6.4 Big Cohen-Macaulay algebras

Although the material in this section is strictly speaking not part of tight closure theory, the development of the latter was germane to the discovery by Hochster and Huneke of Theorem 6.4.1 below.

6.4.1 Big Cohen-Macaulay algebras in prime characteristic.

Recall that the *absolute integral closure* A^+ of a domain A with field of fractions F, is the integral closure of A inside an algebraic closure of F. Since algebraic closure is unique up to isomorphism, so is absolute integral closure. Nonetheless it is not functorial, and we only have the following quasi-functorial property: given a homomorphism $A \to B$ of domains, there exists a (not necessarily unique) homomorphism $A^+ \to B^+$ making the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A^+ & \longrightarrow & B^+ \end{array} \qquad (6.7)$$

commute.

Theorem 6.4.1 ([49]). *For every excellent local domain R in characteristic p, the absolute integral closure R^+ is a balanced big Cohen-Macaulay algebra.*

The condition that a Noetherian local ring is excellent is for instance satisfied when R is either K-affine or complete (see [69, §32]). The proof of the above result is beyond the scope of these notes (see for instance [59, Chapters 7& 8]) although we will present a 'dishonest' proof shortly. It is quite a remarkable fact that the same result is completely false in characteristic zero: in fact any extension of a normal domain is split, and hence provides a counterexample as soon as R is not Cohen-Macaulay. One can use the absolute integral closure to define a closure operation in an excellent local domain R of prime characteristic as follows. For an ideal I, let the *plus closure* of I be the ideal $I^+ := IR^+ \cap R$. One can show that I^+ is a closure operation in the sense of Definition 5.2.5, satisfying the five key properties listed in Theorem 6.2.3. Moreover, unlike tight closure, it is not hard to show that it commutes with localization.

Proposition 6.4.2. *In an excellent local domain R of prime characteristic, the plus closure of an ideal $I \subseteq R$ is contained in its tight closure.*

Proof. Let $z \in I^+$. By definition, there exists a finite extension $R \subseteq S \subseteq R^+$ such that $z \in IS$ (note that R^+ is the direct limit of all finite extensions of R by local domains). Hence $z \in \text{cl}(I)$ by Theorem 5.3.4. □

It was conjectured that plus closure always equals tight closure. In view of [15], this now seems unlikely, since plus closure is easily seen to commute with localization, whereas tight closure apparently does not (see our discussion of (5.5)). Nonetheless, Smith has verified a special case of the conjecture for an important class of ideals:

Theorem 6.4.3 ([107]). *Any ideal generated by part of a system of parameters in an excellent local domain of prime characteristic has the same plus closure as tight closure.*

Remark 6.4.4. The main ingredient in the proof of Proposition 6.4.2 is the following fact, which is immediate from Lemma 5.3.5: the dual of R^+ as an R-module is non-zero, that is to say, there exists a non-zero R-module morphism $R^+ \to R$. Hochster ([45, Theorem 10.5]) has proven this to be true for any big Cohen-Macaulay algebra over a complete local ring of positive characteristic. Using this fact in the same way as in the proof of Theorem 5.3.4, he shows that if B is a balanced big Cohen-Macaulay algebra over a Noetherian local ring of positive characteristic, then $IB \cap R$ is contained in the tight closure of an ideal $I \subseteq R$. In fact, conversely, any element in the tight closure of I lies in IB, for some balanced big Cohen-Macaulay R-algebra B ([45, Theorem 11.1]).

Proof of Theorem 6.4.1 (affine or complete case) assuming Theorem 6.4.3.

The proof we will present here is dishonest in the sense that Smith made heavily use of Theorem 6.4.1 to derive her result. However, here is how the converse direction goes. Let $(x_1,\dots,x_d)$ be a system of parameters in a local domain R of characteristic p which is either affine or complete, and suppose $zx_{i+1} \in IR^+$ for some $z \in R^+$ and $I := (x_1,\dots,x_i)R$. Hence there already exists a finite extension $R \subseteq S \subseteq R^+$ containing z such that $zx_{i+1} \in IS$. Since $R \subseteq S$ is finite, $(x_1,\dots,x_d)$ is also a system of parameters in S. In either case, Colon Capturing applies (see the remark following Theorem 5.3.3) and we get $z \in \text{cl}(IS)$. By Theorem 6.4.3, this implies that z lies in the plus closure of IS, whence in IS^+. However, it is not hard to see that $R^+ = S^+$, proving that $(x_1,\dots,x_d)$ is an R^+-regular sequence. □

6.4.5 *If R is an excellent regular local ring of prime characteristic, then R^+ is faithfully flat over R.*

This follows immediately from Theorem 6.4.1 and the Cohen-Macaulay criterion for flatness (Theorem 3.3.9). Interestingly, it also provides an alternative strategy to prove Theorem 6.4.1:

Proposition 6.4.6. *Let k be a field of positive characteristic. Suppose we can show that any k-affine (respectively, complete) regular local ring has a faithfully flat absolute* *integral closure, then the absolute integral closure of any k-affine (respectively, complete Noetherian) local domain is a balanced big Cohen-Macaulay algebra.*

Proof. I will only treat the affine case and leave the complete case as an exercise. Let R be a local k-affine domain, and let $\mathbf{x}$ be a system of parameters in R. By Noether normalization with parameters ([27, Theorem 13.3]), we can find a k-affine regular local subring $S \subseteq R$ containing x, such that $S \subseteq R$ is finite and $\mathbf{x}S$ is the maximal ideal of S. By assumption, S^+ is faithfully flat over S, and hence $(x_1,\dots,x_d)$ is an S^+-regular sequence by Proposition 3.2.9. Finiteness yields $S^+ = R^+$, and so we are done. □

6.4.2 Big Cohen-Macaulay algebras in characteristic zero.

As already mentioned, if R is a K-affine local domain of characteristic zero, then R^+ will in general not be a big Cohen-Macaulay algebra. However, we can still associate to any such R (in a quasi-functorial way) a canonically defined balanced big Cohen-Macaulay algebra as follows. Let R_p be an approximation of R. By Theorem 4.3.4, almost all R_p are domains. Let $B(R)$ be the ultraproduct of the R_p^+. To show that this is independent from the choice of approximation, we will give an alternative, more intrinsic description of $B(R)$. Let $\mathbb{N}_\natural$ be the ultrapower of the set of natural numbers, and let t be an indeterminate. For an element $f \in U(R[t])$, define its *ultra-degree* $\alpha \in \mathbb{N}_\natural$ (with respect to t) to be the ultraproduct of the t-degrees α_p of the f_p, where f_p is an approximation of f. Call an element $f \in U(R[t])$ *ultra-monic* if there exists $\alpha \in \mathbb{N}_\natural$ such that $f - t^\alpha$ has ultra-degree strictly less than α (see §2.4.4 for ultra-exponentiation). By a *root* of $g \in U(R[t])$ in a Lefschetz field L containing K we mean an element $a \in L$ such that $g \in (t-a)U(R_L[t])$, where $R_L := R \otimes_K L$ and its ultra-hull is taken in the category $\mathfrak{C}_L$. One now easily shows that there exists an algebraically closed Lefschetz field L containing K such that $B(R)$ is isomorphic to the ring of all $a \in L$ that are a root of some ultra-monic element in $U(R_L[t])$. Moreover, this ring is independent from the choice of L.

By Łoś' Theorem, there is a canonical homomorphism $R \to B(R)$.

Theorem 6.4.7. *If R is a K-affine local domain, then $B(R)$ is a balanced big Cohen-Macaulay algebra over R.*

Proof. Since almost each approximation R_p is a K_p-affine (whence excellent) local domain, R_p^+ is a balanced big Cohen-Macaulay R_p-algebra by Theorem 6.4.1. Let $\mathbf{x}$ be a system of parameters of R, with approximation $\mathbf{x}_p$. By Corollary 4.3.8, almost each $\mathbf{x}_p$ is a system of parameters in R_p, whence an R_p^+-regular sequence. By Łoś' Theorem, $\mathbf{x}$ is therefore $B(R)$-regular, as we wanted to show. □

Hochster and Huneke ([52]) arrive differently at balanced big Cohen-Macaulay algebras in characteristic zero, via their lifting method discussed in §5.6. However,