ULTRAPRODUCTS AT THE CROSS-ROADS OF MODEL-THEORY, ALGEBRA AND GEOMETRY

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1. Introduction

In this mini-course, we will touch upon topics from logic, algebra and geometry, and so some background in either of these topics will be helpful. Nonetheless, I will keep things as self-contained as possible. Before I suggest some reading material, let me briefly introduce the topic, which therefore will make it clear what kind of background will be useful.

A standard tool in mathematics for constructing from some given mathematical structures of interest (like vector spaces, topological spaces, manifolds, ...), a new structure of the same type, is by taking their (free) product. This construction is often innocuous when dealing with finitely many structures, but infinitely many may yield structures that are 'too big'. Our main focus is algebraic, dealing mostly with (commutative) rings (with unit); moreover, we seek applications in commutative algebra, or the 'dual' topic, algebraic geometry, for which rings often need to be Noetherian as well, so the latter will be our structures of interest. However, whereas the (Cartesian) product of finitely many Noetherian rings is again Noetherian, this fails miserably for any infinite products. But there are other issues with taking Cartesian products, even finite ones: they do not preserve many other desirable properties, such as being a domain, or being a field, etc.

To our aid will come a construction from model-theory/logic, that of an *ultraproduct*. Given a sequence of Noetherian rings $R_1, R_2, \dots$, we associate to them their ultraproduct $R_\natural$, which should be thought of as a certain 'average' of the R_i and in particular it will again be a ring. Unfortunately, one of our main desiderata is still violated: this ultraproduct is hardly ever Noetherian. Yet, not all is lost, since often it will contain a 'nice' Noetherian subring (or admits a 'nice' Noetherian quotient). Let me elaborate a little with what I mean with 'average' and 'nice' here, without giving precise details.

Ultraproducts as 'averages'. Consider a $0/1$-probability measure on the index set $\mathbb{N}$ (or any infinite index set will actually work). We will define $R_\natural$ as a quotient of the ordinary Cartesian product R_∞, by identifying two sequences $(a_i)_i, (b_i)_i \in R_\infty$ if $a_i = b_i$ with probability one.[1] This quotient $R_\natural$ yields an average of the R_i in the following sense: if an 'algebraic' property holds with probability one for the R_i, then it also holds for their ultraproduct $R_\natural$. For instance, if all—which is a trivial case of having probability one—R_i are fields (respectively, domains, ...), then $R_\natural$ is a field (respectively, domain, ...). This principle, when specifying precisely what 'algebraic properties' are allowed, is called

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[1]Disclaimer: this is not how they are actually defined; instead one needs the notion of an ultrafilter on the index set, as I will explain in the lectures.

Łos' Theorem.[2] Noetherianity, however, is not one of these properties, and hence is not preserved.

One of the key results that I will discuss is the so-called transfer principle form positive to zero characteristic.[3] The 'miracle fact'[4] that we will exploit abundantly is that if we take the ultraproduct of the $\mathbf{F}_p^{\text{alg}}$—the algebraic closure of the p-element field—, then the resulting ultraproduct is not only a field, it is a very familiar field, to wit $\mathbb{C}$, the field of complex numbers!

Transfer through nice subrings. As mentioned above, the ultraproduct is in general not Noetherian (the field case being one notable exception). For instance, when each $R_p = \mathbf{F}_p^{\text{alg}}[x]$, with $x = (x_1, \dots, x_n)$ a fixed tuple of indeterminates, then $R_\natural$ is surely not a polynomial ring. However, by our miracle fact, it will contain $\mathbb{C}$, and even $\mathbb{C}[x]$. Moreover, the inclusion $\mathbb{C}[x] \subseteq R_\natural$ is as nice as one could hope for: it is *faithfully flat*.[5] In the lectures, we will explore what this means, but one way of visualizing this is that the ideals of $\mathbb{C}[x]$ 'live in' $R_\natural$ (although the latter has many more 'weird', i.e., non-finitely generated, ideals as well). In algebraic geometry, faithfully flat descent is an important tool to transfer properties from the larger to the smaller ring. This is how we will achieve our 'transfer from positive to zero characteristic': consider an 'algebraic property' that holds in all polynomial rings R_p of positive characteristic, so that their ultraproduct $R_\natural$ also has this property by Łos' Theorem, and then 'descend' this property to $\mathbb{C}[x]$. Incidentally, $R_\natural$ has also a very nice quotient, to wit $\mathbb{C}[[x]]$, the formal power series ring, and we may also transfer properties in that direction. Both the subring $\mathbb{C}[x]$ and the quotient $\mathbb{C}[[x]]$ will be thought of as certain products of the original R_p, called respectively the *protoproduct* $R_\flat$ and *cataproduct* $R_\sharp$, ultimately leading to the 'musical scale'

$$R_\flat \subseteq R_\natural \to R_\sharp. \tag{1}$$

Applications. In this mini-course, I will mention two types of applications: uniform bounds and transfer from positive to zero characteristic. Both rely heavily on the properties of the 'musical scale' (1), the former extending the groundbreaking work in [18], and the latter greatly simplifying the techniques in [15] that relied on Artin Approximation ([1]). One of my original goals ([19]) was to use the transfer from positive to zero characteristic to obtain an elegant tight closure theory in characteristic zero. Tight closure is a beautiful yet powerful theory for rings of positive characteristic developed in the '90s by Hochster and Huneke (see, for instance, [9]).[6] Tight closure theory exploits properties of the Frobenius map, such as the 'students binomial theorem'

$$(x+y)^p = x^p + y^p, \tag{2}$$

which, of course does not hold in characteristic zero (or does it?).

[2] Spoiler alert: any first-order property.

[3] Recall that the *characteristic* of a ring is the positive generator of the kernel of the canonical map $\mathbb{Z} \to R$; for domains, this is always a prime or zero.

[4] It is no accident that it is on the front page of [20]; I normally refer to it as the *weak Lefschetz Principle*.

[5] In the words of David Mumford, the great geometer: "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers." ([14, p. 214]).

[6] Their version [7] in characteristic zero, unfortunately, is far from elegant.

2. Introductory readings

From this introduction, it is clear that we will be dealing with many topics from commutative algebra, including fields, Noetherian rings, local rings, modules, ideals, Krull dimension, etc. Introductory textbooks like [2] or [17] should suffice, but of course it wouldn't hurt if you have already read some more specialized textbooks like [4, 5, 13], as we will make use of some more advanced notions. All that is needed from commutative algebra can be found in my online notes [21]. We also will make references to some notions from algebraic geometry, but these will be less central. Some introductory course in algebraic geometry, such as [6, Chapter I] or my notes [21, Chapter I] will suffice. Ultraproducts were first studied in model-theory ([11]), and although there were some early applications ([3, 10]) to algebra and other fields, they never really became common knowledge among general mathematicians, in spite of their power.[7] I will introduce them chiefly as algebraic objects, which removes the necessity for notions from model-theory. This perspective, though, is a little disingenuous, since to prove some of the basic results, one does need model-theory. Therefore, some familiarity with the topic is helpful, and to this end, I include in the next section an extremely water-downed summary of model-theory, or rather, an ad hoc version for rings only (for more details, consult any textbook, such as [8, 12, 16]). Of course, ultimately, my book [20] is the main reference for this course, and I will post some excerpts and exercises taken from some earlier versions.

3. Model-theory in rings

Formulae. By a *quantifier free formula without parameters* in the free variables $x = (x_1, \dots, x_n)$, we will mean an expression of the form

$$\varphi(x) := \bigvee_{j=1}^{m} f_{1j} = 0 \wedge \dots \wedge f_{sj} = 0 \wedge g_{1j} \neq 0 \wedge \dots \wedge g_{tj} \neq 0, \tag{3}$$

where each f_{ij} and g_{ij} is a polynomial with integer coefficients in the variables x, and where $\wedge$ and $\vee$ are the logical connectives *and* and *or*. If instead we allow the f_{ij} and g_{ij} to have coefficients in a ring R, then we call $\varphi(x)$ a *quantifier free formula with parameters in* R. We allow all possible degenerate cases as well: there might be no variables at all (so that the formula simply declares that certain elements in $\mathbb{Z}$ or in R are zero and others are non-zero) or there might be no equations or no negations or perhaps no conditions at all. Put succinctly, a quantifier free formula is a Boolean combination of polynomial equations using the connectives $\wedge$, $\vee$ and $\neg$ (negation), with the understanding that we use distributivity and De Morgan's Laws to rewrite this Boolean expression in the (disjunctive normal) form (3).

By a *formula without parameters* in the free variables x, we mean an expression of the form

$$\varphi(x) := (\mathrm{Q}_1\, y_1) \cdots (\mathrm{Q}_p\, y_p)\, \psi(x, y),$$

[7] From the introduction of [20]: ...[ultraproducts] did not leave a lasting impression on the algebraic community though, shunned perhaps because there were conceived as non-algebraic, belonging to the alien universe of set-theory and non-standard arithmetic, a universe in which most mathematicians did not, and still do not feel too comfortable. The present book intends to debunk this common perception of ultraproducts: when applied to algebraic objects, their construction is quite natural, yet very powerful, and requires hardly any knowledge of model-theory.

where $\psi(x, y)$ is a quantifier free formula without parameters in the free variables x and $y = (y_1, \dots, y_p)$ and where Q_i is either the universal quantifier $\forall$ or the existential quantifier $\exists$. If instead $\psi(x, y)$ has parameters from R, then we call $\varphi(x)$ a *formula with parameters in* R. A formula with no free variables is called a *sentence*.

Satisfaction. Let $\varphi(x)$ be a formula in the free variables $x = (x_1, \dots, x_n)$ with parameters from R (this includes the case that there are no parameters by taking $R = \mathbb{Z}$ and the case that there are no free variables by taking $n = 0$). Let A be an R-algebra and let $\mathbf{a} = (a_1, \dots, a_n)$ be a tuple with entries from A. We will give meaning to the expression $\mathbf{a}$ *satisfies the formula* $\varphi(x)$ *in* A (sometimes abbreviated to $\varphi(\mathbf{a})$ *holds in* A or *is true in* A) by induction on the number of quantifiers. Suppose first that $\varphi(x)$ is quantifier free, given by the Boolean expression (3). Then $\varphi(\mathbf{a})$ holds in A, if for some j_0, all $f_{ij_0}(\mathbf{a}) = 0$ and all $g_{ij_0}(\mathbf{a}) \neq 0$. For the general case, suppose $\varphi(x)$ is of the form $(\exists y)\psi(x, y)$ (respectively, $(\forall y)\psi(x, y)$), where the satisfaction relation is already defined for the formula $\psi(x, y)$. Then $\varphi(\mathbf{a})$ holds in A, if there is some $b \in A$ such that $\psi(\mathbf{a}, b)$ holds in A (respectively, if $\psi(\mathbf{a}, b)$ holds in A, for all $b \in A$). The subset of A^n consisting of all tuples satisfying $\varphi(x)$ will be called the *subset defined by* φ, and will be denoted $\varphi(A)$. Any subset that arises in such way will be called a *definable subset* of A^n.

Note that if $n = 0$, then there is no mention of tuples in A. In other words, a sentence is either true or false in A. By convention, we set A^0 equal to the singleton $\{\emptyset\}$ (that is to say, A^0 consists of the empty tuple $\emptyset$). If φ is a sentence, then the set defined by it is either $\{\emptyset\}$ or $\emptyset$, according to whether φ is true or false in A.

Constructible Sets. There is a connection between definable sets and Zariski-constructible sets,[8] where the relationship is the most transparent over algebraically closed fields, as we will explain below. In general, we can make the following observations.

Let R be a ring. Let $\varphi(x)$ be a quantifier free formula with parameters from R, given as in (3). Let $\Sigma_{\varphi(x)}$ denote the constructible subset of $\mathbb{A}^n_R$ consisting of all prime ideals $\mathfrak{p}$ of $\operatorname{Spec}(R[x])$ which, for some j_0, contain all f_{ij_0} and do not contain any g_{ij_0}. In particular, if $n = 0$, so that $\mathbb{A}^0_R$ is by definition $\operatorname{Spec}(R)$, then the constructible subset Σ_φ associated to φ is a subset of $\operatorname{Spec}(R)$.

Let A be an R-algebra and assume moreover that A is a domain (we will never use constructible sets associated to formulae if A is not a domain). For an n-tuple $\mathbf{a}$ over A, let $\mathfrak{p}_{\mathbf{a}}$ be the (prime) ideal in $A[x]$ generated by the $x_i - a_i$, where $x = (x_1, \dots, x_n)$. Since $A[x]/\mathfrak{p}_{\mathbf{a}} \cong A$, we call such a prime ideal an A-*rational point* of $A[x]$. It is not hard to see that this yields a bijection between n-tuples over A and A-rational points of $A[x]$, which we therefore will identify with one another. In this terminology, $\varphi(\mathbf{a})$ holds in A if and only if the corresponding A-rational point $\mathfrak{p}_{\mathbf{a}}$ lies in the constructible set $\Sigma_{\varphi(x)}$ (strictly speaking, we should say that it lies in the base change $\Sigma_{\varphi(x)} \times_{\operatorname{Spec}(R)} \operatorname{Spec}(A)$, but for notational clarity, we will omit any reference to base changes). If we denote the collection of A-rational points of the constructible set $\Sigma_{\varphi(x)}$ by $\Sigma_{\varphi(x)}(A)$, then this latter set corresponds to the definable subset $\varphi(A)$ under the identification of A-rational points of $A[x]$ with n-tuples over A. If φ is a sentence, then Σ_φ is a constructible subset of $\operatorname{Spec}(R)$ and hence its base change to $\operatorname{Spec}(A)$ is a constructible subset of $\operatorname{Spec}(A)$. Since A is a domain, $\operatorname{Spec}(A)$ has a unique A-rational point (corresponding to the zero-ideal) and hence φ holds in A if and only if this point belongs to Σ_φ.

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[8]Recall that a *Zariski-constructible* subset of some affine scheme $\operatorname{Spec}(R)$ is a finite Boolean combination of *Zariski closed* subsets $\mathrm{V}(I)$, where the latter means all prime ideals of R containing the ideal I.

Conversely, if Σ is an R-constructible subset of $\mathbb{A}^n_R$, then we can associate to it a quantifier free formula $\varphi_\Sigma(x)$ with parameters from R as follows. However, here there is some ambiguity, as a constructible set is more intrinsically defined than a formula. Suppose first that Σ is the Zariski closed subset $\mathrm{V}(I)$, where I is an ideal in $R[x]$. Choose a system of generators, so that $I = (f_1, \dots, f_s)R[x]$ and set $\varphi_\Sigma(x)$ equal to the quantifier free formula $f_1(x) = \dots = f_s(x) = 0$. Let A be an R-algebra without zero-divisors. It follows that an n-tuple $\mathbf{a}$ is an A-rational point of Σ if and only if $\mathbf{a}$ satisfies the formula φ_Σ. Therefore, if we make a different choice of generators $I = (f'_1, \dots, f'_s)R[x]$, although we get a different formula φ', it defines in any R-algebra A without zero-divisors the same definable set, to wit, the collection of A-rational points of Σ. To associate a formula to an arbitrary constructible set, we do this recursively by letting $\varphi_\Sigma \wedge \varphi_\Psi$, $\varphi_\Sigma \vee \varphi_\Psi$ and $\neg\varphi_\Sigma$ correspond to the constructible sets $\Sigma \cap \Psi$, $\Sigma \cup \Psi$ and $-\Sigma$ respectively.

We say that two formulae $\varphi(x)$ and $\psi(x)$ in the same free variables $x = (x_1, \dots, x_n)$ are *equivalent over a ring* A, if they hold on exactly the same tuples from A (that is to say, if they define the same subsets in A^n). In particular, if φ and ψ are sentences, then they are equivalent in A if they are simultaneously true or false in A. If $\varphi(x)$ and $\psi(x)$ are equivalent for all rings A in a certain class $\mathcal{K}$, then we say that $\varphi(x)$ and $\psi(x)$ are *equivalent modulo the class* $\mathcal{K}$. In particular, if Σ is a constructible set in $\mathbb{A}^n_R$, then any two formulae associated to it are equivalent modulo the class of all R-algebras without zero-divisors. In this sense, there is a one-one correspondence between constructible subsets of $\mathbb{A}^n_R$ and quantifier free formulae with parameters from R up to equivalence.

Quantifier Elimination. For certain rings (or classes of rings), every formula is equivalent to a quantifier free formula; this phenomenon is known under the name *Quantifier Elimination*. We will only encounter it for the following class.

3.1. **Theorem** (Quantifier Elimination for algebraically closed fields)**.** *If $\mathcal{K}$ is the class of all algebraically closed fields, then any formula without parameters is equivalent modulo $\mathcal{K}$ to a quantifier free formula without parameters.*

More generally, if F is a field and $\mathcal{K}(F)$ the class of all algebraically closed fields containing F, then any formula with parameters from F is equivalent modulo $\mathcal{K}(F)$ to a quantifier free formula with parameters from F.

Sketch of proof. These statements can be seen as translations in model-theoretic terms of Chevalley's Theorem which says that the projection of a constructible set is again constructible. I will only explain this for the first assertion. Let K be an algebraically closed field. As already observed, a quantifier free formula $\varphi(x)$ (without parameters) corresponds to a constructible set $\Sigma_{\varphi(x)}$ in $\mathbb{A}^n_{\mathbb{Z}}$ and the tuples in K^n satisfying $\varphi(x)$ are precisely the K-rational points $\Sigma_{\varphi(x)}(K)$ of $\Sigma_{\varphi(x)}$. The key observation is now the following. Let $\psi(x, y)$ be a quantifier free formula and put $\gamma(x) := (\exists y)\psi(x, y)$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$. Let $\Psi := \psi(K)$ be the subset of K^{n+m} defined by $\psi(x, y)$ and let $\Gamma := \gamma(K)$ be the subset of K^n defined by $\gamma(x)$. Therefore, if we identify K^{n+m} with the collection of K-rational points of $\mathbb{A}^{n+m}_K$, then

$$\Psi = \Sigma_{\psi(x,y)}(K).$$

Moreover, if $p\colon \mathbb{A}^{n+m}_K \to \mathbb{A}^n_K$ is the projection onto the first n coordinates then $p(\Psi) = \Gamma$. By Chevalley's Theorem (see for instance [5, Corollary 14.7] or [6, II. Exercise 3.19]), $p(\Sigma_{\psi(x,y)})$ (as a subset in $\mathbb{A}^n_{\mathbb{Z}}$) is again constructible, and therefore, by our previous discussion, of the form $\Sigma_{\chi(x)}$ for some quantifier free formula $\chi(x)$. Hence $\Gamma = \Sigma_{\chi(x)}(K)$, showing that $\gamma(x)$ is equivalent modulo K to $\chi(x)$. Since $\chi(x)$ does not depend on K, we have in fact an equivalence of formulae modulo the class $\mathcal{K}$. To get rid of an arbitrary chain of quantifiers, we use induction on the number of quantifiers, noting that the complement of a set defined by $(\forall y)\psi(x, y)$ is the set defined by $(\exists y)\neg\psi(x, y)$, where $\neg(\cdot)$ denotes negation. For some alternative proofs, see [8, Corollary A.5.2] or [12, Theorem 1.6]. $\square$

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