A GENERALIZATION OF THE AUSLANDER-BUCHSBAUM FORMULA

HANS SCHOUTENS

ABSTRACT. Let R be a Noetherian local ring and Ω an arbitrary R-module of finite depth and finite projective dimension. The flat dimension of Ω is at least $\operatorname{depth}(R)-\operatorname{depth}(\Omega)$ with equality in the following cases: (i) Ω is finitely generated over some Noetherian local R-algebra S; (ii) $\dim(R)=1$; (iii) $\dim(R)=2$ and Ω is separated; (iv) R is Cohen-Macaulay, $\dim(R)=3$ and Ω is complete.

1. INTRODUCTION

The *Auslander-Buchsbaum formula* states that

$$\operatorname{proj.dim}_R \Omega + \operatorname{depth}_R(\Omega) = \operatorname{depth}(R)$$

for any finitely generated R-module Ω of finite projective dimension over a Noetherian local ring R (see for instance [5, Theorem 19.1]). Recall that the *projective dimension* $\operatorname{proj.dim}(\Omega)$ of Ω is the minimal length of a projective resolution of Ω, and the *depth* $\operatorname{depth}(\Omega)$ of Ω, is the length of a maximal Ω-regular sequence. This formula is no longer true, if we drop the requirement that Ω is finitely generated. The reasons for this failure are threefold: a non-finitely generated module Ω can (i) be flat but not free; (ii) have infinitely many associated primes; and (iii) be non-separated. Regarding (i), projective dimension is in this context simply the wrong invariant and should be replaced by flat dimension. Recall that the *flat dimension* or *weak dimension* of Ω, denoted $\operatorname{fl.dim}_R(\Omega)$, is defined to be the supremum of all i for which $\operatorname{Tor}_i^R(\cdot,\Omega)$ is not identically zero, or, equivalently, the minimal length of a flat resolution of Ω. Note that since a flat module has finite projective dimension, a module has finite flat dimension if and only if it has finite projective dimension. As for (ii), we can no longer define the depth of Ω as the maximal length of an Ω-regular sequence (we will call the latter invariant therefore the *naive depth* of Ω and denote it $\operatorname{n-depth}(\Omega)$). Instead, $\operatorname{depth}(\Omega)$ is defined by means of the vanishing of certain *Ext* functors (see (3) below). Finally, (iii) is a fact of life and is responsible for the additional separatedness constraint on our modules. In particular, Nakayama's lemma does no longer hold and a non-zero module can therefore have infinite depth.

The depth formulas. The following four *depth formulas* (two equalities and two inequalities) will play an important role in this paper; they always hold if $\Omega \neq 0$ is finitely generated:

Auslander-Buchsbaum formula:

$$\operatorname{fl.dim}(\Omega) + \operatorname{depth}(\Omega) = \operatorname{depth}(R);$$

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naive depth formula:

$$\text{n-depth}(\Omega) = \operatorname{depth}(\Omega);$$

Ischebeck inequality: for every prime ideal $\mathfrak{p}$ of R, we have

$$\operatorname{depth}(\Omega) \leq \operatorname{depth}(\mathfrak{p};\Omega) + \dim R/\mathfrak{p};$$

subdimension inequality: for every associated prime ideal $\mathfrak{p}$ of Ω, we have

$$\operatorname{depth}(\Omega) \leq \dim R/\mathfrak{p}.$$

For non-finitely generated modules, any of these four depth formulas may fail to hold, and this paper is for a large part devoted to understanding the relationship between these failures. Among other things, we will establish the following implications:

Auslander-Buchsbaum formula $\Longrightarrow$ *Ischebeck inequality* $\Longrightarrow$ *subdimension inequality*.

As far as the two equalities are concerned, they are in general only inequalities giving a lower bound for depth, namely

$$\text{n-depth}(\Omega) \leq \operatorname{depth}(\Omega) \quad \text{and} \quad \operatorname{depth}(R) - \text{fl.dim}(\Omega) \leq \operatorname{depth}(\Omega).$$

Even if the ring R is regular, there are still plenty of examples of modules of finite depth for which the Auslander-Buchsbaum formula fails: for instance, any R-module Ω which is a big Cohen-Macaulay module but which is not balanced (for a construction of such a module, see 4.13). Indeed, such a module is not flat but has maximal depth. Nonetheless, there are also many situations in which we can prove the Auslander-Buchsbaum formula, and I will now review some of these. Any module of depth zero over a Noetherian local ring of Cohen-Macaulay-defect at most one satisfies the Auslander-Buchsbaum formula; see Theorem 5.1 and Proposition 6.2. The Auslander-Buchsbaum formula holds in low dimensions, under some additional separatedness conditions. Our main result in that regard is:

1.1. **Theorem.** *Let R be a d-dimensional Noetherian local ring. Let Ω be an arbitrary R-module of finite depth and finite projective dimension. The Auslander-Buchsbaum formula holds, that is to say,*

$$\text{fl.dim}_R(\Omega) + \operatorname{depth}_R(\Omega) = \operatorname{depth}(R), \tag{1}$$

under any of the following additional hypotheses: (i) $d = 1$; (ii) $d = 2$ and Ω is separated; (iii) R is Cohen-Macaulay, $d = 3$ and Ω is complete.

This result will be proved in §§5–6. Another class of modules for which the Auslander-Buchsbaum formula holds are the finitely generated modules over local R-algebras (see Theorem 6.1). In particular, we get the following special case.

1.2. **Theorem.** *For any local homomorphism $R \to S$ of Noetherian local rings, we have an equality*

$$\text{fl.dim}_R(S) + \operatorname{depth}_R(S) = \operatorname{depth}(R),$$

provided S has finite projective dimension over R. □

1.3. **Residual homological dimension.** We remind the reader of another formula regarding the depth of an arbitrary module, due to Bartijn-Strooker ([2, Théorème 4.1]): if Ω has finite depth and finite projective dimension, then

$$\operatorname{depth}(R) - \operatorname{depth}(\Omega) = \operatorname{res.dim}(\Omega), \tag{2}$$

where $\operatorname{res.dim}(\Omega)$ is the *residual homological dimension* of Ω defined as the supremum of all $n \geq 0$ for which $\operatorname{Tor}_n^R(k, \Omega) \neq 0$, where k is the residue field of R (if no such n exists, we put $\operatorname{res.dim}_R(\Omega) := \infty$). This formula hinges on the dual nature of $\operatorname{Ext}_R^\bullet(k, \Omega)$ and $\operatorname{Tor}_\bullet^R(k, \Omega)$. By this formula, Ω satisfies the Auslander-Buchsbaum formula if and only if the inequality $\operatorname{res.dim}(\Omega) \leq \operatorname{fl.dim}(\Omega)$ is an equality, thus yielding a more subtle connection between the vanishing of certain *Ext* and *Tor* functors. Put differently, the Auslander-Buchsbaum formula is a generalized local flatness theorem. For instance, the following result, an immediate corollary of (2), generalizes some flatness criteria of [9] when we take $e = 1$ and apply Theorem 1.1.

1.4. **Theorem.** *Let R be a Noetherian local ring with residue field k and let Ω be an R-module of finite projective dimension. Let $q := \operatorname{depth}(R)$ and $p \leq q$. If Ω satisfies the Auslander-Buchsbaum formula and $\operatorname{Tor}_n^R(k, \Omega)$ vanishes for all $n = p, \dots, q$, then Ω has flat dimension strictly less than p.* □

We start in §2 with recalling the definition of depth for non-finitely generated modules. In §3, we derive some local criteria for the vanishing of certain *Ext* and *Tor* functors, which in turn will yield a local characterization of flat dimension. In §4, we study in more detail the four depth formulas discussed above. The last two sections are then devoted to our main results on the Auslander-Buchsbaum formula.

2. Depth and dimension

In this section, we review the notion of depth for non-finitely generated modules. We take the treatment from [3, §9.1], albeit presented without reference to *grade* or Koszul sensitivity (see also [4]) and only over Noetherian local rings. Formulating depth in terms of regular sequences is a delicate matter if the module Ω is not finitely generated, even over a Noetherian local ring $(R, \mathfrak{m})$. Firstly, it is possible that Nakayama's Lemma fails, so that $\Omega = \mathfrak{m}\Omega$ without Ω being zero. When this is the case, we will call Ω *degenerated*. Secondly, even if the depth is positive, this is not necessarily witnessed by the existence of a regular element. Nonetheless, it does so after an appropriate extension. For our purposes it is instrumental that we can detect depth by means of regular sequences after an extension which does not increase the dimension, contrary to what is done in the more usual treatments of the subject. This is accomplished by Lemma 2.3. To this end, we make the following definition.

The extension $R \subseteq R(X)$. For $(R, \mathfrak{m})$ a local ring and X a finite tuple of indeterminates, let $R(X)$ denote the localization of $R[X]$ at $\mathfrak{m}R[X]$. If Ω is an R-module, we will denote $\Omega \otimes_R R(X)$ by $\Omega(X)$ and write its elements as polynomials with coefficients in Ω, that is to say, we write ωX^p for $\omega \otimes X^p$. In the terminology of [8, §4], an extension of the form $R \subseteq R(X)$ is a scalar extension, that is to say, a faithfully flat and unramified extension. We will use the following preservation properties of these extensions.

2.1. **Lemma.** *Let R be a Noetherian local ring, Ω an R-module and X a finite tuple of indeterminates.*

(2.1.1) *R is regular, Gorenstein or Cohen-Macaulay if and only if so is $R(X)$;*

(2.1.2) *$\dim R = \dim R(X)$ and $\operatorname{depth}(R) = \operatorname{depth}(R(X))$;*
(2.1.3) *Ω is separated if and only if so is $\Omega(X)$;*
(2.1.4) *the associated primes of $\Omega(X)$ are precisely the prime ideals of the form $\mathfrak{p}R(X)$ for $\mathfrak{p} \in \operatorname{Ass}(\Omega)$.*

Proof. The first two properties follow from the fact that $R \subseteq R(X)$ is faithfully flat and unramified; see [5, §23]. Next we prove (2.1.3), where one direction is clear since $\Omega \subseteq \Omega(X)$. To prove the other direction, we induct on the number of variables, and hence we may assume that X is a single variable. Set $\Omega[X] := \Omega \otimes_R R[X]$. Towards a contradiction, suppose that $\Omega(X)$ is not separated. Therefore, we can find already an element $\pi := \omega_0 + \omega_1 X + \cdots + \omega_d X^d$ in $\Omega[X]$ which lies in every $\mathfrak{m}^n\Omega(X)$. I claim that $\pi \in \mathfrak{m}^n\Omega[X]$ for all $n > 0$. Assuming the claim, we see that each ω_i lies in all $\mathfrak{m}^n\Omega$ whence must be zero by our separatedness assumption.

To prove the claim, observe that since $\pi \in \mathfrak{m}^n\Omega(X)$, we can find $g \in R[X]$ with not all coefficients in $\mathfrak{m}$, such that $g\pi \in \mathfrak{m}^n\Omega[X]$. Write $g = p + m$ with p a monic polynomial and $m \in \mathfrak{m}R[X]$. In particular, $p\pi = g\pi - m\pi$ lies in $\mathfrak{m}\Omega[X]$. Since p is monic, one readily verifies that then $\pi \in \mathfrak{m}\Omega[X]$. If $n = 1$, we are done, otherwise, $p\pi = g\pi - m\pi$ lies in $\mathfrak{m}^2\Omega[X]$ and the same argument then yields that $\pi \in \mathfrak{m}^2\Omega[X]$. Continuing in this way, we reach after n steps that $\pi \in \mathfrak{m}^n\Omega[X]$, as required.

For the proof of (2.1.4), note that $R(X)/\mathfrak{a}R(X) \cong (R/\mathfrak{a})(X)$ for all ideals $\mathfrak{a} \subseteq R$. In particular, if $\mathfrak{a}$ is prime, then so is $\mathfrak{a}R(X)$. We leave it as an exercise to show that if $\mathfrak{p} := \operatorname{Ann}_R(\omega)$ is an associated prime of Ω, where $\omega \in \Omega$, then $\operatorname{Ann}_{R(X)}(\omega) = \mathfrak{p}R(X)$, showing that the extended ideal $\mathfrak{p}R(X)$ is an associated prime ideal of $\Omega(X)$. So remains to show that given an associated prime $\mathfrak{q} := \operatorname{Ann}_{R(X)}(\pi)$ of $\Omega(X)$ for some $\pi \in \Omega(X)$, then $\mathfrak{p} := \mathfrak{q} \cap R$ is an associated prime of Ω and $\mathfrak{q} = \mathfrak{p}R(X)$. Without loss of generality, we may assume $\pi = \sum \omega_i X^i$ lies in $\Omega[X]$, for some $\omega_i \in \Omega$. It is easy to check that $\mathfrak{p}$ is the intersection of the $\operatorname{Ann}_R(\omega_i)$. Since $\mathfrak{p}$ is a prime ideal, it must therefore be equal to one of them, say $\mathfrak{p} = \operatorname{Ann}_R(\omega_k)$. This already shows that $\mathfrak{p} \in \operatorname{Ass}(\Omega)$. To show that $\mathfrak{q} = \mathfrak{p}R(X)$, let $f \in R[X]$ be in $\mathfrak{q}$. By induction on the degree e of f, we may assume that any polynomial in $\mathfrak{q}$ of degree less than e already belongs to $\mathfrak{p}R(X)$. Write $f = a + Xg$ with $a \in R$ and $g \in R[X]$ of degree $e - 1$. By induction on i, one easily obtains from $f\pi = 0$ that $a^i\omega_i = 0$. In particular, $a^k \in \operatorname{Ann}_R(\omega_k) = \mathfrak{p}$ and hence $a \in \mathfrak{p}$. This in turn implies that $a\omega_i = 0$ for all i, since $\mathfrak{p} \subseteq \operatorname{Ann}_R(\omega_i)$. Hence $Xg\pi = (f - a)\pi = 0$. Since X is a unit, $g \in \mathfrak{q}$, so that by induction $g \in \mathfrak{p}R(X)$ and therefore $f = a + Xg \in \mathfrak{p}R(X)$. □

2.2. **Depth.** Let $(R, \mathfrak{m})$ be a Noetherian local ring and Ω an arbitrary R-module. An element $x \in R$ is called a *zero-divisor* on Ω, if $x\omega = 0$ for some non-zero $\omega \in \Omega$. If x is not a zero-divisor on Ω, then we call it Ω*-regular*. A sequence $(x_1, \dots, x_d)$ is called *weakly* Ω*-regular*, if each x_i lies in $\mathfrak{m}$ and is $\Omega/(x_1, \dots, x_{i-1})\Omega$-regular. Finally, a sequence $(x_1, \dots, x_d)$ is called Ω*-regular*, if it is weakly Ω-regular and $\Omega \neq (x_1, \dots, x_d)\Omega$. By [5, Theorem 6.1], an element x is Ω-regular if and only if it is not contained in an associated prime ideal of Ω. Unfortunately, however, if Ω is not finitely generated, then it might have infinitely many associated primes.

The *naive depth* (also called *classical grade* in [4]) of Ω is defined to be the maximum possible length of a weakly Ω-regular sequence. We do not exclude the case that this length is infinite, for instance, when $\Omega = 0$, or, more generally, when Ω is *degenerated*. The naive depth is denoted by $\operatorname{n-depth}_R(\Omega)$. We define the *depth* of Ω as the maximum of the $\operatorname{n-depth}_{R(X)}(\Omega(X))$, where X runs over all finite tuples of indeterminates. More generally, if $\mathfrak{a}$ is a proper ideal of R, then $\operatorname{n-depth}_R(\mathfrak{a}; \Omega)$ denotes the maximum length of a weakly regular Ω-sequence contained in $\mathfrak{a}$ and $\operatorname{depth}_R(\mathfrak{a},\Omega)$, called the $\mathfrak{a}$-*depth* of Ω, is the maximum of the $\operatorname{n-depth}_{R(X)}(\mathfrak{a}R(X);\Omega(X))$, where X runs over all finite tuples of indeterminates. By definition, $\operatorname{n-depth}(\Omega) = \operatorname{n-depth}(\mathfrak{m};\Omega)$. One easily checks that $\operatorname{n-depth}(\Omega)$ can be at most $\dim R$ whenever it is finite.

To reconcile our definition of depth with the one in [3] or [4], we need the following analog of [3, Theorem 9.1.3].

2.3. **Lemma.** *Let R be a Noetherian local ring, X a single variable and Ω an R-module. Let $\mathfrak{a}$ be an ideal of R with $\Omega \neq \mathfrak{a}\Omega$. If $\mathfrak{a}$ is not contained in any associated prime of Ω, then $\mathfrak{a}R(X)$ contains an $\Omega(X)$-regular element.*

Proof. This is an immediate corollary of the proof of [3, Proposition 9.1.3] (or alternatively, it can also be deduced from (2.1.4)). Namely, suppose $\mathfrak{a} = (x_1,\dots,x_n)R$ and put $f := x_1 + x_2X + \dots + x_nX^{n-1}$. In the proof of [3, Proposition 9.1.3], it is shown that f is $\Omega[X]$-regular (where $\Omega[X] = \Omega \otimes R[X]$). Since $f \in \mathfrak{m}R[X]$, this property is preserved after localization, so that f is $\Omega(X)$-regular. □

By the arguments from for instance [3, §9.1], one can then prove using Lemma 2.3 that

$$\operatorname{depth}_R(\mathfrak{a};\Omega) = \inf\left\{\, i \in \mathbb{N} \mid \operatorname{Ext}^i_R(R/\mathfrak{a},\Omega) \neq 0 \,\right\}. \tag{3}$$

Any non-degenerated module has finite depth. Note that the converse is false in general: for instance, if R is a discrete valuation ring with field of fractions K, then K/R is degenerated, yet has depth zero (since the maximal ideal is an associated prime; in fact, K/R satisfies the Auslander-Buchsbaum formula).

2.4. **Lemma.** *Let R be a Noetherian local ring and Ω an arbitrary R-module. If R and Ω have both depth at least e, then there exists an $R(X)$-regular sequence $(x_1,\dots,x_e)$ which is also $\Omega(X)$-regular, where X is some finite set of variables.*

Proof. An inductive argument reduces to the case $e = 1$ and this case is dealt with by a careful analysis of the proof of [3, Proposition 9.1.3]. Namely, let $x_1 \in \mathfrak{m}$ be R-regular and choose $x_2,\dots,x_n \in \mathfrak{m}$ such that $\mathfrak{m} = (x_1,\dots,x_n)R$. As explained in the proof of Lemma 2.3, the element $f = x_1 + x_2X + \dots + x_nX^{n-1}$ is $\Omega(X)$-regular. I claim that f is $R[X]$-regular. Indeed, suppose $fg = 0$ in $R[X]$ with $g \neq 0$. Since X is regular, we may divide out a power of X and assume that g has non-zero constant term $y \in R$. However, $fg = 0$ yields $x_1y = 0$, so that $y = 0$ by the regularity of x_1, contradiction. Therefore, f is $R[X]$-regular whence $R(X)$-regular, as required. □

2.5. *Remark.* A simple modification of this proof shows that if Ω has positive $\mathfrak{a}$-depth, then we can find $x \in \mathfrak{a}R(X)$ which is simultaneously $R(X)$-regular and $\Omega(X)$-regular.

2.6. **Strong depth.** Call an element $x \in R$ a *strongly Ω-regular* element, if x is Ω-regular and Ω is xR-adically separated. More generally, a sequence $(x_1,\dots,x_e)$ is called *strongly Ω-regular*, if each x_{i+1} is strongly $\Omega/(x_1,\dots,x_i)\Omega$-regular. The maximal length of a strongly Ω-regular sequence will be called the *strong depth* of Ω and will be denoted $\operatorname{s-depth}_R(\Omega)$. For finitely generated modules all three depth variants are equal, but in general we only have inequalities $\operatorname{s-depth}(\Omega) \leq \operatorname{n-depth}(\Omega) \leq \operatorname{depth}(\Omega)$. Even if the latter two are equal, the former can still be smaller: the strong depth of the module Ω from Example 4.13 is zero, whereas its (naive) depth is two. Moreover, we cannot increase the strong depth simply by passing to an extension $\Omega(X)$.

2.7. **Subdimension.** Recall that the *dimension* $\dim(\Omega)$ of an R-module Ω is the dimension of $R/\operatorname{Ann}_R(\Omega)$. (Caveat: when I is an ideal of R, one often calls $\dim(R/I)$, the dimension of the *ideal* I, which is in general different from the dimension of the *module* I.) By the *subdimension* of Ω, we mean the minimum of all $\dim(\Pi)$ for $\Pi \subseteq \Omega$ running over all non-zero submodules of Ω. We denote the subdimension of Ω by $\operatorname{subdim}_R(\Omega)$. The subdimension is completely determined by the associated primes of Ω:

2.8. **Lemma.** *The subdimension of an R-module Ω is the minimum of the $\dim(R/\mathfrak{p})$ for $\mathfrak{p}$ running over all associated prime ideals of Ω.*

Proof. If $\mathfrak{p}$ is an associated prime of Ω, then $R/\mathfrak{p}$ is isomorphic to a submodule of Ω. Conversely, if $\Pi_1 \subseteq \Pi_2$ are submodules of Ω, then $\dim(\Pi_1) \leq \dim(\Pi_2)$. Hence in the definition of subdimension, we may restrict ourselves to non-zero cyclic submodules. Let ω be a non-zero element of Ω and put $H := R\omega$, so that $H \cong R/\operatorname{Ann}_R(\omega)$. By [5, Theorem 6.1], there is an associated prime $\mathfrak{p}$ of Ω containing $\operatorname{Ann}_R(\omega)$ and whence in particular $R/\mathfrak{p}$ has dimension at most $\dim(H)$. □

3. The vanishing of Ext and Tor

Let R be a Noetherian ring and Ω an arbitrary R-module. The flat dimension of Ω is given as the largest n for which $\operatorname{Tor}_n^R(\cdot,\Omega)$ is not identically zero. Therefore, we would like to have some simple criteria for its vanishing. Since *Tor* commutes with direct limits, it suffices to check that $\operatorname{Tor}_n^R(M,\Omega)$ vanishes for all finitely generated R-modules M. It is well-known that a finitely generated R-module M admits a *prime filtration*, that is to say, there is an ascending chain of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$ with each subsequent quotient M_{i+1}/M_i isomorphic to a cyclic module of the form $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Moreover, if $\operatorname{Ann}_R(M)$ has height h (respectively, M has dimension d), then the prime ideals $\mathfrak{p}_i$ occurring in a prime filtration of M all contain $\operatorname{Ann}_R(M)$ whence, in particular, all have height at least h (respectively, all $R/\mathfrak{p}_i$ have dimension at most d). Therefore, we proved the following result.

3.1. **Lemma.** *Let R be a Noetherian ring and Ω an R-module. If for some $e \in \mathbb{N}$ and for all prime ideals $\mathfrak{p}$ of R, we have $\operatorname{Tor}_e^R(R/\mathfrak{p},\Omega) = 0$, then Ω has flat dimension at most $e-1$.*

More generally, if $\operatorname{Tor}_e^R(R/\mathfrak{p},\Omega) = 0$ (respectively, $\operatorname{Ext}_R^e(R/\mathfrak{p},\Omega) = 0$) for all prime ideals $\mathfrak{p}$ with $\dim R/\mathfrak{p} \leq d$ and if M is a finitely generated R-module of dimension d, then $\operatorname{Tor}_e^R(M,\Omega) = 0$ (respectively, $\operatorname{Ext}_R^e(M,\Omega) = 0$). Similarly, if $\operatorname{Tor}_e^R(R/\mathfrak{p},\Omega) = 0$ (respectively, $\operatorname{Ext}_R^e(R/\mathfrak{p},\Omega) = 0$) for all prime ideals $\mathfrak{p}$ of height at least h and if $\mathfrak{a}$ is an ideal of height h, then $\operatorname{Tor}_e^R(R/\mathfrak{a},\Omega) = 0$ (respectively, $\operatorname{Ext}_R^e(R/\mathfrak{a},\Omega) = 0$).

In order to formulate a more local criterion, we need a definition. Let R be a Noetherian ring, $\mathfrak{p}$ a prime ideal of R and Ω an arbitrary R-module. We will denote the residue field of $\mathfrak{p}$ by $k(\mathfrak{p})$, that is to say, $k(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$.

3.2. **Definition.** The n*-th Betti number* of Ω at $\mathfrak{p}$ is the (possibly infinite) dimension of the $k(\mathfrak{p})$-vector space $\operatorname{Tor}_n^{R_\mathfrak{p}}(k(\mathfrak{p}),\Omega_\mathfrak{p})$ and is denoted by $\beta_n^R(\mathfrak{p};\Omega)$, or simply, by $\beta_n(\mathfrak{p};\Omega)$ if the ring is understood. Similarly, the n*-th Bass number* $\mu_R^n(\mathfrak{p};\Omega)$ is defined as the dimension of the $k(\mathfrak{p})$-vector space $\operatorname{Ext}_{R_\mathfrak{p}}^n(k(\mathfrak{p}),\Omega_\mathfrak{p})$.

3.3. **Proposition.** *Let R be a Noetherian ring, Ω an R-module and $e \in \mathbb{N}$. If $\beta_j^R(\mathfrak{p};\Omega) = 0$ for all $j > e$ and all prime ideals $\mathfrak{p}$ of R, then Ω has flat dimension at most e.*

Proof. In view of Lemma 3.1, it suffices to show that $\operatorname{Tor}^R_{e+1}(R/\mathfrak{p},\Omega)=0$ for all prime ideals $\mathfrak{p}$ of R. In fact, we will prove the stronger statement that $\operatorname{Tor}^R_j(R/\mathfrak{p},\Omega)=0$ for all $j>e$ and all $\mathfrak{p}$. To this end, we will perform a downward induction on the height h of $\mathfrak{p}$. Assume first that $h=\dim R$, so that $\mathfrak{p}$ is a maximal ideal of R. Let $j>e$ and let τ be an arbitrary element of $\operatorname{Tor}^R_j(R/\mathfrak{p},\Omega)$. Clearly, $\mathfrak{p}\tau=0$. On the other hand, since $\beta^R_j(\mathfrak{p};\Omega)=0$, we can find some $s\notin\mathfrak{p}$, such that $s\tau=0$. Since $\mathfrak{p}$ is maximal, we can find some $t\in R$ and some $m\in\mathfrak{p}$, such that $st+m=1$. It follows that $\tau=st\tau+m\tau=0$, showing that $\operatorname{Tor}^R_j(R/\mathfrak{p},\Omega)=0$.

Next, suppose the claim proven for all prime ideals of height at least $h+1$ and let $\mathfrak{p}$ be a height h prime ideal of R. Note that by Lemma 3.1, our induction hypothesis actually gives that $\operatorname{Tor}^R_j(R/\mathfrak{a},\Omega)=0$ for each ideal $\mathfrak{a}$ of height at least $h+1$ and each $j>e$. Fix some $j>e$. Let θ be an arbitrary element of $\operatorname{Tor}^R_j(R/\mathfrak{p},\Omega)$. Since by assumption $\beta^R_j(\mathfrak{p};\Omega)=0$, we can find some $x\notin\mathfrak{p}$ such that $x\theta=0$. From the exact sequence

$$0\to R/\mathfrak{p}\xrightarrow{x}R/\mathfrak{p}\to R/\mathfrak{n}\to 0$$

with $\mathfrak{n}:=\mathfrak{p}+xR$, we get part of a long exact sequence

$$\operatorname{Tor}^R_{j+1}(R/\mathfrak{n},\Omega)\to\operatorname{Tor}^R_j(R/\mathfrak{p},\Omega)\xrightarrow{x}\operatorname{Tor}^R_j(R/\mathfrak{p},\Omega).$$

The first of these modules is zero by the inductive hypothesis and the argument above. In other words, we showed that x is not a zero-divisor on $\operatorname{Tor}^R_j(R/\mathfrak{p},\Omega)$. Since $x\theta=0$, this implies that $\theta=0$, as required. □

Note that the proof gives the following more precise result: *if $\beta_j(\mathfrak{p};\Omega)=0$ for all $j>e$ and all prime ideals $\mathfrak{p}$ of R with $\dim R/\mathfrak{p}\leq d$, then $\operatorname{Tor}^R_j(M,\Omega)=0$ for every finitely generated R-module M of dimension at most d.* By the same argument, taking into account the contravariancy of $\operatorname{Ext}^\bullet_R(\cdot,\Omega)$, we get the following criterion for the vanishing of an *Ext* functor.

3.4. **Proposition.** *Let R be a Noetherian ring and Ω an R-module. Let $d\in\mathbb{N}$ and $e\in\mathbb{N}\cup\{\infty\}$. If $\mu^j_R(\mathfrak{p};\Omega)=0$ for all $j<e$ and all prime ideals $\mathfrak{p}$ with $\dim R/\mathfrak{p}\leq d$, then $\operatorname{Ext}^j_R(M,\Omega)=0$ for every $j<e$ and every finitely generated R-module M of dimension at most d.* □

The main result in this section is a local criteria for flat dimension in terms of residual homological dimension (see §1.3).

3.5. **Theorem.** *Let R be a Noetherian local ring and Ω an R-module of finite projective dimension. Then the following numbers are all equal*

- *the flat dimension $\operatorname{fl.dim}(\Omega)$ of Ω;*
- *the maximum of all $\operatorname{res.dim}_{R_\mathfrak{p}}(\Omega_\mathfrak{p})$, where $\mathfrak{p}$ runs over the prime ideals of R;*
- *the maximum of all $\operatorname{depth}(R_\mathfrak{p})-\operatorname{depth}_{R_\mathfrak{p}}(\Omega_\mathfrak{p})$, where $\mathfrak{p}$ runs over the prime ideals for which $\operatorname{depth}_{R_\mathfrak{p}}(\Omega_\mathfrak{p})$ is finite.*

Proof. The equality of the first two numbers is merely a reformulation of Proposition 3.3 in terms of residual homological dimension. In order to prove equality with the third, we have to show in view of (2) that if $\operatorname{fl.dim}(\Omega)=\operatorname{res.dim}(\Omega_\mathfrak{p})$ for some prime $\mathfrak{p}$, then $\operatorname{depth}(\Omega_\mathfrak{p})$ is finite. Suppose not and let q be the depth of $R_\mathfrak{p}$. After possibly taking an extension of the form $R_\mathfrak{p}\subseteq R_\mathfrak{p}(X)$, which is harmless in this case in view of Lemma 2.1, we may assume by Lemma 2.4 that there exists a q-tuple $\mathbf{x}$ which is simultaneously $R_\mathfrak{p}$-regular and $\Omega_\mathfrak{p}$-regular. Since $\mathfrak{p}R_\mathfrak{p}$ is an associated prime of $\mathbf{x}R_\mathfrak{p}$, we get an exact sequence

$$0 \to k(\mathfrak{p}) \to R_\mathfrak{p}/\mathbf{x}R_\mathfrak{p} \to V \to 0$$

for some (cyclic) $R_\mathfrak{p}$-module V. From the *Tor* long exact sequence we get an exact sequence

$$\mathrm{Tor}_{e+1}^{R_\mathfrak{p}}(V, \Omega_\mathfrak{p}) \to \mathrm{Tor}_e^{R_\mathfrak{p}}(k(\mathfrak{p}), \Omega_\mathfrak{p}) \to \mathrm{Tor}_e^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathbf{x}R_\mathfrak{p}, \Omega_\mathfrak{p}),$$

where $e := \mathrm{fl.dim}(\Omega)$. Hence, the left most module is zero, and so is the right most module by the regularity of the sequence $\mathbf{x}$. Therefore also the middle module vanishes, contradicting that $\beta_e^R(\mathfrak{p}; \Omega) \neq 0$. □

This result in conjunction with (2) shows that the Auslander-Buchsbaum formula has a local character: if after localizing the residual homological dimension becomes smaller, then the Auslander-Buchsbaum formula holds. As an immediate corollary we reprove another result of Auslander and Buchsbaum.

3.6. **Corollary** ([1, Theorem 2.4])**.** *A module Ω of finite projective dimension over a Noetherian local ring R has flat dimension at most the maximum of all $\mathrm{depth}(R_\mathfrak{p})$, where $\mathfrak{p}$ runs over the prime ideals of R. In particular, $\mathrm{fl.dim}_R(\Omega) \leq \dim R$ and this inequality is strict if R is not Cohen-Macaulay.* □

That the bound in Corollary 3.6 is sharp, is illustrated by the two-dimensional local ring $R := K[[x,y,z]]/(x^2, xy, xz)K[[x,y,z]]$, with K a field. If $\mathfrak{p} := (x,y)R$, then $R_\mathfrak{p}$ is isomorphic to $K((z))[[y]]$, so that $R_\mathfrak{p}/yR_\mathfrak{p}$ has flat dimension one, although R itself has depth zero. Note that $R_\mathfrak{p}/yR_\mathfrak{p}$ has actually infinite depth.

4. Depth formulas

We now take a closer look at the depth formulas from the introduction. Throughout, $(R, \mathfrak{m})$ is a Noetherian local ring and Ω is an arbitrary R-module.

4.1. **The naive depth formula.** Recall that we said that a module Ω satisfies the *naive depth formula*, if its naive depth equals its depth, that is to say, if there exists an Ω-regular sequence of length $\mathrm{depth}(\Omega)$. Note that if the naive depth formula holds in Ω, then it is not necessarily the case that it also holds in a deformation $\Omega/x\Omega$, for x some Ω-regular element (for a counterexample, consider the module in [9, Example 7.3] and the Ω-regular element x). However, we can always choose an Ω-regular element which does preserve the naive depth formula: simply take the first element in an Ω-regular sequence of length $\mathrm{depth}(\Omega)$. Lemma 2.3 shows that $\Omega(X)$ satisfies the naive depth formula, where X is a tuple of indeterminates of length equal to the depth of Ω.

A sufficient condition for the naive depth formula to hold is for each $\Omega/\mathbf{x}\Omega$ to have only finitely many associated prime ideals, where $\mathbf{x}$ runs over all weakly Ω-regular sequences ([3, Exercise 9.1.10]). This is in particular true if Ω is finitely generated over a Noetherian local R-algebra. In fact, we can prove a stronger property, for which we need another definition.

4.2. **The strong depth formula.** We say that Ω satisfies the *strong depth formula* if its strong depth equals its depth (see §2.6 for the definition of strong depth). In particular, if the strong depth formula holds, then so does the naive depth formula. Recall that a module Ω is called *universally separated* if $\Omega/I\Omega$ is separated for every ideal I of R (equivalently, if each $I\Omega$ is closed in the adic topology). Hence if Ω is universally separated and satisfies the naive depth formula, then it also satisfies the strong depth formula. This is in particular true if Ω is finitely generated over a Noetherian local R-algebra, so that we showed:

4.3. **Corollary.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Then every finitely generated S-module satisfies the strong depth formula (whence the naive depth formula) when viewed as an R-module.* □

4.4. **The subdimension inequality.** We say that an R-module Ω satisfies the *subdimension inequality* if

$$\operatorname{depth}_R(\Omega) \leq \operatorname{subdim}_R(\Omega). \tag{4}$$

(See §2.7 for the definition of subdimension).

4.5. **Proposition.** *If x is a strongly Ω-regular element and $\Omega/x\Omega$ satisfies the subdimension inequality, then so does Ω.*

Proof. Let $e := \operatorname{depth}(\Omega)$, so that $\Omega_1 := \Omega/x\Omega$ has depth $e-1$. According to Lemma 2.8, we need to show that $e \leq h := \dim R/\mathfrak{p}$ for every associated prime $\mathfrak{p}$ of Ω. We may assume that $\mathfrak{p}$ is a maximal associated prime of Ω. Choose $\omega \in \Omega$ such that $\mathfrak{p} = \operatorname{Ann}(\omega)$. Since x is strongly Ω-regular, there exists some n such that $\omega \notin x^n\Omega$. If $\omega = x\theta$, then $x\mathfrak{p}\theta = 0$ whence $\mathfrak{p}\theta = 0$, since x is Ω-regular. Hence $\mathfrak{p}$ is also equal to $\operatorname{Ann}(\theta)$ by maximality. Applying this n times, we may assume from the start that $\omega \notin x\Omega$. Let H be the submodule of Ω generated by ω, so that $h = \dim H$. Let H_1 be the (non-zero) submodule generated by the image of ω in Ω_1. Since there is natural surjective map $H/xH \twoheadrightarrow H_1$, we have $\dim(H_1) \leq \dim(H/xH) \leq h-1$, where the last inequality follows from the fact that x is also H-regular. By assumption, $e-1 \leq \dim(H_1)$, so that putting both inequalities together, we get $e-1 \leq h-1$, as required. □

Without any separatedness assumption, the conclusion is false. For instance, Ω as in Example 4.13 does not satisfy the subdimension inequality, but its deformation $\Omega/x\Omega$ by an Ω-regular element x does.

4.6. **Theorem.** *Let R be a Noetherian local ring and Ω an R-module of finite depth. If Ω satisfies the strong depth formula, then it also satisfies the subdimension inequality.*

Proof. We induct on $s := \operatorname{depth}(\Omega)$. If $s = 0$, the subdimension inequality holds trivially, so we may assume $s > 0$. Let $(x_1, \dots, x_s)$ be a strong Ω-regular sequence and put $\Omega_1 := \Omega/x_1\Omega$. The depth of Ω_1 is $s-1$ by [3, Proposition 9.1.2], and its strong depth is clearly $s-1$, as witnessed by the strong Ω_1-regular sequence $(x_2, \dots, x_s)$. Hence our induction hypothesis implies that Ω_1 satisfies the subdimension inequality, and therefore so does Ω by Proposition 4.5. □

4.7. **Corollary.** *If Ω has finite depth and $\Omega(X)$ is universally separated for every finite tuple of indeterminates X, then Ω satisfies the subdimension inequality.*

Proof. By Lemma 2.3, the naive depth formula holds for some $\Omega(X)$ and hence so does the strong depth formula by the remark preceding Corollary 4.3. Therefore, $\Omega(X)$ satisfies the subdimension inequality by Theorem 4.6. However, it is easy to see using Lemmas 2.1 and 2.8 that $\Omega(X)$ satisfies the subdimension inequality if and only if Ω does. □

4.8. **The Ischebeck inequality.** Our next condition is based on Ischebeck's lemma (see [5, Theorem 17.1]), of which the Ischebeck inequality will be a corollary (see Theorem 4.10 below). Recall that Ω is said to satisfy the *Ischebeck inequality*, if

$$\operatorname{depth}_R(\Omega) \leq \operatorname{depth}_R(\mathfrak{p};\Omega) + \dim R/\mathfrak{p} \tag{5}$$

for every prime ideal $\mathfrak{p}$ of R. By considering an associated prime $\mathfrak{p}$ of Ω, we see that every module satisfying the Ischebeck inequality must have finite depth, and in fact, in view of Lemma 2.8, we get:

4.9. **Corollary.** *Any module satisfying the Ischebeck inequality also satisfies the subdimension inequality.*

Any module of depth zero clearly satisfies the Ischebeck inequality, and so does any module of depth one, since $\mathfrak{m}$ is then not an associated prime. For an example of a module of finite depth in which the Ischebeck inequality fails, see Example 4.13 below. To better understand the failure of Ischebeck's Lemma, we start with showing some equivalent conditions (note that (4.10.2) is the usual formulation of Ischebeck's lemma).

4.10. **Theorem.** *Let R be a Noetherian local ring and Ω an R-module of finite depth. The following properties of Ω are equivalent.*

(4.10.1) *The Ischebeck inequality (5) holds for Ω.*

(4.10.2) *For every finitely generated R-module M and every i less than $\operatorname{depth}_R(\Omega) - \dim M$, we have*

$$\operatorname{Ext}^i_R(M,\Omega) = 0.$$

(4.10.3) *For every ideal $\mathfrak{a}$ of R, we have*

$$\operatorname{depth}_R(\Omega) \leq \operatorname{depth}_R(\mathfrak{a};\Omega) + \dim R/\mathfrak{a}.$$

(4.10.4) *For every prime ideal $\mathfrak{p}$ of R, we have*

$$\operatorname{depth}_R(\Omega) \leq \operatorname{depth}_{R_\mathfrak{p}}(\Omega_\mathfrak{p}) + \dim R/\mathfrak{p}.$$

Proof. Let $s := \operatorname{depth}(\Omega)$ and let $\mathfrak{a}$ be an ideal of R with $h := \dim R/\mathfrak{a}$. Assume first that (4.10.2) holds and apply it with $M := R/\mathfrak{a}$ to conclude that $\operatorname{Ext}^i_R(R/\mathfrak{a},\Omega) = 0$ for all $i < s - h$. However, by (3), this means that $\operatorname{depth}(\mathfrak{a};\Omega)$ is at least $s - h$, showing that (4.10.3) holds. Clearly, (4.10.3) implies the Ischebeck inequality.

In general, for $\mathfrak{p}$ a prime ideal, we have by [3, Proposition 9.1.2] an inequality

$$\operatorname{depth}_R(\mathfrak{p};\Omega) \leq \operatorname{depth}_{R_\mathfrak{p}}(\Omega_\mathfrak{p}).$$

From this it is immediate that the Ischebeck inequality implies (4.10.4).

Finally, assuming that (4.10.4) holds, we want to show that then also (4.10.2) holds. Let h be the dimension of M. By Proposition 3.4, it suffices to show that $\mu^j_R(\mathfrak{p};\Omega) = 0$ for all $j < s - h$ and all prime ideals $\mathfrak{p}$ such that $\dim R/\mathfrak{p} \leq h$. However, this is clear since by (4.10.4), the depth of $\Omega_\mathfrak{p}$ is at least $s - h$, so that by (3), all $\mu^j_R(\mathfrak{p};\Omega) = 0$ for $j < s - h$. □

4.11. *Remark.* Note that (5) is trivially satisfied for the maximal ideal as well as for all minimal prime ideals $\mathfrak{g}$ such that $\dim R = \dim R/\mathfrak{g}$. In particular, if R has dimension one, then every R-module of finite depth satisfies the Ischebeck inequality.

4.12. **Corollary.** *Let R be a Noetherian local ring and Ω an R-module. If Ω satisfies the Ischebeck inequality and $\mathbf{x}$ is an Ω-regular sequence, then $\Omega/\mathbf{x}\Omega$ satisfies the Ischebeck inequality too (viewed either as an R-module or an $R/\mathbf{x}R$-module).*

Proof. By induction, we only need to treat the case of a single Ω-regular element x. Let $\mathfrak{p}$ be a prime ideal of R. If $x \notin \mathfrak{p}$ then $(\Omega/x\Omega)_{\mathfrak{p}}$ is zero whence has infinite depth and (4.10.4) holds at $\mathfrak{p}$. In the remaining case, the two depths in (4.10.4) both have dropped by one, and so also this case holds. □

4.13. **Example.** Let R be a complete two-dimensional regular local ring, for instance $R := K[[x, y]]$ with K a field. Let $\Omega := R \oplus \mathrm{Frac}(R/yR)$, where $\mathrm{Frac}(R/yR)$ denotes the field of fractions of R/yR. Clearly, x is Ω-regular, and since $\Omega/x\Omega \cong R/xR$, the sequence (x, y) is Ω-regular. On the other hand, y is a zero-divisor on Ω, so that (y, x) is not Ω-regular. Therefore, $\mathrm{depth}(\Omega) = 2$, but $\mathrm{depth}(yR; \Omega) = 0$, showing that the Ischebeck inequality fails for Ω. Note that $\text{s-depth}(\Omega) = 0$ so that the strong depth formula fails, whereas the naive depth formula holds.

4.14. **Proposition.** *Let R be an equidimensional, catenary Noetherian local ring. Any balanced big Cohen-Macaulay R-module satisfies the Ischebeck inequality.*

Proof. Let d be the dimension of R and let $\mathfrak{p}$ be a prime ideal of R. Since R is equidimensional and catenary, $\mathfrak{p}$ has height $h := d - \dim R/\mathfrak{p}$. Choose a system of parameters $(x_1, \dots, x_d)$ in R with $x_1, \dots, x_h \in \mathfrak{p}$. Let Ω be a big balanced Cohen-Macaulay module, so that in particular $(x_1, \dots, x_d)$ is Ω-regular. It follows that $\mathrm{depth}(\mathfrak{p}; \Omega)$ is at least h, so that (5) holds. □

Note that a Cohen-Macaulay local ring is automatically equidimensional and catenary, and so is any complete local domain. Corollary 4.9 has the following converse:

4.15. **Theorem.** *Let R be a Noetherian local ring and Ω an R-module. All $\Omega(X)$ satisfy the Ischebeck inequality if and only if all $\Omega(X)/\mathbf{x}\Omega(X)$ satisfy the subdimension inequality, where X is a finite tuple of indeterminates and $\mathbf{x}$ is an $\Omega(X)$-regular sequence with entries in $R(X)$.*

Proof. One direction is immediate from Corollaries 4.12 and 4.9. To prove the converse, it suffices, by induction on the number of variables X, to show that Ω itself satisfies the Ischebeck inequality. We will verify condition (5) for each prime ideal $\mathfrak{p}$ of R. Let $s := \mathrm{depth}(\Omega)$, let $h := \dim R/\mathfrak{p}$ and let $e := \mathrm{depth}(\mathfrak{p}; \Omega)$. By Lemma 2.3, there exists an $\Omega(X)$-regular sequence $(x_1, \dots, x_e)$ inside $\mathfrak{p}R(X)$, for some (e-)tuple of indeterminates X. Let $R' := R(X)/(x_1, \dots, x_e)R(X)$ and $\Omega' := \Omega \otimes R'$. By assumption, the depth of Ω' is at most its subdimension. Since $\mathrm{depth}(\mathfrak{p}R'; \Omega') = 0$, there exists an associated prime $\mathfrak{p}'$ of Ω' with $\mathfrak{p}R' \subseteq \mathfrak{p}'$. In particular, the depth of Ω' is at most $\dim R'/\mathfrak{p}' \leq \dim R/\mathfrak{p} = h$. Moreover, the depth of Ω' is equal to $s - e$ by [3, Proposition 9.1.2], so that we showed the desired inequality $s - e \leq h$. □

Immediately from this, Corollary 4.3 and Theorem 4.6, we get:

4.16. **Corollary.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Then every finitely generated S-module satisfies the Ischebeck inequality when viewed as an R-module.* □

Using residual homological dimension (see §1.3), we may rephrase condition (4.10.4) over an equidimensional, catenary Noetherian local ring R as follows: Ω satisfies the Ischebeck inequality if and only if for all prime ideals $\mathfrak{p}$ for which $\mathrm{depth}_{R_{\mathfrak{p}}}(\Omega_{\mathfrak{p}})$ is finite, we have an inequality

(6) $$\mathrm{res.dim}_{R_{\mathfrak{p}}}(\Omega_{\mathfrak{p}}) + \text{CM-def}(R_{\mathfrak{p}}) \leq \mathrm{res.dim}_R(\Omega) + \text{CM-def}(R),$$

where the *Cohen-Macaulay defect* $\mathrm{CM\text{-}def}(R)$ is the difference between the dimension and the depth of R. In particular, over a Cohen-Macaulay local ring, we get:

4.17. **Corollary.** *Let R be a local Cohen-Macaulay ring and Ω an R-module. If Ω has finite depth and finite projective dimension, then Ω satisfies the Ischebeck inequality if and only if $\mathrm{res.dim}(\Omega_{\mathfrak{p}}) \leq \mathrm{res.dim}(\Omega)$ for every prime ideal $\mathfrak{p}$ for which $\mathrm{depth}_{R_{\mathfrak{p}}}(\Omega_{\mathfrak{p}})$ is finite.* □

4.18. **The Auslander-Buchsbaum formula.** Recall that a module Ω is said to satisfy the *Auslander-Buchsbaum formula* if

$$\mathrm{fl.dim}_R(\Omega) + \mathrm{depth}_R(\Omega) = \mathrm{depth}(R).$$

In particular, any module satisfying the Auslander-Buchsbaum formula must have finite depth and finite projective dimension. In view of (2), satisfying the Auslander-Buchsbaum formula is therefore equivalent with the equality $\mathrm{fl.dim}(\Omega) = \mathrm{res.dim}(\Omega)$.

For an example of a module of finite depth and finite projective dimension for which the Auslander-Buchsbaum formula fails, take Ω from Example 4.13. Indeed, Ω is a big Cohen-Macaulay which is not balanced (and in particular not flat). Moreover, Ω has finite projective dimension whence finite flat dimension, showing that the Auslander-Buchsbaum formula fails. See [2, Exemple 3.11] or [9, Example 8.3] for an example of a non-balanced big Cohen-Macaulay module which is also separated, and hence for which the Auslander-Buchsbaum formula fails (there can be no separated non-balanced big Cohen-Macaulay module over a two-dimensional ring by Theorem 6.4). Nonetheless, one direction in the Auslander-Buchsbaum formula always holds.

4.19. **Lemma.** *If R is a Noetherian local ring and Ω an R-module, then*

$$\mathrm{fl.dim}_R(\Omega) + \mathrm{depth}_R(\Omega) \geq \mathrm{depth}(R).$$

More generally, if $\mathfrak{a}$ is an arbitrary ideal of R, then

$$\mathrm{fl.dim}_R(\Omega) + \mathrm{depth}_R(\mathfrak{a};\Omega) \geq \mathrm{depth}_R(\mathfrak{a};R).$$

Proof. There is nothing to show if the depth or the projective dimension of Ω are infinite, so that we may moreover assume that both are finite. By (2), the first assertion is equivalent with $\mathrm{res.dim}(\Omega) \leq \mathrm{fl.dim}(\Omega)$, which in turn follows from the definition of residual homological dimension (alternatively, the first assertion follows from the second by letting $\mathfrak{a}$ be the maximal ideal of R). As for the second assertion, let s be the $\mathfrak{a}$-depth of Ω and let e be its flat dimension. We will induct on e. If $e = 0$, so that Ω is flat, then any R-regular sequence is Ω-regular, so that in fact $\mathrm{n\text{-}depth}_R(\mathfrak{a};\Omega) \geq s$.

Therefore, assume $e > 0$. Choose a short exact sequence

$$0 \to \Pi \to \Phi \to \Omega \to 0$$

with Φ flat. It follows that Π has flat dimension $e-1$. Therefore, by our induction hypothesis, its $\mathfrak{a}$-depth is at least $s-e+1$. Since $\mathrm{depth}(\mathfrak{a};\Phi) = s$ by the previous argument, Ω has $\mathfrak{a}$-depth at least $s-e$ by (3), as claimed. □

4.20. **Corollary.** *Any flat module of finite depth satisfies the Auslander-Buchsbaum formula.*

Proof. One direction in the equality $\mathrm{depth}(\Omega) = \mathrm{depth}(R)$ follows from Lemma 4.19 and the other from (2). □

4.21. **Proposition.** *Let R be a Noetherian local ring and Ω an R-module. Let $\mathbf{x}$ be a sequence which is both R-regular and Ω-regular. If Ω satisfies the Auslander-Buchsbaum formula, then so does $\Omega/\mathbf{x}\Omega$ when viewed as an $R/\mathbf{x}R$-module.*

Proof. By induction on the length of the sequence, we only need to treat the case that x is an R-regular and Ω-regular element. Let $R_1 := R/xR$ and $\Omega_1 := \Omega/x\Omega$, and let $e := \text{fl.dim}_R(\Omega)$. Since all $\text{Tor}_n^R(R_1, \Omega)$ vanish for $n > 0$, any flat resolution of Ω remains flat after tensoring with R_1. Hence $\text{fl.dim}_{R_1}(\Omega_1) \leq e$. On the other hand, since x is both R-regular and Ω-regular, $\text{Tor}_n^R(k, \Omega) \cong \text{Tor}_n^{R_1}(k, \Omega_1)$ for all $n > 0$, showing that $\text{res.dim}_R(\Omega) = \text{res.dim}_{R_1}(\Omega_1)$. Since Ω satisfies the Auslander-Buchsbaum formula, $e = \text{res.dim}_R(\Omega)$, leading to the inequalities $e = \text{res.dim}_{R_1}(\Omega_1) \leq \text{fl.dim}_{R_1}(\Omega_1) \leq e$ showing that Ω_1 satisfies the Auslander-Buchsbaum formula too. □

4.22. **Proposition.** *Let R be a Noetherian local ring and Ω an R-module. If Ω satisfies the Auslander-Buchsbaum formula, then it also satisfies the Ischebeck inequality and hence the subdimension inequality.*

Proof. To verify condition (4.10.4), let $\mathfrak{p}$ be a prime ideal of R such that $\text{depth}_{R_\mathfrak{p}}(\Omega_\mathfrak{p})$ is finite. Since residual homological dimension never exceeds flat dimension,

$$\text{res.dim}_{R_\mathfrak{p}}(\Omega_\mathfrak{p}) \leq \text{fl.dim}(\Omega) = \text{depth}(R) - \text{depth}(\Omega),$$

where the last equality is just the Auslander-Buchsbaum formula. From (2), we then get

$$\text{depth}(\Omega) \leq \text{depth}(R) - \text{depth}(R_\mathfrak{p}) + \text{depth}_{R_\mathfrak{p}}(\Omega_\mathfrak{p}).$$

On the other hand, since the Ischebeck inequality holds for R itself, we have $\text{depth}(R) \leq \text{depth}(R_\mathfrak{p}) + \dim(R/\mathfrak{p})$. Putting both inequalities together therefore yields (4.10.4). □

The Ischebeck inequality for small depths. We conclude with proving some cases in which the Ischebeck inequality holds. Typically the higher the depth of a module, the stronger the separatedness condition needed.

4.23. **Lemma.** *Let R be a Noetherian local ring and Ω an R-module. Let x be a strongly Ω-regular element and let $\mathfrak{p}$ be a prime ideal of R. If $\text{depth}(\mathfrak{p}; \Omega) = 0$, then $\mathfrak{p}$ is strictly contained in an associated prime of $\Omega/x\Omega$.*

Proof. Since $\mathfrak{p}$ is contained in some associated prime ideal of Ω by [3, Proposition 9.1.4], we may assume without loss of generality that $\mathfrak{p}$ itself is an associated prime of Ω. By the same argument as in the proof of Lemma 4.5, we may choose some $\omega \notin x\Omega$ such that $\mathfrak{p} = \text{Ann}(\omega)$. Since $\mathfrak{p}$ annihilates the non-zero image of ω in $\Omega/x\Omega$, it must be contained in an associated prime $\mathfrak{q}$ of $\Omega/x\Omega$ by [5, Theorem 6.1]. Clearly, $\mathfrak{p} \subsetneq \mathfrak{q}$ since $x \notin \mathfrak{p}$. □

4.24. **Corollary.** *Any module of depth at most one or any separated module of depth two satisfies the Ischebeck inequality.*

Proof. We already argued that any module of depth at most one satisfies the Ischebeck inequality. So assume $\text{depth}(\Omega) = 2$. Suppose (5) does not hold for some (non-maximal) prime ideal $\mathfrak{p}$. The only way this can be the case is if $\text{depth}(\mathfrak{p}; \Omega) = 0$ and $\dim R/\mathfrak{p} = 1$. By Lemmas 2.1 and 2.3 we may assume, after possibly making a base change over $R \subseteq R(X)$, that there exists an Ω-regular element x. Since Ω is separated, x is strongly Ω-regular. Hence, by Lemma 4.23, there is an associated prime $\mathfrak{q}$ of $\Omega/x\Omega$ such that $\mathfrak{p} \subsetneq \mathfrak{q}$. Since $R/\mathfrak{p}$ is one-dimensional, $\mathfrak{q}$ must be the maximal ideal, showing that $\Omega/x\Omega$ has depth zero. On the other hand, x is Ω-regular and hence $\Omega/x\Omega$ has depth one by [3, Proposition 9.1.2], contradiction. □

4.25. **Corollary.** *Let R be a three-dimensional, equidimensional, catenary Noetherian local ring and Ω an R-module. If Ω is complete, then it satisfies the Ischebeck inequality.*

Proof. Since Ω is complete, it is non-degenerated and hence has finite depth. By Corollary 4.24, we only need to treat the case that Ω has depth three. This means that Ω is a big Cohen-Macaulay module. However, since Ω is complete, it is therefore a balanced big Cohen-Macaulay module by [3, Corollary 8.5.3] and hence we are done by Proposition 4.14. □

I do not know whether every complete module of depth three satisfies the Ischebeck inequality.

4.26. **Corollary.** *Let R be a four-dimensional, equidimensional, catenary Noetherian local ring and Ω an R-module of finite projective dimension. If Ω is complete and satisfies the subdimension inequality, then Ω satisfies the Ischebeck inequality.*

Proof. Inspecting the previous proofs, one sees that the only instance in which we have not yet verified (5), is when $\mathfrak{p}$ has height two and Ω has depth three. We need to then show that $\operatorname{depth}(\mathfrak{p};\Omega)$ is at least one. Towards a contradiction, suppose it is zero, so that $\mathfrak{p}$ is contained in an associated prime $\mathfrak{q}$ of Ω by [3, Proposition 9.1.4]. Since $\mathfrak{p}$ has height two, $R/\mathfrak{q}$ has dimension at most two and hence violates the subdimension inequality. □

5. The Auslander-Buchsbaum formula: the Cohen-Macaulay case

In this section, R will always be a local Cohen-Macaulay ring. We investigate conditions which guarantee that a module over R satisfies the Auslander-Buchsbaum formula. We start with an extremal case.

5.1. **Theorem.** *Let R be a local Cohen-Macaulay ring of dimension d. Let Ω be an arbitrary R-module of finite depth and finite projective dimension. If either Ω has depth zero or flat dimension d, then it also satisfies the other condition and hence the Auslander-Buchsbaum formula.*

Proof. Suppose Ω has depth zero. By Lemma 4.19, the flat dimension of Ω is at least d and by Corollary 3.6, it is at most d, and hence the Auslander-Buchsbaum formula holds.

Conversely, towards a contradiction, suppose that Ω has flat dimension d, but positive depth. Since neither flat dimension nor depth change after an extension of the form $R \subseteq R(X)$ by Lemma 2.1, we may assume by Lemma 2.4 that there exists an R-regular, Ω-regular element x. Taking a flat resolution of Ω and tensoring with R/xR yields a flat resolution of $\Omega/x\Omega$ over R/xR. In particular, $\Omega/x\Omega$ has finite flat dimension as an R/xR-module.

Since Ω has flat dimension d, By Proposition 3.3, there is some prime ideal $\mathfrak{p}$ of R for which $\beta_d(\mathfrak{p};\Omega)$ is non-zero. However, if $\mathfrak{p}$ is not maximal, then the $R_\mathfrak{p}$-module $\Omega_\mathfrak{p}$ has flat dimension at most $d-1$ by Corollary 3.6, so that $\beta_d(\mathfrak{p};\Omega) = 0$. Therefore, $\mathfrak{p}$ must be the maximal ideal, so that $\operatorname{Tor}_d^R(k,\Omega) \neq 0$, where k is the residue field of R. Again by Corollary 3.6, the flat dimension of $\Omega/x\Omega$ is at most $\dim(R/xR) = d-1$. It follows that $\operatorname{Tor}_d^{R/xR}(k,\Omega/x\Omega) = 0$. However, the latter *Tor* module is isomorphic to $\operatorname{Tor}_d^R(k,\Omega)$ since x is both R-regular and Ω-regular, contradiction. □

There are two more extremal cases: if the flat dimension is zero, then the Auslander-Buchsbaum formula always holds by Corollary 4.20. On the other hand, if Ω has maximal depth (equal to the dimension of R), then the Auslander-Buchsbaum formula might fail as the non-flat big Cohen-Macaulay module in [9, Example 7.3] shows. However, if we moreover assume that Ω is complete whence a balanced big Cohen-Macaulay module by [3, Corollary 8.5.3], then the Auslander-Buchsbaum formula holds again by an application of Proposition 4.14 and the next theorem.

5.2. **Theorem.** *A module Ω of finite projective dimension over a local Cohen-Macaulay ring satisfies the Ischebeck inequality if and only if it satisfies the Auslander-Buchsbaum formula.*

Proof. Suppose the Ischebeck inequality holds, so that $\operatorname{res.dim}_{R_{\mathfrak{p}}}(\Omega_{\mathfrak{p}}) \leq \operatorname{res.dim}_R(\Omega)$ whenever $\operatorname{depth}_{R_{\mathfrak{p}}}(\Omega_{\mathfrak{p}}) < \infty$ by Corollary 4.17. By Theorem 3.5, the flat dimension of Ω is equal to the maximum of these local residual dimensions, and hence in this case, equal to $\operatorname{res.dim}(\Omega)$. The Auslander-Buchsbaum formula therefore holds by (2). The converse was already shown in Proposition 4.22. □

Since the Auslander-Buchsbaum formula holds for Ω if and only if it holds for $\Omega(X)$ by Lemma 2.1, we obtain from Theorem 4.15 that the Auslander-Buchsbaum formula holds for Ω if and only if the subdimension inequality holds for all $\Omega(X)/\mathbf{x}\Omega(X)$ with $\mathbf{x}$ an $\Omega(X)$-regular sequence. In particular, in view of Corollary 4.7, we proved that if Ω has finite projective dimension and each $\Omega(X)$ is universally separated, then Ω satisfies the Auslander-Buchsbaum formula. Using Theorem 4.6, we may formulate this in terms of the strong depth formula. However, since the latter does not deform well under non-strongly regular sequences, we also have to enforce this:

5.3. **Corollary.** *Let R be a local Cohen-Macaulay ring and Ω an R-module of finite depth and finite projective dimension. If $\Omega/\mathbf{x}\Omega$ satisfies the strong depth formula for each Ω-regular sequence $\mathbf{x}$, then Ω satisfies the Auslander-Buchsbaum formula.* □

The module in [9, Example 7.3], already discussed above, is an example of a module for which the Auslander-Buchsbaum formula fails but the strong depth formula holds. Hence, we cannot drop the requirement on the deformations in Corollary 5.3. Another immediate application of Theorem 5.2 can be derived from Corollary 4.16:

5.4. **Corollary.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. If R is Cohen-Macaulay, then any finitely generated S-module of finite projective dimension over R satisfies the Auslander-Buchsbaum formula over R.* □

In this result, we may drop the Cohen-Macaulay assumption, as we will show in Theorem 6.1 below. Combining Corollary 4.24 with Theorem 5.2, we get:

5.5. **Corollary.** *Let R be local Cohen-Macaulay ring and Ω an R-module of finite projective dimension. If Ω has depth at most one or if Ω is separated and has depth two, then it satisfies the Auslander-Buchsbaum formula.* □

Note that Example 4.13 shows that the separatedness condition cannot be omitted in the depth two case. The same argument, using this time Corollaries 4.25 and 4.26, yields the following result.

5.6. **Corollary.** *Let R be a local Cohen-Macaulay ring and let Ω be a complete R-module of finite projective dimension. If $\dim R = 3$ or if $\dim R = 4$ and Ω satisfies the subdimension inequality, then Ω satisfies the Auslander-Buchsbaum formula.* □

6. The Auslander-Buchsbaum formula: the non Cohen-Macaulay case

We start with proving Corollary 5.4 without the Cohen-Macaulay assumption.

6.1. **Theorem.** *If $R \to S$ is a local homomorphism of Noetherian local rings, then any finitely generated S-module Ω of finite projective dimension over R satisfies the Auslander-Buchsbaum formula over R.*

Proof. Let us first show that $\operatorname{Tor}_n^R(M, \Omega)$ is separated for every finitely generated R-module M and every $n \geq 0$. We induct on n, where the case $n = 0$ follows from the fact that $M \otimes_R \Omega$ is finitely generated as an S-module. For $n > 0$, let N be a first syzygy of M. From the long exact sequence for *Tor*, we have an inclusion $\operatorname{Tor}_n^R(M, \Omega) \hookrightarrow \operatorname{Tor}_{n-1}^R(N, \Omega)$ for every $n > 0$ (and in fact, for $n > 1$ these are isomorphisms). By induction, the second module is separated, whence so is the first.

We now turn to the proof of the assertion. Let e be the flat dimension of Ω and let $\mathfrak{p}$ be maximal among all prime ideals such that $\operatorname{Tor}_e^R(R/\mathfrak{p}, \Omega) \neq 0$. If $\mathfrak{p}$ is not the maximal ideal, then we can choose x so that

$$0 \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \to R/\mathfrak{a} \to 0$$

is an exact sequence for some proper ideal $\mathfrak{a}$ of height strictly larger than $\mathfrak{p}$. Tensoring with Ω yields part of a long exact sequence

$$\operatorname{Tor}_e^R(R/\mathfrak{p}, \Omega) \xrightarrow{x} \operatorname{Tor}_e^R(R/\mathfrak{p}, \Omega) \to \operatorname{Tor}_e^R(R/\mathfrak{a}, \Omega)$$

By the maximality of $\mathfrak{p}$, Lemma 3.1 yields $\operatorname{Tor}_e^R(R/\mathfrak{a}, \Omega) = 0$. Hence multiplication with x on $\operatorname{Tor}_e^R(R/\mathfrak{p}, \Omega)$ is surjective (and in fact bijective). However, as we argued, $\operatorname{Tor}_e^R(R/\mathfrak{p}, \Omega)$ is separated and therefore must be zero. This contradiction shows that $\mathfrak{p}$ has to be the maximal ideal. In particular, $\operatorname{res.dim}(\Omega) = e$, so that we are done by (2). □

6.2. **Proposition.** *If R is a d-dimensional Noetherian local ring of Cohen-Macaulay defect one, then any depth zero R-module of finite projective dimension satisfies the Auslander-Buchsbaum formula.*

Proof. By Lemma 4.19, the flat dimension of Ω is at least $d - 1$, and by Corollary 3.6 it is at most $d - 1$. □

Under the same hypotheses, does the other extremal case, namely when Ω has flat dimension $d - 1$, also imply the Auslander-Buchsbaum formula? And is the assumption on the Cohen-Macaulay defect necessary? We finish by completing the proof of (i) and (ii) in Theorem 1.1.

6.3. **Corollary.** *Any module Ω of finite depth and finite projective dimension over a one-dimensional Noetherian local ring satisfies the Auslander-Buchsbaum formula.*

Proof. If R is Cohen-Macaulay, then the result holds by Corollary 5.5. If R is not Cohen-Macaulay, then Ω is flat by Corollary 3.6 and hence satisfies the Auslander-Buchsbaum formula by Corollary 4.20. □

6.4. **Theorem.** *Any separated module of finite projective dimension over a two-dimensional Noetherian local ring satisfies the Auslander-Buchsbaum formula.*

Proof. The Cohen-Macaulay case was already treated by Corollary 5.5. So we may assume that R has depth $q < 2$. Let Ω be a separated R-module of finite flat dimension e and (necessarily finite) depth s. We need to show that $s + e = q$. In any case, $e \leq 1$ by Corollary 3.6, and $s \leq q$ by (2). Assume first that $q = 1$. If $s = 0$, then Ω cannot be flat whence $e = 1$ and we are done. So assume $s = 1$. Hence $\operatorname{res.dim}(\Omega) = 0$ by (2), meaning that $\operatorname{Tor}_1^R(k, \Omega) = 0$. Therefore, Ω is flat by [9, Theorem 6.5].

So remains the case that $q = 0$. In particular, $s = 0$, and we want to show that also $e = 0$. Since R has depth zero, we have an exact sequence

$$0 \to k \to R \to N \to 0 \tag{7}$$

for some (cyclic) R-module N. Tensoring with Ω yields $\operatorname{Tor}_1^R(k, \Omega) \cong \operatorname{Tor}_2^R(N, \Omega) = 0$. The flatness of Ω now follows from another application of [9, Theorem 6.5]. □

REFERENCES

1. M. Auslander and D. Buchsbaum, *Homological dimension in Noetherian rings II*, Trans. Amer. Math. Soc. **88** (1958), 194–206. 8
2. J. Bartijn and J. Strooker, *Modifications monomiales*, Séminaire d'Algèbre P. Dubreil et M.-P. Malliavin, Lect. Notes in Math., vol. 1029, Springer-Verlag, 1982, pp. 192–217. 3, 12
3. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, 1993. 3, 5, 8, 9, 10, 11, 13, 14, 15
4. M. Hochster, *Grade-sensitive modules and perfect modules*, Proc. London Math. Soc. **29** (1974), 55–76. 3, 4, 5
5. H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986. 1, 4, 6, 10, 13
6. H. Schoutens, *Rigid analytic flatificators*, Quart. J. Math. Oxford Ser. **50** (1999), 321–353.
7. ______, *Projective dimension and the singular locus*, Comm. Algebra **31** (2003), 217–239.
8. ______, *Classifying singularities up to analytic extensions of scalars*, [arXiv.org/pdf/math.AC/0508236], in preparation. 3
9. ______, *A local flatness criterion for complete modules*, Comm. Algebra (to appear). 3, 8, 12, 14, 15, 16, 17

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF NEW YORK, 365 FIFTH AVENUE, NEW YORK, NY 10016 (USA)

E-mail address: hschoutens@citytech.cuny.edu