are $> p$, then the ultraproduct A_∞ of the A_w has characteristic zero and hence is an analytic Lefschetz ring. In this example, A_∞ is a K_∞-algebra in a natural way. In general, if K is a Lefschetz field (with respect to $\mathcal{W}$) and $K \to A$ is a homomorphism of analytic Lefschetz rings (with respect to $\mathcal{W}$), then we call A an *analytic Lefschetz K-algebra* (with respect to $\mathcal{W}$). The analytic Lefschetz K-algebras with respect to $\mathcal{W}$ form a category whose morphisms are the homomorphisms of analytic Lefschetz rings with respect to $\mathcal{W}$ that are also K-algebra homomorphisms.

We will on occasion use the following construction.

1.8. Ultraproducts of polynomials of bounded degree. Let $X = (X_1, \ldots, X_n)$ be a tuple of indeterminates and let B_∞ be the ultraproduct of the polynomial rings $A_w[X]$. Taking the ultraproduct of the natural homomorphisms $\mathbb{Z}[X] \to A_p[X]$ gives a canonical homomorphism $\mathbb{Z}[X]^{\mathcal{W}} \to B_\infty$. We will continue to write X_i for the image of X_i under this homomorphism. On the other hand, A_∞ is a subring of B_∞. Using Łos' Theorem, we see that $X_1, \ldots, X_n$ remain algebraically independent over A_∞, so that we have in fact a canonical embedding $A_\infty[X] \subseteq B_\infty$. Suppose now we are given, for some $d \in \mathbb{N}$ and each w, a polynomial

$$Q_w = \sum_\nu a_{\nu w} X^\nu \in A_w[X] \qquad (a_{\nu w} \in A_w)$$

of degree at most d. Here the sum ranges over all multi-indices $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$ with $d \leq |\nu| := \nu_1 + \cdots + \nu_n$, and as usual X^ν is shorthand for $X_1^{\nu_1} \cdots X_n^{\nu_n}$. Let $a_{\nu\infty} \in A_\infty$ be the ultraproduct of the $a_{\nu w}$ and put

$$Q_\infty := \sum_\nu a_{\nu\infty} X^\nu,$$

a polynomial in $A_\infty[X]$ of degree $\leq d$. (The polynomial Q_∞ has degree d if and only if almost all Q_w have degree d.) We call Q_∞ the *ultraproduct* of the Q_w. This is justified by the fact that the image of Q_∞ under the canonical embedding $A_\infty[X] \subseteq B_\infty$ is the ultraproduct of the Q_w. In contrast, ultraproducts of polynomials of unbounded degree do no longer belong to the subring $A_\infty[X]$.

2. Embeddings and Existential Theories

In this section, we want to address the following question: *given S-algebras A and B, when does there exist an S-algebra homomorphism $A \to B$?* If one is willing to replace B by some ultrapower, then a simple criterion exists (Corollary 2.5 below). Although this does not solve the question raised above, it suffices for showing that a faithfully flat Lefschetz extension exists (see §4). To obtain the desired functoriality, we need a nested version of this result, which we now explain.

2.1. **Nested rings.** A *nested* ring is a ring R together with a *nest* of subrings, that is, an ascending chain of subrings

$$R_0 \subseteq R_1 \subseteq \cdots \subseteq R_n \subseteq \cdots$$

of R whose union equals R. We agree that whenever R is a nested ring, we denote the subrings in the nest by R_n, and we express this by saying that $R = (R_n)$ is a nested ring. Every ring R can be made into a nested ring using the nest with $R_n := R$ for all n. (We say that R is *trivially nested*.)

Let $R = (R_n)$ and $S = (S_n)$ be nested rings. A homomorphism $\varphi \colon S \to R$ is called a *homomorphism of nested rings* if $\varphi(S_n) \subseteq R_n$ for all n. Alternatively, we say that R is a *nested S-algebra* (via φ). Note that in this case, R_n is naturally an S_n-algebra, for every n. An S-algebra homomorphism $R \to R'$ between nested S-algebras R and R' which is a homomorphism of nested rings is called a *homomorphism of nested S-algebras.* If $R \to R'$ is injective, we may identify R with a subalgebra of R', and we refer to this situation by calling R a *nested S-subalgebra* of R'. A bijective homomorphism of nested rings (nested S-algebras) is called an *isomorphism* of nested rings (nested S-algebras, respectively).

2.2. **Example.** Let L be a field and $Y_0, Y_1, \ldots$ an infinite sequence of finite (possibly empty) tuples $Y_n = (Y_{n1}, \ldots, Y_{nk_n})$ of distinct indeterminates, $k_n \in \mathbb{N}$. For each n put

$$S_n := L[[Y_0]][Y_1, \ldots, Y_n],$$
$$R_n := L[[Y_0]][Y_1, \ldots, Y_n]^{\mathrm{alg}},$$
$$A_n := L[[Y_0, \ldots, Y_n]].$$

Here and elsewhere, given a domain D and a finite tuple Y of indeterminates we denote by $D[[Y]]^{\mathrm{alg}}$ the subring of $D[[Y]]$ consisting of all elements which are algebraic over $D[Y]$. (If D is an excellent domain, then $D[[Y]]^{\mathrm{alg}}$ is equal to the Henselization $D[Y]^{\sim}$ of $D[Y]_{1+YD[Y]}$ at the ideal generated by the indeterminates, see [45, p. 126].) We make the subrings $S := \bigcup S_n$, $R := \bigcup_n R_n$ and $A := \bigcup_n A_n$ of $L[[Y_0, Y_1, \ldots]]$ into nested rings with nests (S_n), (R_n) and (A_n), respectively. Then R is a nested S-subalgebra of the nested S-algebra A. (This example will play an important role in §3.)

Let S be a nested ring and V a nested S-algebra with nest (V_n). We say that V is *of finite type* (over S) if each V_n is a finitely generated S_n-algebra, and for some n_0, each V_n with $n \geq n_0$ is the S_n-subalgebra of V generated by V_{n_0}, that is to say, $V_n = S_n[V_{n_0}]$. Choose n_0 minimal with this property. Clearly, all the knowledge about V is already contained in the initial chain $V_0 \subseteq V_2 \subseteq \cdots \subseteq V_{n_0}$, and consequently, we refer to it as the *relevant part* of V, and to n_0 as its *length*.

2.3. **Nested equations and nested algebras of finite type.** In the following let S be a nested ring. A *nested system of polynomial equations* with coefficients from S is a finite

sequence $\mathcal{S}$ of systems of polynomial equations

$$
\begin{array}{rcccrcl}
(2.3.1) & P_{00}(Z_0) & = & \cdots & = & P_{0k}(Z_0) & = & 0, \\
& P_{10}(Z_0, Z_1) & = & \cdots & = & P_{1k}(Z_0, Z_1) & = & 0, \\
& \vdots & & & & \vdots & & \\
& P_{n0}(Z_0, \ldots, Z_n) & = & \cdots & = & P_{nk}(Z_0, \ldots, Z_n) & = & 0,
\end{array}
$$

for some n and $k \in \mathbb{N}$, some tuples $Z_i = (Z_{i1}, \ldots, Z_{ik_i})$ of indeterminates over S (where $k_i \in \mathbb{N}$) and some polynomials $P_{ij} \in S_i[Z_0, \ldots, Z_i]$. Given a nested S-algebra A, a tuple $(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ with $\mathbf{a}_i \in (A_i)^{k_i}$ is called a *nested solution of $\mathcal{S}$ in A* if $P_{ij}(\mathbf{a}_0, \ldots, \mathbf{a}_i) = 0$ for all $i = 0, \ldots, n$ and $j = 0, \ldots, k$. Similarly, given an ideal $\mathfrak{a}$ of the nested S-algebra A, we call $(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ an *approximate nested solution of $\mathcal{S}$ modulo $\mathfrak{a}$ in A*, if $P_{ij}(\mathbf{a}_0, \ldots, \mathbf{a}_i) \equiv 0 \bmod \mathfrak{a}$ for all i, j.

Let V be a nested S-algebra of finite type and let n be the length of its relevant part. For $i \leq n$, choose tuples $\mathbf{a}_i \in V_i^{k_i}$, such that each V_i is generated as an S_i-algebra by $\mathbf{a}_0, \ldots, \mathbf{a}_i$. Let $P_{i1}, \ldots, P_{ik}$ be generators of the kernel of the S_i-algebra homomorphism $S_i[Z_0, \ldots, Z_i] \to V$ given by $Z_0 \mapsto \mathbf{a}_0, \ldots, Z_i \mapsto \mathbf{a}_i$. In particular

$$
V_i \cong S_i[Z_0, \ldots, Z_i]/(P_{i0}, \ldots, P_{ik})S_i[Z_0, \ldots, Z_i].
$$

The system of equations $P_{00} = \cdots = P_{nk} = 0$ form a nested system of polynomial equations with coefficients from S, called a *defining nested system of equations* for V. (It depends on the choice of generators $\mathbf{a}_i$.) Note that the generating tuple $(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ is a nested solution of this system in V. Conversely, any nested system of polynomial equations with coefficients from S together with a nested solution in some nested S-algebra B gives rise to a nested S-subalgebra of B of finite type.

Given an ultraset $\mathcal{U}$ and a nested S-algebra B we consider the ultrapower $B_n^{\mathcal{U}}$ as an S_n-subalgebra of $B^{\mathcal{U}}$ in the natural way. We make the S-subalgebra $\bigcup_n B_n^{\mathcal{U}}$ of the ultrapower $B^{\mathcal{U}}$ into a nested S-algebra by means of the nest $(B_n^{\mathcal{U}})$. We denote this nested S-algebra by $B^{\langle \mathcal{U} \rangle}$. The main result of this section is the following criterion for the existence of a homomorphism of nested S-algebras from a nested S-algebra A to an ultrapower of B.

2.4**. Theorem.** *Let A and B be nested S-algebras. If each S_n is Noetherian, then the following are equivalent:*

(2.4.1) *every nested system of polynomial equations with coefficients from S which has a nested solution in A also has one in B;*

(2.4.2) *for every nested S-subalgebra of finite type V of A, there exists a homomorphism of nested S-algebras $\varphi_V \colon V \to B$;*

(2.4.3) *there exists a homomorphism of nested S-algebras $\eta \colon A \to B^{\langle \mathcal{U} \rangle}$, for some ultraset $\mathcal{U}$.*

Proof. Suppose that (2.4.1) holds, and let V be a nested S-subalgebra of finite type of A. Suppose $V_0 \subseteq \cdots \subseteq V_n$ is the relevant part of V (so that $V_m = S_m[V_n]$ for all $m \geq n$). Let $\mathcal{S}$ be a defining nested system of equations of V and let $(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ with $\mathbf{a}_i \in (A_i)^{k_i}$ be the nested solution in A arising from a generating set of V over S (see §2.3). By assumption, there exists a nested solution $(\mathbf{b}_0, \ldots, \mathbf{b}_n)$ of $\mathcal{S}$ with $\mathbf{b}_i \in (B_i)^{k_i}$ for all i. Hence the S_n-algebra homomorphism $S_n[Z_0, \ldots, Z_n] \to B_n$ given by $Z_i \mapsto \mathbf{b}_i$ for $i = 0, \ldots, n$ factors through an S_n-algebra homomorphism $\varphi_V \colon V_n \to B_n$ with $\varphi_V(V_i) \subseteq B_i$ for all i. Since $V_m = S_m[V_n]$ for $m \geq n$, we can extend this to a homomorphism of nested S-algebras $V \to B$, proving implication (2.4.1) $\Rightarrow$ (2.4.2).

Assume next that (2.4.2) holds. Let $\mathcal{U}$ be the collection of all nested S-subalgebras of finite type of A (an infinite set). For each finite subset $E = \big\{(a_1, n_1), \ldots, (a_k, n_k)\big\}$ of $A \times \mathbb{N}$ let $\langle E \rangle$ be the subset of $\mathcal{U}$ consisting of all nested S-subalgebras $V = (V_n)$ of finite type of A with $a_i \in V_{n_i}$ for all i. Any finite intersection of sets of the form $\langle E \rangle$ is again of that form. Hence we can find a non-principal ultrafilter on $\mathcal{U}$ containing each $\langle E \rangle$, where E runs over all finite subsets of $A \times \mathbb{N}$. For each $V \in \mathcal{U}$, let $\widetilde{\varphi}_V \colon A \to B$ be the map which coincides with φ_V on V and which is identically zero outside V. (This is of course no longer a homomorphism.) Define $\eta \colon A \to B^{\mathcal{U}}$ to be the restriction to A of the ultraproduct of the $\widetilde{\varphi}_V$. In other words,

$$\eta(a) := \operatorname*{ulim}_{V \in \mathcal{U}} \widetilde{\varphi}_V(a) \qquad \text{for } a \in A.$$

It remains to verify that the image of η lies inside $B^{\langle \mathcal{U} \rangle}$ and that the induced homomorphism $A \to B^{\langle \mathcal{U} \rangle}$ is a homomorphism of nested S-algebras. For $a, b \in A_n$, we have for each $V \in \langle \{(a, n), (b, n)\} \rangle$ that $\widetilde{\varphi}_V(a) = \varphi(a)$ and $\widetilde{\varphi}_V(b) = \varphi(b)$ lie in B_n and

$$\widetilde{\varphi}_V(a + b) = \varphi_V(a + b) = \varphi_V(a) + \varphi_V(b) = \widetilde{\varphi}_V(a) + \widetilde{\varphi}_V(b).$$

Since this holds for almost all V, we get that $\eta(a), \eta(b) \in B_n^{\mathcal{U}}$ and $\eta(a + b) = \eta(a) + \eta(b)$. In particular, the image of η lies inside $B^{\langle \mathcal{U} \rangle}$. By a similar argument, one also shows that $\eta(ab) = \eta(a)\eta(b)$ and $\eta(sa) = s\eta(a)$ for $s \in S$. We have shown (2.4.2) $\Rightarrow$ (2.4.3).

Finally, suppose that $\eta \colon A \to B^{\langle \mathcal{U} \rangle}$ is a homomorphism of nested S-algebras, for some ultraset $\mathcal{U}$. Suppose moreover that we are given a nested system $\mathcal{S}$ of polynomial equations with coefficients from S as above, which has a nested solution $(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ in A. Then $\big(\eta(\mathbf{a}_0), \ldots, \eta(\mathbf{a}_n)\big)$ is a nested solution of $\mathcal{S}$ in the nested S-algebra $B^{\langle \mathcal{U} \rangle}$. Using Łoś' Theorem it follows that $\mathcal{S}$ has a nested solution in B. This shows (2.4.3) $\Rightarrow$ (2.4.1). $\quad\square$

Applying the theorem to trivially nested rings we obtain the following partial answer to the question raised at the beginning of this section. It is an incarnation of a model-theoretic principle (originating with Henkin [23]) which has proven to be useful in other situations related to Artin Approximation; for instance, see [5, Lemma 1.4] and [17, Lemma 12.1.3].

2.5. **Corollary.** *Let S be a Noetherian ring and let A and B be S-algebras. The following are equivalent:*

- (2.5.1) *every (finite) system of polynomial equations with coefficients from S which is solvable in A, is solvable in B;*
- (2.5.2) *for each finitely generated S-subalgebra V of A, there exists an S-algebra homomorphism $\varphi_V \colon V \to B$;*
- (2.5.3) *there exists an ultraset $\mathcal{U}$ and an S-algebra homomorphism $\eta \colon A \to B^{\mathcal{U}}$.* $\quad\square$

We finish this sections with some remarks about Theorem 2.4 and its corollary above.

2.6. *Remark.* Only the proof of the implications (2.4.1) $\Rightarrow$ (2.4.2) and (2.5.1) $\Rightarrow$ (2.5.2) used the assumption that each S_n (respectively, S) is Noetherian. These implications do hold without the Noetherian assumption, provided we allow for infinite systems (in finitely many variables) in (2.4.1) and (2.5.1) respectively.

2.7. *Remark.* In the proof of (2.4.2) $\Rightarrow$ (2.4.3) we may replace the underlying set of the ultraset $\mathcal{U}$ by any cofinal collection of nested S-subalgebras of finite type of A.

2.8. *Remark.* We can strengthen (2.4.3) and (2.5.3) by making η canonical, that is to say, independent of the choice of S-algebra homomorphisms φ_V. Let us just give the argument in the non-nested case. Replace the above index set $\mathcal{U}$ by the set $\mathcal{A}$ of all S-algebra homomorphisms $\varphi \colon V \to B$ whose domain V is a finitely generated S-subalgebra of A. Given

a finite subset E of A, let $\langle E \rangle$ be the subset of all $\varphi \in \mathcal{A}$ whose domain contains E. If we assume (2.5.2) and A is not finitely generated, then $\mathcal{A}$ is infinite and no $\langle E \rangle$ is empty, so that we can choose a non-principal ultrafilter on $\mathcal{A}$ which contains all the $\langle E \rangle$, for E a finite subset of A. The remainder of the construction is now the same. Namely, define $\eta \colon A \to B^{\mathcal{A}}$ to be the restriction to A of the ultraproduct of all $\widetilde{\varphi}$, where for each $\varphi \in \mathcal{A}$ we let $\widetilde{\varphi} \colon A \to B$ be the extension by zero of φ. The same argument as above then yields that η is an S-algebra homomorphism.

2.9. *Remark.* We also have criteria for A to *embed* into an ultrapower of B: under the same assumptions as in Corollary 2.5, the following are equivalent:

    (2.9.1) *every* (*finite*) *system of polynomial equations and inequalities with coefficients from S which is solvable in A, is solvable in B;*

    (2.9.2) *given a finitely generated S-subalgebra V of A and finitely many non-zero elements $a_1, \ldots, a_n$ of V there exists an S-algebra homomorphism $V \to B$ sending each a_i to a non-zero element of B;*

    (2.9.3) *there exists an ultraset $\mathcal{U}$ and an embedding $A \to B^{\mathcal{U}}$ of S-algebras.*

In particular, if all the φ_V in (2.5.2) can be taken injective, then so can η in (2.5.3). Similar criteria may be formulated in the general nested case. We leave the proof (which is analogous to the proof of Theorem 2.4) to the reader.

In the next remarks (not essential later) we assume that the reader is familiar with basic notions of model theory; see [13] or [33].

2.10. *Remark.* The language $\mathcal{L}(S)$ of S-algebras (in the sense of first-order logic) consists of the language $\mathcal{L} = \{0, 1, +, -, \cdot\}$ of rings augmented by a unary function symbol $s^{\times}$, for each $s \in S$. We construe each S-algebra as an $\mathcal{L}(S)$-structure by interpreting the ring symbols as usual and $s^{\times}$ as multiplication by s. We can then reformulate (2.5.1) in more model-theoretic terms as:

    (2.5.1)$'$ *B is a model of the positive existential theory of A in the language $\mathcal{L}(S)$.*

Similarly (2.9.1) may be replaced by

    (2.9.1)$'$ *B is a model of the* (*full*) *existential $\mathcal{L}(S)$-theory of A.*

2.11. *Remark.* Suppose that B is $|A|$-saturated (as an $\mathcal{L}(S)$-structure). Then to (2.5.1)–(2.5.3) in Corollary 2.5 we may add the equivalent statement

    (2.5.3)$'$ *There exists an S-algebra homomorphism $A \to B$.*

For a proof see for instance [33, Theorem 10.3.1]. The assumption on B is satisfied if S (and hence $\mathcal{L}(S)$) is countable, A has cardinality at most $\aleph_1$, and B is an ultraproduct of a countable family of S-algebras with respect to a non-principal ultrafilter. (See [13, Theorem 6.1.1].) If, on the other hand, B is $\aleph_0$-saturated, then in Remark 2.9 we may replace (2.9.2) with

    (2.9.2)$'$ *For every finitely generated S-subalgebra V of A there exists an embedding $V \to B$ of S-algebras.*

3. Artin Approximation and Embeddings in Ultraproducts

In this section, K is a field which is the ultraproduct of fields K_p (not necessarily algebraically closed nor of different characteristics) with respect to an ultraset $\mathcal{P}$. In most applications, $\mathcal{P}$ will have as underlying set the set of prime numbers and each K_p will have characteristic p. For a finite tuple $X = (X_1, \ldots, X_n)$ of indeterminates, we put

$$K[[X]]_{\infty} := \operatorname*{ulim}_{p \in \mathcal{P}} K_p[[X]].$$

We start with an important fact about ultraproducts of powers series rings taken from [6, Lemma 3.4]; since we will need a similar argument below (Proposition 4.30), we indicate the proof. The *ideal of infinitesimals* of a local ring $(S, \mathfrak{m})$ is the ideal $\mathrm{Inf}(S) := \bigcap_{d \in \mathbb{N}} \mathfrak{m}^d$ of S. The $\mathfrak{m}$-adic topology on S is separated if and only if $\mathrm{Inf}(S) = 0$, and this is the case if S is Noetherian by Krull's Intersection Theorem.

3.1. **Proposition.** *There is a surjective $K[X]$-algebra homomorphism*

$$\pi \colon K[[X]]_\infty \to K[[X]]$$

whose kernel is $\mathrm{Inf}(K[[X]]_\infty)$.

Proof. We start by defining π. Let $f_\infty \in K[[X]]_\infty$ and choose $f_p \in K_p[[X]]$, for $p \in \mathcal{P}$, whose ultraproduct is f_∞. Write each f_p as

$$f_p := \sum_\nu a_{\nu p} X^\nu$$

with $a_{\nu p} \in K_p$. Here the sum ranges over all multi-indices $\nu \in \mathbb{N}^n$. Let $a_{\nu \infty} \in K$ be the ultraproduct of the $a_{\nu p}$ and define

$$\pi(f_\infty) := \sum_\nu a_{\nu \infty} X^\nu \in K[[X]].$$

It follows from Łos' Theorem that π is a well-defined $K[X]$-algebra homomorphism. Its surjectivity is clear. So it remains to show that the kernel of π is $\mathrm{Inf}(K[[X]]_\infty)$. If $f_\infty \in \mathrm{Inf}(K[[X]]_\infty)$, then by Łos' Theorem, for each $d \in \mathbb{N}$, there is a member U_d of the ultrafilter such that $f_p \in (X_1, \ldots, X_n)^d K_p[[X]]$ for all $p \in U_d$. In particular, for each $\nu \in \mathbb{N}^n$ we have that $a_{\nu p} = 0$, for all $p \in U_{|\nu|+1}$. Therefore $a_{\nu \infty} = 0$, and since this holds for all ν, we see that $f_\infty \in \ker \pi$. The converse holds by reversing the argument. $\qquad\square$

3.2. *Remark.* In fact, we may replace in the above $K[[X]]_\infty$ by its subring $K[X]_\infty$, given as the ultraproduct of the $K_p[X]$. That is to say, π induces a surjective $K[X]$-algebra homomorphism $K[X]_\infty \to K[[X]]$ with kernel equal to the intersection of all $(X_1, \ldots, X_n)^d K[X]_\infty$ for $d \in \mathbb{N}$. Indeed, it suffices to show that π maps $K[X]_\infty$ onto $K[[X]]$. Let us explain this just in case the underlying set of $\mathcal{P}$ is countable and hence, after identification, we may think of it as a subset of $\mathbb{N}$. Given $f = \sum_\nu a_\nu X^\nu \in K[[X]]$, choose $a_{\nu p} \in K_p$ so that their ultraproduct is a_ν and put

$$f_p := \sum_{|\nu| \le p} a_{\nu p} X^\nu \in K_p[X].$$

Then $\pi(f_\infty) = f$, where $f_\infty \in K[X]_\infty$ is the ultraproduct of the polynomials f_p.

3.3. **Artin Approximation.** Recall that a Noetherian local ring $(R, \mathfrak{m})$ is said to satisfy *Artin Approximation* if every system of polynomial equations over R which is solvable in the completion $\widehat{R}$ of R is already solvable in R. In view of Corollary 2.5, this is equivalent with the existence of an ultraset $\mathcal{U}$ and an R-algebra homomorphism

(3.3.1) $$\widehat{R} \to R^{\mathcal{U}}.$$

In fact, if R satisfies Artin Approximation, then R is existentially closed in $\widehat{R}$, that is to say, every system of polynomial equations and inequalities over R which is solvable in $\widehat{R}$ has a solution in R. (Since R is dense in $\widehat{R}$, inequalities, and also congruence conditions, can be incorporated in a system of equations.) Artin proved (in [1] and [2, Theorem 1.10], respectively) that the ring of convergent complex power series $\mathbb{C}\{X\}$ and the ring of algebraic power series $L[[X]]^{\mathrm{alg}}$, with L an arbitrary field, satisfy Artin Approximation.

Artin's Conjecture. A local ring $(R, \mathfrak{m})$ satisfying Artin Approximation is necessarily Henselian, and Artin conjectured that the converse holds if R is excellent. This conjecture was eventually confirmed to be true [43, 60, 61]. In each of these papers, Artin's Conjecture is derived from *generalized Néron Desingularization*, stating that a homomorphism $A \to B$ of Noetherian rings is regular if and only if B is the direct limit of smooth A-algebras. In the development of tight closure in characteristic zero in the sense of Hochster and Huneke [31], this latter theorem plays an essential role. In this paper we give an alternative definition of tight closure relying only on a weaker form of Artin Approximation, to wit, Rotthaus' result [47] on the Artin Approximation property for rings of the form $L[[X]][[Y]]^{\mathrm{alg}}$ with L a field of characteristic zero. (In Theorem 3.15 below, which is not needed anywhere else, we do need generalized Néron Desingularization.)

Strong Artin Approximation. We say that a Noetherian local ring $(R, \mathfrak{m})$ satisfies *Strong Artin Approximation*, if any system of polynomial equations over R which is solvable modulo arbitrary high powers of $\mathfrak{m}$ is already solvable in R. By Corollary 2.5, this amounts to the existence of an ultraset $\mathcal{U}$ and an R-algebra homomorphism

$$(3.3.2) \qquad \prod_{n \in \mathbb{N}} R/\mathfrak{m}^n \to R^{\mathcal{U}}.$$

From (3.3.1) and (3.3.2) it follows that R satisfies Strong Artin Approximation if and only if R satisfies Artin Approximation and $\widehat{R}$ satisfies Strong Artin Approximation. In [6], a very quick proof using ultraproducts is given to show that $L[[X]]$ satisfies Strong Artin Approximation, for every uncountable algebraically closed field L. Using the Cohen Structure Theorem, one then deduces from this and the positive solution of Artin's Conjecture, that every equicharacteristic, excellent, Henselian local ring with an uncountable algebraically closed residue field satisfies Strong Artin Approximation.

Uniform Strong Artin Approximation. Any version in which the same conclusion as in Strong Artin Approximation can be reached just from the solvability modulo a single power $\mathfrak{m}^N$ of $\mathfrak{m}$, where N only depends on (some numerical invariants of) the system of equations, is called *Uniform Strong Artin Approximation*. In [6], using ultraproducts, Uniform Strong Artin Approximation for $R = L[[X]]^{\mathrm{alg}}$ is shown to follow from Artin Approximation for that ring. In more general situations, additional restrictions have to be imposed on the equations (see [2, Theorem 6.1] or [6, Theorem 3.2]) and substantially more work is required [14, 56]. For instance, the proof of the parametric version in [15, Theorem 3.1] uses the positive solution [47] of Artin's Conjecture in the equicharacteristic case.

Nested Conditions. An even more subtle question regarding (Strong or Uniform Strong) Artin Approximation for subrings of $L[[X]]$ is whether one can maintain side conditions on the solutions requiring some of the entries of a solution tuple to depend only on some of the variables, provided the given (approximate) solutions also satisfy such constraints. In [6], several examples are presented to show that this might fail in general (see also §4.33 below). However, Rotthaus' approximation result [47] implies that cylindrical approximation does hold, provided $\operatorname{char} L = 0$. (This was first noted in [6].) Here, by *cylindrical approximation* we mean Artin Approximation for nested systems of polynomial equations in the context of Example 2.2. We refer to Theorem 3.11 below for a precise formulation.

3.4. **Embeddings in ultraproducts.** We now turn to the issue of embedding a power series ring in the ultraproduct of power series rings, which is needed for our construction of a Lefschetz hull in the next section. The existence of a Lefschetz hull is immediate from

the following corollary to Artin's original result on the Artin Approximation property for algebraic power series.

3.5. Proposition. *For every finitely generated $K[X]$-subalgebra V of $K[[X]]$ there exists a $K[X]$-algebra homomorphism $V \to K[[X]]_\infty$. In particular, there exists an ultraset $\mathcal{U}$ and a $K[X]$-algebra homomorphism $K[[X]] \to K[[X]]_\infty^{\mathcal{U}}$.*

Proof. Translating the Artin Approximation property for $K[[X]]^{\mathrm{alg}}$ in the terminology of Corollary 2.5 yields the existence of a $K[X]$-algebra homomorphism $\varphi \colon V \to K[[X]]^{\mathrm{alg}}$. As the Henselian property can be expressed in terms of the solvability of certain systems of polynomial equations, it follows by Łos' Theorem that $K[[X]]_\infty$ is Henselian. By the universal property of Henselizations there exists a unique $K[X]$-algebra homomorphism $K[[X]]^{\mathrm{alg}} \to K[[X]]_\infty$. Composition with φ yields the desired $K[X]$-algebra homomorphism $V \to K[[X]]_\infty$. The last assertion is now clear by Corollary 2.5. $\qquad\square$

The remainder of the section is devoted to enhancements of this, and in particular, the nested version from Theorem 3.10, which we need to obtain functoriality of Lefschetz extensions. In the following L denotes a field and $S = \bigcup_n S_n$ the nested subring of the nested ring $A = \bigcup_n A_n$ as defined in Example 2.2, so

$$S_n = L[[Y_0]][Y_1, \ldots, Y_n], \quad A_n = L[[Y_0, \ldots, Y_n]] \qquad \text{for all } n,$$

where $Y_0, Y_1, \ldots$ is an infinite sequence of finite tuples Y_n of indeterminates. If we need to emphasize the field, we will write S_L and A_L for S and A. We need some further notations concerning nested rings.

3.6. *Notation.* Let $Q = \bigcup_n Q_n$ be a nested ring. We denote by $Q(1)$ the ring Q considered as a nested ring with nest $Q(1)_n := Q_{n+1}$. A homomorphism $\psi \colon Q \to R$ of nested rings is then also a homomorphism $Q(1) \to R(1)$ of nested rings.

If I is an ideal of Q, then we construe Q/I as a nested ring with nest given by $(Q/I)_n := Q_n/I \cap Q_n$ for all n. If $\mathfrak{p}$ is a prime ideal of Q, then the localization $Q_\mathfrak{p}$ is a nested ring with nest given by $(Q_\mathfrak{p})_n := (Q_n)_{\mathfrak{p} \cap Q_n}$ for all n. If each Q_n is a local ring with maximal ideal $\mathfrak{m}_n$, then Q is local with maximal ideal $\mathfrak{m} := \bigcup_n \mathfrak{m}_n$ and residue field $Q/\mathfrak{m} = \bigcup_n Q_n/\mathfrak{m}_n$. In this case we say that $(Q, \mathfrak{m})$ is a *nested local ring*. If moreover $Q_n \cap \mathfrak{m}^k = \mathfrak{m}_n^k$ for every k, then $\mathrm{Inf}(Q) \cap Q_n = \mathrm{Inf}(Q_n)$ for all n.

If Q is a nested R-algebra, for some nested ring $R = \bigcup_n R_n$, and T an R_0-algebra, then we consider $Q \otimes_{R_0} T$ as a nested R-algebra by means of the nest $(Q_n \otimes_{R_0} T)$.

Given a Henselian local ring $(H, \mathfrak{n})$ and a homomorphism $\psi \colon Q \to H$ we denote the Henselization of $Q_{\mathfrak{n} \cap Q}$ by $Q^\sim$ and we let $\psi^\sim \colon Q^\sim \to H$ be the unique extension of ψ given by the universal property of Henselizations. (Often, H is to be understood from the context.) Note that then $Q^\sim$ is a nested local ring with nest $(Q^\sim)_n := (Q_n)^\sim$. For instance, applied to $Q := S$, $H := A$ and ψ the natural inclusion $S \to A$, we get the nested S-subalgebra $S^\sim = R$ of A (see Example 2.2).

The argument in the proof of the following lemma was inspired by [5, Remark 1.5].

3.7. Lemma (Cylindrical Approximation)**.** *If V is a nested S-subalgebra of finite type of A, then there exists a homomorphism of nested S-algebras $\varphi \colon V \to S^\sim$.*

Proof. We proceed by induction on the length n of the relevant part $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n$ of V, where the case $n = 0$ holds trivially since then $V = S$. Consider the nested ring $T := S(1) \otimes_{S_1} A_1$ with nest (T_n) given by

$$T_n := S(1)_n \otimes_{S_1} A_1 = L[[Y_0, Y_1]][Y_2, \ldots, Y_{n+1}].$$

In particular, T is a nested $S(1)$-subalgebra of $A(1)$. Let W be the image of the homomorphism of nested T-algebras $V(1) \otimes_{S_1} A_1 \to A(1)$ induced by the inclusion $V(1) \subseteq A(1)$, so W is a nested T-subalgebra of finite type of $A(1)$. Since its relevant part has length $< n$, we may apply our induction hypothesis to conclude that there is a homomorphism of nested T-algebras $W \to T^\sim$. Via the natural homomorphism $V(1) \to W$, we get a homomorphism of nested $S(1)$-algebras $V(1) \to T^\sim$. Let W' be its image, so that W' is a nested $S(1)$-subalgebra of finite type of $T^\sim$.

For our purposes in §4, we only have to deal with the case that the base field L has characteristic zero. In that case, we can use [47, Theorem 4.2], which implies that $S_1^\sim$ has the Artin Approximation property. In case the characteristic of L is positive, we require the positive solution of Artin's Conjecture by [43, 60]. In any case, by (3.3.1), there exists an ultraset $\mathcal{U}$ and an S_1-algebra homomorphism $A_1 \to (S_1^\sim)^{\mathcal{U}}$. For each n, this S_1-algebra homomorphism extends to an S_n-algebra homomorphism

$$T_n = A_1[Y_2, \ldots, Y_{n+1}] \to (S_1^\sim)^{\mathcal{U}}[Y_2, \ldots, Y_{n+1}] \to (S_{n+1}^\sim)^{\mathcal{U}}.$$

Since the right hand side is Henselian, we may replace the ring on the left by its Henselization. Gathering these homomorphisms for all n yields a homomorphism of nested $S(1)$-algebras $T^\sim \to (S(1)^\sim)^{\langle \mathcal{U} \rangle}$. Applying (2.4.2) to the nested $S(1)$-subalgebra of finite type $W' \subseteq T^\sim$, yields the existence of a homomorphism of nested $S(1)$-algebras $W' \to S(1)^\sim$. Composition with $V(1) \to W'$ gives a homomorphism $V(1) \to S(1)^\sim$ of nested $S(1)$-algebras. Since $S_0 = V_0 = A_0$, this is in fact a homomorphism of nested S-algebras $V \to S^\sim$, as required. $\qquad\square$

From now on we always assume that Y_0 is the empty tuple. (The more general case was only needed for inductive reasons, in the proof of the previous lemma.)

In the following corollary we specialize to $L = K$. Then, in a natural way, $B_n := K[[Y_1, \ldots, Y_n]]_\infty$ is an S_n-algebra, and we may identify B_n with a subring of B_{n+1}, for all n; hence the subring $B := \bigcup_n B_n$ of $\mathrm{ulim}_p K_p[[Y_1, Y_2, \ldots]]$ is a nested S-algebra with nest (B_n).

3.8. **Corollary.** *There exists an ultraset $\mathcal{U}$ and a homomorphism $\eta \colon A \to B^{\langle \mathcal{U} \rangle}$ of nested S-algebras.*

Proof. We only need to verify that condition (2.4.2) in Theorem 2.4 is fulfilled. To this end, let V be a nested S-subalgebra of A of finite type. By Lemma 3.7, there exists a homomorphism of nested S-algebras $V \to S^\sim$. Since B is Henselian, the homomorphism of nested rings $S \to B$ extends to a homomorphism of nested rings $S^\sim \to B$, and the composition $V \to S^\sim \to B$ proves (2.4.2). $\qquad\square$

We denote by $\mathfrak{m}$ the ideal of S generated by all the indeterminates Y_{ni}, for all n and $i = 0, \ldots, k_n$.

3.9. *Remark.* For each n, let π_n be the canonical epimorphism $B_n \to A_n$ given by Proposition 3.1 and let $\pi \colon B \to A$ be the induced homomorphism of nested S-algebras (given as the direct limit of the π_n). Then π induces an isomorphism between $B/\mathfrak{m}^c B$ and $S/\mathfrak{m}^c S$, for all $c \in \mathbb{N}$. On the other hand, for a fixed $c \in \mathbb{N}$, we can realize A as the union of all nested S-subalgebras V of finite type of A such that $V/\mathfrak{m}^c V \cong S/\mathfrak{m}^c S$. For those V, the homomorphism $V \to S^\sim$ given by Lemma 3.7 becomes an isomorphism modulo $\mathfrak{m}^c$, and applying Remark 2.7 with this collection we see that we may take η in Corollary 3.8 so that its composition with π is congruent modulo $\mathfrak{m}^c$ to the diagonal embedding $A \subseteq A^{\langle \mathcal{U} \rangle}$. Without proof, we mention that one can achieve the even stronger condition that $\pi \circ \eta$

is *equal* to the diagonal embedding. (This however, even in characteristic zero, requires generalized Néron Desingularization.)

Applying Corollary 3.8 with each Y_n for $n \geq 1$ equal to a single indeterminate yields the following result, needed for the functorial construction of faithfully flat Lefschetz extensions in the next section.

3.10. Theorem. *There exists an ultraset $\mathcal{U}$ and for each n a $K[X_1, \ldots, X_n]$-algebra homomorphism*

$$\eta_n \colon K[[X_1, \ldots, X_n]] \to K[[X_1, \ldots, X_n]]_\infty^{\mathcal{U}},$$

such that for all $n \leq m$, the diagram

$$
\begin{array}{ccc}
K[[X_1, \ldots, X_n]] & \xrightarrow{\;\eta_n\;} & K[[X_1, \ldots, X_n]]_\infty^{\mathcal{U}} \\
\downarrow & & \downarrow \\
K[[X_1, \ldots, X_m]] & \xrightarrow{\;\eta_m\;} & K[[X_1, \ldots, X_m]]_\infty^{\mathcal{U}}
\end{array}
$$

commutes, where the vertical arrows are the natural inclusion maps. □

Given a nested system of polynomial equations $\mathcal{S}$ over S as in (2.3.1) we call the maximum of $n, k_0, \ldots, k_n$ and the degrees of the polynomials P_{ij} the *complexity* of $\mathcal{S}$. We say that a nested S-algebra V of finite type has *complexity* $\leq d$ if V admits a defining system of nested equations of complexity $\leq d$. (See §2.3.) The proof of the next theorem is a modification of the argument in [6, Theorem 4.3].

3.11. Theorem (Uniform Strong Artin Approximation with Nested Conditions)**.** *Given $c, d \in \mathbb{N}$, there exists $N = N(c, d) \in \mathbb{N}$ with the following property. Let L be a field, let $S := S_L$ and let V be a nested S-algebra of finite type and of complexity at most d. If $\psi \colon V \to S/\mathfrak{m}^N S$ is a homomorphism of S-nested algebras, then there exists a homomorphism $\varphi \colon V \to S^\sim$ of nested S-algebras such that*

$$
\begin{array}{ccc}
V & \xrightarrow{\;\varphi\;} & S^\sim \\
\downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle q} \\
S/\mathfrak{m}^N S & \longrightarrow & S/\mathfrak{m}^c S
\end{array}
$$

commutes, where q is induced by the canonical isomorphism $S^\sim/\mathfrak{m}^c S^\sim \cong S/\mathfrak{m}^c S$.

Proof. Suppose the claim is false for some pair (c, d), so that we have counterexamples for increasing powers of $\mathfrak{m}$. That is to say, for each $p \in \mathbb{N}$ there is a field K_p and a nested S_{K_p}-algebra V_p of finite type with a defining nested equations $\mathcal{S}_p$ of complexity at most d and a homomorphism of nested algebras $V_p \to S_{K_p}/\mathfrak{m}^p S_{K_p}$ which is not congruent modulo $\mathfrak{m}^c$ to a homomorphism $V_p \to S_{K_p}^\sim$. We may assume that there exist $k \in \mathbb{N}$ and indeterminates $Z_i = (Z_{i1}, \ldots, Z_{id})$, for $i = 0, \ldots, d$, such that each $\mathcal{S}_p$ has the form (2.3.1) with $n = d$, for some polynomials $P_{ij} \in (S_{K_p})_i[Z_0, \ldots, Z_i]$ of degree $\leq d$. Let K be the ultraproduct of the K_p with respect to some ultraset with underlying set $\mathbb{N}$. Taking the ultraproduct of the polynomials in $\mathcal{S}_p$ yields a nested system $\mathcal{S}$ of equations with coeffcients in S_K. Let V be the corresponding nested S_K-algebra of

finite type. By [18, (1.8)], V embeds into the ultraproduct of the V_p. Taking ultraproducts of the homomorphisms $V_p \to S_{K_p}/\mathfrak{m}^p S_{K_p} \cong A_{K_p}/\mathfrak{m}^p A_{K_p}$ yields a homomorphism $V \to B_K/\operatorname{Inf}(B_K) \cong A_K$ of nested S_K-algebras, where we used Proposition 3.1 for the last isomorphism. By Lemma 3.7, applied to (the image under) $V \to A_K$, there exists a homomorphism $V \to S_K^{\sim}$ of nested S_K-algebras, which we may assume to be congruent to $V \to A_K$ modulo $\mathfrak{m}^c A_K$, by Remark 3.9. By Łoś' Theorem, the ultraproduct $\tilde{B}$ of the $S_{K_p}^{\sim}$ is Henselian. Since $\tilde{B}$ is a nested S_K-algebra, it is in fact a nested $S_K^{\sim}$-algebra by the universal property of Henselizations. Hence we have a composed homomorphism $V \to \tilde{B}$ of nested S_K-algebras which is congruent to $V \to A_K$ modulo $\mathfrak{m}^c A_K$. Łoś' Theorem then yields for almost all p a homomorphism $V_p \to S_{K_p}^{\sim}$ of nested S_{K_p}-algebras which modulo $\mathfrak{m}^c$ is equal to the original homomorphism $V_p \to S_{K_p}/\mathfrak{m}^c S_{K_p}$, a contradiction. $\square$

3.12. *Remark.* Conversely, Lemma 3.7 is an immediate consequence of Theorem 3.11. Indeed, let V be a nested S-subalgebra of A of finite type. Since $A/\mathfrak{m}^N A \cong S/\mathfrak{m}^N S$, this induces for each N a homomorphism $V \to S/\mathfrak{m}^N S$ of nested S-algebras. For sufficiently large N this yields by Theorem 3.11 a homomorphism $V \to S^{\sim}$ of nested S-algebras.

3.13. *Remark.* Spelling out the previous result in terms of equations yields the following equational form of cylindrical approximation: For all $c, d \in \mathbb{N}$ there exists a bound $N = N(c, d) \in \mathbb{N}$ with the following property. Let L be a field and let $\mathcal{S}$ be a nested system of polynomial equations with coefficients from S_L, of complexity at most d. If $\mathcal{S}$ has an approximate nested solution $\mathbf{a} = (\mathbf{a}_0, \ldots, \mathbf{a}_n)$ in A_L modulo $(Y_1, \ldots, Y_n)^N$, then $\mathcal{S}$ has a nested solution in $S_L^{\sim}$ which is congruent to $\mathbf{a}$ modulo $(Y_1, \ldots, Y_n)^c$.

3.14. *Remark.* Let $\mathcal{L}(n)$ be the language of rings augmented by unary predicate symbols $R_0, \ldots, R_n$. We construe a formal power series ring $S[[X_1, \ldots, X_n]]$ over a ring S as an $\mathcal{L}(n)$-structure by interpreting R_i by the subring $S[[X_1, \ldots, X_i]]$. The previous remark yields the following *existential Lefschetz principle* for nested power series rings: An existential $\mathcal{L}(n)$-sentence holds in $\mathbb{C}[[X_1, \ldots, X_n]]$ if and only if it holds in $\mathbb{F}_p^{\mathrm{alg}}[[X_1, \ldots, X_n]]$ for all but finitely many primes p. For existential sentences not involving the R_i, this has already been noted elsewhere, see [7, Proposition 1]. For $n = 1$ a much stronger transfer principle holds in which not only existential sentences are carried over, but any sentence. (This follows from the Ax-Kochen-Ershov Principle.)

We finish this section by indicating a strengthening of Theorem 3.10 (not needed later). Given a power series $f \in \mathbb{Z}[[X]] = \mathbb{Z}[[X_1, \ldots, X_n]]$, let f_p be its image in $K_p[[X]]$ and let f_∞ be the ultraproduct of the f_p in $K[[X]]_\infty$. One verifies that the map $f \mapsto f_\infty$ is an injective $\mathbb{Z}[X]$-algebra homomorphism $\mathbb{Z}[[X]] \to K[[X]]_\infty$ which extends to an injective $\mathbb{Z}[X]$-algebra homomorphism $\mathbb{Z}[[X]] \otimes_{\mathbb{Z}} K \to K[[X]]_\infty$. We will view $Z := \mathbb{Z}[[X]] \otimes_{\mathbb{Z}} K$ as a subring of $K[[X]]_\infty$ via this embedding. Write $\mathfrak{m}_n$ for the ideal in $\mathbb{Z}[X]$ generated by $X_1, \ldots, X_n$. Since $Z/\mathfrak{m}_n^k Z \cong K[[X]]/\mathfrak{m}_n^k K[[X]]$, for all k, we see that $K[[X]]$ is the $\mathfrak{m}_n Z$-adic completion of Z. In particular, Z is a dense subring of $K[[X]]$, equal to the $K[X]$-subalgebra of $K[[X]]$ generated by all power series with integral coefficients. Inspecting the proof of Proposition 3.1, we see that π is in fact a Z-algebra homomorphism. Let $Z^{\sim}$ be the Henselization of Z at the maximal ideal $\mathfrak{m}_n Z$. By the universal property of Henselizations, the embedding $Z \subseteq K[[X]]_\infty$ extends to a unique embedding $Z^{\sim} \to K[[X]]_\infty$. Henceforth we will view $Z^{\sim}$ as a subring of $K[[X]]_\infty$. Note that since $K[X]$ is a subring of $Z^{\sim}$, so is its Henselization $K[[X]]^{\mathrm{alg}}$ at $\mathfrak{m}_n K[X]$.

3.15. **Theorem.** *The ultraset $\mathcal{U}$ and the $K[X_1, \ldots, X_n]$-algebra homomorphisms η_n in Theorem 3.10 can be chosen so that in addition each η_n is a $Z^{\sim}$-algebra homomorphism.*

Proof. This theorem follows as above from the corresponding extension of Lemma 3.7. To this end, replace the filtered ring S in Lemma 3.7 and its proof by the nested ring $T = \bigcup_n T_n$, where T_n is the Henselization of $\mathbb{Z}[[Y_0, \dots, Y_n]] \otimes_{\mathbb{Z}} K$ at $(Y_0, \dots, Y_n)Z$. Whenever we invoked Rotthaus' result, we now use the positive solution of Artin's Conjecture due to [43, 60, 61] instead. Note that each T_n is excellent. (Use for instance the Jacobian Criterion [39, Theorem 101].) Details are left to the reader. $\qquad\square$

3.16. *Remark.* By the same argument as in Remark 3.9, we can choose the η_n moreover so that its composition with the canonical epimorphism $\pi^{\mathcal{U}} \colon K[[X]]_{\infty}^{\mathcal{U}} \to K[[X]]^{\mathcal{U}}$ is equal to the diagonal embedding $K[[X]] \subseteq K[[X]]^{\mathcal{U}}$, for each n.

3.17. *Remark.* By Theorem 3.15, the existential Lefschetz principle from Remark 3.14 remains true upon augmenting $\mathcal{L}(n)$ by additional constant symbols, one for each power series in $\mathbb{Z}[[X]] = \mathbb{Z}[[X_1, \dots, X_n]]$, to be interpreted in the natural way in $S[[X]]$.

4. LEFSCHETZ HULLS

Our objective in this section is to prove the theorem stated in the introduction, in a more precise form. *Throughout, we fix a Lefschetz field K with respect to some ultraset with underlying set equal to the set of the prime numbers, whose components K_p are algebraically closed fields of characteristic p.* See the remark following Proposition 1.4 on how to obtain such K, of arbitrarily large cardinality.

In obtaining a functorially defined Lefschetz extension, we face the following complication: not every automorphism of K is an ultraproduct of automorphisms of its components K_p. The simplest counterexample is complex conjugation on $\mathbb{C}$, for no algebraically closed field of positive characteristic has a subfield of index 2. In fact, each subfield of K has an automorphism which cannot be extended to an automorphism of K that is an ultraproduct of automorphisms of the K_p. Therefore there cannot exist a functor from the category of equicharacteristic zero Noetherian local rings R whose residue field is contained in K to a category of analytic Lefschetz rings. The way around this problem is to fix some additional data of R, as we will now explain.

4.1. **Quasi-coefficient fields.** Let $(R, \mathfrak{m})$ be a Noetherian local ring which contains the rationals (that is to say, R has equicharacteristic zero). A subfield k of R is called a *quasi-coefficient field* of R if $R/\mathfrak{m}$ is algebraic over the image of k under the residue homomorphism $R \to R/\mathfrak{m}$. Every maximal subfield of R is a quasi-coeffcient field. A quasi-coefficient field is called a *coefficient field* if the natural map $k \to R/\mathfrak{m}$ is an isomorphism. In general, coefficient fields may not exist. If R is Henselian then a subfield of R is a coefficient field if and only if it is maximal. In particular, if R is complete, then R has a coefficient field. Every quasi-coefficient field k of R is contained in a unique coefficient field of $\widehat{R}$, namely, the algebraic closure of k in $\widehat{R}$. For proofs and more details, see [40, §28].

4.2. **The category $\mathbf{Coh}_K$.** In order to state a refined version of the theorem from the introduction, we introduce a category $\mathbf{Coh}_K$ (for "Cohen"). Its objects are quadruples $\Lambda = (R, \mathbf{x}, k, u)$ where

     (a) $(R, \mathfrak{m})$ is a Noetherian local ring (the *underlying ring* of Λ),

     (b) $\mathbf{x}$ is a (finite) tuple of elements of R which generate $\mathfrak{m}$,

     (c) k is a quasi-coefficient field of R, and

     (d) $u \colon R \to K$ is a local homomorphism (that is to say, u is a ring homomorphism with $\ker u = \mathfrak{m}$).

A morphism $\Lambda \to \Gamma$ from Λ to another such quadruple $\Gamma = (S, \mathbf{y}, l, v)$ is given by a local ring homomorphism $\alpha \colon R \to S$ such that

(a) $\alpha(\mathbf{x})$ is an initial segment of $\mathbf{y}$ (if $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$, then $n \le m$ and $y_i = \alpha(x_i)$ for $i = 1, \ldots, n$),

(b) $\alpha(k) \subseteq l$, and

(c) $v \circ \alpha = u$.

We will often identify a morphism $\Lambda \to \Gamma$ with its *underlying homomorphism* $\alpha \colon R \to S$ and hence denote it also by α.

Let **Loc** be the category of (not necessarily Noetherian) local rings, with the local ring homomorphisms as morphisms. Given an ultraset $\mathcal{W}$ we denote by $\mathbf{Lef}_{\mathcal{W}}$ the category of analytic Lefschetz rings with respect to $\mathcal{W}$ as defined in §1. (Its objects are ultraproducts, with respect to $\mathcal{W}$, of complete local rings with algebraically closed residue fields of positive characteristic, and its morphisms are ultraproducts of local homomorphisms.) We stress once more that $\mathbf{Lef}_{\mathcal{W}}$, as a subcategory of **Loc**, is not full. We will denote the *forgetful functor* with values in **Loc** always by *ring* (regardless of the source category). If F and G are functors from a category $\mathbf{C}$ to **Loc**, then we will say that a natural transformation $\eta \colon F \to G$ is *faithfully flat* if the ring homomorphism $\eta_\Lambda \colon F(\Lambda) \to G(\Lambda)$ is faithfully flat, for each object Λ in $\mathbf{C}$.

4.3. Theorem. *There exists an ultraset $\mathcal{W}$, a functor $\mathfrak{D} \colon \mathbf{Coh}_K \to \mathbf{Lef}_{\mathcal{W}}$ and a faithfully flat natural transformation $\eta \colon \underline{ring} \to \underline{ring} \circ \mathfrak{D}$.*

We call $\mathfrak{D}(\Lambda)$ the *Lefschetz hull* of Λ. Let us state in more detail what the above functoriality amounts to. Given a morphism $\Lambda \to \Gamma$ in $\mathbf{Coh}_K$ with underlying homomorphism $\alpha \colon R \to S$, where $R := \underline{ring}(\Lambda)$ and $S := \underline{ring}(\Gamma)$, we get a morphism $\mathfrak{D}(\alpha) \colon \mathfrak{D}(\Lambda) \to \mathfrak{D}(\Gamma)$ in $\mathbf{Lef}_{\mathcal{W}}$ and faithfully flat homomorphisms $\eta_\Lambda \colon R \to \mathfrak{D}(\Lambda)$ and $\eta_\Gamma \colon S \to \mathfrak{D}(\Gamma)$ fitting into a commutative diagram

(4.3.1)
$$
\begin{array}{ccc}
R & \xrightarrow{\ \ \alpha\ \ } & S \\
\downarrow{\scriptstyle \eta_\Lambda} & & \downarrow{\scriptstyle \eta_\Gamma} \\
\mathfrak{D}(\Lambda) & \xrightarrow{\ \mathfrak{D}(\alpha)\ } & \mathfrak{D}(\Gamma).
\end{array}
$$

(Technically speaking we should have written $R \to \underline{ring}(\mathfrak{D}(\Lambda))$, etc., but we'll always identify $\mathfrak{D}(\Lambda)$ with its underlying ring.)

The proof proceeds in two steps. We first prove the theorem for a certain subcategory $\mathbf{An}_K$ of $\mathbf{Coh}_K$ given by quotients of power series rings over K (see §4.4). The existence of the functor $\mathfrak{D}$ for these rings then follows from Theorem 3.10. The second step consists in associating in a functorial way to an object $\Lambda = (R, \mathbf{x}, k, u)$ of $\mathbf{Coh}_K$ a complete local K-algebra which is a faithfully flat R-algebra (see §4.13). This is achieved by making a base change to K using k and u, and then taking completion. By Cohen's structure theorem, $\mathbf{x}$ then determines a unique ring $C(\Lambda)$ in $\mathbf{An}_K$ isomorphic to the latter. After the proof of Theorem 4.3 we discuss a construction of Lefschetz hulls with some additional properties. We finish the section by pointing out (in §4.33) another obstacle which prevented us from outright defining a functor from the category of Noetherian local rings whose residue field embeds into K to a category of analytic Lefschetz rings.

We adopt the following notation for polynomial and power series rings: we fix a countable sequence of indeterminates $X_1, X_2, \ldots$, and for each n and each ring S, we let $S[n]$

and $S[[n]]$ be shorthand for respectively $S[X_1, \ldots, X_n]$ and $S[[X_1, \ldots, X_n]]$. We write $K[[n]]_\infty$ for the ultraproduct of the $K_p[[n]]$.

4.4. **Power series rings.** We first describe in more detail the category of quotients of power series rings over K, which we denote by $\mathbf{An}_K$. Its objects are local rings of the form $K[[n]]/I$, for some n and some ideal I of $K[[n]]$. A morphism in $\mathbf{An}_K$ is a K-algebra homomorphism $\alpha \colon K[[n]]/I \to K[[m]]/J$ where $n \leq m$, $I \subseteq J$, and α is induced by the inclusion $K[[n]] \subseteq K[[m]]$. To each object $K[[n]]/I$ of $\mathbf{An}_K$ we associate the object $(K[[n]]/I, \mathbf{x}, K, \pi_n)$ in $\mathbf{Coh}_K$, where $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_i := X_i + I$ for each i and $\pi_n \colon K[[n]] \to K$ is the residue map. Every $\mathbf{An}_K$-morphism $\alpha \colon K[[n]]/I \to K[[m]]/J$ gives rise to a $\mathbf{Coh}_K$-morphism (with underlying homomorphism α) between the objects corresponding to $K[[n]]/I$ and $K[[m]]/J$, respectively. It is easily verified that via this identification, $\mathbf{An}_K$ becomes a full subcategory of $\mathbf{Coh}_K$.

We now embark on the proof of Theorem 4.3, first for the subcategory $\mathbf{An}_K$. Let $\mathcal{U}$ be the ultraset proclaimed in Theorem 3.10 and set

$$\mathfrak{D}(n) := K[[n]]_\infty^{\mathcal{U}} \qquad \text{for each } n.$$

By that theorem, there exists, for each n, a $K[n]$-algebra homomorphism $\eta_n \colon K[[n]] \to \mathfrak{D}(n)$ such that for each $n \leq m$, the diagram

(4.4.1)
$$
\begin{array}{ccc}
K[[n]] & \longrightarrow & K[[m]] \\
\downarrow{\scriptstyle \eta_n} & & \downarrow{\scriptstyle \eta_m} \\
\mathfrak{D}(n) & \longrightarrow & \mathfrak{D}(m)
\end{array}
$$

commutes, where the horizontal maps are the natural inclusions. We construe $\mathfrak{D}(n)$ as a K-algebra via η_0; then each η_n is a K-algebra homomorphism.

4.5. *Remark.* If we are only interested in constructing a Lefschetz extension for a single $K[[n]]$, then the existence of a $K[n]$-algebra homomorphism $\eta_n \colon K[[n]] \to \mathfrak{D}(n)$ already follows by combining Theorem 2.4 with the more elementary Proposition 3.5.

4.6. *Remark.* Suppose that $K = \mathbb{C}$. If we are willing to weaken the requirement that η_n be a $K[n]$-algebra homomorphism, then under assumption of the Continuum Hypothesis $2^{\aleph_0} = \aleph_1$ the passage to the ultrapower $\mathbb{C}[[n]]_\infty^{\mathcal{U}}$ is superfluous: Let L be a countable subfield of $\mathbb{C}$; then $S_n = L[n]$ is countable, and under the assumption $2^{\aleph_0} = \aleph_1$ it follows along the lines of Remark 2.11 that there exists, for each n, an S_n-algebra homomorphism $\varrho_n \colon \mathbb{C}[[n]] \to \mathbb{C}[[n]]_\infty$ such that ϱ_n is the restriction of ϱ_m to $\mathbb{C}[[n]]$, for all $n \leq m$.

Note that $\mathfrak{D}(n)$, being an ultrapower of the analytic Lefschetz ring $K[[n]]_\infty$, is itself an analytic Lefschetz ring. Indeed, we can construct an ultraset $\mathcal{W}$ with the following property: for each n, the ring $\mathfrak{D}(n)$ is isomorphic to the ultraproduct with respect to $\mathcal{W}$ of the rings $K_w[[n]]$, where $K_w := K_{p(w)}$ for some prime number $p(w)$. (See [13, Proposition 6.5.2].) From now on, we always represent $\mathfrak{D}(n)$ in this way. The Lefschetz ring $\mathfrak{D}(n)$ is a $\mathbf{K}$-algebra, where $\mathbf{K} := \mathfrak{D}(0) = K^{\mathcal{U}}$, via the natural inclusion $\mathfrak{D}(0) \to \mathfrak{D}(n)$, and the natural inclusions $\mathfrak{D}(n) \to \mathfrak{D}(m)$ (for $n \leq m$) are $\mathbf{K}$-algebra homomorphisms. Next we show that $\mathfrak{D}(n)$ gives the desired faithfully flat Lefschetz extension:

4.7. **Proposition.** *For each n, the homomorphism $\eta_n \colon K[[n]] \to \mathfrak{D}(n)$ is faithfully flat.*

In the proof we use the following variant of [12, Corollary 8.5.3]. A module M over a local ring R is called a *big Cohen-Macaulay module* over R if there exists a system of parameters of R which is an M-regular sequence. If every system of parameters of R is an M-regular sequence, then M is called a *balanced big Cohen-Macaulay module* over R. If $(R, \mathfrak{m})$ is a regular local ring and M a balanced big Cohen-Macaulay module over R, then M is flat, see [35, proof of Theorem 9.1].

4.8**. Lemma.** *Let R be a Noetherian local ring and let M be a big Cohen-Macaulay module over R. If every permutation of an M-regular sequence is again M-regular, then M is a balanced big Cohen-Macaulay module over R.*

Proof. We proceed by induction on $d := \dim R$. There is nothing to show if $d = 0$, so assume $d > 0$. By assumption, there exists a system of parameters $(x_1, \ldots, x_d)$ of R which is an M-regular sequence. Let $(y_1, \ldots, y_d)$ be an arbitrary system of parameters. By prime avoidance we find $z \in \mathfrak{m}$ not contained in a minimal prime of $(x_1, \ldots, x_{d-1})R$ and of $(y_1, \ldots, y_{d-1})R$. Hence both $(x_1, \ldots, x_{d-1}, z)$ and $(y_1, \ldots, y_{d-1}, z)$ are systems of parameters of R. Since a power of x_n is a multiple of z modulo $(x_1, \ldots, x_{d-1})R$, the sequence $(x_1, \ldots, x_{d-1}, z)$ is M-regular. Thus, by assumption, the permuted sequence $(z, x_1, \ldots, x_{d-1})$ is also M-regular. In particular, the canonical image of $(x_1, \ldots, x_{d-1})$ in R/zR is M/zM-regular, showing that M/zM is a big Cohen-Macaulay module over R/zR. Moreover, every permutation of an M/zM-regular sequence is again M/zM-regular. By induction hypothesis, the canonical image of $(y_1, \ldots, y_{d-1})$ in R/zR, being a system of parameters in R/zR, is M/zM-regular. Hence $(z, y_1, \ldots, y_{d-1})$ is M-regular, and therefore, using the assumption once more, so is $(y_1, \ldots, y_{d-1}, z)$. As some power of z is a multiple of y_d modulo $(y_1, \ldots, y_{d-1})R$, we get that $(y_1, \ldots, y_d)$ is M-regular, as required. $\qquad\square$

Proof of Proposition 4.7. Since $(X_1, \ldots, X_n)$ is a $K_w[[n]]$-regular sequence for each w, it is a $\mathfrak{D}(n)$-regular sequence by Łos' Theorem. It follows that $\mathfrak{D}(n)$ is a big Cohen-Macaulay $K[[n]]$-algebra via the homomorphism η_n. Using Łos' Theorem once more, one shows that every permutation of a $\mathfrak{D}(n)$-regular sequence is again $\mathfrak{D}(n)$-regular (since every permutation of a $K_w[[n]]$-regular sequence in $K_w[[n]]$ remains $K_w[[n]]$-regular by [12, Proposition 1.1.6]). Therefore, $\mathfrak{D}(n)$ is a balanced big Cohen-Macaulay $K[[n]]$-algebra, by the lemma above. Since $K[[n]]$ is regular, η_n is flat by [35, proof of Theorem 9.1], hence faithfully flat. $\qquad\square$

Below, we write $I\mathfrak{D}(n)$ to denote the ideal of $\mathfrak{D}(n)$ generated by the image of an ideal I of $K[[n]]$ under η_n.

4.9. *Remark.* We have

$$\mathrm{Im}(\eta_n) = \mathrm{Im}(\eta_{n+1}) \cap \mathfrak{D}(n) \qquad \text{for all } n.$$

This follows from $X_{n+1}\mathfrak{D}(n+1) \cap \mathfrak{D}(n) = (0)$ and the injectivity of η_{n+1}.

4.10. **Proof of Theorem 4.3 for the category An$_K$.** The construction of $\mathfrak{D}(n)$ above extends in a natural way to quotients of $K[[n]]$. Namely, if $I = (a_1, \ldots, a_m)K[[n]]$ is an ideal of $K[[n]]$ and $R := K[[n]]/I$, then we choose $b_{iw} \in K_w[[n]]$ whose ultraproduct in $\mathfrak{D}(n)$ is $\eta_n(a_i)$, for each i, and put
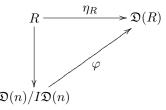
$$\mathfrak{D}(R) := \mathrm{ulim}_w K_w[[n]]/I_w.$$

Here I_w is the ideal of $K_w[[n]]$ generated by $b_{1w}, \ldots, b_{mw}$. The canonical surjections $K_w[[n]] \to K_w[[n]]/I_w$ yield a surjection $\mathfrak{D}(n) \to \mathfrak{D}(R)$ whose kernel is $I\mathfrak{D}(n)$. On

the one hand, this shows that $\mathfrak{D}(R)$ does not depend on the choice of the b_{iw} and that we have an isomorphism $\varphi\colon \mathfrak{D}(n)/I\mathfrak{D}(n) \to \mathfrak{D}(R)$. On the other hand, composing with the homomorphism $\eta_n\colon K[[n]] \to \mathfrak{D}(n)$ we obtain a homomorphism $K[[n]] \to \mathfrak{D}(R)$ whose kernel contains I, and hence an induced K-algebra homomorphism

$$\eta_R\colon R = K[[n]]/I \to \mathfrak{D}(R).$$

(According to this definition $\mathfrak{D}(K[[n]]) = \mathfrak{D}(n)$ and $\eta_{K[[n]]} = \eta_n$, for all n.) We have a commutative diagram

where the arrow on the left is the homomorphism obtained from η_n by base change modulo I. Hence by Proposition 4.7 the homomorphism η_R is faithfully flat. In the following we identify $\mathfrak{D}(n)/I\mathfrak{D}(n)$ and $\mathfrak{D}(R)$ via the isomorphism φ.

Let J be an ideal of $K[[n+m]]$ with $I \subseteq J$. The natural inclusion $K[[n]] \to K[[n+m]]$ induces a morphism $\alpha\colon R \to S := K[[n+m]]/J$ in $\mathbf{An}_K$. (This is the only $\mathbf{An}_K$-morphism $R \to S$.) Choose $J_w \subseteq K_w[[n+m]]$ in the same way as we constructed the I_w; so their ultraproduct is $J\mathfrak{D}(n+m)$ and $\mathfrak{D}(S) \cong \mathfrak{D}(n+m)/J\mathfrak{D}(n+m)$. Since $I\mathfrak{D}(n) \subseteq J\mathfrak{D}(n+m)$, we have $I_w \subseteq J_w$ for almost all w, by Łoś' Theorem. The natural inclusions $K_w[[n]] \to K_w[[n+m]]$ give rise to homomorphisms

$$\alpha_w\colon K_w[[n]]/I_w \to K_w[[n+m]]/J_w.$$

The ultraproduct of the α_w yields a $\mathbf{K}$-algebra homomorphism $\mathfrak{D}(\alpha)\colon \mathfrak{D}(R) \to \mathfrak{D}(S)$ making the diagram

commutative. Together with (4.4.1) this gives a commutative diagram

as required. $\qquad\square$

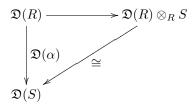This concludes the proof of Theorem 4.3 for the subcategory $\mathbf{An}_K$. Before we turn to the general case, we take a closer look at finite maps. We use the following version of the Weierstrass Division Theorem for $\mathfrak{D}(n+1)$. Let $f, g \in \mathfrak{D}(n+1)$ and suppose g is regular in X_{n+1} of order d, that is,

$$g \equiv X_{n+1}^d(1+\varepsilon) \mod (X_1,\dots,X_n)\mathfrak{D}(n+1)$$

with $\varepsilon \in X_{n+1}\mathfrak{D}(n+1)$. Then there exist unique $q \in \mathfrak{D}(n+1)$ and $r \in \mathfrak{D}(n)[X_{n+1}]$ such that $f = qg + r$ and the degree of r with respect to X_{n+1} is strictly less than d. (Use Łos' Theorem and the Weierstrass Division Theorem in $K_w[[n+1]]$.) A polynomial $P(Y) \in A[Y]$ in a single indeterminate Y with coefficients in a local ring $(A, \mathfrak{n})$ is called a *Weierstrass polynomial* if $P(Y)$ is monic of degree d and $P \equiv Y^d \bmod \mathfrak{n}A[Y]$.

4.11**. Proposition.** *If $\alpha\colon R \to S$ is a* finite *morphism in* $\mathbf{An}_K$ (*that is to say, if S is module-finite over R), then the natural map $\mathfrak{D}(R) \otimes_R S \to \mathfrak{D}(S)$ is an isomorphism, making the diagram*

$$\begin{array}{ccc} \mathfrak{D}(R) & \longrightarrow & \mathfrak{D}(R) \otimes_R S \\ \Big\downarrow {\scriptstyle \mathfrak{D}(\alpha)} & {\scriptstyle \cong} \swarrow & \\ \mathfrak{D}(S) & & \end{array}$$

commutative.

Proof. We keep the notation from above, so that in particular $\alpha\colon R = K[[n]]/I \to S = K[[n+m]]/J$. The case $m = 0$ is clear. By an induction on m, we may reduce to the case $m = 1$. The ideal

$$J_1 := J \cap K[[n]][X_{n+1}]$$

of $K[[n]][X_{n+1}]$ contains a monic polynomial P. Now P (as an element of $K[[n+1]]$) is regular of order at most $d = $ degree of P, hence can be written as $P = uQ$ where $u \in K[[n+1]]$ is a unit and $Q \in K[[n]][X_{n+1}]$ is a Weierstrass polynomial. Replacing P by Q we may assume that P is a Weierstrass polynomial of degree d contained in J_1. The natural inclusion $K[[n]][X_{n+1}] \to K[[n+1]]$ induces an embedding

$$(4.11.1) \qquad\qquad K[[n]][X_{n+1}]/J_1 \to K[[n+1]]/J = S,$$

which is in fact an isomorphism, for given $f \in K[[n+1]]$ we can write $f - r = qP \in J$ where $q \in K[[n+1]]$ and $r \in K[[n]][X_{n+1}]$ of degree $< d$, using Euclidean Division by P. The image of P under η_{n+1}, which we continue to denote by P, lies in $\mathfrak{D}(n)[X_{n+1}]$ and is a Weierstrass polynomial of degree d. The natural inclusion $\mathfrak{D}(n)[X_{n+1}] \to \mathfrak{D}(n+1)$ induces a map

$$(4.11.2) \qquad \mathfrak{D}(n)[X_{n+1}]/P\mathfrak{D}(n)[X_{n+1}] \to \mathfrak{D}(n+1)/P\mathfrak{D}(n+1).$$

From the uniqueness of quotient and remainder in Weierstrass Division by P it follows that (4.11.2) is in fact an isomorphism. Since $J = J_1 K[[n+1]]$ and thus $J\mathfrak{D}(n+1) = J_1\mathfrak{D}(n+1)$, we get an induced isomorphism

$$(4.11.3) \qquad \mathfrak{D}(n)[X_{n+1}]/J_1\mathfrak{D}(n)[X_{n+1}] \to \mathfrak{D}(n+1)/J\mathfrak{D}(n+1) = \mathfrak{D}(S).$$

On the other hand, since $I \subseteq J_1$ and $R = K[[n]]/I$ we have

$$(4.11.4) \qquad\qquad K[[n]][X_{n+1}]/J_1 \cong R[X_{n+1}]/J_1 R[X_{n+1}]$$

and using $I\mathfrak{D}(n) \subseteq J_1\mathfrak{D}(n)[X_{n+1}]$ and $\mathfrak{D}(R) \cong \mathfrak{D}(n)/I\mathfrak{D}(n)$ we get

$$(4.11.5) \qquad \mathfrak{D}(n)[X_{n+1}]/J_1\mathfrak{D}(n)[X_{n+1}] \cong \mathfrak{D}(R)[X_{n+1}]/J_1\mathfrak{D}(R)[X_{n+1}].$$

Therefore, by (4.11.1) and (4.11.3)–(4.11.5):

$$\mathfrak{D}(R) \otimes_R S \cong \mathfrak{D}(R) \otimes_R R[X_{n+1}]/J_1 R[X_{n+1}]$$
$$\cong \mathfrak{D}(R)[X_{n+1}]/J_1 \mathfrak{D}(R)[X_{n+1}]$$
$$\cong \mathfrak{D}(n)[X_{n+1}]/J_1 \mathfrak{D}(n)[X_{n+1}] \cong \mathfrak{D}(S).$$

It is straightforward to check that we have a commutative diagram as claimed. $\square$

Faithful flatness of $\eta_R \colon R \to \mathfrak{D}(R)$ now yields:

4.12. Corollary. *If S is a finite R-module via α, then α is injective if and only if $\mathfrak{D}(\alpha)$ is injective, and α is surjective if and only if $\mathfrak{D}(\alpha)$ is surjective.* $\square$

4.13. Proof of Theorem 4.3. We complete the proof of Theorem 4.3 by defining a functor $C \colon \mathbf{Coh}_K \to \mathbf{An}_K$ and a faithfully flat natural transformation $\gamma \colon \underline{ring} \to C$. The proclaimed $\mathfrak{D}$ and η are then realized as the composite functor $\mathfrak{D} \circ C$ and the natural transformation given by $\eta_{C(\Lambda)} \circ \gamma_\Lambda$, for each object Λ. In essence, C will be a kind of 'completion' functor. (See also §4.16 below.) More precisely, let $\Lambda = (R, \mathbf{x}, k, u)$ be an object in $\mathbf{Coh}_K$ and let k^* be the algebraic closure of k in $\widehat{R}$. Recall from §4.1 that k^* is the unique coefficient field of $\widehat{R}$ containing k. We view $\widehat{R}$ as a k^*-algebra via the inclusion $k^* \subseteq \widehat{R}$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and let $\theta_\Lambda \colon k^*[[n]] \to \widehat{R}$ be the k^*-algebra homomorphism given by $X_i \mapsto x_i$, for $i = 1, \ldots, n$. We denote its kernel by I_Λ. Consequently, we have associated to each Λ a *Cohen presentation* $k^*[[n]]/I_\Lambda \cong \widehat{R}$ of the completion of its underlying ring.

Let $\widehat{u} \colon \widehat{R} \to K$ be the completion of u and denote the restriction of $\widehat{u}$ to k^* by u^*. There is a unique local homomorphism $k^*[[n]] \to K[[n]]$ extending $u^* \colon k^* \to K$ and leaving the variables invariant, which we denote by u_n^*. Define the functor C on objects by the rule

$$C(\Lambda) := K[[n]]/u_n^*(I_\Lambda)K[[n]].$$

As for morphisms, let $\Lambda \to \Gamma = (S, \mathbf{y}, l, v)$ be a morphism with underlying local homomorphism $\alpha \colon R \to S$. Since k^* (respectively, l^*) is the algebraic closure of k in $\widehat{R}$ (respectively, of l in $\widehat{S}$) and since $\alpha(k) \subseteq l$, the completion $\widehat{\alpha} \colon \widehat{R} \to \widehat{S}$ of α maps k^* inside l^*. Let us denote the restriction of $\widehat{\alpha}$ to a field embedding $k^* \to l^*$ by α^*, and the induced map $k^*[[n]] \to l^*[[n+m]]$ leaving the variables $X_1, \ldots, X_n$ invariant by α_n^*. Since $\alpha(\mathbf{x})$ is an initial segment of $\mathbf{y}$, we get $\theta_\Gamma(X_i) = \alpha(x_i) \in \widehat{S}$ for $i = 1, \ldots, n$. Therefore, we have a commutative diagram

(4.13.1)
$$\begin{array}{ccc} k^*[[n]] & \xrightarrow{\theta_\Lambda} & \widehat{R} \\ \downarrow{\alpha_n^*} & & \downarrow{\widehat{\alpha}} \\ l^*[[n+m]] & \xrightarrow{\theta_\Gamma} & \widehat{S}. \end{array}$$

In particular, $\alpha_n^*(I_\Lambda) \subseteq I_\Gamma$. Since $v \circ \alpha = u$, we get $\widehat{v} \circ \widehat{\alpha} = \widehat{u}$ which in turn yields $v_{n+m}^* \circ \alpha_n^* = u_n^*$. Hence $u_n^*(I_\Lambda) \subseteq v_{n+m}^*(I_\Gamma)$ under the inclusion $K[[n]] \to K[[n+m]]$, and this inclusion then induces a K-algebra homomorphism $C(\alpha) \colon C(\Lambda) \to C(\Gamma)$. Note that $C(\alpha)$ is indeed a morphism in $\mathbf{An}_K$. Because every step in this construction is carried out in a canonical way, C is a functor; details are left to the reader. Note that C is the identity on the subcategory $\mathbf{An}_K$ of $\mathbf{Coh}_K$.

To define the natural transformation $\gamma \colon \underline{ring} \to C$ we let γ_Λ be the composite map

$$R \to \widehat{R} \cong k^*[[n]]/I_\Lambda \to K[[n]]/u_n^*(I_\Lambda)K[[n]] = C(\Lambda)$$

where the isomorphism is induced by θ_Λ and the last arrow is the base change of u_n^*. Each map in this composition is canonically defined and faithfully flat. It is now straightforward to check that γ is the desired faithfully flat natural transformation. $\qquad\square$

4.14. *Remark.* It follows from our construction that the maximal ideals of $C(\Lambda)$ and $\mathfrak{D}(\Lambda)$ are $\mathfrak{m}C(\Lambda)$ and $\mathfrak{m}\mathfrak{D}(\Lambda)$, respectively, where $\mathfrak{m}$ is the maximal ideal of the underlying ring of Λ.

4.15. *Remark.* If we do not insist that the ultraproducts are Lefschetz rings, then we can let K be any ultraproduct of arbitrary fields, and Theorem 4.3 above, suitably reformulated, remains true in this more general setting, apart from the Lefschetz condition.

4.16. **Extension of scalars.** On occasion, we need a Lefschetz extension with some additional properties, and to achieve this, we enlarge the category $\mathbf{Coh}_K$ to a category $\mathbf{Coh}_K^*$. To this end, we need a method to extend scalars. Suppose that we have a quasi-coefficient field k of a Noetherian local ring $(R, \mathfrak{m})$ and a local homomorphism $u \colon R \to L$ to a field L. Let k^* be the algebraic closure of k in $\widehat{R}$ (the unique coefficient field of $\widehat{R}$ containing k). We view $\widehat{R}$ and L as k^*-algebras via respectively the inclusion $k^* \subseteq \widehat{R}$ and the restriction of $\widehat{u} \colon \widehat{R} \to L$ to k^*. Let $\widehat{R}_{(k,u)}$ be the completion of the Noetherian local ring $\widehat{R} \otimes_{k^*} L$ with respect to its maximal ideal $\mathfrak{m}(\widehat{R} \otimes_{k^*} L) = \mathfrak{m}\widehat{R} \otimes_{k^*} L$. We view $\widehat{R}_{(k,u)}$ as an R-algebra (respectively, as an L-algebra) via the natural map $R \to \widehat{R} \to \widehat{R} \otimes_{k^*} L \to \widehat{R}_{(k,u)}$ (respectively, $L \to \widehat{R} \otimes_{k^*} L \to \widehat{R}_{(k,u)}$). The image of L in $\widehat{R} \otimes_{k^*} L$ is a coefficient field of $\widehat{R} \otimes_{k^*} L$, and hence of the complete Noetherian local ring $\widehat{R}_{(k,u)}$. The R-algebra $\widehat{R}_{(k,u)}$ is faithfully flat. The following transfer result will be used in the next section:

4.17. **Lemma.** *Suppose that* $\operatorname{char} k = 0$. *Then, for given* $i \in \mathbb{N}$, *the completion* $\widehat{R}$ *of* R *satisfies* (R_i) *(or* (S_i)*) if and only if* $\widehat{R}_{(k,u)}$ *does. In particular,* $\widehat{R}$ *is reduced (regular, normal, or Cohen-Macaulay) if and only if* $\widehat{R}_{(k,u)}$ *has this property. Similarly,* $\widehat{R}$ *is equidimensional if and only if* $\widehat{R}_{(k,u)}$ *is.*

Proof. There is probably a more straightforward way to see this, but we argue as follows: Since $\operatorname{char} k = 0$, the homomorphism $\widehat{u}|k^* \colon k^* \to L$ is separable. Therefore the induced homomorphism $k^*[[n]] \to L[[n]]$ is formally smooth [40, Theorem 28.10] hence regular [40, p. 260]. By the Cohen Structure Theorem we may assume $\widehat{R} \cong k^*[[n]]/I$ (as k^*-algebras) for some ideal I of $k^*[[n]]$; then $\widehat{R}_{(k,u)} \cong L[[n]]/IL[[n]]$ (as L-algebras). Now use [40, Theorem 23.9, and the remark following it] to conclude that $\widehat{R}$ satisfies (R_i) (or (S_i)) if and only if $\widehat{R}_{(k,u)}$ has this property. The local ring $\widehat{R}_{(k,u)}$ is complete and hence catenary. Thus if $\widehat{R}_{(k,u)}$ is equidimensional, then $\widehat{R}$ is equidimensional by [40, Theorem 31.5]. Conversely, if $\widehat{R}$ is equidimensional, then so is $\widehat{R}_{(k,u)}$ by [30, (3.25)]. $\qquad\square$

4.18. *Remark.* Suppose that $\operatorname{char} k = 0$. If R is excellent, then $R \to \widehat{R}$ is regular, hence R satisfies (R_i) (or (S_i)) if and only if $\widehat{R}$ has this property, for every $i \in \mathbb{N}$. Therefore, if R is excellent, then R is reduced (regular, normal, or Cohen-Macaulay) if and only if $\widehat{R}_{(k,u)}$ is. By [44], $\widehat{R}$ is equidimensional if and only if R is equidimensional and universally catenary.

Suppose we are given another Noetherian local ring $(S, \mathfrak{n})$ with quasi-coefficient field l of characteristic zero and local homomorphim $v\colon S \to L$, as well as a local homomorphism $\alpha\colon R \to S$ such that $u = v \circ \alpha$ and $\alpha(k) \subseteq l$. Since $\widehat{\alpha}(k^*) \subseteq l^*$, we get natural maps

$$(4.18.1) \qquad \widehat{R} \otimes_{k^*} L \xrightarrow{\widehat{\alpha} \otimes 1} \widehat{S} \otimes_{k^*} L = \widehat{S} \otimes_{l^*} (l^* \otimes_{k^*} L) \to \widehat{S} \otimes_{l^*} L,$$

where the last map is induced by the map $l^* \otimes_{k^*} L \to L$ given by $a \otimes b \mapsto v(a)b$, for $a \in l$ and $b \in L$. Taking completions yields an L-algebra homomorphism $\widehat{R}_{(k,u)} \to \widehat{S}_{(l,v)}$, which we denote by $\widehat{\alpha}_L$.

4.19. **The category $\mathbf{Coh}_K^*$.** Let us first look at an object $\Lambda = (R, \mathbf{x}, k, u)$ in $\mathbf{Coh}_K$. Applying the above construction with respect to the homomorphism $u\colon R \to K$, we get a K-algebra $\widehat{R}_{(k,u)}$ which is isomorphic with $C(\Lambda)$; the isomorphism is uniquely determined by $\mathbf{x}$. Allowing more general choices for $\mathbf{x}$ leads to the extension $\mathbf{Coh}_K^*$ of $\mathbf{Coh}_K$. Namely, for objects we take the quadruples $\Lambda = (R, \mathbf{x}, k, u)$ where as before $(R, \mathfrak{m})$ is a Noetherian local ring with quasi-coefficient field k and $u\colon R \to K$ is a local homomorphism, but this time $\mathbf{x}$ is a tuple in the larger ring $\widehat{R}_{(k,u)}$ generating its maximal ideal $\mathfrak{m}\widehat{R}_{(k,u)}$. A morphism $\Lambda \to \Gamma = (S, \mathbf{y}, l, v)$ in this extended category is given by a local homomorphism $\alpha\colon R \to S$ such that $u = v \circ \alpha$, $\alpha(k) \subseteq l$, and such that $\widehat{\alpha}_K\colon \widehat{R}_{(k,u)} \to \widehat{S}_{(l,v)}$ sends $\mathbf{x}$ to an initial segment of $\mathbf{y}$. It is clear that $\mathbf{Coh}_K$ is a full subcategory of $\mathbf{Coh}_K^*$.

4.20. *Remark.* Up to isomorphism, the k^*-algebra $\widehat{R}_{(k,u)}$ is independent of the choice of u, since every isomorphism between subfields of K can be extended to an automorphism of K (but not necessarily to an ultraproduct of automorphisms of the K_p). It is also easy to see that $\widehat{R}_{(k,u)}$ is independent of the choice of k, up to local isomorphism of local rings.

We extend C to a functor $\mathbf{Coh}_K^* \to \mathbf{An}_K$ as follows. Let $\mathbf{x} = (x_1, \dots, x_n)$ and let $\widehat{I}_\Lambda$ be the kernel of the K-algebra homomorphism $\widehat{\theta}_\Lambda\colon K[[n]] \to \widehat{R}_{(k,u)}$ with $X_i \mapsto x_i$ for $i = 1, \dots, n$. We now put

$$C(\Lambda) := K[[n]]/\widehat{I}_\Lambda.$$

It follows that $C(\Lambda) \cong \widehat{R}_{(k,u)}$. Note that if Λ is an object of the subcategory $\mathbf{Coh}_K$, then $\widehat{I}_\Lambda = u_n^*(I_\Lambda)K[[n]]$ and $\widehat{\theta}_\Lambda$ is the base change of θ_Λ over u_n^*, showing that $C(\Lambda)$ agrees with the K-algebra defined previously. As for morphisms, let $\alpha\colon \Lambda \to \Gamma$ be as above. We have a commutative diagram

$$
\begin{array}{ccc}
K[[n]] & \xrightarrow{\widehat{\theta}_\Lambda} & \widehat{R}_{(k,u)} \\
\downarrow & & \downarrow{\scriptstyle \widehat{\alpha}_K} \\
K[[n+m]] & \xrightarrow{\widehat{\theta}_\Gamma} & \widehat{S}_{(l,v)}
\end{array}
$$

where the first vertical arrow is the natural inclusion. It follows that $\widehat{I}_\Lambda \subseteq \widehat{I}_\Gamma$, thus giving rise to a morphism $C(\alpha)\colon C(\Lambda) \to C(\Gamma)$ in $\mathbf{An}_K$. It is now straightforward to verify that C is a functor. Furthermore, the composition

$$\gamma_\Lambda\colon R \to \widehat{R}_{(k,u)} \cong C(\Lambda)$$

is faithfully flat and hence yields a faithfully flat natural transformation $\gamma\colon \underline{ring} \to C$ (extending the previously defined natural transformation γ). From this discussion it is clear that we have the following extension of Theorem 4.3:

4.21. Theorem. *There exists a functor* $\mathfrak{D}\colon \mathbf{Coh}_K^* \to \mathbf{Lef}_{\mathcal{W}}$ *and a faithfully flat natural transformation* $\eta\colon \underline{ring} \to \underline{ring} \circ \mathfrak{D}$. $\hfill\square$

4.22. Noether normalizations. To explain the advantages of this extended version, we need to discuss Noether normalizations. Let $(A, \mathfrak{m})$ be a complete Noetherian local ring with coefficient field k. A k-algebra homomorphism $k[[d]] \to A$ which is finite and injective is called a *Noether normalization* of A. (Here necessarily $d = \dim A$.) If $\mathbf{x}$ is an n-tuple generating $\mathfrak{m}$ whose first d entries form a system of parameters of A, then the k-algebra homomorphism $k[[d]] \to A$ given by $X_i \mapsto x_i$ for $i = 1, \ldots, d$ is a Noether normalization of A. (See for instance [40, Theorem 29.4].) However, by choosing $\mathbf{x}$ even more carefully, we can achieve this also for homomorphic images:

4.23. Lemma. *Let* $(A, \mathfrak{m})$ *be a complete Noetherian local ring with an uncountable algebraically closed coefficient field k and let $\mathcal{I}$ be a set of proper ideals of A. If the cardinality of $\mathcal{I}$ is strictly less than that of k, then there exists a surjective k-algebra homomorphism $\theta\colon k[[n]] \to A$ with the property that for every $I \in \mathcal{I}$, the k-algebra homomorphism $k[[d]] \to A/I$ obtained by composing the restriction of θ to the subring $k[[d]]$ with the natural surjection $A \to A/I$ is a Noether normalization of A/I, where $d := \dim A/I$.*

Proof. Choose generators $y_1, \ldots, y_n$ of $\mathfrak{m}$ and let

$$x_i = \sum_{j=1}^n a_{ij} y_j, \qquad i = 1, \ldots, n, \text{ and } a_{ij} \in k$$

be general k-linear combinations of the y_j. By [40, Theorem 14.14] there exists, for every $I \in \mathcal{I}$, a non-empty Zariski open subset U_I of $k^{n \times n}$ such that $x_1, \ldots, x_d$ (where $d = \dim A/I$) is a system of parameters modulo I for all $(a_{ij}) \in U_I$. Since the transcendence degree of k is strictly larger than $|\mathcal{I}|$, the intersection $\bigcap_{I \in \mathcal{I}} U_I$ is non-empty. Choose (a_{ij}) in this intersection and let $(x_1, \ldots, x_n)$ be the corresponding tuple. The k-algebra homomorphism $\theta\colon k[[n]] \to A$ given by $X_i \mapsto x_i$ for all i has the required properties. $\hfill\square$

Let us express the property stated in the lemma by saying that θ is *normalizing with respect to $\mathcal{I}$*. Let $\Lambda = (R, \mathbf{x}, k, u)$ be an object in $\mathbf{Coh}_K^*$ and let i denote the embedding of k in the algebraic closure of $u(k)$ in K induced by u. The natural homomorphism $R \to \widehat{R}_{(k,u)}$ factors as $R \to \widehat{R}_{(k,i)} \to \widehat{R}_{(k,u)}$. We say that Λ is *absolutely normalizing* if $\widehat{\theta}_\Lambda\colon K[[n]] \to \widehat{R}_{(k,u)}$ is normalizing with respect to the set of all ideals of the form $I\widehat{R}_{(k,u)}$ with I an ideal in $\widehat{R}_{(k,i)}$. (This definition will be useful in §5.) By Lemma 4.23 and noting that the cardinality of $\widehat{R}_{(k,i)}$ is at most $2^{|R|}$, we immediately get:

4.24. Corollary. *If we choose K sufficiently large (e.g., so that $2^{|R|} < |K|$), then there exists an absolutely normalizing object in $\mathbf{Coh}_K^*$ with underlying ring R.* $\hfill\square$

We say that Λ is *normalizing* if the entries of the tuple $\mathbf{x} = (x_1, \ldots, x_n)$ are in the maximal ideal $\mathfrak{m}\widehat{R}$ of $\widehat{R}$ and if the k^*-algebra homomorphism $k^*[[n]] \to \widehat{R}$ with $X_i \mapsto x_i$ for $i = 1, \ldots, n$ is normalizing with respect to the collection $\mathcal{I}$ consisting of the zero ideal and all minimal prime ideals of $\widehat{R}$. (As before k^* denotes the algebraic closure of k in $\widehat{R}$.) Since $\widehat{R}$ is Noetherian, $\mathcal{I}$ is finite, and hence:

4.25. Corollary. *If k is uncountable, then there exists a normalizing object in $\mathbf{Coh}_K^*$ with underlying ring R.*   □

4.26. *Remark.* Let us discuss now how we intend to apply Theorem 4.3 and its extension, Theorem 4.21, in practice. With aid of a faithfully flat Lefschetz extension of an equicharacteristic zero Noetherian local ring R, we'll define in §6 a non-standard tight closure relation on R, and in §7, a big Cohen-Macaulay algebra for R. If we only are interested in the ring R itself, then no functoriality is necessary, and we remarked already that the proof in that case is much simpler, as it only relies on Proposition 3.5.

Functoriality comes in when we are dealing with several rings at the same time, and when we need to compare the constructions made in each of these rings. We explain the strategy in the case of a single local homomorphism $\alpha\colon R \to S$ between equicharacteristic zero Noetherian local rings. Choose an algebraically closed Lefschetz field K of sufficiently large cardinality (for instance larger than $2^{|R|}$ and $2^{|S|}$) and choose an embedding of the residue field k_S of S into K. Denote the compostion $S \to k_S \to K$ by v and let $u := v \circ \alpha$. Choose a quasi-coefficient field k of R and then a quasi-coefficient field l of S containing $\alpha(k)$. Finally, choose a tuple $\mathbf{x}$ in R generating its maximal ideal and enlarge the tuple $\alpha(\mathbf{x})$ to a generating tuple $\mathbf{y}$ of the maximal ideal of S. These data yield two objects $\Lambda := (R, \mathbf{x}, k, u)$ and $\Gamma := (S, \mathbf{y}, l, v)$ of $\mathbf{Coh}_K$ and α induces a morphism between them. We take $\mathfrak{D}(\Lambda)$ and $\mathfrak{D}(\Gamma)$ as the faithfully flat Lefschetz extensions of R and S respectively, and use $\mathfrak{D}(\alpha)$ to go from one to the other. Of course, in this way, the closure operations defined on R and S, and similarly, the big Cohen-Macaulay algebras associated to them, depend on the choices made, but this will not cause any serious problems. Therefore, we will often simply denote the Lefschetz extensions by $\mathfrak{D}(R)$ and $\mathfrak{D}(S)$ with $\mathfrak{D}(\alpha)\colon \mathfrak{D}(R) \to \mathfrak{D}(S)$ the homomorphism between them.

For certain α, more adequate choices for the quadruples Λ and Γ (and hence for the Lefschetz extensions $\mathfrak{D}(R)$ and $\mathfrak{D}(S)$) can be made. For instance, this is the case if α is *unramified*, that is to say, if the image of the maximal ideal of R generates the maximal ideal in S and α induces an algebraic extension of the residue fields. In that case, we can take $l = \alpha(k)$ and $\mathbf{y} = \alpha(\mathbf{x})$. It follows that $C(\alpha)\colon C(\Lambda) \to C(\Gamma)$ is also unramified, whence $\mathfrak{D}(\alpha)$ sends the maximal ideal of $\mathfrak{D}(R)$ to the maximal ideal of $\mathfrak{D}(S)$. We'll tacitly assume that whenever α is unramified (for instance if α is surjective), then we choose $\mathfrak{D}(R)$ and $\mathfrak{D}(S)$ with these additional properties. (See also §4.28 below.)
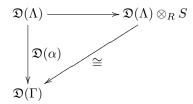
In the above construction of Λ, after we chose k and u, we could have chosen the tuple $\mathbf{x}$ with entries in $\widehat{R}_{(k,u)}$, so that the resulting Λ is only an object in $\mathbf{Coh}_K^*$. This has the following advantage: by an application of Corollary 4.24, we now may choose Λ so that it is absolutely normalizing. We express this by saying that the corresponding Lefschetz extension $\mathfrak{D}(R)$ $(:= \mathfrak{D}(\Lambda))$ is *absolutely normalizing*. Similarly, we also say that $\mathfrak{D}(R)$ is *normalizing* if Λ is normalizing. One easily proves that if $\alpha\colon R \to S$ is a local homomorphism as before, then Λ with underlying ring R and Γ with underlying ring S can be chosen so that α is a morphism $\Lambda \to \Gamma$ and Γ and Λ are absolutely normalizing. If moreover α is surjective and an absolutely normalizing object Λ of $\mathbf{Coh}_K^*$ with underlying ring R is given, then Γ with underlying ring S chosen as above is also absolutely normalizing.

Next, we extend Proposition 4.11. We call a morphism $\Lambda \to \Gamma$ in $\mathbf{Coh}_K^*$ *finite* if the underlying homomorphism $\alpha\colon R \to S$ is finite.

4.27. Proposition. *If $\alpha\colon \Lambda \to \Gamma$ is a finite morphism in $\mathbf{Coh}_K^*$, then $\mathfrak{D}(\alpha)$ is also finite. If in addition α induces an isomorphism on the residue fields, then the natural map*

$$\mathfrak{D}(\Lambda) \otimes_R S \to \mathfrak{D}(\Gamma)$$

is an isomorphism, making the diagram

$$\begin{array}{ccc} \mathfrak{D}(\Lambda) & \longrightarrow & \mathfrak{D}(\Lambda) \otimes_R S \\ \downarrow{\scriptstyle\mathfrak{D}(\alpha)} & \swarrow{\scriptstyle\cong} & \\ \mathfrak{D}(\Gamma) & & \end{array}$$

commutative. In particular, α is injective (respectively, surjective) if and only if $\mathfrak{D}(\alpha)$ is.

Proof. In view of Proposition 4.11 and Corollary 4.12, it suffices to show the analogous statements with $\mathfrak{D}$ replaced by the functor C. If $\alpha\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ is finite, then all the maps in (4.18.1) are finite and hence so is $C(\alpha)$. Assume next that α induces an isomorphism on the residue fields. By the maximality property of coefficient fields we have $\widehat{\alpha}(k^*) = l^*$. Since the canonical map $\widehat{R} \otimes_R S \to \widehat{S}$ is an isomorphism by [40, Theorem 8.7], we get a canonical isomorphism

$$(\widehat{R} \otimes_{k^*} K) \otimes_R S \cong (\widehat{R} \otimes_R S) \otimes_{k^*} K \cong \widehat{S} \otimes_{k^*} K \cong \widehat{S} \otimes_{l^*} K.$$

Moreover, the $\mathfrak{m}(\widehat{S}\otimes_{l^*} K)$-adic topology on $\widehat{S}\otimes_{l^*} K$ is equivalent with its $\mathfrak{n}(\widehat{S}\otimes_{l^*} K)$-adic topology, since $\mathfrak{m}S$ is $\mathfrak{n}$-primary. Hence taking completions and using [40, Theorem 8.7] once more, we get a canonical isomorphism

$$\widehat{R}_{(k,v)} \otimes_R S \cong \widehat{S}_{(l,v)}.$$

This in turn gives rise to a canonical isomorphism $C(\Lambda) \otimes_R S \cong C(\Gamma)$, which fits in a analogous commutative diagram as the above one. $\qquad\square$

4.28. **Quotients.** Given an object $\Lambda = (R, \mathbf{x}, k, u)$ in $\mathbf{Coh}_K^*$ and an ideal I of R, we define the *quotient object* Λ/I as the quadruple $(R/I, \bar{\mathbf{x}}, k, \bar{u})$, where we identify k with its image in R/I, where $\bar{\mathbf{x}}$ denotes the image of $\mathbf{x}$ in $\widehat{R}_{(k,u)}/I\widehat{R}_{(k,u)}$ and where $\bar{u}$ is the factorization of u through R/I. The residue map $\pi\colon R \to R/I$ gives rise to a morphism $\Lambda \to \Lambda/I$. It follows from Proposition 4.27 that π induces a surjective map $\mathfrak{D}(\Lambda) \to \mathfrak{D}(\Lambda/I)$ and one easily checks that its kernel is $I\mathfrak{D}(\Lambda)$. If Λ is absolutely normalizing, then so is Λ/I.

4.29. **Further basic properties.** Recall from §4.4 that $\mathbf{K} = \mathfrak{D}(0)$ is just the ultrapower $K^{\mathcal{U}}$. By construction, $\mathbf{K}$ is a coefficient field of $\mathfrak{D}(\Lambda)$, for every Λ, and $\mathfrak{D}(\alpha)\colon \mathfrak{D}(\Lambda) \to \mathfrak{D}(\Gamma)$ is a morphism of analytic $\mathbf{K}$-algebras with respect to $\mathcal{W}$ (as defined in §1.7), for every morphism $\alpha\colon \Lambda \to \Gamma$ in $\mathbf{Coh}_K^*$. The following is an analogue of Proposition 3.1.

4.30. **Proposition.** *For each n there exists an exact sequence*

$$0 \to \mathrm{Inf}(\mathfrak{D}(n)) \to \mathfrak{D}(n) \xrightarrow{\pi} \mathbf{K}[[n]] \to 0$$

where π is a $K[[n]]$-algebra homomorphism.

Proof. Recall that $\mathrm{Inf}(\mathfrak{D}(n))$ denotes the ideal of infinitesimals of $\mathfrak{D}(n)$, that is to say, the intersection of all $\mathfrak{m}^d\mathfrak{D}(n)$, where $\mathfrak{m} := (X_1, \ldots, X_n)K[[n]]$. Define $\pi\colon \mathfrak{D}(n) \to \mathbf{K}[[n]]$ as follows. Take an element $f \in \mathfrak{D}(n)$ and realize it as an ultraproduct of power series $f_w \in K_w[[n]]$, say of the form

$$f_w = \sum_\nu a_{\nu w} X^\nu$$

with $a_{\nu w} \in K_w$, where ν ranges over $\mathbb{N}^n$. For each such ν let $a_\nu \in \mathbf{K}$ be the ultraproduct of the $a_{\nu w}$. Define now $\pi(f)$ as the power series $\sum_\nu a_\nu X^\nu$. We leave it to the reader to verify that π is a well-defined, surjective $K[n]$-algebra homomorphism, and that its kernel is equal to $\mathrm{Inf}(\mathfrak{D}(n))$. (The argument is the same as in the proof of Proposition 3.1.) It remains to show that it is in fact a $K[[n]]$-algebra homomorphism. Let $f \in K[[n]]$ and choose polynomials $f_l \in K[n]$ so that $f \equiv f_l \bmod \mathfrak{m}^l$. It follows that

$$\eta_n(f) \equiv \eta_n(f_l) = f_l \mod \mathfrak{m}^l \mathfrak{D}(n).$$

Taking the image under π shows that

$$\pi(\eta_n(f)) \equiv f_l \equiv f \mod \mathfrak{m}^l \mathbf{K}[[n]].$$

Since this holds for all l, we get that $\pi(\eta_n(f)) = f$, proving that π is a $K[[n]]$-algebra homomorphism. $\square$

4.31. *Remark.* The ultraproduct of the i-th partial derivative on each $K_w[[n]]$, for $i = 1, \ldots, n$, is a $\mathbf{K}$-linear endomorphism of $\mathfrak{D}(n)$, which we denote again by $\partial/\partial X_i$. It follows that

$$\pi \left(\frac{\partial a}{\partial X_i} \right) = \frac{\partial(\pi(a))}{\partial X_i}$$

for each $a \in \mathfrak{D}(n)$. In particular, for every $f \in K[[n]]$ we have

$$\varepsilon(f) := \eta_n \left(\frac{\partial f}{\partial X_i} \right) - \frac{\partial(\eta_n(f))}{\partial X_i} \in \mathrm{Inf}(\mathfrak{D}(n)).$$

The map $f \mapsto \varepsilon(f) \colon K[[n]] \to \mathrm{Inf}(\mathfrak{D}(n))$ is a derivation which is trivial on $K[n]$. We do not know whether $\varepsilon(f) = 0$ for all $f \in K[[n]]$. (Note that $\Omega_{K[[n]]/K[n]} \neq 0$.)

4.32. **Corollary.** *For each $\Lambda = (R, \mathbf{x}, k, u)$ in $\mathbf{Coh}_K^*$ we have an isomorphism of R-algebras*

$$\mathfrak{D}(R)/\mathrm{Inf}(\mathfrak{D}(R)) \cong \mathbf{K}[[n]]/\widehat{I}_\Lambda \mathbf{K}[[n]] \cong \widehat{R}_{(k, \eta_0 \circ u)},$$

where $\mathfrak{D}(R) := \mathfrak{D}(\Lambda)$. If $\mathfrak{n}$ is an $\mathfrak{m}$-primary ideal of R, then $\mathfrak{n}\mathfrak{D}(R)$ is $\mathfrak{m}\mathfrak{D}(R)$-primary and

$$(4.32.1) \qquad\qquad \mathfrak{D}(R)/\mathfrak{n}\mathfrak{D}(R) \cong (R/\mathfrak{n})^\mathcal{U}.$$

Proof. For the first statement use that the base change modulo $\widehat{I}_\Lambda$ of the $K[[n]]$-algebra homomorphism π from Proposition 4.30 yields an epimorphism $\mathfrak{D}(R) \to \mathbf{K}[[n]]/\widehat{I}_\Lambda \mathbf{K}[[n]]$. One verifies that its kernel is precisely $\mathrm{Inf}(\mathfrak{D}(R))$. The second isomorphism is then clear since $K[[n]]/\widehat{I}_\Lambda \cong \widehat{R}_{(k, u)}$. (Recall that $\eta_0 \colon K \to \mathbf{K} = K^\mathcal{W}$ is the diagonal embedding.)

Now let $\mathfrak{n}$ be an $\mathfrak{m}$-primary ideal of R, say $\mathfrak{m}^l \subseteq \mathfrak{n}$. Then $\mathfrak{m}^l \mathfrak{D}(R) \subseteq \mathfrak{n}\mathfrak{D}(R)$, hence $\mathfrak{n}\mathfrak{D}(R)$ is $\mathfrak{m}\mathfrak{D}(R)$-primary. To establish (4.32.1) we first treat the case that $R = K[[n]]$. By Proposition 4.30, π induces an isomorphism

$$\mathfrak{D}(n)/\mathfrak{n}\mathfrak{D}(n) \cong \mathbf{K}[[n]]/\mathfrak{n}\mathbf{K}[[n]].$$

The natural homomorphism $\mathbf{K}[[n]] \to (K[[n]]/\mathfrak{n})^\mathcal{U}$ has kernel $\mathfrak{n}\mathbf{K}[[n]]$ (use Łoś' Theorem). The general case follows from this by base change. $\square$

4.33. **A note of caution— unnested conditions.** In the following we fix a natural number n. For a field L and $i \in \{1, \ldots, n\}$ let us write

$$L[[\,\widehat{\imath}\,]] := L[[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]].$$

Let $\mathfrak{D}(\,\widehat{\imath}\,)$ be the ultraproduct of the $K_w[[\,\widehat{\imath}\,]]$. The natural inclusion $\mathfrak{D}(\,\widehat{\imath}\,) \subseteq \mathfrak{D}(n)$ is a section of the canonical epimorphism

$$\mathfrak{D}(n) \to \mathfrak{D}(n)/X_i\mathfrak{D}(n) \cong \mathfrak{D}(\,\widehat{\imath}\,).$$

However, it is not true in general that $\eta_n \colon K[[n]] \to \mathfrak{D}(n)$ maps $K[[\,\widehat{\imath}\,]]$ inside $\mathfrak{D}(\,\widehat{\imath}\,)$ for all i (the exception being of course $i = n$ by (4.4.1)). This is rather surprising since after all, η_n sends a power series f to a limit of its truncations in $\mathfrak{D}(n)$ and if f does not involve X_i then neither does each truncation, yet the limit element must involve X_i.

To prove that such inclusions cannot hold in general, we use an example due to Roberts in [46], which was designed to be a counterexample to a question of Hochster on solid closure. Namely, suppose for $n = 6$, we would have inclusions

(4.33.1) $$\eta_6(K[[\,\widehat{\imath}\,]]) \subseteq \mathfrak{D}(\,\widehat{\imath}\,)$$

for $i = 4, 5, 6$. Let $z := X_1^2 X_2^2 X_3^2$ and $a_i := X_i^3$ for $i = 1, 2, 3$. Given a field L, the monomial z lies in the *solid closure* of the ideal $(a_1, a_2, a_3)L[[3]]$ if and only if

$$f := zX_4X_5X_6 + a_1X_5X_6 + a_2X_4X_6 + a_3X_4X_5 \in \mathbb{Z}[6]$$

viewed as an element of $L[[6]]$, has a non-zero multiple inside the L-subspace

$$L[[\,\widehat{4}\,]] + L[[\,\widehat{5}\,]] + L[[\,\widehat{6}\,]]$$

of $L[[6]]$. (See [25, §9].) With Hochster we say that this non-zero multiple of f is *special*. If (4.33.1) holds, then for $L := K$ the image under η_6 of such a non-zero multiple lies in the K-subspace

$$\mathfrak{D}(\,\widehat{4}\,) + \mathfrak{D}(\,\widehat{5}\,) + \mathfrak{D}(\,\widehat{6}\,).$$

of $\mathfrak{D}(6)$. By Łos' Theorem, f, as an element of $K_w[[6]]$, has then a non-zero multiple which is special for almost all w. This in turn means that z, viewed as an element of $K_w[[3]]$, lies in the solid closure of $(a_1, a_2, a_3)K_w[[3]]$. By [25, Theorem 8.6] solid closure is trivial in $K_w[[3]]$ (since $K_w[[3]]$ is regular of positive characteristic). Hence z lies in $(a_1, a_2, a_3)K_w[[3]]$, which is clearly false.

The failure of the existence of inclusions (4.33.1) bears a strong resemblance to the fact that there is no Artin Approximation for unnested conditions (see our discussion in §3.3).

5. Transfer of Structure

Throughout this section $(R, \mathfrak{m})$ denotes an equicharacteristic zero Noetherian local ring, and K is a Lefschetz field with respect to some ultraset with underlying set equal to the set of the prime numbers, whose components K_p are algebraically closed fields of characteristic p. Whenever necessary, we assume that K has cardinality $> 2^{|R|}$. We fix once and for all an object $\Lambda = (R, k, \mathbf{x}, u)$ of $\mathbf{Coh}_K^*$ with underlying ring R. (We might on occasion require some additional properties for Λ, such as being absolutely normalizing.) By abuse of notation, we write $\mathfrak{D}(R)$ for $\mathfrak{D}(\Lambda)$. We view $\mathfrak{D}(R)$ as an R-algebra via the faithfully flat map $\eta_\Lambda \colon R \to \mathfrak{D}(R)$ and often surpress this map in our notation. In particular, given an ideal I in R, we simply write $I\mathfrak{D}(R)$ for the ideal in $\mathfrak{D}(R)$ generated by $\eta_\Lambda(I)$. Moreover, we construct a Lefschetz hull for R/I always by means of the quotient Λ/I, as explained in §4.28. In particular, $\mathfrak{D}(R/I) \cong \mathfrak{D}(R)/I\mathfrak{D}(R)$. The other notations introduced in §4 remain in force.

5.1. **Approximations.** By construction, $\mathfrak{D}(R)$ is an ultraproduct (with respect to some unspecified ultraset) of equicharacteristic complete Noetherian local rings R_w with algebraically closed residue field K_w (of prime characteristic $p(w)$). We think of R_w as an *approximation* of R. Each R_w is of the form $K_w[[n]]/I_w$, where I_w are ideals whose ultraproduct is equal to $\widehat{I}_\Lambda\mathfrak{D}(n)$ (in the notation of §4.19). In this section, we make more precise how the R_w play the role of a reduction modulo p of R. A similar study for affine K-algebras was carried out in [55] and the subsequent papers, using effective bounds and the resulting first-order definability (as established in [18, 49]). Since no such tool is available in the present situation, our arguments are purely algebraic. Here is a first example:

5.2. **Theorem.**

   (5.2.1) *Almost all R_w have the same dimension* (*respectively, embedding dimension or depth*) *as R.*
   (5.2.2) *Almost all R_w are regular* (*respectively, Cohen-Macaulay or Gorenstein*) *if and only if R has the same property.*

   Before we begin the proof, let us introduce some more notations. Given an element $a \in \mathfrak{D}(R)$ choose elements $a_w \in R_w$ whose ultraproduct is a. We call a_w an *approximation* of a. If a'_w is another choice of elements whose ultraproduct is a, then $a_w = a'_w$ for almost all w. We use similar terminology for tuples of elements in $\mathfrak{D}(R)$, and given a finitely generated ideal $I = (a_1, \ldots, a_s)\mathfrak{D}(R)$ of $\mathfrak{D}(R)$, let $I_w := (a_{1w}, \ldots, a_{sw})R_w$, where a_{iw} is an approximation of a_i. The ultraproduct of the I_w is I, and we call I_w an *approximation* of I. If we choose different generators and approximations of these generators and denote the resulting ideals by I'_w, then the ultraproduct of the I'_w is again I and therefore $I_w = I'_w$ for almost all w. With an *approximation* of an ideal I of R we mean an approximation of its extension $I\mathfrak{D}(R)$ to an ideal of $\mathfrak{D}(R)$. Note that then R_w/I_w is an approximation of R/I. By faithful flatness of $R \to \mathfrak{D}(R)$ we have:

5.3. **Lemma.** *If I and J are ideals of R with approximations I_w and J_w, then*

   (5.3.1) $I\mathfrak{D}(R) \cap R = I$,
   (5.3.2) $I\mathfrak{D}(R) \cap J\mathfrak{D}(R) = (I \cap J)\mathfrak{D}(R)$,
   (5.3.3) $\big(I\mathfrak{D}(R) :_{\mathfrak{D}(R)} J\mathfrak{D}(R)\big) = (I :_R J)\mathfrak{D}(R)$,

and the ideals in (5.3.2) *and* (5.3.3) *have approximations $I_w \cap J_w$ and $(I_w :_{R_w} J_w)$, respectively.*

   Let $\mathfrak{m}$ be the maximal ideal of R. As a first step in the proof of Theorem 5.2 we show the following lemma, of interest in its own right:

5.4. **Lemma.** *A d-tuple $\mathbf{z} = (z_1, \ldots, z_d) \in R^d$ is a system of parameters for R if and only if almost every $\mathbf{z}_w$ is a system of parameters for R_w, where $\mathbf{z}_w = (z_{1w}, \ldots, z_{dw})$ is an approximation of $\mathbf{z}$. Similarly, $\mathbf{z}$ is an R-regular sequence if and only if $\mathbf{z}$ is a $\mathfrak{D}(R)$-regular sequence if and only if almost every $\mathbf{z}_w$ is an R_w-regular sequence.*

Proof. Suppose $\mathbf{z}$ is a system of parameters for R, so $d = \dim R$. We claim that almost every $\mathbf{z}_w$ is a system of parameters for R_w. We have $\mathfrak{m}^r \subseteq (z_1, \ldots, z_d)R$ for some r, and since this is preserved in $\mathfrak{D}(R)$, we get by Łos' Theorem that $\mathfrak{m}_w^r \subseteq (z_{1w}, \ldots, z_{dw})R_w$, for almost all w. This shows that almost all R_w have dimension at most d, and it suffices to shows that $\dim R_w = d$ for almost all w. Suppose on the contrary that $\dim R_w < d$ for almost all w. We may assume, after renumbering if necessary, that the ideal $\mathfrak{n}_w := (z_{1w}, \ldots, z_{d-1,w})R_w$ of R_w is $\mathfrak{m}_w$-primary for almost all w. For those w let $r_w \in \mathbb{N}$ be

minimal such that $(z_{dw})^{r_w} \in \mathfrak{n}_w$. By Noetherianity of R, we have for some s that

$$(\mathfrak{n} : z_d^s) = (\mathfrak{n} : z_d^r)$$

for all $r \geq s$, where $\mathfrak{n} := (z_1, \ldots, z_{d-1})R$. By (5.3.3) we get

$$
\begin{aligned}
(5.4.1) \qquad \big(\mathfrak{n}\mathfrak{D}(R) :_{\mathfrak{D}(R)} z_d^r\big) &= (\mathfrak{n} :_R z_d^r)\mathfrak{D}(R) \\
&= (\mathfrak{n} :_R z_d^s)\mathfrak{D}(R) \\
&= \big(\mathfrak{n}\mathfrak{D}(R) :_{\mathfrak{D}(R)} z_d^s\big),
\end{aligned}
$$

for all $r \geq s$. Suppose $r_w > s$ for almost all w, and let $b \in \mathfrak{D}(R)$ equal the ultra-product of the $(z_{dw})^{r_w-s-1}$. By Łos' Theorem, $bz_d^{s+1} \in \mathfrak{n}\mathfrak{D}(R)$. By (5.4.1), we have $bz_d^s \in \mathfrak{n}\mathfrak{D}(R)$ and hence, by Łos' Theorem once more, $(z_{dw})^{r_w-1} \in \mathfrak{n}_w$ for almost all w, contradicting the minimality of r_w. Therefore, $r_w \leq s$ and hence $(z_{dw})^s \in \mathfrak{n}_w$, for almost all w. By Łos' Theorem, this yields $z_d^s \in \mathfrak{n}\mathfrak{D}(R)$ and hence $z_d^s \in \mathfrak{n}$ by faithful flatness of $R \to \mathfrak{D}(R)$, contradicting that $\mathbf{z}$ is a system of parameters for R. Conversely, assume that $\mathbf{z}_w$ is a system of parameters for R_w for almost all w. Then $\dim R_w = d$ for almost all w. We have already shown $\dim R_w = \dim R$ for almost all w, hence $\dim R = d$. Therefore it suffices to show that $(z_1, \ldots, z_d)R$ is $\mathfrak{m}$-primary. Now $(z_{1w}, \ldots, z_{dw})R_w$ is $\mathfrak{m}_w$-primary, hence $\dim R_w/(z_{1w}, \ldots, z_{dw})R_w = 0$ for almost all w. The rings $S_w := R_w/(z_{1w}, \ldots, z_{dw})R_w$ are approximations to $S := R/(z_1, \ldots, z_d)R$. Thus $\dim S = 0$ by what we have shown above, or equivalently, $(z_1, \ldots, z_d)R$ is $\mathfrak{m}$-primary.

If $\mathbf{z}$ is R-regular, then $\mathbf{z}$ is also $\mathfrak{D}(R)$-regular due to faithful flatness of $R \to \mathfrak{D}(R)$, see [40, Exercise 16.4]. By Łos' Theorem, if $\mathbf{z}$ is $\mathfrak{D}(R)$-regular, then almost all $\mathbf{z}_w$ are R_w-regular. Finally, suppose that almost all $\mathbf{z}_w$ are R_w-regular, and let $i \in \{1, \ldots, d-1\}$ and $a \in R$ with $az_{i+1} \in (z_1, \ldots, z_i)R$. Then we have $a_w z_{i+1,w} \in (z_{1w}, \ldots, z_{iw})R_w$ for almost all w, hence $a_w \in (z_{1w}, \ldots, z_{iw})R_w$ for almost all w, and therefore $a \in (z_1, \ldots, z_i)\mathfrak{D}(R)$, by Łos' Theorem. Now (5.3.1) yields $a \in (z_1, \ldots, z_i)R$. Similarly one shows that $1 \notin (z_1, \ldots, z_d)R$. Hence $\mathbf{z}$ is R-regular. $\qquad \square$

Proof of Theorem 5.2. Suppose that R has embedding dimension e, so that we can write $\mathfrak{m} = (z_1, \ldots, z_e)R$ for some $z_1, \ldots, z_e \in R$. Hence $\mathfrak{m}_w = (z_{1w}, \ldots, z_{ew})R_w$, where z_{iw} is an approximation of z_i. If the embedding dimension of almost all R_w would be strictly less than e, then after renumbering if necessary, $\mathfrak{m}_w = (z_{1w}, \ldots, z_{e-1,w})R_w$ for almost all w (by Nakayama's Lemma). Therefore $\mathfrak{m}\mathfrak{D}(R) = (z_1, \ldots, z_{e-1})\mathfrak{D}(R)$ by Łos' Theorem, hence $\mathfrak{m} = (z_1, \ldots, z_{e-1})R$ by faithful flatness of $R \to \mathfrak{D}(R)$, contradiction. From Lemma 5.4 it follows that $\dim R = \dim R_w$ for almost all w. Now suppose R has depth d, and let $\mathbf{z} = (z_1, \ldots, z_d)$, with $z_i \in \mathfrak{m}$ for all i be an R-regular sequence. By Lemma 5.4 almost every approximation $\mathbf{z}_w \in \mathfrak{m}$ of $\mathbf{z}$ is an R_w-regular sequence in $\mathfrak{m}_w$ and hence R_w has depth at least d, for almost all w. On the other hand, since R has depth d, the quotient $R/(z_1, \ldots, z_d)R$ has depth zero, that is to say, $\mathfrak{m}$ is an associated prime of $(z_1, \ldots, z_d)R$. Choose $s \notin (z_1, \ldots, z_d)R$ such that $s\mathfrak{m} \subseteq (z_1, \ldots, z_d)R$ and let s_w be an approximation of s. By Łos' Theorem, $s_w\mathfrak{m}_w \subseteq (z_{1w}, \ldots, z_{dw})R_w$ and $s_w \notin (z_{1w}, \ldots, z_{dw})R_w$ for almost all w. Hence the depth of almost all R_w equals d.

Since R is regular (respectively, Cohen-Macaulay) if and only if its dimension is equal to its embedding dimension (respectively, to its depth), the desired transfer follows from the preservation of these invariants in the approximations. Let $d := \dim R$, and recall that R Gorenstein means that R is Cohen-Macaulay and for some (equivalently, every) R-regular sequence $\mathbf{z} = (z_1, \ldots, z_d)$ in $\mathfrak{m}$, the socle of $R/\mathfrak{n}$ is principal, where $\mathfrak{n} := (z_1, \ldots, z_d)R$; that is, there exists $a \in R$ such that $(\mathfrak{n} : \mathfrak{m}) = \mathfrak{n} + aR$. In order to show that R is

Gorenstein if and only if almost all R_w are, we may assume, by our agument above, that R and hence almost all R_w are Cohen-Macaulay. Suppose that R is Gorenstein. Let $\mathbf{z}$, $\mathfrak{n}$, and a as above, and let a_w and $\mathfrak{n}_w$ be approximations of a and $\mathfrak{n}$ respectively, so $\mathfrak{n}_w$ is generated by an R_w-sequence, for almost all w. By Łos' Theorem, we get

$$(5.4.2) \qquad (\mathfrak{n}_w : \mathfrak{m}_w) = \mathfrak{n}_w + a_w R_w,$$

for almost all w. It follows that almost all R_w are Gorenstein. Conversely, if almost all R_w are Gorenstein, then there exist $a_w \in R_w$ satisfying (5.4.2). By Łos' Theorem,

$$(5.4.3) \qquad \big(\mathfrak{n}\mathfrak{D}(R) :_{\mathfrak{D}(R)} \mathfrak{m}\mathfrak{D}(R)\big) = \mathfrak{n}\mathfrak{D}(R) + a_\infty \mathfrak{D}(R)$$

where $a_\infty \in \mathfrak{D}(R)$ is the ultraproduct of the a_w.

Let f and g be elements in $(\mathfrak{n} : \mathfrak{m})$ but not in $\mathfrak{n}$. From (5.4.3) it follows that $f \equiv a_\infty b_\infty \bmod \mathfrak{n}\mathfrak{D}(R)$ and $g \equiv a_\infty c_\infty \bmod \mathfrak{n}\mathfrak{D}(R)$, for some $b_\infty, c_\infty \in \mathfrak{D}(R)$. By faithful flatness of $R \to \mathfrak{D}(R)$, neither f nor g belongs to $\mathfrak{n}\mathfrak{D}(R)$, so that b_∞ and c_∞ must be units in $\mathfrak{D}(R)$. In particular, $f \in g\mathfrak{D}(R) + \mathfrak{n}\mathfrak{D}(R)$ and $g \in f\mathfrak{D}(R) + \mathfrak{n}\mathfrak{D}(R)$. Therefore, again by faithful flatness, $f \in gR + \mathfrak{n}$ and $g \in fR + \mathfrak{n}$. Since this holds for every choice of f and g, the socle of $R/\mathfrak{n}$ is principal, showing that R is Gorenstein. $\qquad \square$

Since a Noetherian local ring is a discrete valuation ring (DVR) if and only if it has positive dimension and its maximal ideal is principal [40, Theorem 11.2], we get:

5.5. Corollary. *The following are equivalent:*

    (5.5.1) *R is a DVR;*
    (5.5.2) *almost every R_w is a DVR;*
    (5.5.3) *$\mathfrak{D}(R)$ is a valuation ring.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

5.6. Flatness and Noether normalization. Let Γ be an object in $\mathbf{Coh}_K^*$ with underlying ring S, and $\Lambda \to \Gamma$ a morphism in $\mathbf{Coh}_K^*$ with underlying homomorphism $\alpha \colon R \to S$. We denote the induced morphism $\mathfrak{D}(R) \to \mathfrak{D}(S) := \mathfrak{D}(\Gamma)$ by $\mathfrak{D}(\alpha)$. By definition, $\mathfrak{D}(\alpha)$ is an ultraproduct of K_w-homomorphisms $\alpha_w \colon R_w \to S_w$, where S_w is an approximation of S.

5.7. Proposition. *If $\alpha \colon R \to S$ is finite, then so are almost all α_w. If α moreover induces an isomorphism on the residue fields, then the following are equivalent:*

    (5.7.1) *α is flat;*
    (5.7.2) *$\mathfrak{D}(\alpha)$ is flat;*
    (5.7.3) *almost all α_w are flat.*

Proof. The first assertion and the implication $(5.7.1) \Rightarrow (5.7.2)$ are immediate by Proposition 4.27. From the commutative diagram (4.3.1) and the faithful flatness of $\eta_R := \eta_\Lambda$ and $\eta_S := \eta_\Gamma$ we get $(5.7.2) \Rightarrow (5.7.1)$. Hence remains to show that $(5.7.1)$ and $(5.7.3)$ are equivalent. We use the local flatness criterion [40, Theorem 22.3]: *a finitely generated module M over a local Noetherian ring $(A, \mathfrak{n})$ is flat if and only if $\mathrm{Tor}_1^A(A/\mathfrak{n}, M) = 0$.* Since η_R is flat we have an isomorphism of $\mathfrak{D}(R)$-modules

$$\mathfrak{D}(R) \otimes_R \mathrm{Tor}_1^R(R/\mathfrak{m}, S) \cong \mathrm{Tor}_1^{\mathfrak{D}(R)}\big(\mathfrak{D}(R) \otimes_R (R/\mathfrak{m}), \mathfrak{D}(R) \otimes_R S\big).$$

Moreover $\mathfrak{D}(R) \otimes_R (R/\mathfrak{m}) \cong \mathfrak{D}(R/\mathfrak{m}) = \mathfrak{D}(K)$, and $\mathfrak{D}(R) \otimes_R S \cong \mathfrak{D}(S)$ by Proposition 4.27. The finitely generated R-module S has a free resolution by finitely generated free R-modules (since R is Noetherian). Hence by the faithful flatness of $R \to \mathfrak{D}(R)$,

the finitely generated $\mathfrak{D}(R)$-module $\mathfrak{D}(S)$ has a free resolution by finitely generated free $\mathfrak{D}(R)$-modules. Since $\mathfrak{D}(K)$ is a field and hence coherent, Proposition 1.2 yields

$$\mathfrak{D}(R) \otimes_R \operatorname{Tor}_1^R(R/\mathfrak{m}, S) \cong \operatorname*{ulim}_w \operatorname{Tor}_1^{R_w}(K_w, S_w).$$

The Noetherian local ring R_w has residue field K_w, and S_w is finitely generated as a module over R_w, for almost all w. The claim now follows from the local flatness criterion and faithful flatness of $R \to \mathfrak{D}(R)$. $\qquad\square$

5.8. **Proposition.** *Let I be an ideal in R with approximations $I_w \subseteq R_w$, and $d = \dim R/I$. If $\mathfrak{D}(R)$ is absolutely normalizing, then the composition*

$$K_w[[d]] \subseteq K_w[[n]] \to R_w \to R_w/I_w$$

(*where the first map is given by inclusion and the remaining maps are the natural surjections*) *is a Noether normalization of R_w/I_w, for almost all w.*

Proof. By Remark 4.26, the natural K-algebra homomorphism

$$K[[d]] \to K[[n]] \xrightarrow{\widehat{\theta}_\Lambda} C(\Lambda) \to C(\Lambda)/IC(\Lambda) = C(\Lambda/I)$$

is injective and finite, hence a Noether normalization. By Proposition 4.11, applying $\mathfrak{D}$ yields a finite and injective homomorphism $\mathfrak{D}(d) \to \mathfrak{D}(R/I)$. By Łos' Theorem, the maps in the statement are therefore almost all injective and finite, since their ultraproduct is precisely $\mathfrak{D}(d) \to \mathfrak{D}(R/I)$. $\qquad\square$

5.9. *Remark.* Suppose $I = (0)$. Then the conclusion of the proposition holds if $\mathfrak{D}(R)$ is only assumed to be normalizing.

Let us elaborate some more on the Proposition 5.8. Let $T := K[[d]]$, where $0 \le d \le n$. We have a commutative diagram

$$
\begin{array}{ccc}
T[X_{d+1}, \dots, X_n] & \longrightarrow & \mathfrak{D}(d)[X_{d+1}, \dots, X_n] \\
\downarrow & & \downarrow \\
K[[n]] & \xrightarrow{\ \eta_n\ } & \mathfrak{D}(n).
\end{array}
$$

Hence given $f \in T[X_{d+1}, \dots, X_n]$ we may choose approximations f_w of f in the subring $T_w[X_{d+1}, \dots, X_n]$ of $K_w[[n]]$. Note that $T_w := K_w[[d]]$ is an approximation of T, so that f is the ultraproduct of the polynomials f_w of bounded degree, in the sense of §1.8. Given generators $f_1, \dots, f_r$ of an ideal J of $T[X_{d+1}, \dots, X_n]$ we let J_w be the ideal of $T_w[X_{d+1}, \dots, X_n]$ generated by $f_{1w}, \dots, f_{rw}$. (We think of J_w as an approximation of the ideal J.)

Suppose that $\mathfrak{D}(R)$ is normalizing. Recall that we denote the kernel of $\widehat{\theta}_\Lambda$ by $\widehat{I}_\Lambda$, and let $J := T[X_{d+1}, \dots, X_n] \cap \widehat{I}_\Lambda$, where as above $T = K[[d]]$, with $d := \dim R$. Then $J \cap T = (0)$, and the natural inclusion $T[X_{d+1}, \dots, X_n] \to K[[n]]$ induces an isomorphism

$$T[X_{d+1}, \dots, X_n]/J \to K[[n]]/\widehat{I}_\Lambda = C(\Lambda).$$

Hence for every ideal M of $T[X_{d+1}, \dots, X_n]$ containing J, we have $M = MK[[n]] \cap T[X_{d+1}, \dots, X_n]$. By the remark following the proposition above, we see that then $M_w = M_w K_w[[n]] \cap T_w[X_{d+1}, \dots, X_n]$ for almost all w. This fact is used in the proof of Theorem 5.31 below.

5.10. *Remark.* Because of its importance, let us give alternative arguments for (5.2.2) using Noether normalization. These arguments work if $\Lambda = (R, \mathbf{x}, k, u)$ is absolutely normalizing. (In fact, it is enough that $\widehat{\theta}_\Lambda$ be normalizing with respect to the zero ideal.) First, in all three cases we may replace R by $\widehat{R}_{(k,u)}$ and hence assume that R is in $\mathbf{An}_K$. (See Theorem 23.7, Corollary to Theorem 23.3, and Theorem 23.4, respectively, in [40].) Say $R = K[[n]]/I$ for some n and some ideal I of $K[[n]]$. The restriction of the K-algebra homomorphism $K[[n]] \to R$ with $X \mapsto \mathbf{x}$ to $T := K[[d]]$, where $d := \dim R$, is a Noether normalization of R. By Proposition 5.8, $T_w \to R_w$ is a Noether normalization of R_w, for almost all w. The proof of [40, Theorem 29.4] shows that R is regular if and only if $T \to R$ is surjective. Hence R is regular if and only if $\mathfrak{D}(T) \to \mathfrak{D}(R)$ is surjective (by Corollary 4.12) if and only if $T_w \to R_w$ is surjective for almost all w. Therefore R is regular if and only if almost each R_w is regular. By [12, Proposition 2.2.11], R is Cohen-Macaulay if and only if $T \to R$ is flat. By Proposition 5.7 this is equivalent with the flatness of almost all $T_w \to R_w$, which in turn is equivalent with almost all R_w being Cohen-Macaulay. Finally, for the Gorenstein property, observe that $R/\mathfrak{n}$ is Artinian, where $\mathfrak{n}$ is a parameter ideal of R, and so is $\mathfrak{D}(R/\mathfrak{n})$, as it is an ultrapower of $(R/\mathfrak{n}) \otimes_k K$ by (4.32.1). Since being Gorenstein is first order definable for Artinian local rings by [48], we get that $R/\mathfrak{n}$ is Gorenstein if and only if $\mathfrak{D}(R/\mathfrak{n})$ is if and only if almost all $R_w/\mathfrak{n}_w$ are. Since almost every $\mathfrak{n}_w$ is generated by an R_w-sequence, this is equivalent with R_w Gorenstein for almost all w.

5.11. **Hilbert-Samuel functions.** We now want to strengthen (5.2.2) and show that almost all R_w have the same Hilbert-Samuel function as R. For this, we assume that the reader is familiar with the fundamentals of the theory of standard bases in power series rings; for example, see [8]. We fix $n > 0$, and we denote by $\preceq$ the degree-lexicographic ordering on $\mathbb{N}^n$, that is, $\nu \preceq \mu$ if and only if $|\nu| < |\mu|$, or $|\nu| = |\mu|$ and $\nu \leq \mu$ lexicographically. Let L be a field. For every non-zero

$$f = \sum_\nu a_\nu X^\nu \in L[[n]] \qquad \text{(with } a_\nu \in L \text{ for all } \nu \in \mathbb{N}^n\text{)}$$

there exists a $\preceq$-smallest $\lambda \in \mathbb{N}^n$ with $a_\lambda \neq 0$, and we put $c(f) := a_\lambda$ and $v(f) := \lambda$. It is convenient to define $c(0) := 0$ and $v(0) := \infty$ with $\infty + \nu = \nu + \infty = \infty$ for all $\nu \in \mathbb{N}^n \cup \{\infty\}$. We extend $\preceq$ to $\mathbb{N}^n \cup \{\infty\}$ by $\mathbb{N}^n \prec \infty$. Note that v is a valuation on $L[[n]]$ with values in the ordered semigroup $(\mathbb{N}^n, \preceq)$, that is, for all $f, g \in L[[n]]$:

   (5.11.1) $v(f) = \infty \Longleftrightarrow f = 0$,
   (5.11.2) $v(fg) = v(f) + v(g)$, and
   (5.11.3) $v(f + g) \succeq \min\{v(f), v(g)\}$.
Given a subset $\mathfrak{s}$ of $L[[n]]$ we put

$$v(\mathfrak{s}) := \{v(f) : f \in \mathfrak{s}L[[n]]\} \subseteq \mathbb{N}^n \cup \{\infty\}$$

where $\mathfrak{s}L[[n]]$ denotes the ideal generated by $\mathfrak{s}$. Let $f, g_1, \ldots, g_m \in L[[n]]$. We call an expression

$$f = \sum_{i=1}^m q_i g_i \qquad \text{(where } q_1, \ldots, q_m \in L[[n]]\text{)}$$

such that $v(f) \preceq v(q_i) + v(g_i)$ for all i a *standard representation* of f with respect to $\mathfrak{s} = \{g_1, \ldots, g_m\}$ (in $L[[n]]$). Note that then $v(f)$ equals the ($\preceq$-) minimum of the $v(q_i) + v(g_i)$. If $L \subseteq L'$ is a field extension, and $f \in L[[n]]$ has a standard representation with respect to $\mathfrak{s}$ in $L'[[n]]$, then f has a standard representation with respect to $\mathfrak{s}$ in $L[[n]]$. (Since $L[[n]] \to L'[[n]]$ is faithfully flat.) Moreover:

5.12. Lemma. *An element f of $K[[n]]$ has a standard representation with respect to a subset $\mathfrak{s} = \{g_1, \ldots, g_m\}$ of $K[[n]]$ if and only if almost every f_w has a standard representation with respect to $\mathfrak{s}_w := \{g_{1w}, \ldots, g_{mw}\}$, where f_w and g_{iw} are approximations of f and g_i respectively.*

Proof. We may assume $f \neq 0$. Writing $f = f_0 + \varepsilon$ where $f_0 \in K[n]$ is homogeneous of degree $d := |v(f)|$ and $\varepsilon \in \mathfrak{m}^{d+1}$ we see that $c(f)_w = c(f_w)$ and $v(f) = v(f_w)$ for almost all w. Hence if $f = \sum_{i=1}^m q_i g_i$ is a standard representation of f with respect to $\mathfrak{s}$, then $f_w = \sum_{i=1}^m q_{iw} g_{iw}$ is a standard representation of f_w in terms of $\mathfrak{s}_w$, where q_{iw} is an approximation of q_i. Conversely, suppose that almost every f_w has a standard representation $f_w = \sum_{i=1}^m q_{iw} g_{iw}$ with respect to $\mathfrak{s}_w$, where $q_{iw} \in K_w[[n]]$. Since v is a valuation, there is some i such that $v(f_w) = v(q_{iw}) + v(g_{iw}) \preceq v(q_{jw}) + v(g_{jw})$ for all j and almost all w. Therefore, if we let q_j be the ultraproduct of the q_{jw} and π as in Proposition 4.30, then $v(f) = v(\pi(q_i)) + v(g_i) \preceq v(\pi(q_j)) + v(g_j))$ for all j, showing that $f = \sum_{i=1}^m \pi(q_i) g_i$ is a standard representation of f with respect to $\mathfrak{s}$ in $\mathfrak{D}(K)[[n]]$. Hence f has a standard representation with respect to $\mathfrak{s}$ in $K[[n]]$ by faithful flatness. $\square$

Every ideal I of $L[[n]]$ has a *standard basis,* that is, a finite subset $\mathfrak{s}$ of I such that every element of I has a standard representation with respect to $\mathfrak{s}$, or equivalently, such that $v(\mathfrak{s}) = v(I)$. (See [8, Theorem on p. 219].)

5.13. Proposition. *A subset $\mathfrak{s}$ of an ideal $I \subseteq K[[n]]$ is a standard basis for I if and only if its approximation $\mathfrak{s}_w$ is a standard basis for the approximation $I_w \subseteq K_w[[n]]$ of I, for almost all w. In particular we have $v(I) = v(I_w)$ for almost all w.*

Proof. We use the Buchberger criterion for standard bases: for non-zero $f, g \in L[[n]]$ we define

$$s(f, g) := c(g)X^\mu f - c(f)X^\nu g \in L[[n]]$$

where μ, ν are the multiindices in $\mathbb{N}^n$ such that $X^{\mu + v(f)} = X^{\nu + v(g)} =$ the least common multiple of $X^{v(f)}$ and $X^{v(g)}$. Then a finite subset $\mathfrak{e}$ of $L[[n]]$ is a standard basis of the ideal it generates if and only if $s(f, g)$ has a standard representation with respect to $\mathfrak{e}$, for all $0 \neq f, g \in \mathfrak{e}$ [8, Theorem 4.1]. The claim follows from this and Lemma 5.12, since if $f, g \in \mathfrak{s}$ are non-zero then their approximations f_w, g_w are non-zero and $s(f_w, g_w)$ is an approximation of $s(f, g)$ for almost all w. $\square$

Given a Noetherian local ring $(S, \mathfrak{n})$ we use χ_S to denote the Hilbert-Samuel function $d \mapsto \text{length}(S/\mathfrak{n}^{d+1})$ of S. By Corollary 4.32 we see that for fixed $d \in \mathbb{N}$, we have $\chi_R(d) = \chi_{R_w}(d)$ for almost all w. Proposition 5.13 implies the following stronger version:

5.14. Corollary. *For almost all w, we have $\chi_R = \chi_{R_w}$ (that is, $\chi_R(d) = \chi_{R_w}(d)$ for all d).*

Proof. Since $\widehat{R}_{(k,u)}/\mathfrak{m}^{d+1}\widehat{R}_{(k,u)} = (R/\mathfrak{m}^{d+1}) \otimes_k K$ for all d, we have $\chi_R = \chi_{\widehat{R}_{(k,u)}}$. For an ideal I of $L[[n]]$, the Hilbert-Samuel functions of $L[[n]]/I$ and $L[[n]]/I'$, where I' is the ideal generated by all X^ν with $\nu \in v(I)$, agree. Hence by Proposition 5.13 we obtain that $\chi_{\widehat{R}_{(k,u)}} = \chi_{R_w}$ for almost all w. $\square$

In particular, almost all R_w have the same Hilbert-Samuel polynomial as R, hence the same multiplicity, and we see once more that almost all R_w have the same dimension and the same embedding dimension as R. For later use we also show:

5.15. Lemma. *Let $f_1, \ldots, f_r \in K[[n]]$ and $\varepsilon_1, \ldots, \varepsilon_r \in \mathrm{Inf}(\mathfrak{D}(n))$, and consider the ideals $I = (f_1, \ldots, f_r)K[[n]]$ and $I_\varepsilon = (f_1 + \varepsilon_1, \ldots, f_r + \varepsilon_r)\mathfrak{D}(n)$ with respective approximations I_w and $I_{\varepsilon,w}$. Then $v(I_w) \subseteq v(I_{\varepsilon,w})$ and hence $\dim(K_w[[n]]/I_w) \geq \dim(K_w[[n]]/I_{\varepsilon,w})$, for almost all w.*

Proof. We may assume $r > 0$ and $f_i \neq 0$ for all i. Let $d := \max_i |v(f_i)|$. Then for almost all w we have $\varepsilon_{iw} \in \mathfrak{m}_w^{d+1}$ and hence $v(f_{iw} + \varepsilon_{iw}) = v(f_{iw}) = v(f)$ for almost all w. Let $\mathfrak{s} = \{g_1, \ldots, g_m\}$ be a standard basis for I. Then its approximation $\mathfrak{s}_w$ is a standard basis for I_w by Proposition 5.13, and thus $v(I_w) = v(\mathfrak{s}_w)$, for almost all w. For every $j \in \{1, \ldots, m\}$ there exists $i \in \{1, \ldots, r\}$ and $\nu \in \mathbb{N}^n$ with $v(g_j) = v(f_i) + \nu$. Hence $v(g_{jw}) = v(f_{iw}) + \nu = v(f_{iw} + \varepsilon_{iw}) + \nu$ for almost all w. This shows $v(I_w) = v(\mathfrak{s}_w) \subseteq v(I_{\varepsilon,w})$ for almost all w. $\qquad\square$

5.16. Irreducibility. Suppose that $\Lambda = (R, \mathbf{x}, k, u)$, and recall from the discussion before 4.24 that $i = u|k$ is the embedding of k into the algebraic closure of $u(k)$ inside K. We call R *absolutely analytically irreducible* if $\widehat{R}_{(k,i)}$ is a domain. This does not depend on the choice of k and u. (Cf. Remark 4.20.) *From now on up to and including §5.27 we assume that $\mathfrak{D}(R)$ is absolutely normalizing.*

5.17. Theorem. *The following statements are equivalent:*

(5.17.1) *R is absolutely analytically irreducible;*

(5.17.2) *$\mathfrak{D}(R)$ is a domain;*

(5.17.3) *almost all R_w are domains.*

We first establish some auxiliary facts needed in the proof. Let T be a domain with fraction field $F = \mathrm{Frac}(T)$. Let $Y = (Y_1, \ldots, Y_m)$ be a tuple of indeterminates, and let I be a finitely generated ideal of $T[Y]$. There exists a non-zero $\delta \in T$ with the following property: for all domains T' extending T, with fraction field $F' = \mathrm{Frac}(T')$, and all $f \in T'[Y]$ we have $f \in IF'[Y]$ if and only if $\delta f \in IT'[Y]$. (See, e.g., [4, Corollary 3.5].) In other words,

$$IF'[Y] \cap T'[Y] = \left(IT'[Y] :_{T'[Y]} \delta\right)$$

and therefore:

5.18. Lemma. *If T' is a domain extending T, with fraction field F', and T' is flat over T, then*

$$IF'[Y] \cap T'[Y] = \left(IF[Y] \cap T[Y]\right)T'[Y].$$

In the following proposition and lemma let $T = K[[d]]$ and $T^* = \mathfrak{D}(d)$.

5.19. Proposition. *If I is a prime ideal of $T[Y]$ with $I \cap T = (0)$, then $IT^*[Y]$ is a prime ideal of $T^*[Y]$ with $IT^*[Y] \cap T^* = (0)$.*

For the proof we need:

5.20. Lemma. *The fraction field F^* of T^* is a regular extension of F.*

Proof. Since $\mathrm{char}\, F = 0$ we only need to show that F is algebraically closed in F^*. Let $y \in F^*$ be algebraic over F. To show that $y \in F$ we may assume that y is integral over T. Since T^* is integrally closed it follows that $y \in T^*$. Let $P(Y) \in T[Y]$ be a monic polynomial of minimal degree such that $P(y) = 0$. Then $\pi(y)$ is a zero of P in $\mathbf{K}[[d]]$, where $\pi : T^* \to \mathbf{K}[[d]]$ is the surjective $K[[d]]$-algebra homomorphism from Proposition 4.30 and $\mathbf{K} = \mathfrak{D}(K)$. Since K is algebraically closed it follows (using Hensel's Lemma) that P has a zero in $K[[d]] = T$. By minimality of P, this zero is y, so $y \in T$ as required. $\quad\square$

Proof of Proposition 5.19. Suppose that I is prime and $I \cap T = (0)$, or equivalently, $IF[Y]$ is prime and $IF[Y] \cap T[Y] = I$. By Lemma 5.20, $IF^*[Y]$ is a prime ideal of $IF[Y]$. (See [9], Chapitre V, §15, Proposition 5 and §17, Corollaire to Proposition 1.) In particular $IT^*[Y] \cap T^* = (0)$. Since T^* is flat over T, by Lemma 5.18 we have

$$IF^*[Y] \cap T^*[Y] = \big(IF[Y] \cap T[Y]\big)T^*[Y] = IT^*[Y].$$

It follows that $IT^*[Y]$ is prime. $\square$

Proof of Theorem 5.17. By Łos' Theorem, almost all R_w are domains if and only if $\mathfrak{D}(R)$ is. Moreover, if this is the case, then every subring of the domain $\mathfrak{D}(R)$ is also a domain. Hence we only have to prove that if R is absolutely analytically irreducible, then $\mathfrak{D}(R)$ is a domain. Let us first assume that $\widehat{R}_{(k,u)}$ is a domain. Put $T := K[[d]]$ and let $J := T[X_{d+1}, \ldots, X_n] \cap \widehat{I}_\Lambda$, where $d = \dim R$. Since $\widehat{\theta}_\Lambda$ is a Noether normalization of $C(\Lambda)$,

$$\widehat{R}_{(k,u)} \cong T[X_{d+1}, \ldots, X_n]/J$$

and

$$\mathfrak{D}(R) \cong \mathfrak{D}(T)[X_{d+1}, \ldots, X_n]/J\mathfrak{D}(T)[X_{d+1}, \ldots, X_n].$$

(See Proposition 4.11 and the discussion following Proposition 5.8.) Now $\widehat{R}_{(k,u)}$ is a domain if and only if J is a prime ideal, and in this case, by Proposition 5.19, the expansion $J\mathfrak{D}(T)[X_{n+1}, \ldots, X_{n+m}]$ of J to an ideal of $\mathfrak{D}(T)[X_{n+1}, \ldots, X_{n+m}]$ remains prime. Hence $\mathfrak{D}(R)$ is a domain, as required. The proof of Theorem 5.17 is now completed by Lemma 5.21 below. $\square$

5.21. Lemma. *If $\widehat{R}_{(k,i)}$ is an integral domain then so is $\widehat{R}_{(k,u)}$.*

Proof. We write l for the algebraic closure of $u(k)$ inside K. It is easy to see that the unique extension of a Noether normalization $l[[d]] \to \widehat{R}_{(k,i)}$ of $\widehat{R}_{(k,i)}$ to a K-algebra homomorphism $K[[d]] \to \widehat{R}_{(k,u)}$ is a Noether normalization of $\widehat{R}_{(k,u)}$. Hence the argument above, which allowed us to transfer integrality from $\widehat{R}_{(k,u)}$ to $\mathfrak{D}(R)$, can be used to transfer integrality of $\widehat{R}_{(k,i)}$ to $\widehat{R}_{(k,u)}$, provided we know that the fraction field of $K[[d]]$ is a regular extension of the fraction field of $l[[d]]$. This is shown as in Lemma 5.20. $\square$

A prime ideal $\mathfrak{p}$ of R is called *absolutely analytically prime* if $R/\mathfrak{p}$ is absolutely analytically irreducible, that is to say, if $\mathfrak{p}\widehat{R}_{(k,i)}$ is prime. Since $\mathfrak{D}(R)$ is absolutely normalizing, so is $\mathfrak{D}(\Lambda/I) = \mathfrak{D}(R)/I\mathfrak{D}(R)$ for every ideal I of R. Hence the theorem implies:

5.22. Corollary. *The following statements are equivalent, for a prime ideal $\mathfrak{p}$ of R:*

   (5.22.1) $\mathfrak{p}$ *is absolutely analytically prime;*
   (5.22.2) $\mathfrak{p}\mathfrak{D}(R)$ *is prime;*
   (5.22.3) *almost all approximations $\mathfrak{p}_w$ of $\mathfrak{p}$ are prime.* $\square$

5.23. Reducedness. A local ring A is called *analytically unramified* (or, *analytically reduced*), if its completion is reduced (that is to say, without non-zero nilpotent elements).

5.24. Theorem. *The following statements are equivalent:*

   (5.24.1) R *is analytically unramified;*
   (5.24.2) $\mathfrak{D}(R)$ *is reduced;*
   (5.24.3) *almost all R_w are reduced.*

Proof. The implication (5.24.3) $\Rightarrow$ (5.24.2) is a consequence of Łos' Theorem, and the implication (5.24.2) $\Rightarrow$ (5.24.1) is trivial. Hence we only need to show that if $\widehat{R}$ is reduced, then almost all R_w are reduced. If $\widehat{R}$ is reduced, then so is $\widehat{R}_{(k,i)}$, by Lemma 4.17. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals of $\widehat{R}_{(k,i)}$. Since $\widehat{R}_{(k,i)}$ is reduced, their intersection is zero, and hence so is the intersection of their approximations $\mathfrak{p}_{iw}$ for almost all w. Since $\mathfrak{D}(R)$ is absolutely normalizing, almost all $\mathfrak{p}_{iw}$ are prime ideals by Corollary 5.22, proving that almost all R_w are reduced. $\square$

5.25. Corollary. *Suppose that R is excellent. For an ideal I of R the following are equivalent:*

(5.25.1) *I is radical;*
(5.25.2) *$I\mathfrak{D}(R)$ is radical;*
(5.25.3) *almost all approximations I_w of I are radical.*

In particular, we have $\sqrt{I}\mathfrak{D}(R) = \sqrt{I\mathfrak{D}(R)}$, and $(\sqrt{I})_w = \sqrt{I_w}$ for almost all w. $\square$

A Noetherian ring is called *equidimensional* if all its minimal primes have the same dimension. A Noetherian local ring is called *formally equidimensional* if its completion is equidimensional.

5.26. Corollary. *If R is complete and k is algebraically closed, then the following are equivalent, for a prime ideal $\mathfrak{p}$ of R:*

(5.26.1) *$\mathfrak{p}$ is a minimal prime ideal of R;*
(5.26.2) *$\mathfrak{p}\mathfrak{D}(R)$ is a minimal prime ideal of $\mathfrak{D}(R)$;*
(5.26.3) *for almost all w the approximation $\mathfrak{p}_w$ of $\mathfrak{p}$ is a minimal prime ideal of R_w.*

If R is arbitrary, then R is formally equidimensional if and only if almost all R_w are equidimensional.

Proof. The intersection of the minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ of R equals the (nil-) radical of R. By Corollary 5.22 the $\mathfrak{p}_i\mathfrak{D}(R)$ are prime ideals of $\mathfrak{D}(R)$, and almost all $\mathfrak{p}_{iw}$ are prime ideals of R_w. By the previous corollary and (5.3.2), the intersection $\mathfrak{p}_1\mathfrak{D}(R)\cap\cdots\cap\mathfrak{p}_s\mathfrak{D}(R)$ equals the radical of $\mathfrak{D}(R)$, and hence $\mathfrak{p}_{1w} \cap \cdots \cap \mathfrak{p}_{sw}$ is the radical of R_w for almost all w. This yields the equivalence of (5.26.1)–(5.26.3). It remains to show that when R is arbitrary, it is formally equidimensional if and only if almost all R_w are equidimensional. Using Lemma 4.17 we reduce to the case that R is complete and k is algebraically closed, and then the claim follows from the earlier statements and (5.2.1). $\square$

Given a ring A and ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ of A, the canonical homomorphism

$$A \to A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_s$$

is an isomorphism if and only if $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_s = (0)$ and $1 \in \mathfrak{a}_i + \mathfrak{a}_j$ for all $i \neq j$. Hence by Theorem 5.24 and Corollary 5.26 we get:

5.27. Corollary. *Suppose that R is complete and k is algebraically closed. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals of R. The following statements are equivalent:*

(5.27.1) *the canonical homomorphism $R \to R/\mathfrak{p}_1 \times \cdots \times R/\mathfrak{p}_s$ is bijective;*
(5.27.2) *the canonical homomorphism*

$$\mathfrak{D}(R) \to \mathfrak{D}(R)/\mathfrak{p}_1\mathfrak{D}(R) \times \cdots \times \mathfrak{D}(R)/\mathfrak{p}_s\mathfrak{D}(R)$$

   is bijective;

(5.27.3) *the canonical homomorphism*

$$R_w \to R_w/\mathfrak{p}_{1w} \times \cdots \times R_w/\mathfrak{p}_{sw}$$

is bijective for almost all w. ◻

5.28. *Remark.* The proof of Theorem 5.17 shows that if $\widehat{\theta}_\Lambda$ is normalizing with respect to a prime ideal $\mathfrak{p}$ of $\widehat{R}_{(k,u)}$, then almost all approximations of $\mathfrak{p}$ are prime. Hence by the proof of Theorem 5.24: if $\widehat{\theta}_\Lambda$ is normalizing for all minimal primes of an ideal I of $\widehat{R}_{(k,u)}$, then $(\sqrt{I})_w = \sqrt{I_w}$ for almost all w. (This will be used in §5.30 below.)

5.29. *Remark.* Suppose that Λ is normalizing (see §4.25). Then Theorem 5.17 above remains true, with the same proof. Moreover, if k is algebraically closed and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are the minimal primes of $\widehat{R}$, then for almost all w, the approximations $\mathfrak{p}_{1w}, \ldots, \mathfrak{p}_{sw}$ are the minimal primes of R_w, and Theorem 5.24 and Corollary 5.26 also remain true. (This will be used in Sections 6 and 7.)

5.30. **Normality.** Recall that a domain is called *normal* if it is integrally closed in its fraction field. By Łos' Theorem, R is a normal domain if and only if almost all R_w are normal domains, and in this case R is a normal domain, by faithful flatness of $R \to \mathfrak{D}(R)$.

5.31. **Theorem.** *Suppose that R is a complete normal domain with algebraically closed residue field. Then Λ with underlying ring R can be chosen such that $\mathfrak{D}(R) = \mathfrak{D}(\Lambda)$ is a normal domain.*

The proof is based on the following criterion for normality due to Grauert and Remmert [21, pp. 220–221]; see also [37]. Let B be a Noetherian domain, and $N(B)$ be the non-normal locus of B, that is, the set of all prime ideals $\mathfrak{p}$ of B such that $B_\mathfrak{p}$ is not normal.

5.32. **Proposition.** *Let H be a non-zero radical ideal of B such that every $\mathfrak{p} \in N(B)$ contains H, and $0 \neq f \in H$. Then*

$$B \text{ is normal} \qquad \Longleftrightarrow \qquad fB = (fH :_B H).$$

Let A be a ring and B an A-algebra of finite type, that is, B is of the form $B = A[Y]/J$ where $J = (f_1, \ldots, f_r)A[Y]$ is an ideal of the polynomial ring $A[Y] = A[Y_1, \ldots, Y_m]$. Given a tuple $\mathbf{g} = (g_1, \ldots, g_s)$ with entries in $\{f_1, \ldots, f_r\}$ we write $\Delta\mathbf{g}$ for the ideal of $A[Y]$ generated by all the $s \times s$-minors of the $s \times m$-matrix $\left(\frac{\partial g_i}{\partial Y_j}\right)$, with the understanding that $\Delta\emptyset := A$. We let $H_{B/A}$ denote the nilradical of the ideal in $A[Y]$ generated by J and by the $\Delta\mathbf{g} \cdot \big(\mathbf{g}A[Y] : J\big)$, for $\mathbf{g}$ ranging over all tuples with entries in $\{f_1, \ldots, f_r\}$. The image in B of the ideal $H_{B/A}$ does not depend on the chosen presentation $B \cong A[Y]/J$ of the A-algebra B. (See [60, Property 2.13].) If A is Noetherian and $\mathfrak{p} \supseteq J$ a prime ideal of $A[Y]$, then $B_\mathfrak{p}$ is smooth over A if and only if $H_{B/A} \not\subseteq \mathfrak{p}$. In this case, $A \to B_\mathfrak{p}$ is regular [60, Corollary 2.9]. In particular, if A is regular, then so is $B_\mathfrak{p}$ [40, Theorem 23.7] and, since a regular ring is normal, the canonical image of $H_{B/A}$ in B is then a non-zero radical ideal which is contained in every element of $N(B)$. Therefore Proposition 5.32 implies:

5.33. **Corollary.** *Let B be an integral domain, of finite type over a regular ring A, and let f be a non-zero element of the canonical image H of $H_{B/A}$ in B. Then B is normal if and only if $fB = (fH :_B H)$.* ◻

Proof of Theorem 5.31. The desired object Λ has the form $(R, \mathbf{x}, k, u)$, where k is an arbitary coefficient field of R, where $u \colon R \to K$ is an arbitrary local homomorphism, and $\mathbf{x}$ is determined as follows. Choose a Noether normalization $\theta \colon K[[n]] \to \widehat{R}_{(k,u)}$ of $\widehat{R}_{(k,u)}$.

Let $I := \ker \theta$ and put $J := A[X_{d+1}, \ldots, X_n] \cap I$, where $d := \dim R$ and $A := K[[d]]$. Let $B := A[X_{d+1}, \ldots, X_n]/J$, so that $B \cong \widehat{R}_{(k,u)}$ as A-algebras, and let H be the image of the ideal $H_{B/A}$ of $A[X_{d+1}, \ldots, X_n]$ in $\widehat{R}_{(k,u)}$. We already remarked that H does not depend on the choice of θ. By Lemma 4.23 we can choose θ normalizing for all minimal prime ideals of H and for all ideals $\mathfrak{a}\widehat{R}_{(k,u)}$, where $\mathfrak{a}$ is an ideal of $\widehat{R}_{(k,i)}$. Now put $x_i := \theta(X_i)$ for $i = 1, \ldots, n$ and $\mathbf{x} := (x_1, \ldots, x_n)$. It follows that $\theta = \widehat{\theta}_\Lambda$ (hence $I = \widehat{I}_\Lambda$) for the thus constructed Λ, and Λ is absolutely normalizing.

We claim that $\mathfrak{D}(R) = \mathfrak{D}(\Lambda)$ is a normal domain. By Lemma 5.21 and [40, Theorem 23.9], $\widehat{R}_{(k,u)}$ is a normal domain, and therefore, $\mathfrak{D}(R)$ is a domain, by Theorem 5.17. With $A_w := K_w[[d]]$, Proposition 5.8 yields for almost all w an isomorphism of A_w-algebras

$$R_w \cong B_w := A_w[X_{d+1}, \ldots, X_n]/J_w$$

where J_w is an approximation of the ideal J of $A[X_{d+1}, \ldots, X_n]$. We claim that H_{B_w/A_w} is an approximation of $H_{B/A}$, for almost all w. This implies that for almost all w the canonical image of H_{B_w/A_w} in B_w is an approximation of H. Lemma 5.3 and Corollary 5.33 then show that almost all B_w are normal, as required.

To establish the claim, note that since a radical ideal of $A[X_{d+1}, \ldots, X_n]$ remains radical upon extension to $K[[n]]$, the ideal $H_{B/A}K[[n]]$ is the radical of the ideal

$$I + \sum_{\mathbf{g}} \Delta\mathbf{g} \cdot \left(\mathbf{g}K[[n]] :_{K[[n]]} I\right)$$

where $\mathbf{g}$ ranges over all tuples with entries in a fixed set of generators of J, and similarly the ideal $H_{B_w/A_w}K_w[[n]]$ is the radical of

$$I_w + \sum_{\mathbf{g}} \Delta\mathbf{g}_w \cdot \left(\mathbf{g}_w K_w[[n]] :_{K_w[[n]]} I_w\right)$$

where I_w and $\mathbf{g}_w$ are approximations of I and $\mathbf{g}$ respectively. Note that the ideal $\Delta\mathbf{g}_w$ of $A_w[X_{d+1}, \ldots, X_n]$ is an approximation of the ideal $\Delta\mathbf{g}$ of $A[X_{d+1}, \ldots, X_n]$. It follows that $H_{B_w/A_w}K_w[[n]]$ is an approximation of $H_{B/A}K[[n]]$, for almost all w, by Lemma 5.3, Remark 5.28, and the choice of θ. Moreover

$$H_{B/A} = H_{B/A}K[[n]] \cap A[X_{d+1}, \ldots, X_n]$$

and similarly, using the remarks preceding §5.10:

$$H_{B_w/A_w} = H_{B_w/A_w}K_w[[n]] \cap A_w[X_{d+1}, \ldots, X_n].$$

This yields that H_{B_w/A_w} is an approximation of $H_{B/A}$, as claimed. $\qquad\square$

A ring A is called *normal* if $A_\mathfrak{p}$ is a normal domain for every prime ideal $\mathfrak{p}$ of A. If A has finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ then A is normal if and only if $A \cong A/\mathfrak{p}_1 \times \cdots \times A/\mathfrak{p}_s$ and each domain $A/\mathfrak{p}_i$ is normal. A local ring A is called *analytically normal* if $\widehat{A}$ is normal.

5.34. Corollary. *Suppose that R is analytically normal. Then the object Λ with underlying ring R can be chosen such that $\mathfrak{D}(R) = \mathfrak{D}(\Lambda)$ is normal and almost all R_w are normal.*

Proof. As in the proof of Theorem 5.24 reduce to the case that R is complete and k is algebraically closed. The claim now follows from Corollary 5.27 and Theorem 5.31. $\quad\square$

Recall that Serre's condition (R_i) for a Noetherian ring A signifies that $A_{\mathfrak{p}}$ is regular for all prime ideals $\mathfrak{p}$ of A of height at most i, see [40, §23]. In the transfer of property (R_i), the fact that we do not know whether η_n commutes with partial differentiation (see Remark 4.31) poses a technical difficulty. We confine ourselves to showing:

5.35. Theorem. *Suppose that R is equidimensional and excellent and $\mathfrak{D}(R)$ is absolutely normalizing. Then for each i, if R satisfies (R_i) then so do almost all R_w.*

To show this note that if R is excellent, then R satisfies (R_i) if and only if $S := \widehat{R}_{(k,u)}$ does (see Remark 4.18). By Corollary 5.26, if R is equidimensional and $\mathfrak{D}(R)$ is absolutely normalizing, then almost all approximations R_w of R are equidimensional. Note that the R_w are also approximations of S. Now apply the following lemma to S:

5.36. Lemma. *Suppose that $R \in \mathbf{An}_K$, and R and almost all its approximations R_w are equidimensional. Then for each i, if R satisfies (R_i) then so do almost all R_w.*

Proof. Let $f_1, \ldots, f_r \in K[[n]]$ be generators of the ideal $I := \widehat{I}_\Lambda$, and let h be the height of I. Let J be the Jacobian ideal of I, that is to say the ideal of $K[[n]]$ generated by I and all $h \times h$-minors of the matrix with entries $\partial f_i / \partial X_j$. By the Jacobian criterion for regularity for power series rings in characteristc zero [40, Theorem 30.8], given a prime ideal $\mathfrak{p}$ of R, the localization of R at $\mathfrak{p}$ is regular if and only if $JR \not\subseteq \mathfrak{p}$. Hence R satisfies (R_i) if and only if JR has height at least $i+1$. Since R is equidimensional, this is equivalent with J having height at least $h+i+1$, and hence with $K[[n]]/J$ having dimension at most $n-(h+i+1)$. By (5.2.1) this is in turn is equivalent with $\dim K_w[[n]]/J_w \leq n-(h+i+1)$ for almost all w, where J_w is an approximation of J. Now for every w let $\widetilde{J}_w$ be the Jacobian ideal of I_w. By Remark 4.31 and Lemma 5.15 we have $\dim K_w[[n]]/\widetilde{J}_w \leq \dim K_w[[n]]/J_w$ for almost all w. Hence if R satisfies (R_i), then almost all $\widetilde{J}_w$ have height $\geq h+i+1$ and thus, since almost all R_w are equidimensional, almost all $\widetilde{J}_w R_w$ have height $\geq i+1$. Hence by the Jacobian criterion for regularity for power series rings over the algebraically closed fields K_w of positive characteristic [40, Theorem 30.10], almost all R_w satisfy (R_i). $\qquad\square$

5.37. Affine approximations and localization. One of the main drawbacks of the present theory is the fact that there is no a priori way to compare the $\mathfrak{D}$-extension of a local ring with the $\mathfrak{D}$-extension of one of its localizations. For example, suppose that R is complete and k algebraically closed, and let $\mathfrak{p}$ be a prime ideal of R. From Theorem 5.17, we know that we can choose $\mathfrak{D}(R)$ such that $\mathfrak{p}\mathfrak{D}(R)$ is a prime ideal, and then $\mathfrak{D}(R)_{\mathfrak{p}\mathfrak{D}(R)}$ is a faithfully flat extension of $R_{\mathfrak{p}}$. However, it is not clear how this compares with a Lefschetz extension $\mathfrak{D}(R_{\mathfrak{p}})$ of $R_{\mathfrak{p}}$: there is no obvious homomorphism from $\mathfrak{D}(R)$ to $\mathfrak{D}(R_{\mathfrak{p}})$, since the homomorphism $R \to R_{\mathfrak{p}}$ is not local. (This problem is already apparent in the simplest possible situation that $R = k[[n]]$ with $n > 1$, and $\mathfrak{p}$ is generated by a single variable.)

We have to take these considerations into account when comparing the affine approximations defined in [55] with the present version of approximations. Therefore, we restrict our attention to the case that $R = C_{\mathfrak{m}}$ is a localization of a finitely generated k-algebra C at a maximal ideal $\mathfrak{m}$. Here k is a Lefschetz field, realized as an ultraproduct of algebraically closed subfields k_p of K_p, with respect to the same ultraset as used for K. We consider k as a subfield of K in the natural way. Suppose $C = k[n]/I$ where I is an ideal of $k[n]$. As explained in the introduction, the non-standard hull of C is
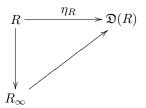
$$C_\infty := k[n]_\infty / I k[n]_\infty$$

where $k[n]_\infty$ is the ultraproduct of the $k_p[n]$. By [55, Corollary 4.2], the ideal $\mathfrak{m}C_\infty$ is again prime and by definition [55, §4.3], the non-standard hull of R is then

$$R_\infty := (C_\infty)_{\mathfrak{m}C_\infty}.$$

If $C' = k[n']/I'$ is another k-algebra and $\mathfrak{m}'$ a maximal ideal of C' such that $R' := C'_{\mathfrak{m}'}$ is isomorphic to R as k-algebras, then $R_\infty \cong (R')_\infty$ as Lefschetz rings [55, §4.3]. In particular, since k is algebraically closed we can make a translation and assume that $\mathfrak{m} = (X_1, \ldots, X_n)k[n]$. The embedding $k[n]_\infty \subseteq k[[n]]_\infty$ factors through $(k[n]_\infty)_{\mathfrak{m}k[n]_\infty}$, where we denote the ultraproduct of the $k_p[[n]]$ by $k[[n]]_\infty$. Composing with the natural embedding $k[[n]]_\infty \subseteq K[[n]]_\infty$ followed by the diagonal embedding $K[[n]]_\infty \to K[[n]]_\infty^{\mathcal{U}} = \mathfrak{D}(n)$ yields a $k[n]$-algebra homomorphism

$$\big(k[n]_\infty\big)_{\mathfrak{m}k[n]_\infty} \to \mathfrak{D}(n).$$

Taking reduction modulo I gives a homomorphism $R_\infty \to \mathfrak{D}(R)$ making

$$
\begin{array}{ccc}
R & \xrightarrow{\ \eta_R\ } & \mathfrak{D}(R) \\
\downarrow & \nearrow & \\
R_\infty & &
\end{array}
$$

commutative, where $R \to R_\infty$ is the canonical embedding. Let R_p^{aff} be approximations of R in the affine sense, that is to say, p ranges over the set of prime numbers and the ultraproduct of the R_p^{aff} is equal to R_∞. Recall that for almost all p, we can obtain R_p^{aff} as the localization of $k_p[n]/I_p^{\text{aff}}$ at the prime ideal $\mathfrak{m}_p^{\text{aff}}$, where I_p^{aff} and $\mathfrak{m}_p^{\text{aff}}$ are respective approximations of I and $\mathfrak{m}$ in the sense of [55]. Let $p(w) := \operatorname{char} K_w$, so $k_{p(w)}$ is a subfield of K_w, for each w. Let R_w be the completion of $R_{p(w)}^{\text{aff}} \otimes_{k_{p(w)}} K_w$ at the ideal generated by the X_i. Hence there is a canonical map $R_{p(w)}^{\text{aff}} \to R_w$ and this is faithfully flat. Alternatively, with the notation from §4.16, we have that

$$R_w = (\widehat{R_{p(w)}^{\text{aff}}})_{(k_{p(w)}, u_w)}$$

where $u_w \colon R_{p(w)}^{\text{aff}} \to K_w$ is the composition of the residue map $R_{p(w)}^{\text{aff}} \to k_{p(w)}$ with the inclusion $k_{p(w)} \subseteq K_w$. It follows that the ultraproduct of the R_w is equal to $\mathfrak{D}(R)$, showing that the R_w are approximations of R in the present sense. Moreover, if $c \in R$, then approximations c_w of c in the present sense are obtained by taking approximations c_p^{aff} of c in the sense of [55] and setting $c_w := c_{p(w)}^{\text{aff}}$ (as an element of R_w). Put succinctly, an approximation of R is obtained by the process of taking an approximation of R in the sense of [55], extending scalars and completing. We use this below to compare results between the affine and the complete case.

5.38. Proposition. *With the notations just introduced, the homomorphism $R_\infty \to \mathfrak{D}(R)$ is pure, and it is flat if R has dimension at most* 2.

A homomorphism $M \to N$ between modules over a ring A is *pure* if it is injective (so M can be regarded as a submodule of N) and every finite system of linear equations with constants in M which admits a solution in N admits a solution in M. For a module M

over a ring A let $\mu(M) \in \mathbb{N} \cup \{\infty\}$ be the least number of elements in a generating set for M, and put

$$\mu_A(m) := \sup\{\mu(\ker \varphi) : \varphi \in \mathrm{Hom}_A(A^m, A)\} \in \mathbb{N} \cup \{\infty\} \qquad \text{for all } m.$$

The ring A is called *uniformly coherent* if $\mu_A(m) < \infty$ for all m. If A is a finitely generated algebra over a field then A is uniformly coherent if and only if $\dim A \leq 2$, and in this case $\mu_A(m) \leq m + 2$ for all m. (See [20, Corollary 6.1.21].)

5.39. Lemma. *For each v in an ultraset $\mathcal{V}$, let $C_v \to D_v$ be a flat homomorphism, with each C_v a two-dimensional algebra over a field, and let $C_\infty \to D_\infty$ be their ultraproduct. If $D_\infty \to D^*$ is any elementary map, then the composition $C_\infty \to D_\infty \to D^*$ is flat.*

Proof. We have to show that for every linear form $L \in C_\infty[Y]$ where $Y = (Y_1, \ldots, Y_m)$, the solution set of $L = 0$ in $(D^*)^m$ is generated by the solution set of $L = 0$ in $(C_\infty)^m$. Let L_v be an approximation of L. For each v, there exist $m + 2$ tuples $\mathbf{a}_{1v}, \ldots, \mathbf{a}_{m+2,v}$ with entries in C_v which generate the solution set of $L_v = 0$ in $(C_v)^m$, by uniform coherence. These same tuples generate the solution set of $L_v = 0$ in $(D_v)^m$, by flatness. The ultraproducts $\mathbf{a}_{1\infty}, \ldots, \mathbf{a}_{m+2,\infty}$ of these $m + 2$ tuples are then solutions of $L = 0$ and generate the solution set of $L = 0$ in $(D_\infty)^m$, by Łoś' Theorem. Since $D_\infty \to D^*$ is elementary, the $\mathbf{a}_{i\infty}$ also generate the solution set of $L = 0$ in $(D^*)^m$, as required. $\qquad\square$

Proof of Proposition 5.38. We keep the notation from above. The inclusions

$$(5.39.1) \qquad\qquad R_p^{\mathrm{aff}} \to K_p[[n]]/I_p^{\mathrm{aff}} K_p[[n]]$$

are faithfully flat and hence pure. Their ultraproduct $R_\infty \to K[[n]]_\infty/IK[[n]]_\infty$ is also pure. The diagonal embedding

$$(5.39.2) \qquad K[[n]]_\infty/IK[[n]]_\infty \to \big(K[[n]]_\infty/IK[[n]]_\infty\big)^{\mathcal{U}} = \mathfrak{D}(R)$$

is pure, hence so is the composition

$$(5.39.3) \qquad\qquad\qquad R_\infty \to \mathfrak{D}(R).$$

Assume next that R has dimension at most 2. We may choose a finitely generated k-algebra C such that $R = C_\mathfrak{m}$ with C of dimension at most 2. It suffices to show that the composition $C_\infty \to R_\infty \to \mathfrak{D}(R)$ is flat. Almost all C_p^{aff} have dimension at most 2 by [55, Theorem 4.5]. Since (5.39.2) is elementary, and since $C_p^{\mathrm{aff}} \to R_p^{\mathrm{aff}}$ and (5.39.1) are flat, Lemma 5.39 yields that (5.39.3) is flat, as required. $\qquad\square$

We do not know in general whether $R_\infty \to \mathfrak{D}(R)$ is flat (and hence faithfully flat).

5.40. The non-local case. Let $A \supseteq \mathbb{Q}$ be a Noetherian ring of cardinality at most the cardinality of K. Let $\mathrm{Max}\,A$ be the set of all maximal ideals of A, and for every $\mathfrak{n} \in \mathrm{Max}\,A$ choose a faithfully flat Lefschetz extension $\eta_{A_\mathfrak{n}} : A_\mathfrak{n} \to \mathfrak{D}(A_\mathfrak{n})$ of the Noetherian local ring $A_\mathfrak{n}$ of equicharacteristic zero. The product of the $\eta_{A_\mathfrak{n}}$ yields a faithfully flat embedding

$$(5.40.1) \qquad\qquad A \to A^* := \prod_{\mathfrak{n} \in \mathrm{Max}\,A} \mathfrak{D}(A_\mathfrak{n}).$$

In general, A^* is not a Lefschetz ring, but it is so if A is semi-local. Thus:

5.41. Proposition. *Every semi-local Noetherian ring containing $\mathbb{Q}$ admits a faithfully flat Lefschetz extension.* $\qquad\square$

For arbitrary A, in spite of the fact that A^* is not Lefschetz, it still admits a non-standard Frobenius, so that the constructions in the next two sections can be generalized to the non-local case as well; see §6.17 for a further discussion.

Part 2. Applications

The standing assumption for the rest of this paper is that $(R, \mathfrak{m})$ is an equicharacteristic zero Noetherian local ring and K is an algebraically closed Lefschetz field with respect to an ultraset whose underlying set is the set of all prime numbers, whose approximations K_p are algebraically closed fields of characteristic p (as in Section 5). We take K of uncountable cardinality, as large as necessary. (Most of the time $|K| > 2^{|R|}$ will suffice.) We fix a Lefschetz extension $\mathfrak{D}(R)$ of R as defined in Part 1, and we let $(R_w, \mathfrak{m}_w)$ be the corresponding approximation of R. In other words, we fix some $\Lambda = (R, \mathbf{x}, k, u)$ in $\mathbf{Coh}_K^*$ with underlying ring R and put $\mathfrak{D}(R) := \mathfrak{D}(\Lambda)$. Where necessary, we'll make some additional assumptions on Λ (for instance so that $\mathfrak{D}(R)$ is absolutely normalizing; see §4.22). If $\alpha \colon R \to S$ is a local homomorphism, then we choose an object Γ of $\mathbf{Coh}_K^*$ so that α induces a morphism $\Lambda \to \Gamma$, and hence a local homo $\mathfrak{D}(\alpha) \colon \mathfrak{D}(R) \to \mathfrak{D}(S)$. In the sequel, we often will use a subscript w to indicate a choice of approximation of a certain object without explicitly mentioning this. For instance, S_w will stand for some approximation of S, etc. We now discuss non-standard tight closure and big Cohen-Macaulay algebras, and indicate several applications of these notions.

6. NON-STANDARD TIGHT CLOSURE

Every Lefschetz ring comes with a canonical endomorphism obtained by taking the ultraproduct of the Frobenii on each component: let $\mathbf{F}_w \colon R_w \to R_w$ be the Frobenius $x \mapsto x^{p(w)}$ on R_w, where $p(w)$ denotes the characteristic of R_w, and let $\mathbf{F}_\infty$ be the ultraproduct of the $\mathbf{F}_w$, that is to say,

$$\mathbf{F}_\infty \colon \mathfrak{D}(R) \to \mathfrak{D}(R) \colon \operatorname*{ulim}_w a_w \mapsto \operatorname*{ulim}_w \mathbf{F}_w(a_w).$$

We call $\mathbf{F}_\infty$ the *non-standard Frobenius* on $\mathfrak{D}(R)$. More generally, for each w let l_w be a positive integer and let l_∞ be its ultraproduct in the ultrapower $\mathbb{Z}^{\mathcal{W}}$ of $\mathbb{Z}$. We let $\mathbf{F}_\infty^{l_\infty}$ denote the ultraproduct of the $\mathbf{F}_w^{l_w}$, and call it an *ultra-Frobenius* on $\mathfrak{D}(R)$. In this paper, we are only concerned with the (powers of the) non-standard Frobenius $\mathbf{F}_\infty$; for an application of ultra-Frobenii, see [58]. Note that if $\alpha \colon R \to S$ is a local homomorphism, then for each l_∞, we have a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{D}(R) & \xrightarrow{\ \mathfrak{D}(\alpha)\ } & \mathfrak{D}(S) \\
\downarrow {\scriptstyle \mathbf{F}_\infty^{l_\infty}} & & \downarrow {\scriptstyle \mathbf{F}_\infty^{l_\infty}} \\
\mathfrak{D}(R) & \xrightarrow{\ \mathfrak{D}(\alpha)\ } & \mathfrak{D}(S).
\end{array}
$$

Given an ideal $\mathfrak{a}$ of R, we use $\mathbf{F}_\infty(\mathfrak{a})\mathfrak{D}(R)$ to denote the ideal in $\mathfrak{D}(R)$ generated by all $\mathbf{F}_\infty(a)$ with $a \in \mathfrak{a}$ (and a similar notation for powers of $\mathbf{F}_\infty$). Note that in general, $\mathbf{F}_\infty$ does not leave the subring R invariant. In fact, we have an inclusion

(6.0.1)
$$\mathbf{F}_\infty(\mathfrak{m})\mathfrak{D}(R) \subseteq \operatorname{Inf}(\mathfrak{D}(R)).$$

It follows that $\mathbf{F}_\infty(\mathfrak{m})\mathfrak{D}(R) \cap R = (0)$, by the faithful flatness of $R \to \mathfrak{D}(R)$.

Below we often make use of the important fact (easily checked using Łos' Theorem and [40, Theorem 16.1]) that the image of a $\mathfrak{D}(R)$-regular sequence in $\mathfrak{D}(R)$ under $\mathbf{F}_\infty$, and hence under each of its powers $\mathbf{F}_\infty^m$, is $\mathfrak{D}(R)$-regular. In particular, by Lemma 5.4, the image under $\mathbf{F}_\infty^m$ of any R-regular sequence in R is $\mathfrak{D}(R)$-regular.

6.1. **Non-standard tight closure.** Let $\mathfrak{a}$ be an ideal of R. We say that $z \in R$ belongs to the *non-standard tight closure* of $\mathfrak{a}$ if there exists $c \in R$ not contained in any minimal prime of R such that

$$(6.1.1) \qquad\qquad c\,\mathbf{F}_\infty^m(z) \in \mathbf{F}_\infty^m(\mathfrak{a})\mathfrak{D}(R),$$

for all sufficiently big m. We denote the non-standard tight closure of an ideal $\mathfrak{a}$ by $\mathrm{cl}(\mathfrak{a})$. A priori, this notion depends on the choice of $\mathfrak{D}(R)$, that is to say, on the choice of Λ. If we want to make this dependence explicit, we write $\mathrm{cl}_\Lambda(\mathfrak{a})$. It is an interesting (and probably difficult) question to determine whether different choices of Λ give rise to the same closure operation. Here we take a pragmatic approach: we are primarily interested in using non-standard tight closure to prove statements (about R) which do not mention it, and for this, we are free to choose Λ to suit our needs.

The next proposition shows that $\mathrm{cl}(\,\cdot\,)$ shares some basic properties with characteristic p tight closure. We denote the set of all elements of a ring A which are not contained in a minimal prime of A by A° (a multiplicatively closed subset of A).

6.2. **Proposition.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R. Then the following hold:*
- (6.2.1) $\mathrm{cl}(\mathfrak{a})$ *is an ideal of R and $\mathfrak{a} \subseteq \mathfrak{b}$ implies that $\mathrm{cl}(\mathfrak{a}) \subseteq \mathrm{cl}(\mathfrak{b})$;*
- (6.2.2) *there exists $c \in R^\circ$ such that $c\,\mathbf{F}_\infty^m(\mathrm{cl}(\mathfrak{a}))\mathfrak{D}(R) \subseteq \mathbf{F}_\infty^m(\mathfrak{a})\mathfrak{D}(R)$ for all sufficiently large m;*
- (6.2.3) $\mathfrak{a} \subseteq \mathrm{cl}(\mathfrak{a}) = \mathrm{cl}(\mathrm{cl}(\mathfrak{a}))$;
- (6.2.4) $\mathrm{cl}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathrm{cl}(\mathfrak{a}) \cap \mathrm{cl}(\mathfrak{b})$, $\mathrm{cl}(\mathfrak{a} + \mathfrak{b}) = \mathrm{cl}(\mathrm{cl}(\mathfrak{a}) + \mathrm{cl}(\mathfrak{b}))$, *and* $\mathrm{cl}(\mathfrak{a}\mathfrak{b}) = \mathrm{cl}(\mathrm{cl}(\mathfrak{a})\,\mathrm{cl}(\mathfrak{b}))$;
- (6.2.5) *if R is reduced and the residue class of $z \in R$ lies in $\mathrm{cl}_{\Lambda/\mathfrak{p}}\big(\mathfrak{a}(R/\mathfrak{p})\big)$ for each minimal prime $\mathfrak{p}$ of R, then $z \in \mathrm{cl}_\Lambda(\mathfrak{a})$.*

Proof. The proofs of the first four properties are as in the case of tight closure in positive characteristic. Suppose that R is reduced. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be all the minimal prime ideals of R, and for each j choose an element c_j inside all minimal primes except $\mathfrak{p}_j$. In particular, $c_j\mathfrak{p}_j = 0$. By assumption, for each j there exists an element $d_j \notin \mathfrak{p}_j$ such that

$$d_j\mathbf{F}_\infty^m(z) \in \mathbf{F}_\infty^m(\mathfrak{a})\mathfrak{D}(R/\mathfrak{p}_j),$$

for all large m. By the discussion in §4.28, this means that

$$(6.2.1) \qquad\qquad d_j\mathbf{F}_\infty^m(z) \in \mathbf{F}_\infty^m(\mathfrak{a})\mathfrak{D}(R) + \mathfrak{p}_j\mathfrak{D}(R),$$

for all large m. Put $c := c_1 d_1 + \cdots + c_s d_s$; note that c does not lie in any minimal prime of R. Multiplying (6.2.1) with c_j and taking the sum over all j, we get that $c\mathbf{F}_\infty^m(z)$ lies in $\mathbf{F}_\infty^m(\mathfrak{a})$, for all large m, showing that $z \in \mathrm{cl}(\mathfrak{a})$. $\square$

Next we derive versions of some other well-known results about tight closure in prime characteristic. We say that an ideal of R is *non-standard tightly closed* if it is equal to its non-standard tight closure.

6.3. **Theorem.** *If R is regular, then every ideal of R is non-standard tightly closed.*

Proof. The image under $\mathbf{F}_\infty^m$ of any regular system of parameters of R is $\mathfrak{D}(R)$-regular, and by Łos' Theorem and [12, Proposition 1.1.6] every permutation of a $\mathfrak{D}(R)$-regular sequence in $\mathfrak{D}(R)$ is $\mathfrak{D}(R)$-regular. Hence the R-algebra structure on $\mathfrak{D}(R)$ given by

$$(6.3.1) \qquad R \to \mathfrak{D}(R)\colon a \mapsto \mathbf{F}_\infty^m(a)$$

is that of a balanced big Cohen-Macaulay algebra. Since R is regular, this implies that the homomorphism (6.3.1) is flat. (See the remarks preceding the proof of Proposition 4.7.) Suppose towards a contradiction that z lies in the non-standard tight closure of an ideal $\mathfrak{a}$ in R but not in $\mathfrak{a}$. For some non-zero $c \in R$, we have relations (6.1.1) for m sufficiently large. Thus

$$c \in \big(\mathbf{F}_\infty^m(\mathfrak{a})\mathfrak{D}(R) :_{\mathfrak{D}(R)} \mathbf{F}_\infty^m(z)\big) = \mathbf{F}_\infty^m(\mathfrak{a} :_R z)\mathfrak{D}(R)$$

where we used flatness of (6.3.1) for the last equality. Since $z \notin \mathfrak{a}$, the colon ideal $(\mathfrak{a} :_R z)$ is contained in $\mathfrak{m}$. Therefore, c is zero by (6.0.1), contradiction. $\qquad \square$

6.4. *Remark.* For this argument to work, it suffices that (6.1.1) only holds for $m = 1$; the ensuing notion is the analogue of what was called *non-standard closure* in [55].

In the next result, we require that R is a homomorphic image of a Cohen-Macaulay local ring S, say $R = S/I$. In order to get a induced map on the Lefschetz hulls, we tacitly assume that Λ is equal to a quotient Γ/I for some object Γ in $\mathbf{Coh}_K^*$ whose underlying ring is S (see §4.28).

6.5. **Theorem** (Colon Capturing). *Suppose that R is a homomorpic image of a Cohen-Macaulay local ring and that R is equidimensional. If $\mathbf{z} = (z_1, \dots, z_d)$ is a system of parameters of R, then for each $i = 1, \dots, d$, we have an inclusion*

$$\big((z_1, \dots, z_{i-1})R :_R z_i\big) \subseteq \mathrm{cl}\big((z_1, \dots, z_{i-1})R\big).$$

Proof. Write $R = S/I$ with S a Cohen-Macaulay local ring and consider $\mathbf{z}$ already as a tuple in S. Suppose I has height e. By prime avoidance, we can find $y_1, \dots, y_e \in I$, such that for each i, the ideal $J + (z_1, \dots, z_i)S$ has height $e + i$, where $J := (y_1, \dots, y_e)S$. In particular, $(y_1, \dots, y_e, z_1, \dots, z_d)$ is a system of parameters in S, whence S-regular. By the Unmixedness Theorem (see for instance [40, Theorem 17.6]), the ideal J has no embedded associated primes. We can now use the same argument as in the proof of [55, Theorem 8.1], to get $c \in S$ not contained in any minimal prime of I and $N \in \mathbb{N}$ such that

$$(6.5.1) \qquad cI^N \subseteq J.$$

Let $a \in S$ be such that its image in R lies in $\big((z_1, \dots, z_{i-1})R : z_i\big)$, and hence az_i lies in $I + (z_1, \dots, z_{i-1})S$. For a fixed m, applying $\mathbf{F}_\infty^m$ yields

$$\mathbf{F}_\infty^m(a)\mathbf{F}_\infty^m(z_i) \in \mathbf{F}_\infty^m(I)\mathfrak{D}(S) + \big(\mathbf{F}_\infty^m(z_1), \dots, \mathbf{F}_\infty^m(z_{i-1})\big)\mathfrak{D}(S).$$

Multiplying this with c and using that $\mathbf{F}_\infty(I)\mathfrak{D}(S) \subseteq I^N\mathfrak{D}(S)$, we get from (6.5.1) that

$$c\mathbf{F}_\infty^m(a)\mathbf{F}_\infty^m(z_i) \in J\mathfrak{D}(S) + \big(\mathbf{F}_\infty^m(z_1), \dots, \mathbf{F}_\infty^m(z_{i-1})\big)\mathfrak{D}(S).$$

By the remark before §6.1, the sequence

$$\big(y_1, \dots, y_e, \mathbf{F}_\infty^m(z_1), \dots, \mathbf{F}_\infty^m(z_d)\big)$$

is $\mathfrak{D}(S)$-regular, so that the previous equation can be simplified to

$$c\mathbf{F}_\infty^m(a) \in J\mathfrak{D}(S) + \big(\mathbf{F}_\infty^m(z_1), \dots, \mathbf{F}_\infty^m(z_{i-1})\big)\mathfrak{D}(S).$$

By our choice of Λ we have $\mathfrak{D}(R) = \mathfrak{D}(S)/I\mathfrak{D}(S)$. Taking the reduction modulo $I\mathfrak{D}(S)$ we get equations exhibiting a as an element of the non-standard tight closure of the ideal $(z_1, \dots, z_{i-1})R$. (Note that the image of c lies in R°.) $\qquad \square$

6.6. *Remark.* Every complete Noetherian local ring R is a homomorphic image of a Cohen-Macaulay (in fact, regular) local ring by Cohen's Structure Theorem, hence Colon Capturing holds for R. If we were able to prove that

$$(6.6.1) \qquad\qquad \mathrm{cl}(\mathfrak{a}) \overset{?}{=} \mathrm{cl}(\mathfrak{a}\widehat{R}) \cap R,$$

for every ideal $\mathfrak{a}$ in R, then we get Colon Capturing for every equidimensional and universally catenary Noetherian local ring R. Note that the inclusion $\subseteq$ in (6.6.1) is immediate. On the other hand, even for tight closure in characteristic p, the other inclusion is still open. Below (see Lemma 6.27), we prove Colon Capturing for complete reduced R with algebraically closed residue field.

Using the previous theorem, we get a direct proof of the celebrated Hochster-Roberts Theorem [32]. A ring homomorphism $A \to B$ is called *cyclically pure* if $\mathfrak{a} = \mathfrak{a}B \cap A$, that is to say, if $A/\mathfrak{a} \to B/\mathfrak{a}B$ is injective, for every ideal $\mathfrak{a}$ of A. A cyclically pure homomorphism $A \to B$ between local rings A and B is automatically local. Moreover:

6.7. **Lemma.** *Let A and B be Noetherian local rings with respective completions $\widehat{A}$ and $\widehat{B}$. The completion $\widehat{A} \to \widehat{B}$ of a cyclically pure homomorphism $A \to B$ is cyclically pure.*

Proof. The homomorphism $B \to \widehat{B}$ is faithfully flat, hence cyclically pure; thus the composition $A \to B \to \widehat{B}$ is cyclically pure. So from now on we may suppose that $B = \widehat{B}$. It suffices to show that $\widehat{A} \to B$ is injective, since the completion of $A/\mathfrak{a}$ is equal to $\widehat{A}/\mathfrak{a}\widehat{A}$ for any ideal $\mathfrak{a}$ in A. Let $a \in \widehat{A}$ be such that $a = 0$ in B, and for each i choose $a_i \in A$ such that $a \equiv a_i \bmod \mathfrak{p}^i\widehat{A}$, where $\mathfrak{p}$ is the maximal ideal of A. Then a_i lies in $\mathfrak{p}^iB$, hence by cyclical purity, in $\mathfrak{p}^i$. Therefore $a \in \mathfrak{p}^i\widehat{A}$ for all i, showing that $a = 0$ in $\widehat{A}$ by Krull's Intersection Theorem. $\qquad\square$

6.8. **Theorem** (Hochster-Roberts). *If there exists a cyclically pure homomorphism $R \to S$ into a regular local ring S, then R is Cohen-Macaulay.*

Proof. By Lemma 6.7 we reduce to the case that R and S are complete. Let $(z_1, \ldots, z_d)$ be a system of parameters in R. We need to show that $(z_1, \ldots, z_d)$ is R-regular. To this end assume that

$$a z_i \in \mathfrak{a} := (z_1, \ldots, z_{i-1})R,$$

for some i and some $a \in R$. Since R is a complete domain, we can apply Theorem 6.5, to get that $a \in \mathrm{cl}_\Lambda(\mathfrak{a})$, for a suitable choice of Λ with underlying ring R. So for some $c \neq 0$ in R we have relations (6.1.1) for all sufficiently large m. Now $R \to S$ induces a homomorphism $\mathfrak{D}(R) \to \mathfrak{D}(S)$. Applying this homomorphism to (6.1.1) we get that a lies in $\mathrm{cl}(\mathfrak{a}S)$. (Note that c remains non-zero in S since $R \to S$ is injective.) Hence $a \in \mathfrak{a}S \cap R = \mathfrak{a}$ by Theorem 6.3 and cyclic purity. $\qquad\square$

6.9. *Remark.* We say that R is *weakly non-standard F-regular* if every ideal is non-standard tightly closed, for every choice of Λ with underlying ring R. The argument in the proof above actually gives two independent results. Firstly, if S is weakly non-standard F-regular, then S is Cohen-Macaulay. Secondly, if $R \to S$ is cyclically pure and S is weakly non-standard F-regular, then so is R.

For some more proofs of this theorem, see Remarks 6.28 and 7.5 below. By the argument in the beginning of the proof of [30, (2.3)], the theorem implies the following global version; for further discussion, see Conjecture A in the next section.

6.10. Corollary. *If $A \to B$ is a pure homomorphism of Noetherian rings containing $\mathbb{Q}$ and if B is regular, then A is Cohen-Macaulay.* □

The *integral closure* of an ideal $J \subseteq S$ of a ring S will be denoted by $\overline{J}$. It is the set of all $z \in S$ which are *integral over J,* that is, which satisfy a relation

$$(6.10.1) \qquad z^d + a_1 z^{d-1} + \cdots + a_{d-1} z + a_d = 0$$

with $a_i \in J^i$ for each i. See [12, §10.2] for a proof that $\overline{J}$ is an ideal of S, and other basic properties of $\overline{J}$. The following is a useful characterization of integral closure:

6.11. Lemma. *Let S be a Noetherian local ring and J an ideal of S. An element $z \in S$ is integral over J if and only if $z \in JV$ for every local homomorphism $S \to V$ to a discrete valuation ring V whose kernel is a minimal prime of S.*

See [34, Lemma 3.4] for the proof in the case where S is a domain; the general case easily reduces to this one; see for instance [26, Lemma 3.2].

Before we state the next property of tight closure, we make a general remark:

6.12. Lemma. *Let J be an ideal of a ring S and suppose that $z \in S$ satisfies an integral relation (6.10.1). Then $J^{d-1} z^N \in J^N$ for all $N \in \mathbb{N}$.*

Proof. We claim that $z^{d+k} \in J^{k+1}$ for all $k \in \mathbb{N}$. We show this by induction on k, the case $k = 0$ being trivial. For the inductive step note that by (6.10.1) we have

$$z^{d+k+1} = -(a_1 z^{d+k} + \cdots + a_{k+1} z^d + a_{k+2} z^{d-1} + \cdots + a_d z^{k+1}).$$

Since $a_i z^{d+k+1-i} \in J^{k+2}$ for $i = 1, \ldots, k+1$ (by the inductive hypothesis) and $a_i \in J^i \subseteq J^{k+2}$ for $i = k+2, \ldots, d$, we get that $z^{d+k+1} \in J^{k+2}$ as required. Now clearly $J^{d-1} z^N \subseteq J^N$ if $N < d$, and by the claim we get

$$J^{d-1} z^N = J^{d-1} z^{d+k} \subseteq J^{d-1} J^{k+1} = J^N$$

for all $N \geq d$, where $k := N - d$. □

6.13. Theorem (Briançon-Skoda). *For every ideal $\mathfrak{a}$ of R we have $\mathrm{cl}(\mathfrak{a}) \subseteq \overline{\mathfrak{a}}$. Moreover, if $\mathfrak{a}$ has positive height and is generated by at most m elements, then the integral closure of $\mathfrak{a}^m$ is contained in $\mathrm{cl}(\mathfrak{a})$.*

Proof. Let $z \in \mathrm{cl}(\mathfrak{a})$; so we have a relation (6.1.1) for some $c \in R^\circ$ and all sufficiently large m. In order to prove that $z \in \overline{\mathfrak{a}}$, we apply Lemma 6.11. Let V be a discrete valuation ring and let $R \to V$ be a local homomorphism with kernel a minimal prime of R. This induces a homomorphism $\mathfrak{D}(R) \to \mathfrak{D}(V)$, and applying this homomorphism to the relations (6.1.1) shows that $z \in \mathrm{cl}(\mathfrak{a}V)$. (Note that by assumption $c \neq 0$ in V.) By Theorem 6.3, the latter ideal is just $\mathfrak{a}V$, and we are done.

Suppose now that $\mathfrak{a}$ has positive height and is generated by $\leq m$ elements, and let z lie in the integral closure of $\mathfrak{a}^m$. Then z satisfies a relation

$$z^d + a_1 z^{d-1} + \cdots + a_d = 0$$

with $a_i \in \mathfrak{a}^{im}$. By Łos' Theorem, we have for almost all w an integral relation

$$z_w^d + a_{1w} z_w^{d-1} + \cdots + a_{dw} = 0$$

with $a_{iw} \in \mathfrak{a}_w^{im}$ for all i. For those w, we get for all N that

$$\mathfrak{a}_w^{m(d-1)} z_w^N \subseteq \mathfrak{a}_w^{Nm}$$

by Lemma 6.12. For N equal to the lth power of the characteristic of R_w we get $\mathfrak{a}_w^{Nm} \subseteq$ $\mathbf{F}_w^l(\mathfrak{a}_w)R_w$, since $\mathfrak{a}_w$ is generated by at most m elements by Łos' Theorem. Hence taking ultraproducts, we get

$$\mathfrak{a}^{m(d-1)}\mathbf{F}_\infty^l(z) \subseteq \mathbf{F}_\infty^l(\mathfrak{a})\mathfrak{D}(R).$$

Since this holds for all l and since we assumed that $\mathfrak{a}$ has positive height (hence $R^\circ \cap \mathfrak{a}^{m(d-1)} \neq \emptyset$), we get that $z \in \mathrm{cl}(\mathfrak{a})$. $\qquad\square$

6.14. *Remark.* The same argument together with [35, Remark 5.8.2] proves under the hypothesis of the theorem that the integral closure of $\mathfrak{a}^{m+l}$ lies in the non-standard tight closure of $\mathfrak{a}^{l+1}$, for all l.

6.15. *Remark.* It follows that $\mathrm{cl}(\mathfrak{a}) = \overline{\mathfrak{a}}$ for each principal ideal $\mathfrak{a}$ in R. Hence a domain R is normal if and only if every principal ideal is equal to its non-standard tight closure. In particular, using Remark 6.9, we see that a cyclically pure subring of a regular local ring (and more generally, a weakly non-standard F-regular local ring) is normal.

We immediately obtain the following classical version of the Briançon-Skoda Theorem from [38]. (For the ring of convergent power series over $\mathbb{C}$ this was first proved in [11]; see [35, §5] or [54] for some more background.)

6.16. **Theorem** (Briançon-Skoda for regular rings). *If A is a regular ring containing $\mathbb{Q}$ and $\mathfrak{a}$ an ideal of A generated by at most m elements, then the integral closure of $\mathfrak{a}^m$ is contained in $\mathfrak{a}$. In particular, if f is a formal power series in n variables over a field of characteristic zero with $f(0) = 0$, then f^n lies in the ideal generated by the partial derivatives of f.*

Proof. Since this is a local property, we may assume that A is local. By Theorem 6.13, the integral closure of $\mathfrak{a}^m$ is contained in $\mathrm{cl}(\mathfrak{a})$, hence in $\mathfrak{a}$, by Theorem 6.3. It is an exercise on the chain rule to show, using Lemma 6.11, that f lies in the integral closure of the ideal J generated by the partials of f. (See [35, Exercise 5.1].) Hence f^n lies in $\overline{J^n} \subseteq J$ by our first assertion. $\qquad\square$

6.17. **Tight closure—non-local case.** Although of minor use, one can extend the notion of non-standard tight closure to an arbitrary Noetherian $\mathbb{Q}$-algebra A as follows. For every maximal ideal $\mathfrak{n}$ of A choose a Lefschetz hull $\mathfrak{D}(A_\mathfrak{n})$ of the equicharacteristic zero Noetherian local ring $A_\mathfrak{n}$, and write $\mathrm{cl}_\mathfrak{n}$ for the ensuing notion of non-standard tight closure for ideals of $A_\mathfrak{n}$. We define the *non-standard tight closure* of an ideal $\mathfrak{a}$ of A as the intersection

$$\mathrm{cl}(\mathfrak{a}) := \bigcap_{\mathfrak{n}\in\mathrm{Max}\,A} \mathrm{cl}_\mathfrak{n}(\mathfrak{a}A_\mathfrak{n}) \cap A.$$

We invite the reader to check that this is indeed a closure operation, admitting similar properties as in the local case: for instance, the analogues of Theorems 6.3 and 6.13 hold.

If A^* is the product of all $\mathfrak{D}(A_\mathfrak{n})$ as in (5.40.1), then each of its factors admits the action of a non-standard Frobenius. Let us denote the product of these Frobenii again by $\mathbf{F}_\infty$. We can now define directly a tight closure operation on ideals in A by mimicking the definition in the local case, that is to say: $z \in A$ belongs to the 'global' non-standard tight closure of an ideal $\mathfrak{a}$ if there is some $c \in A^\circ$ such that $c\mathbf{F}_\infty^m(z) \in \mathbf{F}_\infty^m(\mathfrak{a})A^*$, for all sufficiently large m. It is immediate that an element in the 'global' non-standard tight closure of $\mathfrak{a}$ belongs to $\mathrm{cl}(\mathfrak{a})$ as defined above. In case A is semi-local, the converse also holds, but this is no longer clear for arbitrary A, for we do not have yet an appropriate notion of uniform test elements for non-standard tight closure (see also Proposition 6.24 below). This is presumably not an easy problem, and we will not further investigate it here.

6.18. **Comparison with affine non-standard tight closure.** We confine ourselves to the geometric case, that is, where R is the local ring at a closed point on a scheme of finite type over an algebraically closed Lefschetz field $k \subseteq K$ as in §5.37. In such a ring, non-standard tight closure was defined in [55] in a similar fashion, using the non-standard hull R_∞ instead of $\mathfrak{D}(R)$. More precisely, an element $z \in R$ lies in the (affine) non-standard tight closure of an ideal $\mathfrak{a}$ of R if there exists $c \in R^\circ$ such that

$$(6.18.1) \qquad c \, \mathbf{F}_\infty^m(z) \in \mathbf{F}_\infty^m(\mathfrak{a}) R_\infty$$

for all sufficiently large m, where we also write $\mathbf{F}_\infty$ for the non-standard Frobenius on the Lefschetz ring R_∞. As discussed in §5.37, we have a natural embedding $R_\infty \to \mathfrak{D}(R)$, and this is compatible with the non-standard Frobenii defined on each ring. In particular, taking the image of the relations (6.18.1) via this homomorphism shows that $z \in \mathrm{cl}(\mathfrak{a})$ in the present sense. Conversely, suppose there exists $c \in R^\circ$ such that (6.1.1) holds in $\mathfrak{D}(R)$ for all sufficiently large m. By Łos' Theorem, for those m we have that

$$(6.18.2) \qquad c_w \, \mathbf{F}_w^m(z_w) \in \mathbf{F}_w^m(\mathfrak{a}_w) R_w$$

where c_w, z_w, $\mathfrak{a}_w$ and R_w are approximations of c, z, $\mathfrak{a}$ and R respectively. By our discussion in §5.37, we can realize these approximations as follows. If c_p^{aff}, z_p^{aff}, $\mathfrak{a}_p^{\mathrm{aff}}$ and R_p^{aff} are approximations of c, z, $\mathfrak{a}$ and R in the sense of [55], then we may take R_w to be the completion of $R_{p(w)}^{\mathrm{aff}} \otimes_{k_{p(w)}} K_w$ and c_w, z_w and $\mathfrak{a}_w$ the corresponding image of $c_{p(w)}^{\mathrm{aff}}$, $z_{p(w)}^{\mathrm{aff}}$ and $\mathfrak{a}_{p(w)}^{\mathrm{aff}}$ in this completion. (Recall that $p(w) = \mathrm{char}\, K_w$.) Therefore, by faithful flatness, relation (6.18.2) already holds in the subring $R_{p(w)}^{\mathrm{aff}}$, for almost all w, hence for almost all characteristics. Taking ultraproducts of this relation in almost all R_p^{aff} yields (6.18.1), and since this is true for any sufficiently large choice of m, we showed that z lies in the non-standard tight closure of $\mathfrak{a}$ in the sense of [55]. In conclusion, we showed that for localizations of finitely generated k-algebras at maximal ideals, both notions of tight closure coincide.

6.19. **Generic tight closure.** We finish this section with studying a related closure operation, which also played an important role in the affine case. Let $\mathfrak{a}$ be an ideal of R. We say that an element $z \in R$ lies in the *generic tight closure* of $\mathfrak{a}$ if z_w lies in the (positive characteristic) tight closure of $\mathfrak{a}_w$ for almost all w. We denote the generic tight closure of $\mathfrak{a}$ by $\mathrm{cl}^*(\mathfrak{a})$. Again, this depends on the choice of Λ with underlying ring R; if we want to stress this dependence, we write $\mathrm{cl}_\Lambda^*(\mathfrak{a})$. From [35, Appendix 1] recall Hochster-Huneke's notion of tight closure in equicharacteristic 0. Here and below, given a ring S and a prime p we let $S(p) := S \otimes_{\mathbb{Z}} \mathbb{F}_p$, and for an ideal I of S we let $I(p)$ be the image of I in $S(p)$ under the map $z \mapsto z(p) := z \otimes 1 \colon S \to S(p)$.

6.20**. Definition.** An element z of R is in the (*equational*) *tight closure* $\mathfrak{a}^*$ of $\mathfrak{a}$ if there exists a finitely generated subring S of R with $z \in S$ such that $z(p)$ is in the (characteristic p) tight closure of $(\mathfrak{a} \cap S)(p)$ in $S(p)$, for all but finitely many primes p.

Let $\mathbf{y} = (y_1, \ldots, y_m) \in R^m$, and let J be the kernel of the ring homomorphism

$$\mathbb{Z}[Y] = \mathbb{Z}[Y_1, \ldots, Y_m] \to R$$

given by $Y_j \mapsto y_j$ for all j. We get an induced embedding $\mathbb{Z}[Y]/J \to R$, and we identify $\mathbb{Z}[Y]/J$ with its image, the subring S of R generated by $\mathbf{y}$. Given a prime p we then have

$$S(p) = S \otimes_{\mathbb{Z}} \mathbb{F}_p = \mathbb{F}_p[Y]/J(p)$$

where $J(p)$ is the image of J under the canonical surjection $\mathbb{Z}[Y] \to \mathbb{F}_p[Y]$. We let S_∞ denote the ultraproduct of the $S(p)$ with respect to the same ultraset that builds K (and whose underlying set is the set of prime numbers). The canonical maps $S \to S(p)$ combine to give a ring homomorphism $S \to S_\infty$. Composing with the diagonal embedding $S_\infty \to S_\infty^{\mathcal{U}}$, where $\mathcal{U}$ is the ultraset constructed in §4, we obtain an S-algebra structure on $S_\infty^{\mathcal{U}}$. We also get an S-algebra structure on $\mathfrak{D}(R)$ via the restriction of η_R to S.

6.21. Lemma. *There exists an S-algebra homomorphism $\varphi \colon S_\infty^{\mathcal{U}} \to \mathfrak{D}(R)$.*

Proof. For every w let $\tilde{S}_w$ be the subring of R_w generated by the approximations $\mathbf{y}_w = (y_{1w}, \ldots, y_{mw})$ of $\mathbf{y}$, and let $\tilde{S}_\infty$ be the ultraproduct of the $\tilde{S}_w$. If $P(Y) \in J$, so $P(\mathbf{y}) = 0$, then $P(\mathbf{y}_w) = 0$ for almost all w. Therefore, since J is finitely generated, we have for almost all w a surjection $S(p(w)) \to \tilde{S}_w$ with $y_j(p(w)) \mapsto y_{jw}$ for all j. Let $\varphi_w \colon S(p(w)) \to R_w$ denote the composition of this surjection with the embedding $\tilde{S}_w \subseteq R_w$ and let φ be the ultraproduct of the φ_w. One easily checks that φ is an S-algebra homomorphism $S_\infty^{\mathcal{U}} \to \mathfrak{D}(R)$. $\qquad\square$

6.22. *Remark.* This means in particular that for every $z \in S$, the $z_w := \varphi_w(z(p(w)))$ are an approximation of z. Indeed, $z = \varphi(z)$ is by construction the ultraproduct of the $\varphi_w(z(p(w)))$.

6.23. Corollary. *For every ideal $\mathfrak{a}$ of R, we have $\mathfrak{a}^* \subseteq \mathrm{cl}^*(\mathfrak{a})$.*

Proof. Let $z \in \mathfrak{a}^*$, and choose $S = \mathbb{Z}[\mathbf{y}]$, where $\mathbf{y} = (y_1, \ldots, y_m) \in R^m$, which contains z and such that $z(p)$ is in the tight closure of $(\mathfrak{a} \cap S)(p)$ in $S(p)$, for all but finitely many p. Then $z(p(w))$ is in the tight closure of $I_w := (\mathfrak{a} \cap S)(p(w))$ in $S(p(w))$, for almost all w. By [35, Theorem 2.3], almost each $z_w := \varphi_w(z(p(w)))$ is in the tight closure of $\varphi_w(I_w)R_w$. By Remark 6.22, the z_w and the $\varphi_w(I_w)R_w$ are approximations of z and $(\mathfrak{a} \cap S)R$ respectively. In particular, if $\mathfrak{a}_w$ is an approximation of $\mathfrak{a}$, then almost each z_w lies in the tight closure of $\mathfrak{a}_w$, showing that $z \in \mathrm{cl}^*(\mathfrak{a})$. $\qquad\square$

The relation between generic tight closure and non-standard tight closure is more subtle. We need a result on test elements. (See [35, Chapter 2] for the notion of test element.)

6.24. Proposition. *Suppose that $\mathfrak{D}(R)$ is normalizing and R is absolutely analytically irreducible. There exists an element of $\widehat{R}$ almost all of whose approximations are test elements.*

Proof. The assumption on $\mathfrak{D}(R)$ implies that the homomorphism $T_0 := k^*[[d]] \to \widehat{R}$ given by $X_i \mapsto x_i$ is a Noether Normalization, where $d := \dim R$ and k^* is the algebraic closure of k in $\widehat{R}$ (whence a coefficient field of $\widehat{R}$). Moreover, this homomorphism induces (by extension of scalars) the restriction of $\widehat{\theta}_\Lambda \colon K[[n]] \to \widehat{R}_{(k,u)}$ to $T := K[[d]]$ (see §4.22). Let c be a non-zero element in the *relative Jacobian* $J_{\widehat{R}/T_0}$. (Recall that $J_{\widehat{R}/T_0}$ is the 0-th Fitting ideal of the relative module of Kähler differentials $\Omega_{\widehat{R}/T_0}$.) By Remark 5.29, almost each R_w is a domain, and by Proposition 4.27, almost each R_w is a finite extension of T_w, of degree e. In particular, for almost all w, the field of fractions of R_w is separably algebraic over the field of fractions of T_w. Since $J_{\widehat{R}/T_0} \subseteq J_{\widehat{R}_{(k,u)}/T}$, almost each c_w is a non-zero element of J_{R_w/T_w}, hence a test element for R_w by [35, Exercise 2.9]. $\qquad\square$

6.25. *Remark.* From this we can also derive the same result for R analytically unramified with k algebraically closed and $\mathfrak{D}(R)$ normalizing. Namely, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals of $\widehat{R}$. By Remark 5.29 the approximations $\mathfrak{p}_{1w}, \ldots, \mathfrak{p}_{sw}$ are the minimal

prime ideals of R_w, for almost all w, and almost all R_w are reduced. For each j, choose $t_j \in \widehat{R}$ inside all minimal prime ideals except $\mathfrak{p}_j$. By Proposition 6.24, there exists $c_j \in \widehat{R} \setminus \mathfrak{p}_j$ whose approximation is a test element for almost all $R_w/\mathfrak{p}_{jw}$. Using Łos' Theorem, one shows that $c := c_1 t_1 + \cdots + c_s t_s$ has the desired properties. (See for instance [35, Exercise 2.10] for more details.)

6.26. **Theorem.** *Suppose that R is complete and $\mathfrak{D}(R)$ is normalizing. If R is either absolutely analytically irreducible or otherwise reduced with k algebraically closed, then* $\mathrm{cl}^*(\mathfrak{a}) \subseteq \mathrm{cl}(\mathfrak{a})$ *for every ideal $\mathfrak{a}$ of R.*

Proof. Let $z \in \mathrm{cl}^*(\mathfrak{a})$, that is, z_w is in the tight closure of $\mathfrak{a}_w^*$ for almost all w. By either Proposition 6.24 or the remark following it, there exists an element c of R whose approximation c_w is a test element in R_w, for almost all w. Hence for almost w and for all m:

$$c_w \mathbf{F}_w^m(z_w) \in \mathbf{F}_w^m(\mathfrak{a}_w) R_w.$$

Taking ultraproducts, we get for all m that

$$c \, \mathbf{F}_\infty^m(z) \in \mathbf{F}_\infty^m(\mathfrak{a}) \mathfrak{D}(R)$$

showing that $z \in \mathrm{cl}(\mathfrak{a})$. $\qquad\qquad\square$

For the Hochster-Huneke notion of tight closure in equicharacteristic zero, Colon Capturing is only known to be true in locally excellent rings. Since Colon Capturing holds for every complete Noetherian local ring of positive characteristic, hence for every approximation of R, Łos' Theorem in conjunction with Lemma 5.4 immediately yields:

6.27. **Lemma** (Colon Capturing for generic tight closure). *If $(z_1, \ldots, z_d)$ is a system of parameters of R, then*

$$\big((z_1, \ldots, z_{i-1})R :_R z_i\big) \subseteq \mathrm{cl}^*\big((z_1, \ldots, z_{i-1})R\big)$$

for each $i = 1, \ldots, d$. $\qquad\qquad\square$

In particular, combining this lemma with Theorem 6.26 yields Colon Capturing for non-standard tight closure in case R is reduced and complete, with algebraically closed k and $\mathfrak{D}(R)$ normalizing.

6.28. *Remark.* It follows from Theorem 5.2 that every ideal in an equicharacteristic zero regular local ring is equal to its generic tight closure. Together with Lemma 6.27, we get an even easier proof of the Hochster-Roberts Theorem (including the global version of Corollary 6.10), using cl^* in place of cl.

7. BALANCED BIG COHEN-MACAULAY ALGEBRAS

Recall that an R-algebra B is called a *balanced big Cohen-Macaulay R-algebra*, if any system of parameters of R is a B-regular sequence. (If we only know this for a single system of parameters, we call B a *big Cohen-Macaulay R-algebra*.) The key result on big Cohen-Macaulay algebras was proved by Hochster-Huneke in [29]: if S is an excellent local domain of prime characteristic p, then its absolute integral closure S^+ is a balanced big Cohen-Macaulay algebra. (Incidentally, this is false in equicharacteristic zero if $\dim S \geq 3$, see [29].) The *absolute integral closure A^+* of a domain A is defined to be the integral closure of A in an algebraic closure of its field of fractions. (We put $A^+ := 0$ if A is not a domain.) In [57], this is used to give a canonical construction of a balanced big Cohen-Macaulay algebra for a local domain S essentially of finite type over $\mathbb{C}$, by taking the ultraproduct of the S_p^+, where S_p is an approximation of S in the sense of [55].

The S_p are local domains, for almost all p, by [55, Corollary 4.2], so that the construction makes sense. In view of the restrictions imposed by Theorem 5.17, we cannot directly generalize this to arbitrary domains. We first consider the case that $\mathfrak{D}(\Lambda)$ is a domain, or equivalently, that almost all approximations R_w of R are domains. This is the case if R is absolutely irreducible and Λ is absolutely normalizing or normalizing (by Theorem 5.17 and Remark 5.29, respectively), but also if R is a DVR (by Corollary 5.5).

7.1**. Definition.**

$$\mathfrak{B}(\Lambda) := \operatorname*{ulim}_w R_w^+.$$

We often write $\mathfrak{B}(R)$ for $\mathfrak{B}(\Lambda)$, keeping in mind that $\mathfrak{B}(R)$ depends on the choice of Λ.

The canonical homomorphism $\eta_R \colon R \to \mathfrak{D}(R)$ induces a homomorphism $R \to \mathfrak{B}(R)$, turning $\mathfrak{B}(R)$ into an R-algebra. (Note that this is no longer an integral extension.) Since the R_w are complete (hence Henselian), the R_w^+ are local, whence so is $\mathfrak{B}(R)$. Moreover, the canonical homomorphism $R \to \mathfrak{B}(R)$ is local.

7.2**. Theorem.** *The R-algebra $\mathfrak{B}(R)$ is a balanced big Cohen-Macaulay algebra. If $\alpha \colon \Lambda \to \Gamma$ is a morphism in $\mathbf{Coh}_K^*$ with underlying ring homomorphism $R \to S$, where $\mathfrak{D}(\Gamma)$ is a domain, then there exists a (non-unique) homomorphism $\widetilde{\alpha} \colon \mathfrak{B}(R) \to \mathfrak{B}(S)$ giving rise to a commutative diagram*

(7.2.1)
$$
\begin{array}{ccc}
R & \xrightarrow{\ \ \alpha\ \ } & S \\
\downarrow & & \downarrow \\
\mathfrak{B}(R) & \xrightarrow{\ \ \widetilde{\alpha}\ \ } & \mathfrak{B}(S).
\end{array}
$$

Moreover, if α is finite, injective, and induces an isomorphism on the residue fields, then $\mathfrak{B}(R) = \mathfrak{B}(S)$.

Proof. Let $\mathbf{z}$ be a system of parameters in R with approximations $\mathbf{z}_w$. By Lemma 5.4 almost each $\mathbf{z}_w$ is a system of parameters in R_w, hence is R_w^+-regular by [29]. By Łos' Theorem, $\mathbf{z}$ is $\mathfrak{B}(R)$-regular. From the homomorphism $\mathfrak{D}(\alpha) \colon \mathfrak{D}(R) \to \mathfrak{D}(S)$ we get homomorphisms $R_w \to S_w$ for almost all w, where S_w is an approximation of S. These extend (non-uniquely) to homomorphisms $R_w^+ \to S_w^+$ whose ultraproduct is the required $\widetilde{\alpha}$. If α is finite, injective, and induces an isomorphism on the residue fields, then almost all $R_w \to S_w$ are finite and injective by Proposition 4.27, and hence $R_w^+ = S_w^+$. The last assertion is now clear. $\qquad\square$

7.3. *Remark.* Incidentally, the argument at the end of the proof shows that there is essentially only one ring in each dimension d playing the role of a big Cohen-Macaulay algebra: Indeed, suppose that the restriction of $\widehat{\theta}_\Lambda$ to $K[[d]]$ is a Noether normalization of $\widehat{R}_{(k,u)}$, where $d = \dim R$. (This is satisfied, for example, if Λ is absolutely normalizing.) Then $\mathfrak{B}(R)$ is isomorphic (non-canonically) to $\mathfrak{B}(K[[d]])$.

7.4**. Corollary.** *If R is regular, then the R-algebra $\mathfrak{B}(R)$ is faithfully flat.*

Proof. We already mentioned that a balanced big Cohen-Macaulay algebra over a regular local ring is automatically flat; see the remarks before Lemma 4.8. Since $R \to \mathfrak{B}(R)$ is local, it is therefore faithfully flat. $\qquad\square$

7.5. *Remark.* This gives us a second direct proof of the Hochster-Roberts Theorem (Theorem 6.8): with notation from the theorem, we may reduce again to the case that R and S are complete and that R has algebraically closed residue field. Suppose $(z_1, \ldots, z_d)$ is a system of parameters in R and let $az_i \in \mathfrak{a} := (z_1, \ldots, z_{i-1})R$. Since $(z_1, \ldots, z_d)$ is $\mathfrak{B}(R)$-regular by Theorem 7.2, we get $a \in \mathfrak{a}\mathfrak{B}(R)$. Choose absolutely normalizing objects Λ and Γ of $\mathbf{Coh}_K^*$ with underlying rings R and S, respectively, such that $R \to S$ becomes an $\mathbf{Coh}_K^*$-morphism, hence induces a homomorphism $\mathfrak{B}(R) \to \mathfrak{B}(S)$ which makes diagram (7.2.1) commutative. Then $a \in \mathfrak{a}\mathfrak{B}(S)$. Since $S \to \mathfrak{B}(S)$ is faithfully flat by Corollary 7.4, we get $a \in \mathfrak{a}S$ and hence, by cyclical purity, $a \in \mathfrak{a}$.

As in positive characteristic, the ring $\mathfrak{B}(R)$ has many additional properties (which fail to hold for the big Cohen-Macaulay algebras in equicharacteristic zero constructed by Hochster-Huneke in [30]). For instance, $\mathfrak{B}(R)$ is absolutely integrally closed, hence in particular quadratically closed, and therefore, the sum of any number of prime ideals is either the unit ideal or again a prime ideal (same argument as in [57, §3]). Moreover:

7.6. **Proposition.** *The canonical map* $\operatorname{Spec}\mathfrak{B}(R) \to \operatorname{Spec} R$ *is surjective.*

Proof. Let $\mathfrak{p}$ be a prime ideal in R and let $\mathfrak{q}$ be a prime ideal in $\widehat{R}_{(k,i)}$ lying over $\mathfrak{p}$. By Theorem 5.17, almost all approximations $\mathfrak{q}_w$ of $\mathfrak{q}$ are prime ideals. Since $R_w \subseteq R_w^+$ is integral, there exists a prime ideal $\mathfrak{Q}_w$ in R_w^+ whose contraction to R_w is $\mathfrak{q}_w$. The ultraproduct of the $\mathfrak{Q}_w$ is then a prime ideal in $\mathfrak{B}(R)$ whose contraction to R is $\mathfrak{p}$. $\qquad\square$

7.7. **Big Cohen-Macaulay algebras— general case.** We now define $\mathfrak{B}(R) = \mathfrak{B}(\Lambda)$ for an arbitrary equicharacteristic zero Noetherian local ring $(R, \mathfrak{m})$, under the assumption that Λ is absolutely normalizing:

$$\mathfrak{B}(R) := \bigoplus_{\mathfrak{P}} \mathfrak{B}(\widehat{R}_{(k,i)}/\mathfrak{P})$$

where $\mathfrak{P}$ runs over all prime ideals of $\widehat{R}_{(k,i)}$ of maximal dimension (that is to say, such that $\dim(\widehat{R}_{(k,i)}/\mathfrak{P}) = \dim R$). Note that this agrees with our former definition in case $\mathfrak{D}(\Lambda)$ (and hence $\widehat{R}_{(k,i)}$) is a domain. Clearly, $\mathfrak{B}(R)$ inherits an R-algebra structure via the $\widehat{R}_{(k,i)}/\mathfrak{P}$-algebra structure on each summand. We claim that $\mathfrak{B}(R)$ is a balanced big Cohen-Macaulay algebra. Indeed, if $\mathbf{z}$ is a system of parameters in R, then it remains so in $\widehat{R}_{(k,i)}$ and hence in each $\widehat{R}_{(k,i)}/\mathfrak{P}$ since the $\mathfrak{P}$ have maximal dimension. Therefore, by Theorem 7.2, for each $\mathfrak{P}$, the sequence $\mathbf{z}$ is $\mathfrak{B}(\widehat{R}_{(k,i)}/\mathfrak{P})$-regular, hence $\mathfrak{B}(R)$-regular. All the properties previously stated in the case that $\mathfrak{D}(\Lambda)$ is a domain remain true in this more general setup.

As in the Hochster-Huneke construction, there is a weak form of functoriality. We need a definition taken from [30] (see also [35, §9]).

7.8. **Definition.** We say that a local homomorphism $R \to S$ of Noetherian local rings is *permissible* if for each prime ideal $\mathfrak{q}$ in $\widehat{S}$ of maximal dimension, we can find a prime ideal $\mathfrak{p}$ in $\widehat{R}$ of maximal dimension such that $\mathfrak{p} \subseteq \mathfrak{q} \cap \widehat{R}$. A $\mathbf{Coh}_K^*$-morphism is called *permissible* if its underlying ring homomorphism is permissible.

As remarked in [35, §9], any local homomorphism with source an equidimensional and universally catenary local ring is permissible. Moreover:

7.9. **Lemma.** *If* $\Lambda \to \Gamma = (S, \mathbf{y}, l, v)$ *is a permissible* $\mathbf{Coh}_K^*$-morphism then the homomorphism $\widehat{R}_{(k,i)} \to \widehat{S}_{(l,j)}$ is permissible.

Proof. Recall that i and j denote the respective embedding of k and l into the algebraic closures $\tilde{k}$ and $\tilde{l}$ of $u(k)$ and $v(l)$ inside K. Let $\mathfrak{Q}$ a prime ideal of maximal dimension in $\widehat{S}_{(l,j)}$ and let $\mathfrak{q}$ be its contraction to $\widehat{S}$. We have inequalities

$$(7.9.1) \qquad \dim(\widehat{S}_{(l,j)}) = \dim\big(\widehat{S}_{(l,j)}/\mathfrak{Q}\big) \le \dim(\widehat{S}/\mathfrak{q}) \le \dim(\widehat{S})$$

where the middle inequality follows from [40, Theorem 15.1], since the closed fiber is trivial. As $\widehat{S}_{(l,j)}$ has the same dimension as $\widehat{S} \otimes_l \tilde{l}$ and therefore as $\widehat{S}$, all inequalities in (7.9.1) are equalities, so that $\mathfrak{q}$ is a prime ideal of maximal dimension. By assumption, there is a prime ideal $\mathfrak{p}$ in $\widehat{R}$ of maximal dimension contained in $\mathfrak{q}$. By faithful flatness, $\widehat{R}_{(k,i)}/\mathfrak{p}\widehat{R}_{(k,i)}$ has dimension $\dim(\widehat{R}) = \dim(\widehat{R}_{(k,i)})$. Since $\widehat{R}/\mathfrak{p}$ is universally catenary and equidimensional, so is $\widehat{R}_{(k,i)}/\mathfrak{p}\widehat{R}_{(k,i)}$. Therefore, if $\mathfrak{P}$ is a minimal prime of $\mathfrak{p}\widehat{R}_{(k,i)}$ contained in $\mathfrak{Q}$, then it has maximal dimension, as required. $\qquad \square$

We turn to the definition of $\mathfrak{B}(R) \to \mathfrak{B}(S)$ for a permissible homomorphism $R \to S$:

7.10. Corollary. *Given a permissible $\mathbf{Coh}_K^*$-morphism $\alpha\colon \Lambda \to \Gamma$ with Γ absolutely normalizing, there exists a homomorphism $\widetilde{\alpha}\colon \mathfrak{B}(\Lambda) = \mathfrak{B}(R) \to \mathfrak{B}(\Gamma) = \mathfrak{B}(S)$ making (7.2.1) commutative.*

Proof. By the lemma, for each prime ideal $\mathfrak{Q}$ in $\widehat{S}_{(l,j)}$ of maximal dimension we can choose a prime ideal $\mathfrak{Q}'$ of maximal dimension in $\widehat{R}_{(k,i)}$ such that $\mathfrak{Q}' \subseteq \mathfrak{Q}$. Fix one such prime ideal $\mathfrak{Q}'$ for each $\mathfrak{Q}$. The homomorphism

$$\widehat{R}_{(k,i)}/\mathfrak{Q}' \to \widehat{S}_{(l,j)}/\mathfrak{Q}$$

induces by Theorem 7.2 a homomorphism

$$j_{\mathfrak{Q}}\colon \mathfrak{B}(\widehat{R}_{(k,i)}/\mathfrak{Q}') \to \mathfrak{B}(\widehat{S}_{(l,j)}/\mathfrak{Q}).$$

Define $\mathfrak{B}(R) \to \mathfrak{B}(S)$ now by sending a tuple $(a_{\mathfrak{P}})$ with $a_{\mathfrak{P}} \in \mathfrak{B}(\widehat{R}_{(k,i)}/\mathfrak{P})$ and $\mathfrak{P}$ a prime ideal in $\widehat{R}_{(k,i)}$ of maximal dimension, to the tuple $(j_{\mathfrak{Q}}(a_{\mathfrak{Q}'}))$, where $\mathfrak{Q}$ runs over all prime ideals in $\widehat{S}_{(l,j)}$ of maximal dimension. It is easy to see that this gives rise to a commutative diagram (7.2.1). $\qquad \square$

It is also easy to see that if $\alpha\colon \Lambda \to \Gamma$ is a permissible $\mathbf{Coh}_K^*$-morphism where $\mathfrak{D}(\Gamma)$ is a domain, then there exists a homomorphism $\widetilde{\alpha}$ making (7.2.1) commutative. Calling Γ *permissible* if $\mathfrak{D}(\Gamma)$ is a domain or absolutely normalizing (so $\mathfrak{B}(\Gamma)$ is defined), we therefore have:

7.11. Corollary. *Given a permissible $\mathbf{Coh}_K^*$-morphism $\alpha\colon \Lambda \to \Gamma$ between permissible $\mathbf{Coh}_K^*$-objects, there exists a homomorphism $\widetilde{\alpha}\colon \mathfrak{B}(\Lambda) \to \mathfrak{B}(\Gamma)$ making (7.2.1) commutative.* $\qquad \square$

To show the strength of the existence of big Cohen-Macaulay algebras, let us give a quick proof of the Monomial Conjecture.

7.12. Corollary (Monomial Conjecture)**.** *Given a system of parameters $(z_1, \dots, z_d)$ in the equicharacteristic zero Noetherian local ring R, we have for all $t \in \mathbb{N}$ that*

$$(7.12.1) \qquad (z_1 z_2 \cdots z_d)^t \notin (z_1^{t+1}, \dots, z_d^{t+1})R.$$

Proof. The sequence $(z_1, \dots, z_d)$ is $\mathfrak{B}(R)$-regular and so (7.12.1) holds in $\mathfrak{B}(R)$, hence *a fortiori* in R. $\qquad \square$

The above proof does rely on the result of Hochster and Huneke that absolute integral closure in positive characteristic yields big Cohen-Macaulay algebras. A more elementary argument is obtained by using Lemma 5.4 together with the observation that the Monomial Conjecture admits an elementary proof in positive characteristic [12, Remark 9.2.4(b)]. An equally quick proof, which we will not produce here, relying also on the weak functoriality property of $\mathfrak{B}$, can be given for the Vanishing Theorem of maps for Tor [30, Theorem 4.1].

7.13. $\mathfrak{B}$-closure. As in [57], we can use our construction of a big Cohen-Macaulay algebra to define yet another closure operation on ideals of R as follows. Suppose that Λ is permissible, and let $\mathfrak{a}$ be an ideal of R. The $\mathfrak{B}$-*closure* of $\mathfrak{a}$ in R is by definition

$$\mathfrak{a}^+ := \mathfrak{a}\mathfrak{B}(R) \cap R.$$

We next show that the analogues of Theorems 6.3, 6.5 and 6.13 hold for $\mathfrak{a}^+$ in place of $\mathrm{cl}(\mathfrak{a})$. As for the last property in the next theorem, persistence, it is not immediately clear that it also holds for non-standard tight closure. We also remind the reader that if R is equidimensional and universally catenary (for instance, an excellent domain), then every local R-algebra is permissible.

7.14. Theorem. *Let $\mathfrak{a}$ be an ideal of R.*

(7.14.1) *If R is regular, then $\mathfrak{a} = \mathfrak{a}^+$.*

(7.14.2) *If $(z_1, \ldots, z_d)$ is a system of parameters in R, then*

$$\left((z_1, \ldots, z_{i-1}) :_R z_i\right) \subseteq \left((z_1, \ldots, z_{i-1})R\right)^+$$

for all i (Colon Capturing).

(7.14.3) *We have $\mathfrak{a}^+ \subseteq \overline{\mathfrak{a}}$, and if $\mathfrak{a}$ is generated by m elements, then*

$$\overline{\mathfrak{a}^{l+m}} \subseteq (\mathfrak{a}^{l+1})^+$$

for all l (Briançon-Skoda).

(7.14.4) *If $\Lambda \to \Gamma = (S, \ldots)$ is a permissible morphism between permissible objects in $\mathbf{Coh}_K^*$, then $\mathfrak{a}^+ S \subseteq (\mathfrak{a}S)^+$ (Persistence).*

Proof. For (7.14.1), observe that $\widehat{R}_{(k,i)}$ is again regular (Lemma 4.17), so that the composition $R \to \widehat{R}_{(k,i)} \to \mathfrak{B}(R)$ is faithfully flat, by Corollary 7.4, hence cyclically pure. For (7.14.2), let $I := (z_1, \ldots, z_{i-1})R$ and suppose $az_i \in I$. Since $(z_1, \ldots, z_d)$ is $\mathfrak{B}(R)$-regular, we get $a \in I\mathfrak{B}(R)$, and hence $a \in I^+$. The argument in [57, §6.1] (the affine case) can be copied almost verbatim to prove the second assertion in (7.14.3); for the first assertion, we use Lemma 6.11 together with (7.14.1) in the same way as in the proof of Theorem 6.13. (Note that $R \to V$ is automatically permissible, where V is as in Lemma 6.11, and every $\mathbf{Coh}_K^*$-object with underlying ring V is permissible, so that we get a homomorphism $\mathfrak{B}(R) \to \mathfrak{B}(V)$, by Corollary 7.11.) Persistence is immediate from weak functoriality of $\mathfrak{B}$. $\square$

Conjecturally, in characteristic p, plus closure and tight closure coincide. A characteristic zero analogue of this is that $\mathfrak{B}$-closure and generic tight closure should be the same. We have at least the following analogue of [30, Theorem 5.12]. (The second statement relies on Smith's work [59]).

7.15. Proposition. *Suppose R is formally equidimensional. For each ideal $\mathfrak{a}$ of R, we have $\mathfrak{a}^+ \subseteq \mathrm{cl}^*(\mathfrak{a})$. If $\mathfrak{a}$ is generated by a system of parameters, then $\mathfrak{a}^+ = \mathrm{cl}^*(\mathfrak{a})$.*

Proof. We give the proof in the case that $\mathfrak{D}(R)$ is absolutely normalizing, the case that $\mathfrak{D}(R)$ is a domain being similar (and simpler). In view of Lemma 4.17, passing from R to $\widehat{R}_{(k,i)}$ reduces the problem to the case that R is complete and equidimensional, with k algebraically closed. (Note that both $\mathfrak{B}$-closure and generic tight closure commute with such an extension of scalars). By Corollary 5.26, almost all R_w are equidimensional, and their minimal primes $\mathfrak{p}_{jw}$ are approximations of the minimal primes $\mathfrak{p}_j$ of R. By definition, $\mathfrak{B}(R)$ is the direct sum of the $\mathfrak{B}(R/\mathfrak{p}_j)$. Suppose $z \in \mathfrak{a}^+$, so that $z \in \mathfrak{a}\mathfrak{B}(R/\mathfrak{p}_j)$ for each j. Hence $z_w \in \mathfrak{a}_w(R_w/\mathfrak{p}_{jw})^+$ for all j and almost all w. If B is an integral extension of a Noetherian domain A of positive characteristic and I is an ideal of A, then $IB \cap A$ is contained in the tight closure of I [35, Theorem 1.7]. Thus almost all z_w lie in the tight closure of $\mathfrak{a}_w(R_w/\mathfrak{p}_{jw})$, hence in the tight closure of $\mathfrak{a}_w$ (since this holds for all minimal primes). This means that $z \in \mathrm{cl}^*(\mathfrak{a})$.

Suppose that $\mathfrak{a}$ is generated by a system of parameters. By Lemma 5.4, almost all $\mathfrak{a}_w$ are generated by a system of parameters, and this remains true in the homomorphic images $R_w/\mathfrak{p}_{jw}$. By [59], the tight closure of $\mathfrak{a}_w(R_w/\mathfrak{p}_{jw})$ is contained in $\mathfrak{a}_w(R_w/\mathfrak{p}_{jw})^+$. Taking ultraproducts yields $\mathrm{cl}^*(\mathfrak{a}) \subseteq \mathfrak{a}\mathfrak{B}(R)$. $\qquad\square$

7.16. *Remark.* Suppose that R is complete and $\mathfrak{D}(R)$ is normalizing. If R is either absolutely analytically irreducible or reduced, equidimensional with k algebraically closed, then the previous result in combination with Theorem 6.26 yields an inclusion $\mathfrak{a}^+ \subseteq \mathrm{cl}(\mathfrak{a})$.

7.17. **Comparison with big Cohen-Macaulay algebras for affine local domains.** We want to compare the present construction with the one from [57] discussed in the introduction of this section. We restrict ourselves once more to the case that R is the localization of a finitely generated k-algebra at a maximal ideal, with k an algebraically closed Lefschetz field contained in K as in §5.37; we continue to use the notations introduced there. Let R_p^{aff} denote an approximation of R in the sense of [55]. Recall that the approximations R_w in the sense of the present paper are defined as

$$R_w := \text{completion of } R_{p(w)}^{\mathrm{aff}} \otimes_{k_{p(w)}} K_w, \qquad \text{where } p(w) = \mathrm{char}\, K_w.$$

Suppose that R is an integral domain; then almost every R_p^{aff} is a domain. In general, $\widehat{R}$ and the R_w, though reduced and equidimensional, will no longer be domains. So let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of $\widehat{R}$. Suppose that Λ is absolutely normalizing. It follows from Corollary 5.26 that for almost all w, the $\mathfrak{p}_{jw}$ are the minimal prime ideals of R_w, and from Theorem 5.2, that they have maximal dimension. By definition, $\mathfrak{B}(R)$ is the direct sum of all $\mathfrak{B}(\widehat{R}/\mathfrak{p}_j)$. The ultraproduct $B(R)$ of the $(R_p^{\mathrm{aff}})^+$ is a big Cohen-Macaulay R-algebra; see [57]. For each w and each j, the composition

$$R_{p(w)}^{\mathrm{aff}} \to R_w \to R_w/\mathfrak{p}_{jw}$$

is injective and can be extended (non-uniquely) to a homomorphism

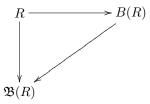$$\left(R_{p(w)}^{\mathrm{aff}}\right)^+ \to (R_w/\mathfrak{p}_{jw})^+.$$

By construction

$$\mathop{\mathrm{ulim}}_w \left(R_{p(w)}^{\mathrm{aff}}\right)^+ \cong B(R)^{\mathcal{U}}$$

where $\mathcal{U}$ is the ultraset from §4. Therefore, the composition of the diagonal embedding with the sum of the ultraproducts of the homomorphisms

$$\left(R_{p(w)}^{\mathrm{aff}}\right)^+ \to (R_w/\mathfrak{p}_{iw})^+$$

yields a homomorphism $B(R) \to \mathfrak{B}(R)$. The reader can verify that this fits in a commutative diagram

$$
\begin{array}{ccc}
R & \longrightarrow & B(R) \\
\downarrow & \swarrow & \\
\mathfrak{B}(R) & &
\end{array}
$$

In [57], the *B-closure* of an ideal $\mathfrak{a}$ of R is defined as the ideal $\mathfrak{a}B(R) \cap R$. Clearly, we have $\mathfrak{a}B(R) \cap R \subseteq \mathfrak{a}^+$ and we suspect that both are equal. For this to be true, it would suffice to show that the homomorphism $B(R) \to \mathfrak{B}(R)$ is cyclically pure. (Note, however, that $B(R)$ is not local.) We leave it to the reader to verify that the discussion in §6.18 also applies to generic tight closure, that is to say, the two notions, the present one and the 'affine' one from [55], coincide for localizations of finitely generated k-algebras at maximal ideals. Using this together with Proposition 7.15 and [57, Corollary 4.5], we get an equality $\mathfrak{a}B(R) \cap R = \mathfrak{a}^+$ for $\mathfrak{a}$ an ideal generated by a system of parameters.

7.18. **Rational singularities.** The main merit of the present approach to tight closure in equicharacteristic zero and to the construction of balanced big Cohen-Macauly modules, via $\mathfrak{D}(R)$, is its flexibility. We want to finish with a brief discussion of one possible application of our construction of $\mathfrak{B}(R)$, which we formulate in two Conjectures.

Let us return to the situation of the Hochster-Roberts Theorem, that is to say, a cyclically pure homomorphism from a Noetherian local ring R into a regular local ring S. We already showed that R (and also its completion $\widehat{R}$) is Cohen-Macaulay and normal (see Theorem 6.8 and Remarks 6.15 and 7.5). In case R and S are of finite type over $\mathbb{C}$, Boutot has shown in [10], using deep Vanishing Theorems, that R has rational singularities. In fact, he proves an even stronger result in that he only needs to assume that S has rational singularities. Recall that an equicharacteristic zero excellent local domain R has *rational singularities* (or, more correctly, is *pseudo-rational*) if it is normal, analytically unramified and Cohen-Macaulay, and the canonical embedding

$$
H_0(W, \omega_W) \to H_0(X, \omega_X)
$$

is surjective (it is always injective), where $W \to X := \operatorname{Spec} R$ is a desingularization, and where in general, ω_Y denotes the canonical sheaf on a scheme Y.

In the affine case, the methods of the second author (via non-standard tight closure in [50], and via big Cohen-Macaulay algebras in [57]) yield more elementary arguments for the fact that a cyclically pure subring of an affine regular ring has rational singularities. Moreover, in the second paper, a more general version is proven, where S is only assumed to have rational singularities and be Gorenstein. However, for this stronger version, one needs a result of Hara in [22], which itself uses deep Vanishing Theorems. In any case, we expect that one can generalize Boutot's result by removing the condition that the rings are finitely generated over a field. (Note that no Vanishing Theorems are known to hold for arbitrary excellent schemes.)

Conjecture A. *Every equicharacteristic zero excellent local ring R which admits a cyclically pure homomorphism into a regular local ring S is pseudo-rational.*

In fact, we suspect that an excellent local domain is pseudo-rational if there exists a system of parameters $\mathbf{z}$ such that $\mathbf{z}R = (\mathbf{z}R)^+$ (= generic tight closure of $\mathbf{z}R$, by Proposition 7.15). It is clear by (7.14.1) how this implies the Conjecture. If R is in addition

$\mathbb{Q}$-Gorenstein, then in the affine case it has log-terminal singularities by [58, Theorem B and Remark 3.13]. (Here again we can weaken the assumption on S to be only log-terminal, provided we use Hara's result; see that article for the terminology.) In view of this, we postulate the following generalization.

Conjecture B. *Every equicharacteristic zero excellent local $\mathbb{Q}$-Gorenstein ring which admits a cyclically pure homomorphism into a regular local ring has log-terminal singularities.*

The conjecture would follow from (7.14.1), if one can show that a $\mathbb{Q}$-Gorenstein excellent local ring R in which each ideal is equal to its $\mathfrak{B}$-closure, or, equivalently, for which $R \to \mathfrak{B}(R)$ is cyclically pure, has log-terminal singularities.

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