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The use of ultraproducts in commutative algebra

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Chapter 1
Ultraproducts and Łos' Theorem

In this chapter, $\mathbb{W}$ always denotes an infinite set.

1.1 Ultraproducts

Non-principal ultrafilters. By a *non-principal ultrafilter* $\mathscr{U}$ on $\mathbb{W}$, we mean a collection of infinite subsets of $\mathbb{W}$ closed under finite intersection, with the property that for any subset F of $\mathbb{W}$, either F or its complement $-F$ belongs to $\mathscr{U}$. In particular, the empty-set does not belong to $\mathscr{U}$ and if $D \in \mathscr{U}$ and E is an arbitrary set containing D, then also $E \in \mathscr{U}$, for otherwise $-E \in \mathscr{U}$, whence $\emptyset = D \cap -E \in \mathscr{U}$, contradiction. Since every set in $\mathscr{U}$ must be infinite, it follows that any co-finite set belongs to $\mathscr{U}$. The existence of non-principal ultrafilters is equivalent with the Axiom of Choice, and we make this set-theoretic assumption henceforth. It follows that for any infinite subset of $\mathbb{W}$, we can find a non-principal ultrafilter containing this set.

In the remainder of these notes, we also fix a non-principal ultrafilter $\mathscr{U}$ on $\mathbb{W}$, and (almost always) omit reference to this fixed ultrafilter from our notation. No extra property of the ultrafilter is assumed, with the one exception described in Remark 11.1.5, which is nowhere used in the rest of our work anyway. Non-principal ultrafilters play the role of a decision procedure on the collection of subsets of $\mathbb{W}$ by declaring some subsets 'large' (those belonging to $\mathscr{U}$) and declaring the remaining ones 'small'. More precisely, let o_w be elements indexed by $w \in \mathbb{W}$, and let $\mathscr{P}$ be a property. We will use the expressions *almost all o_w satisfy property $\mathscr{P}$* or *o_w satisfies property $\mathscr{P}$ for almost all w* as an abbreviation of the statement that there exists a set D in the ultrafilter $\mathscr{U}$, such that property $\mathscr{P}$ holds for the element o_w, whenever $w \in D$. Note that this is also equivalent with the statement that the set of all $w \in \mathbb{W}$ for which o_w has property $\mathscr{P}$, lies in the ultrafilter (read: *is large*). Similarly, we say that the o_w *almost never satisfy property $\mathscr{P}$* (or *almost no o_w satisfies $\mathscr{P}$*), if almost all o_w do not satisfy property $\mathscr{P}$.

Ultraproducts. Let O_w be sets, for $w \in \mathbb{W}$. We define an equivalence relation on the Cartesian product $\prod O_w$, by calling two sequences $(a_w|w \in \mathbb{W})$ and $(b_w|w \in \mathbb{W})$ equivalent, if a_w and b_w are equal for almost all w. In other words, if the set of indices $w \in \mathbb{W}$ for which $a_w = b_w$ belongs to the ultrafilter. We will denote the equivalence class of a sequence $(a_w|w \in \mathbb{W})$ by

$$\operatorname*{ulim}_{w\to\infty} a_w, \quad \text{or} \quad \operatorname{ulim} a_w, \quad \text{or} \quad a_\natural.$$

The set of all equivalence classes on $\prod O_w$ is called the *ultraproduct* of the O_w and is denoted

$$\operatorname*{ulim}_{w\to\infty} O_w, \quad \text{or} \quad \operatorname{ulim} O_w, \quad \text{or} \quad O_\natural.$$

Note that the element-wise and set-wise notations are reconciled by the fact that

$$\operatorname*{ulim}_{w\to\infty}\{o_w\} = \{\operatorname*{ulim}_{w\to\infty} o_w\}.$$

The more common notation for an ultraproduct one usually finds in the literature is O^*; in the past, I also have used O_∞. The reason for using the particular notation $O_\natural$ in these notes is because we will also introduce the remaining "chromatic" products $O_\flat$ and $O_\sharp$ (at least for certain local rings; see Chapters 12 and 11 respectively).

We wil also often use the following terminology: if o is an element in an ultraproduct $O_\natural$, then any choice of elements $o_w \in O_w$ with ultraproduct equal to o will be called an *approximation* of o. Although, an approximation is not uniquely determined by the element, any two agree almost everywhere. Below we will extend our usage of the term approximation to include other objects as well.

Properties of ultraproducts. For the following properties, the easy proofs of which are left as an exercise, let O_w be sets with ultraproduct $O_\natural$.

1.1.1 *If Q_w is a subset of O_w for each w, then $\operatorname{ulim} Q_w$ is a subset of $\operatorname{ulim} O_w$.*

In fact, $\operatorname{ulim} Q_w$ consists of all elements of the form $\operatorname{ulim} o_w$, with almost all o_w in Q_w.

1.1.2 *If each O_w is the graph of a function $f_w\colon A_w \to B_w$, then $O_\natural$ is the graph of a function $A_\natural \to B_\natural$, where $A_\natural$ and $B_\natural$ are the respective ultraproducts of A_w and B_w. We will denote this function by*

$$\operatorname*{ulim}_{w\to\infty} f_w \quad \text{or} \quad f_\natural.$$

Moreover, we have an equality

$$\operatorname*{ulim}_{w\to\infty}(f_w(a_w)) = (\operatorname*{ulim}_{w\to\infty} f_w)(\operatorname*{ulim}_{w\to\infty} a_w), \tag{1.1}$$

for $a_w \in A_w$.

1.1.3 *If each O_w comes with an operation $*_w\colon O_w \times O_w \to O_w$, then*

$$*_\natural := \operatorname*{ulim}_{w\to\infty} *_w$$

is an operation on $O_\natural$. If all (or, almost all) O_w are groups with multiplication $*_w$ and unit element 1_w, then $O_\natural$ is a group with multiplication $*_\natural$ and unit element $1_\natural := \operatorname{ulim} 1_w$. If almost all O_w are Abelian groups, then so is $O_\natural$.

1.1.4 *If each O_w is a (commutative) ring with addition $+_w$ and multiplication $\cdot_w$, then $O_\natural$ is a (commutative) ring with addition $+_\natural$ and multiplication $\cdot_\natural$.*

In fact, in that case, $O_\natural$ is just the quotient of the product $\prod O_w$ modulo the ideal consisting of all sequences $(o_w | w \in \mathbb{W})$ for which almost all o_w are zero. From now on, we will drop subscripts on the operations and just write $+$ and $\cdot$ for the ring operations on the O_w and on $O_\natural$.

1.1.5 *If almost all O_w are fields, then so is $O_\natural$.*

Just to give an example of how to work with ultraproducts, let me give the proof: if $a \in O_\natural$ is non-zero, with approximation a_w (recall that this means that $\operatorname{ulim} a_w = a$), then by the previous description of the ring structure on $O_\natural$, almost all a_w will be non-zero. Therefore, letting b_w be the inverse of a_w whenever this makes sense, and zero otherwise, one verifies that $\operatorname{ulim} b_w$ is the inverse of a.

1.1.6 *If C_w are rings and O_w is an ideal in C_w, then $O_\natural$ is an ideal in $C_\natural := \operatorname{ulim} C_w$. In fact, $O_\natural$ is equal to the subset of all elements of the form $\operatorname{ulim} o_w$ with almost all $o_w \in O_w$. Moreover, the ultraproduct of the C_w/O_w is isomorphic to $C_\natural/O_\natural$.*

1.1.7 *If $f_w\colon A_w \to B_w$ are ring homomorphisms, then the ultraproduct $f_\natural$ is again a ring homomorphism. In particular, if σ_w is an endomorphism on A_w, then the ultraproduct $\sigma_\natural$ is a ring endomorphism on $A_\natural := \operatorname{ulim} A_w$.*

1.2 Model-theory in rings

The previous examples are just instances of the general principle that 'algebraic structure' carries over to the ultraproduct. The precise formulation of this principle is called *Łos' Theorem* (Łos is pronounced 'wôsh') and requires some terminology from model-theory. However, for our purposes, a weak version of Łos' Theorem (namely Theorem 1.3.1 below) suffices in almost all cases, and its proof is entirely algebraic. Nonetheless, for a better understanding, the reader is invited to indulge in some elementary model-theory, or rather, an ad hoc version for rings only (if this not satisfies him/her, (s)he should consult any textbook, such as [34, 39, 48]).

Formulae. By a *quantifier free formula without parameters* in the free variables $\xi = (\xi_1, \dots, \xi_n)$, we will mean an expression of the form

$$\varphi(\xi) := \bigvee_{j=1}^{m} f_{1j} = 0 \wedge \dots \wedge f_{sj} = 0 \wedge g_{1j} \neq 0 \wedge \dots \wedge g_{tj} \neq 0, \tag{1.2}$$

where each f_{ij} and g_{ij} is a polynomial with integer coefficients in the variables ξ, and where $\wedge$ and $\vee$ are the logical connectives *and* and *or*. If instead we allow the f_{ij} and g_{ij} to have coefficients in a ring R, then we call $\varphi(\xi)$ a *quantifier free formula with parameters in* R. We allow all possible degenerate cases as well: there might be no variables at all (so that the formula simply declares that certain elements in $\mathbb{Z}$ or in R are zero and others are non-zero) or there might be no equations or no negations or perhaps no conditions at all. Put succinctly, a quantifier free formula is a Boolean combination of polynomial equations using the connectives $\wedge$, $\vee$ and $\neg$ (negation), with the understanding that we use distributivity and De Morgan's Laws to rewrite this Boolean expression in the (disjunctive normal) form (1.2).

By a *formula without parameters* in the free variables ξ, we mean an expression of the form

$$\varphi(\xi) := (\mathrm{Q}_1\, \zeta_1) \cdots (\mathrm{Q}_p\, \zeta_p)\, \psi(\xi, \zeta),$$

where $\psi(\xi, \zeta)$ is a quantifier free formula without parameters in the free variables ξ and $\zeta = (\zeta_1, \dots, \zeta_p)$ and where Q_i is either the universal quantifier $\forall$ or the existential quantifier $\exists$. If instead $\psi(\xi, \zeta)$ has parameters from R, then we call $\varphi(\xi)$ a *formula with parameters in* R. A formula with no free variables is called a *sentence*.

Satisfaction. Let $\varphi(\xi)$ be a formula in the free variables $\xi = (\xi_1, \dots, \xi_n)$ with parameters from R (this includes the case that there are no parameters by taking $R = \mathbb{Z}$ and the case that there are no free variables by taking $n = 0$). Let A be an R-algebra and let $\mathbf{a} = (a_1, \dots, a_n)$ be a tuple with entries from A. We will give meaning to the expression $\mathbf{a}$ *satisfies the formula* $\varphi(\xi)$ *in* A (sometimes abbreviated to $\varphi(\mathbf{a})$ *holds in* A or *is true in* A) by induction on the number of quantifiers. Suppose first that $\varphi(\xi)$ is quantifier free, given by the Boolean expression (1.2). Then $\varphi(\mathbf{a})$ holds in A, if for some j_0, all $f_{ij_0}(\mathbf{a}) = 0$ and all $g_{ij_0}(\mathbf{a}) \neq 0$. For the general case, suppose $\varphi(\xi)$ is of the form $(\exists \zeta)\, \psi(\xi, \zeta)$ (respectively, $(\forall \zeta)\, \psi(\xi, \zeta)$), where the satisfaction relation is already defined for the formula $\psi(\xi, \zeta)$. Then $\varphi(\mathbf{a})$ holds in A, if there is some $b \in A$ such that $\psi(\mathbf{a}, b)$ holds in A (respectively, if $\psi(\mathbf{a}, b)$ holds in A, for all $b \in A$). The subset of A^n consisting of all tuples satisfying $\varphi(\xi)$ will be called the *subset defined by* φ, and will be denoted $\varphi(A)$. Any subset that arises in such way will be called a *definable subset* of A^n.

Note that if $n = 0$, then there is no mention of tuples in A. In other words, a sentence is either true or false in A. By convention, we set A^0 equal to the singleton $\{\emptyset\}$ (that is to say, A^0 consists of the empty tuple $\emptyset$). If φ is a sentence, then the set defined by it is either $\{\emptyset\}$ or $\emptyset$, according to whether φ is true or false in A.

Constructible Sets. There is a connection between definable sets and Zariski-constructible sets, where the relationship is the most transparent over algebraically closed fields, as we will explain below. In general, we can make the following observations. Note, however, that the material in this section already assumes the terminology from Chapter 2 below.

Let R be a ring. Let $\varphi(\xi)$ be a quantifier free formula with parameters from R, given as in (1.2). Let $\Sigma_{\varphi(\xi)}$ denote the constructible subset of $\mathbb{A}^n_R$ (see page 32) consisting of all prime ideals $\mathfrak{p}$ of $\mathrm{Spec}(R[\xi])$ which for some j_0 contain all f_{ij_0} and do not contain any g_{ij_0}. In particular, if $n = 0$, so that $\mathbb{A}^0_R$ is by definition $\mathrm{Spec}(R)$, then the constructible subset Σ_φ associated to φ is a subset of $\mathrm{Spec}(R)$.

Let A be an R-algebra and assume moreover that A is a domain (we will never use constructible sets associated to formulae if A is not a domain). For an n-tuple $\mathbf{a}$ over A, let $\mathfrak{p}_{\mathbf{a}}$ be the (prime) ideal in $A[\xi]$ generated by the $\xi_i - a_i$, where $\xi = (\xi_1, \dots, \xi_n)$. Since $A[\xi]/\mathfrak{p}_{\mathbf{a}} \cong A$, we call such a prime ideal an *A-rational point*. It is not hard to see that this yields a bijection between n-tuples over A and A-rational points in $A[\xi]$, which we therefore will identify with one another. In this terminology, $\varphi(\mathbf{a})$ holds in A if and only if the corresponding A-rational point $\mathfrak{p}_{\mathbf{a}}$ lies in the constructible set $\Sigma_{\varphi(\xi)}$ (strictly speaking, we should say that it lies in the base change $\Sigma_{\varphi(\xi)} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(A)$, but for notational clarity, we will omit any reference to base changes). If we denote the collection of A-rational points of the constructible set $\Sigma_{\varphi(\xi)}$ by $\Sigma_{\varphi(\xi)}(A)$, then there is a one-one correspondence between this latter set and the definable subset $\varphi(A)$. If φ is a sentence, then Σ_φ is a constructible subset of $\mathrm{Spec}(R)$ and hence its base change to $\mathrm{Spec}(A)$ is a constructible subset of $\mathrm{Spec}(A)$. Since A is a domain, $\mathrm{Spec}(A)$ has a unique A-rational point (corresponding to the zero-ideal) and hence φ holds in A if and only if this point belongs to Σ_φ.

Conversely, if Σ is an R-constructible subset of $\mathbb{A}^n_R$, then we can associate to it a quantifier free formula $\varphi_\Sigma(\xi)$ with parameters from R as follows. However, here there is some ambiguity, as a constructible set is more intrinsically defined than a formula. Suppose first that Σ is the Zariski closed subset $\mathrm{V}(I)$, where I is an ideal in $R[\xi]$. Choose a system of generators, so that $I = (f_1, \dots, f_s)R[\xi]$ and set $\varphi_\Sigma(\xi)$ equal to the quantifier free formula $f_1(\xi) = \cdots = f_s(\xi) = 0$. Let A be an R-algebra without zero-divisors. It follows that an n-tuple $\mathbf{a}$ is an A-rational point of Σ if and only if $\mathbf{a}$ satisfies the formula φ_Σ. Therefore, if we make a different choice of generators $I = (f'_1, \dots, f'_s)R[\xi]$, although we get a different formula φ', it defines in any R-algebra A without zero-divisors the same definable set, to wit, the collection of A-rational points of Σ. To associate a formula to an arbitrary constructible set, we do this recursively by letting $\varphi_\Sigma \wedge \varphi_\Psi$, $\varphi_\Sigma \vee \varphi_\Psi$ and $\neg\varphi_\Sigma$ correspond to the constructible sets $\Sigma \cap \Psi$, $\Sigma \cup \Psi$ and $-\Sigma$ respectively.

We say that two formulae $\varphi(\xi)$ and $\psi(\xi)$ in the same free variables $\xi = (\xi_1, \dots, \xi_n)$ are *equivalent over a ring* A, if they hold on exactly the same tuples from A (that is to say, if they define the same subsets in A^n). In particular, if φ and ψ are sentences, then they are equivalent in A if they are simultaneously true or false in A. If $\varphi(\xi)$ and $\psi(\xi)$ are equivalent for all rings A in a certain class $\mathscr{K}$, then we say that $\varphi(\xi)$ and $\psi(\xi)$ are *equivalent modulo the class* $\mathscr{K}$. In particular, if Σ is a constructible set in $\mathbb{A}^n_R$, then any two formulae associated to it are equivalent modulo the class of all R-algebras without zero-divisors. In this sense, there is a one-one correspondence between constructible subsets of $\mathbb{A}^n_R$ and quantifier free formulae with parameters from R modulo the above equivalence relation.

Quantifier Elimination. For certain rings (or classes of rings), every formula is equivalent to a quantifier free formula; this phenomenon is known under the name *Quantifier Elimination*. We will only encounter it for the following class.

Theorem 1.2.1 (Quantifier Elimination for algebraically closed fields). *If $\mathscr{K}$ is the class of all algebraically closed fields, then any formula without parameters is equivalent modulo $\mathscr{K}$ to a quantifier free formula without parameters.*

More generally, if F is a field and $\mathscr{K}(F)$ the class of all algebraically closed fields containing F, then any formula with parameters from F is equivalent modulo $\mathscr{K}(F)$ to a quantifier free formula with parameters from F.

Proof (Sketch of proof). These statements can be seen as translations in model-theoretic terms of Chevalley's Theorem which says that the projection of a constructible set is again constructible. I will only explain this for the first assertion. As already observed, a quantifier free formula $\varphi(\xi)$ (without parameters) corresponds to a constructible set $\Sigma_{\varphi(\xi)}$ in $\mathbb{A}^n_{\mathbb{Z}}$ and the tuples in K^n satisfying $\varphi(\xi)$ are precisely the K-rational points $\Sigma_{\varphi(\xi)}(K)$ of $\Sigma_{\varphi(\xi)}$. The key observation is now the following. Let $\psi(\xi,\zeta)$ be a quantifier free formula and put $\gamma(\xi) := (\exists\zeta)\,\psi(\xi,\zeta)$, where $\xi = (\xi_1,\dots,\xi_n)$ and $\zeta = (\zeta_1,\dots,\zeta_m)$. Let $\Psi := \psi(K)$ be the subset of K^{n+m} defined by $\psi(\xi,\zeta)$ and let $\Gamma := \gamma(K)$ be the subset of K^n defined by $\gamma(\xi)$. Therefore, if we identify K^{n+m} with the collection of K-rational points of $\mathbb{A}_K^{n+m}$, then

$$\Psi = \Sigma_{\psi(\xi,\zeta)}(K).$$

Moreover, if $p\colon \mathbb{A}_K^{n+m} \to \mathbb{A}_K^n$ is the projection onto the first n coordinates then $p(\Psi) = \Gamma$. By Chevalley's Theorem (see for instance [18, Corollary 14.7] or [24, II. Exercise 3.19]), $p(\Sigma_{\psi(\xi,\zeta)})$ (as a subset in $\mathbb{A}_{\mathbb{Z}}^n$) is again constructible, ands therefore, by our previous discussion, of the form $\Sigma_{\chi(\xi)}$ for some quantifier free formula $\chi(\xi)$. Hence $\Gamma = \Sigma_{\chi(\xi)}(K)$, showing that $\gamma(\xi)$ is equivalent modulo K to $\chi(\xi)$. Since $\chi(\xi)$ does not depend on K, we have in fact an equivalence of formulae modulo the class $\mathscr{K}$. To get rid of an arbitrary chain of quantifiers, we use induction on the number of quantifiers, noting that the complement of a set defined by $(\forall\zeta)\,\psi(\xi,\zeta)$ is the set defined by $(\exists\zeta)\,\neg\psi(\xi,\zeta)$, where $\neg(\cdot)$ denotes negation.

For some alternative proofs, see [34, Corollary A.5.2] or [39, Theorem 1.6]. □

1.3 Łos' Theorem

Thanks to Quantifier Elimination (Theorem 1.2.1), when dealing with algebraically closed fields, we may forget altogether about formulae and use constructible sets instead. However, we will not always be able to work just in algebraically closed fields and so we need to formulate a general transfer principle for ultraproducts. For most of our purposes, the following version suffices:

Theorem 1.3.1 (Equational Łos' Theorem). *Suppose each A_w is an R-algebra, and let $A_\natural$ denote their ultraproduct. Let ξ be an n-tuple of variables, let $f \in R[\xi]$, and let $\mathbf{a}_w$ be n-tuples in A_w with ultraproduct $\mathbf{a}_\natural$. Then $f(\mathbf{a}_\natural) = 0$ in $A_\natural$ if and only if $f(\mathbf{a}_w) = 0$ in A_w for almost all w.*

Moreover, instead of a single equation $f = 0$, we may take in the above statement any system of equations and negations of equations over R.

Proof. Let me only sketch a proof of the first assertion. Suppose $f(\mathbf{a}_\natural) = 0$. One checks (do this!), making repeatedly use of (1.1), that $f(\mathbf{a}_\natural)$ is equal to the ultraproduct of the $f(\mathbf{a}_w)$. Hence the former being zero simply means that almost all $f(\mathbf{a}_w)$ are zero. The converse is proven by simply reversing this argument. □

On occasion, we might also want to use the full version of Łos' Theorem, which requires the notion of a formula as defined above. Recall that a sentence is a formula without free variables.

Theorem 1.3.2 (Łos' Theorem). *Let R be a ring and let A_w be R-algebras. If φ is a sentence with parameters from R, then φ holds in almost all A_w if and only if φ holds in the ultraproduct $A_\natural$.*

More generally, let $\varphi(\xi_1,\dots,\xi_n)$ be a formula with parameters from R and let $\mathbf{a}_w$ be an n-tuple in A_w with ultraproduct $\mathbf{a}_\natural$. Then $\varphi(\mathbf{a}_w)$ holds in almost all A_w if and only if $\varphi(\mathbf{a}_\natural)$ holds in $A_\natural$.

The proof is tedious but not hard; one simply has to unwind the definition of formula (see [34, Theorem 9.5.1] for a more general treatment). Note that $A_\natural$ is naturally an R-algebra, so that it makes sense to assert that φ is true or false in $A_\natural$. Applying Łos' Theorem to a quantifier free formula proves Theorem 1.3.1.

1.4 Ultra-rings

An *ultra-ring* is simply an ultraproduct of rings. These rings will form the main tool in these notes, but for the moment we only establish some very basic facts about them.

Ultra-fields. Let K_w be a collection of fields and $K_\natural$ their ultraproduct, which is again a field by 1.1.5 (or by an application of Łos' Theorem). Any field which arises in this way is called an *ultra-field*.[1]. Since an ultraproduct is either finite or uncountable, $\mathbb{Q}$ is an example of a field which is not an ultra-field.

1.4.1 *If for each prime number p, only finitely many K_w have characteristic p, then $K_\natural$ has characteristic zero.*

Indeed, for every prime number p, the equation $p\xi - 1 = 0$ has a solution in all but finitely many of the K_w and hence it has a solution in $K_\natural$, by Theorem 1.3.1. We will call an ultra-field $K_\natural$ of characteristic zero which arises as an ultraproduct of fields of positive characteristic, a *Lefschetz field* (the name is inspired by Theorem 1.4.3 below); and more generally, an ultra-ring of characteristic zero given as the ultraproduct of rings of positive characteristic will be called a *Lefschetz ring* (see page 159 for more).

1.4.2 *If almost all K_w are algebraically closed fields, then so is $K_\natural$.*

The quickest proof is by means of Łos' Theorem, although one could also give an argument using just Theorem 1.3.1 (which is no surprise in light of Exercise 1.5.13).

Proof. For each $n \geq 2$, consider the sentence σ_n given by

$$(\forall \zeta_0, \dots, \zeta_n)\,(\exists \xi)\, \zeta_n = 0 \,\vee\, \zeta_n \xi^n + \cdots + \zeta_1 \xi + \zeta_0 = 0.$$

This sentence is true in any algebraically closed field, whence in almost all K_w, and therefore, by Łos' Theorem, in $K_\natural$. However, a field in which every σ_n holds is algebraically closed. □

We have the following important corollary which can be thought of as a model theoretic Lefschetz Principle (here $\mathbb{F}_p^{\mathrm{alg}}$ denotes the algebraic closure of the p-element field).

[1] In case the K_w are finite but of unbounded cardinality, their ultraproduct $K_\natural$ is also called a *pseudo-finite field*; in these notes, however, we prefer the usage of the prefix *ultra-*, and so we would call such fields instead *ultra-finite fields*

Theorem 1.4.3 (Lefschetz Principle). *Let $\mathbb{W}$ be the set of prime numbers, endowed with some non-principal ultrafilter. The ultraproduct of the fields $\mathbb{F}_p^{alg}$ is isomorphic with the field $\mathbb{C}$ of complex numbers, that is to say, we have an isomorphism*

$$\operatorname*{ulim}_{p\to\infty} \mathbb{F}_p^{alg} \cong \mathbb{C}.$$

Proof. Let $\mathbb{F}_\natural$ denote the ultraproduct of the fields $\mathbb{F}_p^{\mathrm{alg}}$. By 1.4.2, the field $\mathbb{F}_\natural$ is algebraically closed, and by 1.4.1, its characteristic is zero. Using elementary set theory, one calculates that the cardinality of $\mathbb{F}_\natural$ is equal to that of the continuum. The theorem now follows since any two algebraically closed fields of the same uncountable cardinality are (non-canonically) isomorphic by Steinitz's Theorem (see [34] or Theorem 1.4.5 below). □

Remark 1.4.4. We can extend the above result as follows: any algebraically closed field K of characteristic zero and cardinality 2^κ, for some infinite cardinal κ, is a Lefschetz field. Indeed, for each p, choose an algebraically closed field K_p of characteristic p and cardinality κ. Since the ultraproduct of these fields is then an algebraically closed field of characteristic zero and cardinality 2^κ, it is isomorphic to K by Steinitz's Theorem (Theorem 1.4.5). Under the generalized Continuum Hypothesis, any uncountable cardinal is of the form 2^κ, and hence any uncountable algebraically closed field of characteristic zero is then a Lefschetyz field.

Theorem 1.4.5 (Steinitz's Theorem). *If K and L are algebraically closed fields of the same characteristic and the same uncountable cardinality, then they are isomorphic.*

Proof (Sketch of proof). Let k be the common prime field of K and L (that is to say, either $\mathbb{Q}$ in characteristic zero, or $\mathbb{F}_p$ in positive characteristic p). Let Γ and Δ be respective transcendence bases of K and L over k. Since K and L have the same uncountable cardinality, Γ and Δ have the same cardinality, and hence there exists a bijection $f\colon \Gamma \to \Delta$. This naturally extends to a field isomorphism $k(\Gamma) \to k(\Delta)$. Since K is the algebraic closure of $k(\Gamma)$, and similarly, L of $k(\Delta)$, this isomorphism then extends to an isomorphism $K \to L$. □

Ultra-rings. Let A_w be a collection of rings. Their ultraproduct $A_\natural$ will be called, as already mentioned, an *ultra-ring*.

1.4.6 *If each A_w is local with maximal ideal $\mathfrak{m}_w$ and residue field $k_w := A_w/\mathfrak{m}_w$, then $A_\natural$ is local with maximal ideal $\mathfrak{m}_\natural := \operatorname{ulim} \mathfrak{m}_w$ and residue field $k_\natural := \operatorname{ulim} k_w$.*

Indeed, a ring is local if and only if the sum of any two non-units is again a non-unit. This statement is clearly expressible by means of a sentence, so that by Łos' Theorem, (Theorem 1.3.2) $A_\natural$ is local. Again we can prove this also directly, or using the equational version, Theorem 1.3.1. The remaining assertions now follow easily from 1.1.6. In fact, the same argument shows that the converse is also true: if $A_\natural$ is local, then so are almost all A_w.

1.4.7 *If A_w are local rings of embedding dimension e, then so is $A_\natural$.*

Recall that the *embedding dimension* of a local ring is the minimal number of generators of its maximal ideal. Hence, by assumption almost all $\mathfrak{m}_w$ are generated by e elements x_{iw}. It follows from 1.1.6 that $\mathfrak{m}_\natural$ is generated by the e ultraproducts $x_{i\natural}$.

1.4.8 *Almost all A_w are domains (respectively, reduced) if and only if $A_\natural$ is a domain (respectively, reduced).*

Indeed, being a domain is captured by the fact that the equation $\xi\zeta = 0$ has no solution by non-zero elements; and being reduced by the fact that the equation $\xi^2 = 0$ has no non-zero solutions. In particular, using 1.1.6, we see that an ultraproduct of ideals is a prime (respectively, radical, maximal) ideal if and only if almost all ideals are prime (respectively, reduced, maximal).

1.4.9 *If I_w are ideals in the local rings $(A_w, \mathfrak{m}_w)$, such that in $(A_\natural, \mathfrak{m}_\natural)$, their ultraproduct $I_\natural$ is $\mathfrak{m}_\natural$-primary, then almost all I_w are $\mathfrak{m}_w$-primary.*

Recall that an ideal I in a local ring $(R, \mathfrak{m})$ is called $\mathfrak{m}$-*primary* if its radical is equal to $\mathfrak{m}$. Note that here the converse may fail to hold: not every ultraproduct of $\mathfrak{m}_w$-primary ideals need to be $\mathfrak{m}_\natural$-primary (see Exercise 1.5.10).

As will become apparent later on, the following ideal plays an important role in the study of local ultra-rings.

Definition 1.4.10 (Ideal of infinitesimals). For an arbitrary local ring $(R, \mathfrak{m})$, define its *ideal of infinitesimals*, denoted $\mathfrak{I}_R$, as the intersection

$$\mathfrak{I}_R := \bigcap_{n \geq 0} \mathfrak{m}^n.$$

The $\mathfrak{m}$-adic topology (see page 91) on R is Hausdorff (=separated) if and only if $\mathfrak{I}_R = 0$. Therefore, we will refer to the residue ring $R/\mathfrak{I}_R$ as the *separated quotient* of R. In commutative algebra, the ideal of infinitesimals hardly ever appears simply because of:

Theorem 1.4.11 (Krull's Intersection Theorem). *If R is a Noetherian local ring, then $\mathfrak{I}_R = 0$.*

Proof. This is an immediate consequence of the Artin-Rees Lemma (for which see [41, Theorem 8.5] or [7, Proposition 10.9]), or of its weaker variant proven in Theorem 11.2.1 below. Namely, for $x \in \mathfrak{I}_R$, there exists some c such that $xR \cap \mathfrak{m}^c \subseteq x\mathfrak{m}$. Since $x \in \mathfrak{m}^c$ by assumption, we get $x \in x\mathfrak{m}$, that is to say, $x = ax$ with $a \in \mathfrak{m}$. Hence $(1-a)x = 0$. As $1-a$ is a unit in R, we get $x = 0$. □

Corollary 1.4.12. *In a Noetherian local ring $(R, \mathfrak{m})$, every ideal is the intersection of $\mathfrak{m}$-primary ideals.*

Proof. For $I \subseteq R$ an ideal, an application of Theorem 1.4.11 to the ring R/I shows that I is the intersection of all $I + \mathfrak{m}^n$, and the latter are indeed $\mathfrak{m}$-primary. □

Almost all local ultra-rings have a non-zero ideal of infinitesimals.

1.4.13 *If R_w are local rings with non-nilpotent maximal ideal, then the ideal of infinitesimals of their ultraproduct $R_\natural$ is non-zero. In particular, $R_\natural$ is not Noetherian.*

Indeed, by assumption, we can find non-zero $a_w \in \mathfrak{m}^w$ (let us for the moment assume that the index set is equal to $\mathbb{N}$) for all w. Hence their ultraproduct $a_\natural$ is non-zero and lies inside $\mathfrak{I}_{R_\natural}$.

Ultra-exponentation. Let $A_\natural$ be an ultra-ring, given as the ultraproduct of rings A_w. Let $\mathbb{N}_\natural$ be the ultrapower of the natural numbers, and let $\alpha \in \mathbb{N}_\natural$ with approximations α_w. The *ultra-exponentation map* on A with exponent α is given by sending $x \in A$ to the ultraproduct, denoted x^α, of the $x_w^{\alpha_w}$, where x_w is an approximation of x. One easily verifies that this definition does not depend on the choice of approximation of x or α. If A is local and x a non-unit, then x^α is an infinitesimal for any α in $\mathbb{N}_\natural$ not in $\mathbb{N}$. In these notes, the most important instance will be the ultra-exponentation map obtained as the ultra-product of Frobenius maps. More precisely, let $A_\natural$ be a Lefschetz ring, say, realized as the ultraproduct of rings A_p of characteristic p (here we assumed for simplicity that the underlying index set is just the set of prime numbers, but this is not necessary). On each A_p, we have an action of the *Frobenius*, given as $\mathbf{F}_p(x) := x^p$ (for more, see §8.1).

Definition 1.4.14 (Ultra-Frobenius). The ultraproduct of these Frobenii yields an endomorphism $\mathbf{F}_\infty$ on $A_\natural$, called the *ultra-Frobenius*, given by $\mathbf{F}_\infty(x) := x^\pi$, where $\pi \in \mathbb{N}_\natural$ is the ultraproduct of all prime numbers.

1.5 Exercises

Ex 1.5.1

Prove properties 1.1.1–1.1.7.

Ex 1.5.2

Prove 1.4.6 in detail, using only Theorem 1.3.1. Show that if $\mathfrak{p}_w$ are prime ideals in A_w, then their ultraproduct $\mathfrak{p}_\natural$ is a prime ideal in $A_\natural$, and the ultraproduct of the $(A_w)_{\mathfrak{p}_w}$ is equal to $(A_\natural)_{\mathfrak{p}_\natural}$.

Ex 1.5.3

Show that an ultrafilter on $\mathbb{W}$ is the same as a filter which is maximal (with respect to inclusion) among all filters containing the Frechet filter. Recall that a filter *on a set $\mathbb{W}$ is a collection of non-empty sets closed under finite intersection and supersets, and that*

the Frechet filter *is the collection of all co-finite subsets, that is to say, all subsets whose complement is finite.*
Also, describe the maximal filters not containing the Frechet filter.

Ex 1.5.4

In the statement of 1.4.1, we tacitly assume that the underlying set is countable. Prove the following more general version which works over an arbitrary infinite index set: if for each prime number p, almost no field K_w has characteristic p, then their ultraproduct $K_\natural$ has characteristic zero, whence is a Lefschetz field.

***Ex 1.5.5**

Fill in the details in the proof of the following result due to Ax ([8]):

Theorem. If a polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is injective, then it is surjective.

Here we call a map $\phi\colon \mathbb{C}^n \to \mathbb{C}^n$ polynomial *if there exist n polynomials $p_1(\xi),\dots,p_n(\xi) \in \mathbb{C}[\xi]$ in the n variables $\xi := (\xi_1,\dots,\xi_n)$ such that $\phi(\mathbf{u}) = (p_1(\mathbf{u}),\dots,p_n(\mathbf{u}))$ for all $\mathbf{u} \in \mathbb{C}^n$ (in the language of Chapter 2 this is just a morphism of affine space $\mathbb{A}^n_{\mathbb{C}}$ to itself).*

Proof. By the Pigeon Hole Principle, the result is true if we replace $\mathbb{C}$ by any finite field; since $\mathbb{F}_p^{alg}$ is a union of finite fields, the assertion also holds upon replacing $\mathbb{C}$ by $\mathbb{F}_p^{alg}$; hence we are done by Theorem 1.4.3. □

Ex 1.5.6

True or false: any homomorphic image of an ultra-ring is again an ultra-ring (you may want to take a peek at the next exercise).

Ex 1.5.7

Suppose $I_w \subseteq A_w$ are ideals, and let $I_\natural \subseteq A_\natural$ be their ultraproduct. Show that if H_w is a set of generators of I_w, then the ultraproduct $H_\natural := \operatorname{ulim} H_w$ generates $I_\natural$. Suppose next that all H_w are finite, say $H_w = \{f_{1w},\dots,f_{m(w),w}\}$, and for each $i \in \mathbb{N}$, let $f_{i\natural}$ be the ultraproduct of the f_{iw}, where we put $f_{iw} := 0$ whenever $m(w) < i$. Let m be the supremum of all $m(w)$ (allowing $m = \infty$). Show that if $m < \infty$, then the $f_{i\natural}$ for $i = 1,\dots,m$ generate $I_\natural$. Use the example $I_w := (\xi,\zeta)^w A_w$ (with $\mathbb{W} = \mathbb{N}$) where $A_w := K[\xi,\zeta]$, to show that the same statement is false if $m = \infty$.
Conclude that if I is a finitely generated ideal in a ring A, then its ultrapower in the ultrapower $A_\natural$ of A is equal to $IA_\natural$. Give a counterexample to this assertion if I is not finitely generated.

Ex 1.5.8

Prove the following more general version of the last assertion in Exercise 1.5.7: let $N \subseteq M$ be modules and let $N_\natural$ and $M_\natural$ be their ultrapowers. If N is finitely generated, then $N_\natural$ is equal to the submodule of $M_\natural$ generated by N.

Ex 1.5.9

Let $A \to B$ be a finite, injective homomorphism. Show, using induction on the number of A-algebra generators, that if A is an ultraring, then so is B.

Ex 1.5.10

Show that the ultraproduct of rings of length l is again a ring of length l (see page 44 for the notion of length). Use this to prove 1.4.9. Give a counterexample to the converse of 1.4.9.

Ex 1.5.11

Show the following sharper version of 1.4.13: if R_w are local rings whose length is unbounded (meaning that for every N, the length $\ell(R_w) > N$ for almost all w–see page 44 for the notion of length), then, and only then, their ultraproduct $R_\natural$ has a non-zero ideal of infinitesimals.

Ex 1.5.12

By an ultra-discrete valuation ring, *we mean an ultraproduct of discrete valuation rings. Show that the ideal of infinitesimals $\mathfrak{I}_V$ of an ultra-discrete valuation ring V is an infinitely generated prime ideal. Show that an ultra-discrete valuation ring is a* valuation domain *(=a domain such that for all a in the field of fractions of V, at least one of a or $1/a$ belongs to V). Show that the separated quotient $V/\mathfrak{I}_V$ is a discrete valuation ring–in Chapter 11 we will call this a* cataproduct *of discrete valuation rings.*

Additional exercises.

Ex 1.5.13

Derive Łos' Theorem (Theorem 1.3.2) from its equational version, Theorem 1.3.1.

Ex 1.5.14

Give a counterexample to Theorem 1.4.5 if we allow the common cardinality to be countable. Can you formulate a version which also works in the countable case?

Ex 1.5.15

Give a detailed proof of Theorem 1.4.5.

Ex 1.5.16

Let k be a field and $k_\natural$ its ultrapower. Use Maclane's criterion for separability (see for instance [41, Theorem 26.4] or [18, Theorem A1.3]) to show that the natural extension $k \to k_\natural$ is separable.

Ex 1.5.17

Recall from model-theory that a class of structures over a language L is axiomatizable *or* first-order definable, *if there exists a theory T in the language L whose models are precisely the members of this class. Show that an axiomatizable class is closed under ultraproducts. Deduce from this and 1.4.13 that the class of Noetherian rings is not first order-definable in the language of rings.*

1.6 Project: ultrarings as stalks

Prerequisites: sheaf-theory (for instance, [24, II.1], or the rudimentary discussion on page 28).

Let $\mathbb{W}$ be an infinite set and give it the discrete topology (in which all sets are open). Let $\mathbb{W}^\vee$ be the *Stone-Čech compactification* of $\mathbb{W}$ consisting of all ultrafilters on $\mathbb{W}$. Embed $\mathbb{W}$ in $\mathbb{W}^\vee$ (and henceforth view it as a subset) by sending an element to the principal ultrafilter it generates.

1.6.1 *Show that taking for open sets all sets of the form $\tau(\mathbb{U})$ for $\mathbb{U} \subseteq \mathbb{W}$, where $\tau(\mathbb{U})$ consists of all ultrafilters containing $\mathbb{U}$, constitutes a topology on $\mathbb{W}^\vee$. Show that $\mathbb{W}$ is dense in $\mathbb{W}^\vee$, that $\mathbb{W}^\vee$ is compact Hausdorff, and that any continuous map $\mathbb{W} \to X$ into a compact Hausdorff space X factors through $\mathbb{W}^\vee$ (this then justifies $\mathbb{W}^\vee$ being called a 'compactification').*

1.6.2 *Show that $\tau(\mathbb{U})$ is homeomorphic to $\mathbb{U}^\vee$, for any infinite subset $\mathbb{U} \subseteq \mathbb{W}$.*

Let A_w be rings, indexed by $w \in \mathbb{W}$. Define a sheaf of rings $\mathscr{A}$ on $\mathbb{W}$ by taking for stalk $\mathscr{A}_w := A_w$ in each point $w \in \mathbb{W}$ (note that since $\mathbb{W}$ is discrete, this completely determines the sheaf $\mathscr{A}$). Let $i\colon \mathbb{W} \to \mathbb{W}^\vee$ be the above embedding and let $\mathscr{A}^\vee := i_*\mathscr{A}$ be the direct image sheaf of $\mathscr{A}$ under i. By general sheaf theory, this is a sheaf on $\mathbb{W}^\vee$.

1.6.3 *Show that the stalk of $\mathscr{A}^\vee$ in a boundary point $\mathscr{U} \in \mathbb{W}^\vee \setminus \mathbb{W}$ is isomorphic to the ultraproduct $\operatorname{ulim} A_w$ with respect to the non-principal ultrafilter $\mathscr{U}$.*

Part II
Toccata in C minor[2]

[2] Being a minor introduction into Commutative Algebra. . .

Chapter 2
Commutative Algebra versus Algebraic Geometry

Historically, algebraic geometry was developed over the complex numbers $\mathbb{C}$. However, because of its algebraic nature, it can be carried out over any algebraically closed field. Therefore, in this chapter, we fix an algebraically closed field K, and we let $A := K[\xi]$ be the polynomial ring in n indeterminates $\xi := (\xi_1, \dots, \xi_n)$. Let us first take a look at classical or 'naive' algebraic geometry. Gradually we will move to an algebraization of the concepts, which we then will study by means of the algebraic theory developed in the subsequent chapters.

2.1 Classical algebraic geometry

Affine space. One defines *affine n*-space over K to be the topological space whose underlying set is K^n, and in which the closed sets are the algebraic sets. Recall that by an *algebraic set* we mean any solution set of a system of polynomial equations. More precisely, given a subset $\Sigma \subseteq A$, let $\mathrm{V}(\Sigma)$ be the collection of all tuples $\mathbf{u}$ such that $p(\mathbf{u}) = 0$ for all $p \in \Sigma$. Note that if $I := \Sigma A$ denotes the ideal generated by Σ, then $\mathrm{V}(I) = \mathrm{V}(\Sigma)$, so that in the definition, we may already assume that Σ is an ideal. In particular, if $p_1, \dots, p_s$ are generators of I, then $\mathrm{V}(I) = \mathrm{V}((p_1, \dots, p_s)A) = \mathrm{V}(p_1, \dots, p_s)$. A subset of the form $\mathrm{V}(I)$, for some ideal $I \subseteq A$, is then what is called an *algebraic* set (also called a *Zariski closed* subset). That this forms indeed a topology on K^n, called the *Zariski topology*, is an immediate consequence of the next lemma (the proof of which is deferred to the exercises):

Lemma 2.1.1. *Given ideals $I, J, I_n \subseteq A$, we have*

1. $\mathrm{V}(1) = \emptyset, \quad \mathrm{V}(0) = K^n$;
2. $\mathrm{V}(I) \cup \mathrm{V}(J) = \mathrm{V}(I \cdot J) = \mathrm{V}(I \cap J)$;
3. $\mathrm{V}(I_1) \cap \mathrm{V}(I_2) \cap \dots = \mathrm{V}(I_1 + I_2 + \dots)$,

where in the last equality, the intersection and the sum are allowed to be infinite as well. □

Conversely, given a closed subset $V \subseteq K^n$, we define the *ideal of definition* of V, denoted $\mathfrak{I}(V)$, to be the collection of all $p \in A$ such that p is identical zero on V. We have:

2.1.2 *The set $\mathfrak{I}(V)$ is a radical ideal, $\mathrm{V}(\mathfrak{I}(V)) = V$, and $\mathfrak{I}(V)$ is maximal among all ideals I such that $\mathrm{V}(I) = V$.*

Recall that an ideal $I \subseteq R$ is called *radical* if $x^n \in I$ implies $x \in I$. This is equivalent with R/I being *reduced*, that is to say, without nilpotent elements. The *radical* of an ideal I, denoted $\mathrm{rad}(I)$, is the ideal of all $x \in R$ such that some power belongs to I. Immediately from 2.1.2 we get:

2.1.3 *Every singleton in K^n is closed, and its ideal of definition is a maximal ideal.*

Indeed, let $\mathbf{u} := (u_1, \dots, u_n) \in K^n$. Let $\mathfrak{m}_{\mathbf{u}}$ be the ideal in A generated by the linear polynomials $\xi_i - u_i$. One verifies that the "evaluation at $\mathbf{u}$" map $A \to K \colon p \mapsto p(\mathbf{u})$ is surjective and has kernel equal to $\mathfrak{m}_{\mathbf{u}}$. Hence $A/\mathfrak{m}_{\mathbf{u}} \cong K$, showing that $\mathfrak{m}_u$ is a maximal ideal. Clearly, $\mathrm{V}(\mathfrak{m}_{\mathbf{u}}) = \{\mathbf{u}\}$.

Noetherian spaces. A topological space X is called *Noetherian* if there are no infinite strictly descending chains of closed subsets (one says: *X admits the descending chain condition on closed subsets*). A topological space X is called *irreducible* if it is not the union of two proper closed subsets. We call a subset $V \subseteq X$ *irreducible* if it so in the topology induced from X. An easy but important fact of Noetherian spaces is:

Proposition 2.1.4. *Any closed subset V of a Noetherian space X is a finite union of irreducible closed subsets.*

Proof. The argument is typical for Noetherian spaces, and often is therefore referred to as *Noetherian induction*. Namely, in a Noetherian space, every collection of closed subsets has a minimal element (prove this!). Now, if the assertion is false, let V be a minimal closed counterexample. In particular, V cannot be irreducible, and hence can be written as $V = V_1 \cup V_2$, with $V_1, V_2 \subsetneq V$ closed. By minimality, each V_i is a finite union of irreducible closed subsets, but then so is their union $V = V_1 \cup V_2$, contradiction. □

Hence any closed subset V admits an *irreducible decomposition* $V = V_1 \cup \cdots \cup V_s$ with the V_i irreducible closed subsets. We may always omit any V_i that is contained in some other V_j, and hence arrive at a *minimal* irreducible decomposition. One can show (see Exercise 2.8.2) that such a decomposition is unique (up to a renumbering of its components), and the V_i in this decomposition are then called the *irreducible components* of V.

Definition 2.1.5 (Dimension). The *dimension* of a Noetherian space X is the maximal length[1] of a chain of irreducible closed subsets (this can be infinite), and is denoted $\dim(X)$.

[1] Whenever one talks about the *length of a chain* one means one less than the number of distinct sets in the chain.

2.2 Hilbert-Noether theory

To develop (classical) algebraic geometry, three results are of crucial importance. We will prove them after first reformulating them as algebraic problems.

Hilbert's basis theorem. Hilbert proved the following result by a constructive method. We will provide a more streamlined version of this below.

Theorem 2.2.1. *Affine n-space is a Noetherian space of dimension n.*

In particular, any collection of Zariski closed subsets has a minimal element, any chain of irreducible Zariski closed subsets has length at most n, and any Zariski closed subset is the finite union of irreducible closed subsets. In order to prove Hilbert's basis theorem, we will translate it to an algebraic result (Theorem 2.3.5 below).

Nullstellensatz. We have already seen that a closed subset is given by an ideal as the locus $\mathrm{V}(I)$, and conversely, to a closed subset V is associated its ideal of definition $\mathfrak{I}(V)$. The next result, also due to Hilbert, describes the precise correspondence:

Theorem 2.2.2. *The operator $\mathfrak{I}(\cdot)$ induces an (order-reversing) bijection between (singletons of) K^n and maximal ideals of A; between closed subsets of K^n and radical ideals of A; and between irreducible closed subsets of K^n and prime ideals of A.*

More generally, if $V \subseteq K^n$ is a closed subset, and $I := \mathfrak{I}(V)$ its ideal of definition, then under the above correspondence, points in V correspond to maximal ideals containing I; closed subsets in V to radical ideals containing I; and irreducible closed subsets of V to prime ideals containing I.

Affine varieties and coordinate rings. The 'algebraic leap' to make now is that the three collections of ideals described in the second part of Theorem 2.2.2 correspond naturally to respectively the maximal, radical and prime ideals of the ring A/I (verify this!). We call A/I the *coordinate ring* of V and denote it $K[V]$ (see Exercise 2.8.4 for a justification of this notation). But this then again prompts us to view V as an object on its own, without immediate reference to its ambient affine space. Therefore, we will call any closed subset of K^n, for some n, an *affine variety*[2] over K, and we view it as a topological space via the induced topology.

The previous definition brings to the fore an algebraic object closely associated to a variety, to wit, its coordinate ring. To study it, we introduce some further terminology. By an *affine algebra* over K, or a K*-affine ring* or *algebra*, we mean a finitely generated K-algebra. Later on, we will work over other base rings than just fields, so it is apt to generalize this definition already now: let Z be an arbitrary ring. By a Z*-affine* ring or algebra we mean a finitely presented Z-algebra, that is to say, a Z-algebra of the form $Z[\xi]/I$ with ξ a finite tuple of indeterminates and I a finitely generated ideal. It follows from (the algebraic version of) Theorem 2.2.1 that both

[2] Be aware that some authors, unlike me, insist that varieties should also be irreducible.

our definitions agree in the case Z is a field. If Z is moreover a local ring with maximal ideal $\mathfrak{p}$, then by a *local Z-affine* ring (or algebra) R we mean a localization of a Z-affine ring with respect to a prime ideal containing $\mathfrak{p}$, that is to say $R \cong (Z[\xi]/I)_{\mathfrak{P}}$ with I finitely generated and $\mathfrak{P}$ a prime ideal of $Z[\xi]$ containing $\mathfrak{p}$. In particular, $Z \to R$ is a local homomorphism. By a *homomorphism* of Z-affine rings $A \to B$, we mean a Z-algebra homomorphism making B into an A-affine algebra (that is to say, the homomorphism $A \to B$ itself is of finite type). Similarly, by a *local homomorphism* of local Z-affine rings $R \to S$, we mean a local homomorphism of Z-algebras making S into a local R-affine ring.

Returning to our discussion about coordinate rings, we see that each $K[V]$ is a reduced K-affine ring. In Exercise 2.8.6, you will show that every reduced K-affine ring arises as a coordinate ring, and that different affine varieties have different coordinate rings. Hence we established the following 'duality' between geometric and algebraic objects:

2.2.3 *Associating the coordinate ring to an affine variety yields a one-one correspondence between affine varieties over K and reduced K-affine rings.*

To make this into an equivalence of categories, we must define morphisms between affine varieties. First off, a *morphism* between affine spaces is a polynomial map $\phi\colon K^n \to K^m$, that is to say, a map given by m polynomials $p_1(\xi), \dots, p_m(\xi) \in A$, sending an n-tuple $\mathbf{u}$ to the m-tuple

$$\phi(\mathbf{u}) := (p_1(\mathbf{u}), \dots, p_m(\mathbf{u})).$$

Note that ϕ also induces a K-algebra homomorphism $\varphi\colon B \to A$ by mapping ζ_i to p_i, where $B =: K[\zeta]$ and $\zeta := (\zeta_1, \dots, \zeta_m)$ are the indeterminates on K^m. Now, let V and W be affine varieties, that is to say, V is a closed subset of K^n and W a closed subset of K^m, say. Then a *morphism* $V \to W$ is the restriction of a polynomial map $\phi\colon K^n \to K^m$ for which $\phi(V) \subseteq W$, which we will just denote again as $\phi\colon V \to W$. Let $I := \mathfrak{I}(V) \subseteq A$ and $J := \mathfrak{I}(W) \subseteq B$ be the respective ideals of definition. We already noticed that ϕ induces a K-algebra homomorphism $\varphi\colon B \to A$. One verifies that if $\phi\colon V \to W$ is a morphism, then $\varphi(J) \subseteq I$, so that we get an induced K-algebra homomorphism $K[W] = B/J \to K[V] = A/I$. With this notion of morphism, 2.2.3 gives an anti-equivalence of categories ('anti' since the morphisms $V \to W$ yield homomorphisms $K[W] \to K[V]$ going the other way). An *isomorphism* of affine varieties, as always, is a morphism admitting an inverse which is also a morphism.

The *Krull dimension* of a ring R is by definition the maximal length of a chain of prime ideals in R (see §3.1). Using Theorem 2.2.2, we therefore get:

Corollary 2.2.4. *For every affine variety V, its dimension is equal to the Krull dimension of its coordinate ring $K[V]$.* □

Noether normalization. To formulate the last of our 'great' theorems, we call a morphism of affine varieties $V \to W$ *finite* if the induced homomorphism $K[W] \to K[V]$ is finite (meaning that $K[V]$ is finitely generated as a module over $K[W]$).

Theorem 2.2.5. *Each variety V admits a finite and surjective morphism onto some affine space K^d.*

Proof. We will actually prove the slightly stronger algebraic form of this statement: any K-affine ring C (not necessarily reduced) admits a finite and injective homomorphism $K[\zeta_1,\dots,\zeta_d] \subseteq C$ (see 3.3.7 below). We prove this by induction on n, the number of variables ξ used to define C. Write C as A/I for some ideal I with $A := K[\xi]$. There is nothing to show if I is zero, so assume f is a non-zero polynomial in I. The trick is to find a change of coordinates such that f becomes monic in the last coordinate ξ_n, that is to say, when viewed as a polynomial in $A'[\xi_n]$, the highest degree term of f is equal to ξ_n^s, where $A' := K[\xi']$ and $\xi' := (\xi_1,\dots,\xi_{n-1})$. Such a change of coordinates does indeed exist (Exercise 2.8.23), and in fact, can be taken to be linear in case K is infinite (which is the case if K is algebraically closed). So we may assume f is monic in ξ_n of degree s. By Euclidean division in $A'[\xi_n]$, any polynomial g can be written as $g = fq + r$ with $q, r \in A$ such that the ξ_n-degree of r is at most $s-1$. This means that A/fA is generated as an A'-module by $1, \xi_n, \dots, \xi_n^{s-1}$. Let $I' := I \cap A'$. It follows that the extension $A'/I' \subseteq A/I$ is again finite. By induction, A'/I' is a finite $K[\zeta]$-module for some tuple of variables $\zeta := (\zeta_1,\dots,\zeta_d)$. Hence the composition $K[\zeta] \subseteq A'/I' \subseteq A/I = C$ is the desired *Noether normalization* of C. □

We will see later (in Corollary 3.3.9) that d is actually equal to the dimension of V. In particular, this then proves the second statement in Theorem 2.2.1 (see also Corollary 3.3.3); the first statement will be covered in Theorem 2.3.5 below. Let us next prove the Nullstellensatz. We start with:

Proposition 2.2.6 (Weak Nullstellensatz). *If $E \subseteq F$ is an extension of fields such that F is finitely generated as an E-algebra, then $E \subseteq F$ is a finite extension.*

Proof. By Theorem 2.2.5, we can find a finite, injective homomorphism $E[\zeta] \subseteq F$. The result now follows from Lemma 2.2.7, since the only way $E[\zeta]$ can be a field is for ζ to be the empty tuple of variables, showing that $E \subseteq F$ itself is finite, as claimed. □

Lemma 2.2.7. *If $R \subseteq F$ is a finite, injective homomorphism (or more generally, an integral extension) with F a field, then R is also a field.*

Proof. Let a be a non-zero element of R. By assumption, $1/a \in F$ is integral over R, whence satisfies an equation

$$(1/a)^d + r_1(1/a)^{d-1} + \cdots + r_d = 0$$

with $r_i \in R$. Multiplying with a^d, we get $1 + a(r_1 + r_2 a + \cdots + r_d a^{d-1}) = 0$, showing that a has an inverse in R. □

Proof of the Nullstellensatz, Theorem 2.2.2

We already observed (in 2.1.3) that $\mathfrak{I}(\mathbf{u}) = \mathfrak{m}_{\mathbf{u}}$ is a maximal ideal of A. So we need to prove conversely that any maximal ideal of A is realized in this way. Let $\mathfrak{m}$ be a maximal ideal. By Proposition 2.2.6, the field $A/\mathfrak{m}$ is a finite extension of K, and since K is algebraically closed, it must in fact be equal to it. If u_i denotes the image of ξ_i under the composition $A \to A/\mathfrak{m} \cong K$, then $\mathfrak{m}_{\mathbf{u}} \subseteq \mathfrak{m}$ for $\mathbf{u} := (u_1,\dots,u_n)$, whence both ideals must be equal as they are maximal. This proves the one-one correspondence between K^n and the maximal ideals of A. By 2.1.2, the operator $\mathfrak{I}$ is injective. To prove it is surjective, we have to show that $I = \mathfrak{I}(\mathrm{V}(I))$ for any radical ideal $I \subseteq A$. In fact, the stronger equality

$$\mathfrak{I}(\mathrm{V}(I)) = \mathrm{rad}(I), \tag{2.1}$$

holds for any ideal $I \subseteq A$. Equality (2.1) translates (do this!) into the fact that $\mathrm{rad}(I)$ is equal to the intersection of all maximal ideals containing I. Replacing A by $A/\mathrm{rad}(I)$, we reduce to showing that the Jacobson radical of a reduced K-affine ring C is zero (one says that C is a *Jacobson* ring), where the *Jacobson radical* of C is by definition the intersection of all of its maximal ideals. This amounts to showing that given any non-zero element f of C, there exists a maximal ideal not containing f. By Theorem 2.2.5, we can find a finite, injective homomorphism $B := K[\zeta] \subseteq C$. Let

$$f^s + b_1 f^{s-1} + \cdots + b_s = 0 \tag{2.2}$$

be an integral equation of minimal degree with all $b_i \in B$. By minimality, $b_s \neq 0$. By Exercise 2.8.23, there exists $\mathbf{v}$ such that $b_s(\mathbf{v}) \neq 0$. In other words, $\mathfrak{m}_{\mathbf{v}}$ is a maximal ideal of B not containing b_s. Since $\mathfrak{m}_{\mathbf{v}}C$ is not the unit ideal (this follows for instance from Theorem 3.3.8, or can be proven directly), we can find a maximal ideal $\mathfrak{m}$ of C containing $\mathfrak{m}_{\mathbf{v}}$. In particular, $\mathfrak{m}_{\mathbf{v}} \subseteq \mathfrak{m} \cap B$ and hence this must be an equality by maximality. In particular, it follows then from (2.2) that $f \notin \mathfrak{m}$.

This establishes the one-one correspondence between closed subsets and radical ideals. In Exercise 2.8.2 you are asked to show that $\mathfrak{I}(V)$ is a prime ideal if and only if V is irreducible. This then concludes the proof of the first part of Theorem 2.2.2. The second part, however, simply follows from this by identifying ideals of A/I with the ideals of A containing I. □

2.3 Affine schemes

There are several motivations for generalizing the classical perspective, by introducing a larger class of ‘geometric’ objects. Let us look at two of these motivations.

Generic points. Firstly, geometers often reason by ‘general’, or ‘generic’, points. They will for instance say that a “general point on a variety is non-singular” (see 2.6.5 below for the exact meaning of this phrase). But what is a ‘generic’ point. We can give a topological definition:

Definition 2.3.1 (Generic point). A point x of an irreducible topological space X is called *generic* if the closure of $\{x\}$ is all of X.

More generally, for X an arbitrary Noetherian topological space, one calls $x \in X$ generic, if its closure (or more accurately, the closure of the singleton determined by x) is an irreducible component (see page 20) of X.

In view of 2.1.3, the only closed subsets of K^n having a generic point are the singletons themselves. So how do we get generic points? There is a simple topological construction. Given a Noetherian space X, let $\mathfrak{Irr}(X)$ be the collection of all irreducible closed subsets of X. Define a topology on $\mathfrak{Irr}(X)$ by taking for closed subsets the sets of the form $\mathfrak{Irr}(V)$ for $V \subseteq X$ closed. There is a continuous map

$X \to \mathfrak{Irr}(X)$ sending a point $x \in X$ to its closure (note that the closure of a singleton is always irreducible). Exercise 2.8.7 explores how this creates plenty of generic points.

If we apply this construction to K^n, then by Theorem 2.2.2, the resulting space $\mathfrak{Irr}(K^n)$ is equal to $|\mathrm{Spec}(A)|$, the collection of all prime ideals of A.[3] A (*Zariski*) closed subset of $|\mathrm{Spec}(A)|$ is then a closed subset in the above defined topology, and hence is of the form $\mathrm{V}(I)$, for some ideal I, where $\mathrm{V}(I)$ denotes the collection of all prime ideals containing I. In particular, if $\mathfrak{p}$ is a prime ideal, then $\mathfrak{p}$ is the unique generic point of $\mathrm{V}(\mathfrak{p})$.

More generally, given a ring R, let $|\mathrm{Spec}(R)|$ be the collection of all its prime ideals and make this into a topological space by taking for closed subsets the $\mathrm{V}(I)$ for $I \subseteq R$. Note that each $\mathrm{V}(I)$ is naturally identified with $|\mathrm{Spec}(R/I)|$, and often we will equate both subsets. That this forms indeed a topology, the so-called *Zariski topology*, follows by the same argument that proves Lemma 2.1.1. We call $\mathfrak{Irr}(K^n)$ the *enhanced affine n-space*. It has a unique generic point given by the zero ideal (check this). This extends by Theorem 2.2.2 to any affine variety:

2.3.2 *Given an affine variety V with coordinate ring $K[V]$, the space $\mathfrak{Irr}(V)$ is homeomorphic to $|\mathrm{Spec}(K[V])|$, where the latter carries the Zariski topology. The generic points of the* enhanced affine variety $\mathfrak{Irr}(V)$ *then correspond to the minimal primes of $K[V]$.*

Henceforth, we will therefore identify $\mathfrak{Irr}(V)$ with $|\mathrm{Spec}(K[V])|$. The canonical map $V \to \mathfrak{Irr}(V) = |\mathrm{Spec}(K[V])|$ is given by identifying a point $\mathbf{u} \in V$ with its (maximal) ideal of definition $\mathfrak{m}_{\mathbf{u}}$; it is easily seen to be injective. A point in $|\mathrm{Spec}(K[V])|$ coming from V is called a *closed* point. Indeed, these are the only points which are equal to their closure. Note that the intersection of the minimal primes of $K[V]$ is equal to the zero ideal (recall that $K[V]$ is reduced). At this point, there is no need to stick to K-affine rings, and so we call any topological space of the form $|\mathrm{Spec}(R)|$ with R any ring, an *enhanced affine variety*. A *closed* point then corresponds to a maximal ideal of R.

Base change Coming back to our discussion of generic points, 2.3.2 shows that every enhanced affine variety has only finitely many generic points, which is not what we would expect of a 'general' point. To get around this obstruction, we need to work over a larger algebraically closed field L containing K. The *base change* of an affine variety V over K to L is defined as the (Zariski) closure V_L of V in L^n. One shows (Exercise 2.8.10) that if V has ideal of definition $I \subseteq A$, then IA_L is the ideal of definition of V_L, where $A_L := L[\xi]$. In particular, V_L is an affine variety over L, and its coordinate ring is

$$L[V_L] = A_L/IA_L = K[V] \otimes_K L.$$

We use:

[3] The reason for the awkward notation will become clear in the next section.

2.3.3 *If $R \to S$ is a (ring) homomorphism, then $|\mathrm{Spec}(S)| \to |\mathrm{Spec}(R)|$ given by the rule $\mathfrak{q} \mapsto \mathfrak{q} \cap R$ is a continuous map of topological spaces.*

Note that we have used the slightly misleading notation for the *contraction* of an ideal $J \subseteq S$ in R as $J \cap R$ (even if R is not a subset of S); by definition $J \subseteq S$ is the ideal of all $r \in R$ such that the image of r in S lies inside J. Hence if φ denotes the homomorphism $R \to S$, then $J \cap S$ is actually $\varphi^{-1}(J)$. Returning to our discussion on generic points, the natural homomorphism $A \to A_L$ (called the *base change*) induces a homomorphism $K[V] \to L[V_L]$, whence a map of enhanced affine varieties

$$\mathfrak{Irr}(V_L) = |\mathrm{Spec}(L[V_L])| \to \mathfrak{Irr}(V) = |\mathrm{Spec}(K[V])|\,.$$

Now, a point $\mathbf{v} \in V_L$ is *generic with respect to* K if its image under the above map is a generic point of $\mathfrak{Irr}(V)$. This is equivalent with $\mathfrak{m}_{\mathbf{v}} \cap K[V]$ being a minimal prime of $K[V]$.

Example 2.3.4. The point with coordinates (e, π) is (probably) a generic point of the affine plane over $\mathbb{Q}^{\mathrm{alg}}$. Similarly, the point $(0, \pi)$ is a generic point over $\mathbb{Q}^{\mathrm{alg}}$ of the y-axis.

Using 2.3.2, we can now also prove Theorem 2.2.1 as it translates immediately to the following algebraic result (recall that a ring is *Noetherian* if there exists no infinite strictly ascending chain of ideals, or equivalently, if every ideal is finitely generated):

Theorem 2.3.5 (Hilbert Basis Theorem–algebraic form). *The polynomial ring A over a field K in n variables is Noetherian.*

Proof. We induct on n, where the case $n = 0$ is trivial, so that we may assume $n > 0$. Let $\mathfrak{a}$ be a non-zero ideal of A and let $p \in \mathfrak{a}$ be non-zero. By Theorem 2.2.5, there exists a finite extension $B := K[\zeta] \subseteq A/pA$, where ζ is a tuple of variables of length at most $n-1$ (and in fact equal to $n-1$). By induction, B is Noetherian. Since A/pA is a finite B-module, it too is Noetherian (see for instance [7, Proposition 6.5]). In particular, $\mathfrak{a}(A/pA)$ is finitely generated, and hence so is $\mathfrak{a}$ (by the liftings of the generators of $\mathfrak{a}(A/pA)$ together with p). □

Nilpotent structure. A second draw-back of the classical approach is that if we intersect two closed subsets, the resulting closed subset does not take into account the finer structure of this intersection. For instance, a circle C in the affine plane with radius one and center $(0,1)$ intersects the x-axis L in a single point, the origin O. However, if we look at equations (or, equivalently, ideals of definitions), where C is given by $I := (\xi^2 + \zeta^2 - 2\zeta)A$, and L by $J := \zeta A$, then we get a system of equations which reduces to $\xi^2 = 0, \zeta = 0$ (equivalently, the ideal $I + J = (\xi^2, \zeta)A$), which suggests that we should count the intersection point O twice (accounting for the tangency of L to C). Hence, instead of looking at the ideal $\mathrm{rad}(I+J) = \mathrm{rad}(\xi^2, \zeta) = (\xi, \zeta)A$, or equivalently, to the coordinate ring $K[O] = A/(\xi, \zeta)A = K$, we should not 'forget' the nilpotent structure of $A/(I+J)$. However, enhanced affine varieties cannot capture this phenomenon. Namely, if B is an arbitrary K-affine ring, then as a topological spaces $|\mathrm{Spec}(B)|$ and $|\mathrm{Spec}(B_{\mathrm{red}})|$ are homeomorphic, where $B_{\mathrm{red}} := B/\mathrm{nil}(B)$ and $\mathrm{nil}(B) := \mathrm{rad}(0)$ is the *nil-radical* of B. In particular, $|\mathrm{Spec}(A/(I+J))|$ and $|\mathrm{Spec}(K)|$ are the same. To resolve this problem, we have to resort to a finer structure, that of an (affine) scheme. Roughly speaking, an affine scheme is an enhanced affine variety X together with a sheaf of functions $\mathscr{O}_X$. I will only provide a sketch of the general definitions. To this end, we must first discuss Zariski open subsets.

Open subsets. Let R be a ring and f an element in R. The localization of R at f, denoted R_f or $R[1/f]$, is the ring $R[\xi]/(f\xi-1)R[\xi]$ obtained by inverting f (this includes the degenerate case that $f=0$ in which case R_f is the zero ring). Equivalently, it is the collection of all fractions r/f^n with $r\in R$ up to the equivalence relation identifying two fractions r/f^n and s/f^m if there exists some k such that $f^{k-n}r = f^{k-m}s$ in R. This definition becomes much more straightforward if we assume $f\neq 0$ and R to be a domain: R_f is then the subring of the field of fractions $\mathrm{Frac}(R)$ of R consisting of all fractions r/f^n with $r\in R$. Let $V := |\mathrm{Spec}(R)|$ be an enhanced affine variety and let $f\in R$. Let $\mathrm{D}(f)$ be the complement of the closed subset $\mathrm{V}(fR) = |\mathrm{Spec}(R/fR)|$ of V. We refer to $\mathrm{D}(f)$ as a *basic* open subset. Indeed, given an arbitrary open subset U, say given as the complement of a closed subset $\mathrm{V}(I)$, we have

$$U = V - \mathrm{V}(I) = \bigcup_{f\in I} \mathrm{D}(f).$$

In particular, if R is Noetherian, then any open subset is a finite union of basic open subsets.

2.3.6 *The basic open* $\mathrm{D}(f)$ *is homeomorphic with* $\left|\mathrm{Spec}(R_f)\right|$, *whence in particular is an enhanced affine variety.*

See Exercise 2.8.15. Note that not every open subset can be realized as an (enhanced) affine variety: for instance the plane with the origin removed is an open which is not affine (see Exercise 2.8.5). Here is an example of a basic open subset with some additional structure.

Example 2.3.7. Let $\mathrm{GL}(K,n)$ be the *general linear group* consisting of all invertible $n\times n$-matrices over K. If we identify an $n\times n$-matrix with a tuple in K^{n^2}, then $\mathrm{GL}(K,n)$ is the open subset $\mathrm{D}(\det)$, where $\det(\cdot)$ is the polynomial representing the determinant function. In particular, we may view $\mathrm{GL}(K,n)$ as an enhanced affine variety. In Exercise 2.8.16, you will show that the multiplication map $\mathrm{GL}(K,n)\times\mathrm{GL}(K,n)\to\mathrm{GL}(K,n)$ is a morphism, and so is the map sending a matrix to its inverse.

2.4 SEC

Sections. To define sections, let us first look at these on an affine variety $V \subseteq K^n$. We already observed that any $f \in K[V]$ induces a function $\sigma_f\colon V \to K\colon \mathbf{u} \mapsto f(\mathbf{u})$. We call such a map a *section* on V. If f is identically zero, or more generally, if $f \in \mathfrak{I}(V)$, then σ_f is just the zero section. So assume $f \notin \mathfrak{I}(V)$, that is to say, f is non-zero in $K[V]$. If $f(\mathbf{u}) \neq 0$, then $1/f(\mathbf{u})$ is defined. Hence $1/f$ can be viewed as a section on $\mathrm{D}(f) \cap V$. More generally, we see that every element of R_f is a section on $\mathrm{D}(f)$.

For an arbitrary enhanced affine variety $V := |\mathrm{Spec}(R)|$, the definition of a section is more involved. We need a definition:

Definition 2.4.1 (Residue field). Given a point $x \in V$ with corresponding prime ideal $\mathfrak{p}_x \subseteq R$, its *residue field* $\kappa(x)$ is by definition the field of fractions of the domain $R/\mathfrak{p}_x$.

Note that if R is a K-affine ring, and x a closed point, then $\kappa(x) = K$ by Theorem 2.2.2. However, in general the various residue fields are no longer the same (they even may have different characteristic; see Exercise 2.8.12). Hence we cannot expect a section to take values in a fixed field. Let $Q(V)$ be the disjoint union of all $\kappa(x)$ where x runs over all points $x \in V$.

A (*reduced*) *section* $\sigma\colon V \to Q(V)$ is a map such that $\sigma(x) \in \kappa(x)$ for every point $x \in V$. Let us denote the collection of all sections on an enhanced affine variety V by $\mathrm{Sect}(V)$, which we may view as a ring, since we can add and multiply sections. Any element $f \in R$ induces a section σ_f on V, simply by letting $\sigma_f(x)$ be the image of f in $\kappa(x)$. More generally, any element of R_f induces a section on $\mathrm{D}(f)$, since f is invertible in $\kappa(x)$ for $x \in \mathrm{D}(f)$. In particular, we have a homomorphism $R_f \to \mathrm{Sect}(\mathrm{D}(f))$. However, in general this map can have a kernel (see Exercise 2.8.15):

2.4.2 *The kernel of* $R \to \mathrm{Sect}(|\mathrm{Spec}(R)|)$ *is the nil-radical of* R.

To define a scheme structure on V, we now have to declare, for each open subset $U \subseteq V$, which sections are to be viewed as 'continuous' sections on U. But we also want to incorporate nilpotent elements, which are 'invisible' in $\mathrm{Sect}(U)$ by 2.4.2. So for each open U, we define a ring $\Gamma(U, \mathscr{O}_V)$ (also denoted $\mathscr{O}_V(U)$) and a surjective homomorphism $\Gamma(U, \mathscr{O}_V) \to \mathrm{Sect}(U)$. Without given all the details, we declare $\Gamma(V, \mathscr{O}_V)$ to be R (the so-called *global sections* of V), and we put

$$\Gamma(\mathrm{D}(f), \mathscr{O}_V) := R_f \tag{2.3}$$

(note that the first case is just a special case of (2.3), by taking $f = 1$). For each open U the elements of $\Gamma(U, \mathscr{O}_V)$ are still called *sections* on U (in fact, this is the correct terminology in view of our discussion on page 37).

Sheafs.

Of course, the sections on the various open subsets of V have to be related to one another. The correct definition is that $\mathscr{O}_V$ has to be a *sheaf* on X. In general, a *sheaf of rings* (or of groups, sets, ...) $\mathscr{A}$ on a topological space X is a functor associating to each open subset $U \subseteq X$ a ring (group, set, etc.) $\mathscr{A}(U)$, and to each inclusion $U \subseteq U'$ a *restriction homomorphism* sending $f \in \mathscr{A}(U')$ to an element $f|_U \in \mathscr{A}(U)$ (being a *functor* means, among other things, that if $U \subseteq U' \subseteq U''$ then the composition of the restriction maps $\mathscr{A}(U'') \to \mathscr{A}(U') \to \mathscr{A}(U)$ is equal to the restriction map $\mathscr{A}(U'') \to \mathscr{A}(U)$), satisfying the following two properties for every open subset $U \subseteq X$ and every open covering $\{U_i\}$ of U:

1. if $f, g \in \mathscr{A}(U)$ are such that their restriction to each U_i is the same, then $f = g$;
2. if $f_i \in \mathscr{A}(U_i)$ are given such that the restriction of f_i and f_j to $U_i \cap U_j$ coincide, for all i, j, then there exists $f \in \mathscr{A}(U)$ such that $f|_{U_i} = f_i$ for all i.

One can show that there exists a unique sheaf $\mathscr{O}_V$ on $V = |\mathrm{Spec}(R)|$ for which conditions (2.3) hold, that is to say, such that $\Gamma(\mathrm{D}(f), \mathscr{O}_V) = R_f$. Moreover, each $g \in \Gamma(U, \mathscr{O}_V)$ then induces a section on U, that is to say, we have a homomorphism $\Gamma(U, \mathscr{O}_V) \to \mathrm{Sect}(U)$. In fact, this gives rise to a natural transformation $\Gamma(\cdot, \mathscr{O}_V) \to \mathrm{Sect}(\cdot)$ of functors. For the 'official' definition of $\mathscr{O}_V$, see page 37 below.

The category of affine schemes. An *affine scheme* $X = \mathrm{Spec}(R)$, therefore, is an enhanced affine variety $|\mathrm{Spec}(R)|$ (with R an arbitrary ring) together with a sheaf of sections $\mathscr{O}_X$ on $|\mathrm{Spec}(R)|$ satisfying (2.3), called the *structure sheaf* of X. Note that we can recover R from its associated affine scheme as the ring of global sections $R = \Gamma(X, \mathscr{O}_X)$. We often refer to R still as the *coordinate ring* of X. A *morphism* $Y \to X$ between affine schemes $X := \mathrm{Spec}(R)$ and $Y := \mathrm{Spec}(S)$ is given by a ring homomorphism $R \to S$: it induces a continuous map $\phi\colon\ |\mathrm{Spec}(S)| \to |\mathrm{Spec}(R)|$ by 2.3.3, as well as ring homomorphisms $\mathscr{O}_X(U) \to \mathscr{O}_Y(\phi^{-1}(U))$, for every open $U \subseteq |\mathrm{Spec}(R)|$. To define the latter, it suffices to do this on a basic open subset $\mathrm{D}(f)$, where it just the induced homomorphism $R_f \to S_f$, for any $f \in R$. In particular, on X, the induced ring homomorphism between global sections is the original homomorphism $R \to S$. Moreover, these homomorphisms are compatible with the restriction maps. The morphism $Y \to X$ is called of *finite type* if the corresponding homomorphism $A \to B$ is of finite type, that is to say, if B is finitely generated as an A-algebra. Note that any K-affine ring R induces a morphism $X := \mathrm{Spec}(R) \to \mathrm{Spec}(K)$ of finite type, sometimes called the *structure map* of X. Note that the underlying set of $\mathrm{Spec}(K)$ is just a singleton, and hence $|X| \to |\mathrm{Spec}(K)|$ is the trivial map. One additional advantage to this formalism is that there is no need anymore to have K algebraically closed: we can define affine schemes of finite type over any field. Generalizing 2.2.3 we now get:

2.4.3 *Associating to an affine scheme X its ring of global sections $\Gamma(X, \mathscr{O}_X)$ induces an anti-equivalence of categories between the category of affine schemes and the category of rings. Under this anti-equivalence, affine schemes of finite type over a field K correspond to K-affine rings.*

Here is one more reason why we should work with the enhanced space of all prime ideals of a ring, not just its maximal ideals: namely, in general the contraction of a maximal ideal, although prime, need not be maximal. For instance in $K[[\xi]][\zeta]$ the ideal generated by $\xi\zeta - 1$ is maximal as its residue ring is the field $K((\xi))$ of Laurent series. However, its contraction to $K[[\xi]]$ is the zero ideal. In classical algebraic geometry, this complication however is absent:

Proposition 2.4.4. *If $Y \to X$ is a morphism of finite type of affine schemes of finite type over K, then the image of a closed point is again closed.*

Proof. The algebraic translation says that if $C \to D$ is a K-algebra homomorphism of K-affine rings, and if $\mathfrak{n} \subseteq D$ is a maximal ideal, then so is $\mathfrak{m} := \mathfrak{n} \cap C$. To prove this, note that $D/\mathfrak{n}$ is again a K-affine ring, whence $K \subseteq D/\mathfrak{n}$ is finite by Proposition 2.2.6. Since $A/\mathfrak{m}$ is a subring of $D/\mathfrak{n}$, it is also finite over K, whence Artinian, whence a field. □

Intersections of closed subschemes. Returning to our discussion on intersections, the correct way of viewing the intersection of two affine varieties $V, W \subseteq K^n$ with respective ideals of definition $I := \mathfrak{I}(V)$ and $J := \mathfrak{I}(W)$ is as the affine scheme $\operatorname{Spec}(A/(I+J))$. To define this also for arbitrary affine schemes, we must make precise what it means to be a 'subscheme'. The next result gives an indication of what this should mean (its proof is relegated to Exercise 2.8.17).

Lemma 2.4.5. *Let $X := \operatorname{Spec}(R)$ be an affine scheme and let V be a closed subset of $|X|$. If $I \subseteq R$ is an ideal such that $\operatorname{V}(I) = V$, then $\operatorname{Spec}(R/I)$ is an affine scheme with underlying set equal to V.*

The 'smallest' scheme structure on V is given by the ideal I_V obtained by intersecting all prime ideals in V. More precisely, if Y is an affine scheme with $|Y| = V$, then there exists an injective morphism $\operatorname{Spec}(R/I_V) \to Y$. □

One refers to $\operatorname{Spec}(R/I_V)$ as the *induced reduced scheme structure* on V. Note that I_V is a radical ideal, and that any ideal I such that $\operatorname{V}(I) = V$ satisfies $\operatorname{rad}(I) = I_V$. More generally, a *closed subscheme* of an affine scheme $X := \operatorname{Spec}(R)$ is an affine scheme of the form $Y := \operatorname{Spec}(R/I)$ for some ideal $I \subseteq R$. By the previous lemma, the underlying set $|Y|$ is a closed subvariety of the underlying set $|X|$. Moreover, the inclusion $Y \subseteq X$ is a morphism of affine schemes, called a *closed immersion*. In analogy with vector spaces, we call the collection of all closed subschemes of an affine scheme X the *Grassmanian* of X and denote it $\operatorname{Grass}(X)$. We can define a (partial) order on $\operatorname{Grass}(X)$ by letting $Y \subseteq Z$ stand for 'Y is a closed subscheme of Z'. It is important to note that in spite of the notation, $Y \subseteq Z$ does not just mean an inclusion of underlying sets. In fact, if I and J are the ideals of R such that $Y = \operatorname{Spec}(R/I)$ and $Z = \operatorname{Spec}(R/J)$, then $Y \subseteq Z$ if and only if $J \subseteq I$. For this reason, we also define the *Grassmanian* $\operatorname{Grass}(R)$ of a ring R as the collection of all its ideals, ordered by reverse inclusion. Hence there is a one-one correspondence between $\operatorname{Grass}(R)$ and $\operatorname{Grass}(\operatorname{Spec}(R))$.

Given two closed subschemes $Y_k := \operatorname{Spec}(R/I_k)$ of X, for $k = 1, 2$, we now define their *scheme-theoretic intersection* $Y_1 \cap Y_2$ as the closed subscheme $\operatorname{Spec}(R/(I_1 + I_2))$. In particular, $Y_1 \cap Y_2 \subseteq Y_1, Y_2$. In fact, intersection is the minimum (or join) operation in the Grassmanian $\operatorname{Grass}(X)$. Note that we have an identity

$$R/(I_1 + I_2) \cong R/I_1 \otimes_R R/I_2.$$

This prompts a further definition:

Fiber products. Given two morphisms of affine schemes $Y_1 \to X$ and $Y_2 \to X$, we define the *fiber product* of Y_1 and Y_2 over X to be the affine scheme

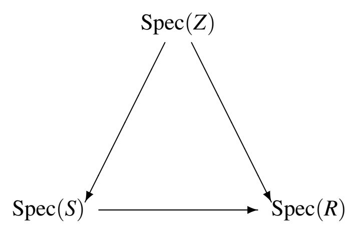
$$Y_1 \times_X Y_2 := \operatorname{Spec}(S_1 \otimes_R S_2)$$

where $R = \Gamma(\mathscr{O}_X, X)$ and $S_k = \Gamma(Y_k, \mathscr{O}_{Y_k})$ are the corresponding rings. By Exercise 2.8.25, the fiber product is in fact a product (in the categorical sense) on the category of affine schemes over X (see below for more on this category). Note that our previous definition of scheme-theoretic intersection is a special case, where the two morphisms are just the closed immersions $Y_k \subseteq X$. Put differently, the intersection of two closed subschemes $Y_k \subseteq X$ is just their fiber product:

$$Y_1 \cap Y_2 = Y_1 \times_X Y_2.$$

Relative schemes.

The formalism of schemes immediately allows one to relativize the notion of a scheme in the following sense. Let Z be a ring. An *affine scheme over* Z or *affine* Z*-scheme* is then simply an affine scheme $\operatorname{Spec}(R)$ given by a Z-algebra R, together with the canonical morphism $\operatorname{Spec}(R) \to \operatorname{Spec}(Z)$ (induced by the natural homomorphism $Z \to R$). A *morphism* of affine Z-schemes $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$, for some Z-algebra S, is then determined by a Z-algebra homomorphism $R \to S$. Note that this gives rise to a commutative diagram



(2.4)

of morphisms of affine schemes. Of course, if we take $Z = \mathbb{Z}$, we recover the category of all affine schemes (since any ring homomorphism is a $\mathbb{Z}$-algebra homomorphism). We say that an affine scheme $\operatorname{Spec}(R)$ is of *finite type* over Z, if the morphism $\operatorname{Spec}(R) \to \operatorname{Spec}(Z)$ is of finite type, that is to say, if R is of the form $Z[\xi]/I$ for some finite tuple of indeterminates ξ and some ideal I. Recall that we called such an algebra Z-affine if I is moreover finitely generated. This double usage of the term 'affine' will hopefully not cause too much confusion.

Fibers. A morphism of affine schemes $\phi\colon Y \to X$ can also be viewed as a family of affine schemes: for each point $x \in X$, the *fiber* $\phi^{-1}(x)$ admits the structure of an affine scheme as follows. If $R \to S$ is the corresponding ring homomorphism and $\mathfrak{p}$ the prime ideal corresponding to x, then

$$\phi^{-1}(x) \cong |\operatorname{Spec}(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p})|. \tag{2.5}$$

In view of this, we call $\operatorname{Spec}(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p})$ the (*scheme-theoretic*) *fiber* of ϕ at $\mathfrak{p}$. Reformulated in the terminology of fiber products, (2.5) says that

$$\phi^{-1}(x) = Y \times_X \operatorname{Spec}(\kappa(x)) \tag{2.6}$$

(recall that $\kappa(x)$ is the residue field of x); see Exercise 2.8.13 for the proofs.

Example 2.4.6. The family of all circles is encoded by the following morphism. Let Y be the *hypersurface* in $\mathbb{A}^5_K$ given by the equation

$$p := (\xi - u)^2 + (\zeta - v)^2 - w^2 = 0,$$

let $X := \mathbb{A}^3_K$, and let $\phi\colon Y \to X$ be induced by the projection $K^5 \to K^3\colon \mapsto (\xi, \zeta, u, v, w) \mapsto (u, v, w)$, that is to say, given by the natural K-algebra homomorphism

$$K[u, v, w] \to K[\xi, \zeta, u, v, w]/pK[\xi, \zeta, u, v, w].$$

If P is a closed point of X corresponding to a triple $(a, b, r) \in K^3$, that is to say, given by the maximal ideal $\mathfrak{m}_P = (u-a, v-b, w-r)K[u, v, w]$, then $\phi^{-1}(P)$ is isomorphic to the circle with center (a, b) and radius r.

Constructible subsets.

Recall that a subset Σ of a toplogical space X is called *constructible* if it is a finite Boolean combination of closed subsets. It follows that any constructible set is a finite union of locally closed subsets, where we call a subset *locally closed* if it is of the form $V \cap U$ with V closed and U open.

If R is any ring, then we can now easily define *affine n-space* over R as the affine scheme $\mathbb{A}^n_R := \operatorname{Spec} R[\xi]$ with $\xi = (\xi_1, \dots, \xi_n)$ indeterminates. We argued on page 6 that any quantifier free formula in the variables ξ, with parameters from R, defines a constructible subset of $\mathbb{A}^n_R$, and conversely.

Rational points.

Recapitulating, given an affine variety $V \subseteq K^n$, we have embedded it as a dense subset of the enhanced affine variety $\mathfrak{Irr}(V)$, which in turn is the underlying set of the affine scheme $X := \operatorname{Spec}(K[V])$. Since $K[V]$ is a K-algebra, X is in fact an affine K-scheme. We can recover V from X as the collection of K-rational points, defined as follows. Let $X := \operatorname{Spec}(R)$ be an affine Z-scheme and let S be a Z-algebra. An *S-rational point* of X over Z is by definition a morphism $\operatorname{Spec}(S) \to X$ of Z-schemes, that is to say, an element of $\operatorname{Mor}_Z(\operatorname{Spec}(S), X)$. We denote the set of all S-rational points of X over Z also by $X_Z(S)$, or $X(S)$, when Z is clear from the context. In other words, we actually view X as a functor, namely $\operatorname{Mor}_Z(\cdot, X)$, on the category of Z-algebras (see Exercise 2.8.26). By definition of a morphism, we have an equality

$$X_Z(S) = \operatorname{Mor}_Z(\operatorname{Spec}(S), X) = \operatorname{Hom}_Z(R, S)$$

where the latter set denotes the collection of Z-algebra homomorphisms $R \to S$. Returning to our example, where we take $S = Z = K$ and $R = K[V] = A/I$ with $I := \mathfrak{I}(V)$, a K-rational point $x \in X(K)$ then corresponds to a K-algebra homomorphism $R \to K$. Now, any K-algebra homomorphism is completely determined by the image of the variables, say $\xi_i \mapsto u_i$, since the image of a polynomial p is then simply $p(\mathbf{u})$ where $\mathbf{u} = (u_1, \dots, u_n)$. To be well-defined, we must have $p(\mathbf{u}) = 0$ for all $p \in I$, that is to say, $\mathbf{u} \in \mathrm{V}(I) = V$. Conversely, substitution by any element of V induces a K-algebra homomorphism $R \to K$ whence a K-rational point of X. We therefore showed that $V = X(K)$, as claimed.

In the sequel, we will sometimes confuse the underlying set $|\mathrm{Spec}(R)|$ of an affine scheme with the scheme itself, and denote it also by $\mathrm{Spec}(R)$.

2.5 Projective schemes

Most schemes we will encounter are affine, and in fact, often we work with the associated ring of global sections, or with their local rings (see §2.6). Nonetheless, we also will need projective schemes, which are a special case of a general scheme.

The category of schemes. Roughly speaking, a *scheme* X is a topological space $|X|$ together with a structure sheaf $\mathscr{O}_X$ of sections on $|X|$, with the property that there exists an open covering $\{X_i\}$ of X by affine schemes $\mathrm{Spec}(R_i)$ (for short, an *open affine covering*) such that $\Gamma(X_i, \mathscr{O}_X) = R_i$. Put differently, a scheme is obtained by *gluing* together affine schemes (for a more precise definition, consult any textbook in algebraic geometry, such as [24] or [37]). A *morphism* of schemes $f\colon Y \to X$ is a continuous map $|Y| \to |X|$ of underlying spaces which is 'locally a morphism of affine schemes' in the sense that there exist open affine coverings $\{Y_i\}$ and $\{X_i\}$ of Y and X respectively such that f maps each $|Y_i|$ inside $|X_i|$ thereby inducing for each i a morphism $Y_i \to X_i$ of affine schemes. If $U \subseteq X$ is any open, then we define a *sheaf of sections* $\mathscr{O}_U := \mathscr{O}_X|_U$ on U by restriction: for $W \subseteq U$ open, let $\Gamma(W, \mathscr{O}_U)$ be the ring of all sections $\mathscr{O}_X(W)$ on W. From the definitions (not all of which have been stated here), the next result follows almost immediately.

2.5.1 *An open $U \subseteq X$ in a scheme X together with the restriction $\mathscr{O}_U$ is again a scheme, and the embedding $U \subseteq X$ is a morphism of schemes, called an* open immersion.

For example, the 'punctured plane' $D \subseteq \mathbb{A}^2_K$ obtained by removing the origin, is a scheme. One can show that $\Gamma(D, \mathscr{O}_D) = K[\xi, \zeta]$, showing that D is not affine (see Exercise 2.8.5).

Here is an example of an actual gluing together of two affine schemes. Let $X_k := \mathbb{A}^1_K$ for $k = 1, 2$ be two copies of the affine line, and let $U \subseteq X_k$ be the open obtained by removing the origin. Note that U is again affine, namely equal to $\mathrm{Spec}(K[\xi, \xi^{-1}])$. Let X be the result of gluing together X_1 and X_2 along their common open subset U. The resulting scheme is called the *affine line with the origin doubled*. It requires some more properties of schemes to see that it is in fact not affine. A more clever choice of gluing the above data together leads to the projective line, as we will now explain.

Projective varieties. To discuss projective schemes, let us first introduce *projective n-space* over K as the set of equivalence classes $K^{n+1} \setminus \{0\}/\approx$, where $\mathbf{u} \approx \mathbf{v}$ if and only if there exists a non-zero $k \in K$ such that $\mathbf{u} = k\mathbf{v}$. An equivalence class of an $n+1$-tuple $\mathbf{u} = (u_0, \dots, u_n)$, that is to say, a point in projective n-space, will be denoted $\tilde{\mathbf{u}} = (u_0 : u_1 : \cdots : u_n)$. Alternatively, we may view projective n-space as the collection of lines in affine $n+1$-space going through the origin. The relevant algebraic counterpart, in fact the *homogeneous coordinate ring* of projective n-space, is the polynomial ring $\tilde{A} := K[\zeta_0,\ldots,\zeta_n]$. However, $\tilde{A}$ cannot be viewed as ring of sections, for given $p \in \tilde{A}$, we can no longer unambiguously evaluate it at a projective point $\tilde{\mathbf{u}}$. Nonetheless, if p is homogeneous, say of degree m, then $p(k\mathbf{u}) = k^m p(\mathbf{u})$, so that p vanishes on some $n+1$-tuple if and only if it vanishes on all $n+1$-tuples $\approx$-equivalent to it. Hence, for a given projective point $\tilde{\mathbf{u}}$, it makes sense to say that it is a *zero* of the homogeneous polynomial p, if $p(\mathbf{u}) = 0$.

We can now make projective n-space into a topological space by taking for closed subsets the sets of the form $\tilde{\mathrm{V}}(I)$, where $\tilde{\mathrm{V}}(I)$ is the collection of all projective points $\tilde{\mathbf{u}}$ that are a zero of each homogeneous polynomial in the ideal I. The analogue of Lemma 2.1.1 also holds, so that we get indeed a topology. Any closed subset of projective n-space is called a *projective variety*. Given such a closed subset V of projective n-space, we define its *ideal of definition* $\tilde{\mathfrak{I}}(V)$ as the ideal generated by all homogeneous forms $p \in \tilde{A}$ that vanish on V, and we call $\tilde{A}/\tilde{\mathfrak{I}}(V)$ the *homogeneous coordinate* ring of V, denoted $\widetilde{K[V]}$. Note that $\tilde{\mathfrak{I}}(V)$ is a homogeneous ideal (an ideal I is called *homogeneous*, if $p \in I$ implies that every homogeneous component of p lies in I too).

2.5.2 *The homogeneous coordinate ring $\widetilde{K[V]}$ of a projective variety V is a graded ring, and V has dimension equal to* $\dim(\widetilde{K[V]}) - 1$.

Recall that a ring S is called *graded*, if it admits a direct sum decomposition $S = \oplus_i S_i$ with each S_i an additive subgroup (called the i-th *graded part* of S) with the additional condition that $S_i \cdot S_j \subseteq S_{i+j}$ (meaning that if $a \in S_i$ and $b \in S_j$, then $ab \in S_{i+j}$). Here the index set of all i can in principal be any ordered, Abelian (semi-)group, but for our purposes, we will only work with $\mathbb{N}$-graded rings (with an occasional occurrence of a $\mathbb{Z}$-graded ring). In an $\mathbb{N}$-graded ring S, the zero-th part S_0 is always a subring of S, and $S_+ := \oplus_{i>0} S_i$ is an ideal such that $S/S_+ \cong S_0$. In case $S = \widetilde{K[V]}$, then $S_0 = K$, and S is generated over S_0 by finitely many linear forms. An $\mathbb{N}$-graded ring with these two properties is called a *standard graded (K-)algebra* (also called a *homogeneous* graded ring). In particular, S_+ is then a maximal ideal, called the *irrelevant maximal ideal*. The terminology comes from the fact that $\tilde{\mathrm{V}}(S_+) = \emptyset$. For example, if $S = \tilde{A}$ viewed as a (standard) graded K-algebra, then $(\zeta_0,\ldots,\zeta_n)S$ is its irrelevant maximal ideal.

Projective schemes. To define *enhanced projective varieties*, let $S = \oplus_i S_i$ be a standard graded K-algebra (for this construction to work, $K = S_0$ need not be algebraically closed—although we will not treat this, S_0 does not even need to be a field), and define $|\mathrm{Proj}(S)|$ to be the collection of all homogeneous prime ideals of S not containing S_+. In analogy with the affine case, we get a topological space by taking as closed subsets the subsets $\tilde{\mathrm{V}}(I)$ of all homogeneous prime ideals containing the ideal I, for various (homogeneous) ideals I. If V is a projective variety and $S := \widetilde{K[V]}$ its projective coordinate ring, then V embeds in $|\mathrm{Proj}(S)|$ by mapping a projective point $\tilde{\mathbf{u}}$ to its ideal of definition $\tilde{\mathfrak{I}}(\tilde{\mathbf{u}})$. The latter is indeed a (homogeneous) prime ideal, generated by the linear forms $u_i\zeta_j - u_j\zeta_i$ for all $i < j$. As before, (the image of) V is dense in $|\mathrm{Proj}(S)|$, so that any projective variety determines a unique enhanced projective variety. Conversely, every (enhanced) projective variety is a closed subset of some (enhanced) projective space, since any standard graded K-algebra is of the form $\tilde{A}/I$ for some homogeneous ideal I (and some appropriate choice of n). Unfortunately, unlike the affine case, non-isomorphic standard graded algebras might give rise to isomorphic (enhanced) projective varieties.

Finally, we define the *projective scheme* associated to S, denoted as $\operatorname{Proj} S$, as the scheme with underlying set $|\operatorname{Proj}(S)|$ and with structure sheaf $\mathscr{O}_X$, roughly speaking, 'induced by S'. Let me only explain this, and then still omitting most details, for projective n-space $\mathbb{P}^n_K := \operatorname{Proj}(\tilde{A})$. Once more we must turn our attention to open subsets. Similarly as in the affine case, given a homogeneous element $f \in \tilde{A}$ of degree m, we define the *basic open* $\tilde{\mathrm{D}}(f)$ as the complement of $\tilde{\mathrm{V}}(f\tilde{A})$. As before, these basic opens form a basis for the topology. Define $\Gamma(\tilde{\mathrm{D}}(f), \mathscr{O}_{\mathbb{P}^n_K})$ to be the *graded localization* $\tilde{A}_{(f)}$, defined as the collection of all fractions of the form $s := p/f^l$ with p homogeneous of degree ml. Put differently, $\tilde{A}_{(f)}$ is the degree zero part of the $\mathbb{Z}$-graded localization $\tilde{A}_f$. Since we are trying to construct a structure sheaf, it should consist of sections, and this is indeed the case. Namely, given $\tilde{\mathbf{u}}$ such that $f(\tilde{\mathbf{u}}) \neq 0$, the value $s(\mathbf{u})$ is independent from the choice of representative of the projective point $\tilde{\mathbf{u}}$, for s a section as above: if $\mathbf{v} \approx \mathbf{u}$, say $\mathbf{v} = k\mathbf{u}$, then $s(\mathbf{v}) = k^{ml}p(\mathbf{u})/(k^m f(\mathbf{u}))^l = s(\mathbf{u})$. Hence we can define $s(\tilde{\mathbf{u}}) := s(\mathbf{u})$, so that $\Gamma(\tilde{\mathrm{D}}(f), \mathscr{O}_{\mathbb{P}^n_K})$ consists indeed of sections on $\tilde{\mathrm{D}}(f)$.

2.5.3 *Each basic open $\tilde{\mathrm{D}}(f)$ with f a non-zero homogeneous form is homeomorphic to the enhanced affine variety $\left|\operatorname{Spec}(\tilde{A}_{(f)})\right|$.*

Indeed, define a map $\phi\colon \tilde{\mathrm{D}}(f) \to \left|\operatorname{Spec}(\tilde{A}_{(f)})\right|$ by sending a homogeneous prime ideal $\mathfrak{p}$ not containing f to the ideal $\phi(\mathfrak{p}) := \mathfrak{p}\tilde{A}_f \cap \tilde{A}_{(f)}$. One checks that $\phi(\mathfrak{p})$ is indeed a prime ideal. We leave it as an exercise (see 2.8.15) to show that this map is a homeomorphism. In particular, if we let f be one of the variables, say ζ_0 to make our notation easy, then one checks that $A \cong \tilde{A}_{(\zeta_0)}$ by sending ξ_i to ζ_i/ζ_0. Hence each $\tilde{\mathrm{D}}(\zeta_i)$ has affine n-space as underlying set. We can now make $\mathbb{P}^n_K$ into a scheme by gluing together the $n+1$ affine schemes $\operatorname{Spec}(\tilde{A}_{(\xi_i)}) \cong \mathbb{A}^n_K$ (again we must leave details to more specialized works). A similar construction applies to any standard graded K-algebra S, thus defining the scheme structure on $\operatorname{Proj}(S)$.

Proposition 2.5.4. *For any projective scheme $X := \operatorname{Proj}(S)$ and any homogeneous element $f \in S$, we have $\Gamma(\tilde{\mathrm{D}}(f), \mathscr{O}_X) = S_{(f)}$. Moreover, $\Gamma(X, \mathscr{O}_X) = K$.*

Proof. The last assertion is a special case of the first by taking $f = 1$, since then $S_{(1)} = S_0 = K$. The first assertion is basically how we defined the scheme structure on X. □

The last assertion shows that unlike in the affine case, the global sections on a scheme in general do not determine the scheme. In fact, two non-isomorphic graded K-algebras can give rise to isomorphic projective schemes, so that even the 'coordinate ring' S is not determined by the scheme (but also depends on the embedding of X as a closed subscheme of some $\mathbb{P}^n_K$). We will have more to say about projective schemes, and their relation to affine schemes, when we discuss singularities: see page 57.

2.6 Local theory

We have now associated to each geometric object (be it an affine variety, a projective variety or a scheme) an algebraic object, its coordinate ring, or more precisely, a collection of rings, the sheaf of sections on each open subset. If x is a closed point (that is to say, $\{x\}$ is closed) of an affine scheme $X := \operatorname{Spec}(R)$, then $\{x\}$ itself is an affine scheme by Lemma 2.4.5, with associated ring $\kappa(x) = R/\mathfrak{m}_x$, the residue field of x. Put pedantically, $x = \operatorname{Spec}(\kappa(x))$. Clearly, this point of view ignores the embedding $\{x\} \subset X$, and hence gives us no information on the nature of X in the neighborhood of x.

Local rings. We therefore introduce the *local ring* of X at an arbitrary point x, denoted $\mathscr{O}_{X,x}$, as the ring of germs of sections at x. This means that a typical element of $\mathscr{O}_{X,x}$ is a pair (U,σ) with U an open containing x and $\sigma \in \Gamma(U,\mathscr{O}_X)$, modulo the equivalence relation $(U,\sigma) \approx (U',\sigma')$ if and only if there exists a common open $x \in U'' \subseteq U \cap U'$ such that σ and σ' agree on U''.

Recall from page 28 that part of $\mathscr{O}_X$ being a sheaf is the fact that for each inclusion $U' \subseteq U$, we have a restriction homomorphism $\Gamma(U,\mathscr{O}_X) \to \Gamma(U',\mathscr{O}_X)$. Hence the $\Gamma(U,\mathscr{O}_X)$ together with the restriction homomorphisms form a direct system, and we can now state the previous definition more elegantly as

$$\mathscr{O}_{X,x} = \varinjlim_{x \in U} \Gamma(U,\mathscr{O}_X). \tag{2.7}$$

Unlike the ring of sections on an arbitrary open, the local ring at a point has a very concrete description:

Proposition 2.6.1. *If $X := \operatorname{Spec}(R)$ is an affine scheme, and x a point in X with corresponding prime ideal $\mathfrak{p}_x \subseteq R$, then $\mathscr{O}_{X,x} = R_{\mathfrak{p}_x}$. In particular, $\mathscr{O}_{X,x}$ is a local ring with residue field equal to the residue field $\kappa(x)$ of x.*

Proof. To simplify the proof, I will assume that R is moreover a domain (the general case is not much harder; see Exercise 2.8.30). In this case, each $\Gamma(U,\mathscr{O}_X)$ is a subring of the field of fractions $\operatorname{Frac}(R)$ and the direct limit (2.7) is simply a union. Since the $\mathrm{D}(f)$ are a basis of opens, it suffices to only consider the contributions in this union given by the U of the form $\mathrm{D}(f)$ with $f \notin \mathfrak{p}_x$. Hence, in view of (2.3), the local ring $\mathscr{O}_{X,x}$ is the union of all R_f with $f \notin \mathfrak{p}_x$, which is easily seen to be the localization $R_{\mathfrak{p}_x}$. The last assertion is immediate from the definition of the residue field (see Definition 2.4.1). □

The maximal ideal of $\mathscr{O}_{X,x}$, that is to say, $\mathfrak{p}_x\mathscr{O}_{X,x}$, will be denoted $\mathfrak{m}_{X,x}$.

Tangent spaces. The local ring of a point x captures quite a lot of information of the geometry of X near x. For instance, one might formally define the *tangent space* $T_{X,x}$ at x as the the dual of the $\kappa(x)$-vector space $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$. Without proof we state the following (for a proof see for instance [37, Lemma 6.3.10] or [24, I. Theorem 5.3]):

Theorem 2.6.2. *Let $X := \mathrm{Spec}(R)$ be an affine scheme of finite type over K and assume R is a domain (whence X is irreducible). Then there exists a non-empty open $U \subseteq X$ such that the tangent space $T_{X,x}$ has dimension equal to the dimension of X, for every closed point $x \in U$.*

Under the stated conditions, the local ring $\mathcal{O}_{X,x}$ of x has the same dimension as X (see Exercise 3.4.14). The dimension of this local ring, even if x is not assumed to be a closed point, is called the *local dimension of X at x*, or put less accurately, the dimension of X in the neighbourhood of x. Immediate from Nakayama's lemma, we get:

2.6.3 *The embedding dimension of the local ring $\mathcal{O}_{X,x}$ of a point x on an affine scheme X, that is to say, the local dimension of X at x, is equal to the dimension of its tangent space $T_{X,x}$.*

It follows that the dimension of the tangent space of an arbitrary point is always at least the local dimension at that point. Points were this is an equality are special enough to deserve a name (we shall return to this concept and study it in more detail in §4 below):

Definition 2.6.4 (Non-singular point). A point x on an affine scheme $X := \mathrm{Spec}(R)$ is called *non-singular* if its tangent space $T_{X,x}$ has the same dimension as the local dimension of X at the point. A point where the dimension inequality is strict is called *singular*.

Returning to a phrase quoted on page 24, we can now prove:

2.6.5 *An affine variety is non-singular at its generic points.*

Indeed, by 2.3.2, a generic point P of V corresponds to a minimal prime ideal $\mathfrak{g}$ of $B := K[V]$. Since B is reduced, $B_{\mathfrak{g}}$ is a reduced local ring of dimension zero, whence a field (see our discussion on page 44). Hence the maximal ideal of $\mathcal{O}_{V,P} = B_{\mathfrak{g}}$ is zero, whence $T_{V,P} = 0$, and the embedding dimension of $B_{\mathfrak{g}}$ is also zero. More generally, this proves that if B is a reduced ring, then the generic points of $\mathrm{Spec}(B)$ are non-singular. This also implies that any K-generic point of V_L, where V_L denotes the base change of V over an algebraically closed overfield L of K (see page 25), is non-singular, but the proof requires some deeper results beyond the scope of these notes.

2.7 CTU

Continuous sections.

We can now give a better definition of a section on an open of an affine scheme $X := \mathrm{Spec}(R)$. Instead of letting a section take values in $Q(|X|)$, the disjoint union of all residue fields, we should take for target the disjoint union $\mathrm{Loc}(X)$ of all local rings $\mathscr{O}_{X,x}$ with $x \in X$: a (*generalized*) *section* on an open $U \subseteq X$ is then a map $\sigma\colon U \to \mathrm{Loc}(X)$ such that $\sigma(x) \in \mathscr{O}_{X,x}$ for all $x \in X$. With this new notion we can now formally define $\Gamma(U, \mathscr{O}_X)$ for an arbitrary open U as the set of all continuous sections on U, where we call a section σ *continuous* if it is locally represented by a fraction, that is to say, if for each $x \in U$, we can find an open $U' \subseteq U$ containing x, and elements $a, f \in R$ such that, for all $y \in U'$, in $\mathscr{O}_{X,y}$, the element f is a unit and $\sigma(y) = a/f$.

Stalks.

One can extend the concept of a local ring to arbitrary schemes. This is just a special case of a *stalk* $\mathscr{A}_x$ of a sheaf $\mathscr{A}$ at a point x on a topological space X, defined similarly as

$$\mathscr{A}_x := \varinjlim_{x \in U} \Gamma(U, \mathscr{A}).$$

However, even if $\mathscr{A}$ is a sheaf of rings, $\mathscr{A}_x$ need not be a local ring, but it is so if X is a scheme and $\mathscr{A} = \mathscr{O}_X$ its structure sheaf. An argument similar to the one in the proof of Proposition 2.6.1 yields:

Proposition 2.7.1. *Let $X := \mathrm{Proj}(S)$ be a projective scheme and let x be a point of X corresponding to the homogeneous prime ideal $\mathfrak{p}_x$. The local ring $\mathscr{O}_{X,x}$ is equal to the degree zero part $S_{(\mathfrak{p}_x)}$ of the localization $S_{\mathfrak{p}_x}$.*

2.8 Exercises

Ex 2.8.1

Verify Lemma 2.1.1. Show that the same properties hold for the operation $\mathrm{V}(\cdot)$ on any affine scheme, and for the operation $\tilde{\mathrm{V}}(\cdot)$ on any projective scheme.

Ex 2.8.2

Show that if $V_1 \cup \cdots \cup V_s = V'_1 \cup \cdots \cup V'_t$ are two minimal irreducible decompositions of a Noetherian space V, then $s = t$, and after renumbering, $V_i = V'_i$ for all i.
Show that for a closed subset $V \subseteq K^n$, its ideal of definition $\mathfrak{I}(V)$ is prime if and only if V is irreducible.

Ex 2.8.3

Show that the Zariski topology on K^n is compact Hausdorff. More generally, any affine variety is compact Hausdorff. Hint: you could use 2.3.6.

Ex 2.8.4

Let $V \subseteq K^n$ be a variety and let $I := \mathfrak{I}(V)$ be its ideal of definition. Every $p \in A$ induces a polynomial map $K^n \to K$ by the rule $\mathbf{u} \mapsto p(\mathbf{u})$. Show that the collection of restrictions $p|_V$ of polynomial maps on V is in one-one correspondence with the coordinate ring $K[V]$ of V.

Ex 2.8.5

Show that the punctured plane $K^2 \setminus \{O\}$ *(where* O *denotes the origin), is not an affine variety, for if it were, then its ideal of definition would be zero, contradiction. In fact, by the discussion on page 33 there is a scheme* D *with underlying set this punctured plane. It can be realized as the union of the two affine opens* $\mathrm{D}(\xi)$ *and* $\mathrm{D}(\zeta)$ *of* $\mathbb{A}^2_K$*, where* $A := K[\xi, \zeta]$ *is the coordinate ring of* $\mathbb{A}^2_K$*. Show that* $\Gamma(D, \mathscr{O}_D) = A_\xi \cap A_\zeta = A$*. Conclude that* D *is not affine.*

Ex 2.8.6

Prove 2.2.3 in detail. In particular, given a reduced K*-affine ring* B*, construct an affine variety whose coordinate ring is* B*. Prove that the correspondence in 2.2.3 induces an anti-equivalence of categories. In particular, show that if two affine varieties are isomorphic, then so are their coordinate rings. Using this equivalence, show that a parabola is isomorphic to a straight line.*

Ex 2.8.7

Show that if X *is Noetherian, then* $\mathfrak{Irr}(X)$ *is a toplogical space in which every irreducible closed subset has a generic point; if* X *is moreover Hausdorff, then every irreducible closed subset has a unique generic point. In particular, in the latter case, the map* $X \to \mathfrak{Irr}(X)$ *is an embedding, and (the image of)* X *is dense in* $\mathfrak{Irr}(X)$*.*

Ex 2.8.8

Let $K \subseteq L$ *be an extension of algebraically closed fields. Show that a point* $\mathbf{u} \in L^n$ *is generic over* K *if and only if* $K(\mathbf{u})$ *has transcendence degree* n *over* K*. This shows that generic points are plentiful. Explain now the enigmatic adverb 'probably' used in Example 2.3.4.*

Ex 2.8.9

Show that if R *is Noetherian, then the associated enhanced affine variety* $|\mathrm{Spec}(R)|$ *is also Noetherian. It is irreducible if and only if* R *has a unique minimal prime ideal (and if* R *is moreover reduced, this is then equivalent to* R *being a domain). The Krull dimension of* R *is equal to the dimension of* $|\mathrm{Spec}(R)|$.

Can you give an example where $|\mathrm{Spec}(R)|$ *is Noetherian, yet* R *is not Noetherian?*

Ex 2.8.10

Show that if $K \subseteq L$ *is an extension of algebraically closed fields and* $V \subseteq K^n$ *is an affine variety over* K*, then its closure in* L^n *is an affine variety over* L *with coordinate ring* $K[V] \otimes_K L$*.*

Ex 2.8.11

Let R *be a domain and* $X := \mathrm{Spec}(R)$ *the associated affine scheme. Let* η *be the (unique) generic point of* X*. Show that the residue field* $\kappa(\eta)$*, the local ring* $\mathscr{O}_{X,\eta}$ *at* η*, and the field of fractions* $\mathrm{Frac}(R)$ *are all equal. This field is often called the* function field *of the scheme.*

Ex 2.8.12

Calculate all residue fields of $\mathrm{Spec}(\mathbb{Z})$*. What are the residue fields of* $\mathrm{Spec}(\mathbb{R}[\xi])$ *for* ξ *a single variable?*

Ex 2.8.13

Prove that (2.5) is a homeomorphism. Use this to prove (2.6).

Ex 2.8.14

Show that a finite morphism of affine schemes has finite fibers.

Ex 2.8.15

Prove 2.3.6, 2.4.2 and 2.5.3.

Ex 2.8.16

Work out Example 2.3.7 in detail.

Ex 2.8.17

Prove Lemma 2.4.5.

Ex 2.8.18

Show that an ideal I in a graded ring S is homogeneous if and only if it is generated by homogeneous elements. For an arbitrary ideal I, let $\tilde{I}$ be the ideal generated by all homogeneous components of all elements in I. Show that $\tilde{\mathrm{V}}(I) = \tilde{\mathrm{V}}(\tilde{I})$.

Ex 2.8.19

Prove 2.5.2 (where you might need some results from Chapter 3 to prove the dimension equality).

Ex 2.8.20

Let V be a projective variety over K, with homogeneous coordinate ring $S := \widetilde{K[V]}$. Show that $\mathfrak{Irr}(V) = |\mathrm{Proj}(S)|$.

Ex 2.8.21

Let C be the affine scheme determined by the ring

$$R := K[\xi,\zeta]/(\xi^2 - \zeta^3)K[\xi,\zeta],$$

a so-called cusp *(see page 54). Let x be the origin, that is to say, the (closed) point determined by the maximal ideal $(\xi,\zeta)R$. Show that the tangent space $T_{C,x}$ has dimension two, whereas C itself has dimension one (showing that x is* singular*). What about the point y given by the maximal ideal $(\xi-1,\zeta-1)R$?*

Additional exercises

Ex 2.8.22

Show that the geometric form of the Noether normalization as stated in Theorem 2.2.5 is indeed equivalent to the algebraic form formulated in the proof.

Ex 2.8.23

We want to prove the assertion in the proof of Theorem 2.2.5 that states that after a change of coordinates, a polynomial becomes monic in one of the variables. Let $p \in A$ be a non-constant polynomial of degree s, and let $p_s(\xi)$ be its homogeneous part of degree s. Put $p' := p(\xi', 1)$ where $\xi' := (\xi_1, \ldots, \xi_{n-1})$. Show that if K is infinite, then there exists $\mathbf{u}' := (u_1, \ldots, u_{n-1}) \in K^{n-1}$ such that $p'(\mathbf{u}') \neq 0$. This is clear if $n-1 = 1$ since a non-zero polynomial has only finitely many roots. Reason by induction to show this also for more variables. Now define a change of coordinates $\xi_i \mapsto \xi_i - u_i\xi_n$ and show that the image of p under this map is monic in ξ_n.
If K is arbitrary, show that the change of variables $\xi_i \mapsto \xi_i - \xi_n^{e^i}$ for $i < n$ also transforms p into a monic polynomial if $e > s$ (examine the transforms of each monomial in p).

Ex 2.8.24

Prove the following generalization of Lemma 2.2.7: if $R \subseteq S$ is a finite (or integral) extension of domains, then R is a field if and only if S is.

Ex 2.8.25

The product *of two objects M and N in a category $\mathfrak{C}$ is the (necessarily unique) object $M \times N$ together with two morphisms $M \times N \to M$ and $M \times N \to N$ (called* projections*), satisfying the following* universal property*: if $K \to M$ and $K \to N$ are morphisms, then there exists a unique morphism $K \to M \times N$ which composed with the two projections yield the original morphisms $K \to M$ and $K \to N$. Show that in the category of affine schemes over a fixed affine scheme X, the fiber product $\cdot \times_X \cdot$ is a product in the above sense.*

Ex 2.8.26

Show that given an (affine) Z-scheme X, the rule assigning to a Z-algebra S the set $X_Z(S)$ of S-rational points of X over Z, constitutes a functor on the category of Z-algebras.

Ex 2.8.27

Show that the definition of $\Gamma(U, \mathcal{O}_X)$ as all continuous sections given on page 37 makes $\mathcal{O}_X$ into a sheaf.

Ex 2.8.28

Prove Proposition 2.7.1.

Ex 2.8.29

Let $S := K[\zeta]/\zeta^2 K[\zeta]$ *be the ring of* dual numbers *over* K *(where* ζ *is a single variable). Let* X *be an affine variety of finite type over* K*. Show that to give an* S*-rational point of* X *over* K *is the same as to give a* K*-rational point* x *of* X *together with an element of the tangent space* $T_{X,x}$*.*

Ex 2.8.30

Show, without relying on Proposition 2.6.1, that if Y *is a closed subscheme of* $X := \operatorname{Spec}(R)$ *with corresponding ideal* $I \subseteq R$*, then* $\mathcal{O}_{Y,y} = \mathcal{O}_{X,y}/I\mathcal{O}_{X,y}$ *for every* $y \in Y$*. Use this then to derive the non-domain case in the proposition.*

Chapter 3
Dimension theory

3.1 Krull dimension

Height. The *height* of a prime ideal $\mathfrak{p}$ in a ring R is by definition the maximal length of a proper chain of prime ideals inside $\mathfrak{p}$, and is often denoted $\mathrm{ht}(\mathfrak{p})$. Hence a prime ideal is minimal if and only if its height is zero. The supremum of the heights of all prime ideals in R is called the (*Krull*) *dimension* of R and is denoted $\dim(R)$. More generally, the *height* $\mathrm{ht}(I)$ of an ideal I is the minimum of the heights of all prime ideals containing I. The following inequality is almost immediate from the definitions (see Exercise 3.4.1).

3.1.1 *For every prime ideal $\mathfrak{p} \subseteq R$, we have an inequality*

$$\dim(R/\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}) \leq \dim(R).$$

Almost immediate from the definitions (see Exercise 2.8.9), we get the following generalization of Corollary 2.2.4:

3.1.2 *The Krull dimension of a ring R is equal to the dimension of the associated enhanced affine variety $|\mathrm{Spec}(R)|$.*

Dimension, although seemingly a global invariant, has a strong local character:

3.1.3 *The height of a prime ideal $\mathfrak{p} \subseteq R$ is equal to the dimension of $R_\mathfrak{p}$. In particular, the dimension of R is equal to the supremum of the dimensions of its localizations $R_\mathfrak{m}$ at maximal ideals $\mathfrak{m}$. Similarly, the dimension of an affine variety $X := \mathrm{Spec}(R)$ is equal to the supremum of the dimensions of its local rings $\mathscr{O}_{X,x}$ at (closed) points $x \in X$.*

The first assertion is proven in Exercise 3.4.5, and the second is an immediate consequence of this (since maximal ideals have the largest height). The last assertion then follows from Proposition 2.6.1.

Artinian rings. Recall that a ring is called respectively *Noetherian* or *Artinian* if the collection of ideals satisfies the ascending or the descending chain condition respectively. Without proof we state the following structure theorem for Artinian rings (for a proof see for instance [41, Theorem 3.2] or [7, Theorems 8.5 and 8.7]):

3.1.4 *Any Artinian ring R is Noetherian, and has only finitely many prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. Each $\mathfrak{p}_i$ is moreover maximal, so that R has dimension zero, and $R = R_{\mathfrak{p}_1} \oplus \cdots \oplus R_{\mathfrak{p}_s}$.*

In fact, a ring R is Artinian if and only if it has finite *length* $l = \ell(R)$, meaning that any proper chain of ideals has length at most l, and there is a chain with this length. It follows that any finitely generated R-module M also has finite length, denoted $\ell(M)$, and defined as the maximal length of a proper chain of submodules. An Artinian ring of length one is a field. Length is a generalization of vector space dimension; for instance, you will be asked to prove the following characterization of length in Exercise 3.4.3:

3.1.5 *If R is finitely generated (as a module) over an algebraically closed field K, then $\ell(R)$ is equal to the vector space dimension of R over K.*

3.2 Hilbert series

Although we are interested in the study of local rings, it turns out that graded rings play an important role in dimension theory. The connection between the two is provided by the graded ring $\mathrm{Gr}(R)$ associated to a local ring R (see page 45). So we first study the graded case.

Let R be an Artinian local ring and let S be a standard graded R-algebra. Recall that this means that $S = \oplus_{i \in \mathbb{N}} S_i$ is $\mathbb{N}$-graded, the degree zero part S_0 is equal to R, and S is generated as an R-algebra by finitely many linear forms (=elements in S_1). Let M be a finitely generated $\mathbb{N}$-graded S-module, meaning that $M = \oplus_{i \in \mathbb{N}} M_i$ and $S_i M_j \subseteq M_{i+j}$ for all i, j.

3.2.1 *Every M_n is a finitely generated R-module, whence in particular has finite length.*

Indeed, we may choose homogenous generators $\mu_1, \dots, \mu_s$ of M as an S-module. If k_i is the degree of μ_i, then $M_n = S_{n-k_1}\mu_1 + \cdots + S_{n-k_s}\mu_s$ (with the understanding that $S_j = 0$ for $j < 0$). Furthermore, if $a_1, \dots, a_s$ are the linear forms generating S as an R-algebra, then S_n is generated as an R-module by all monomials of degree n in the a_i. Therefore, M_n is finitely generated over R, and therefore has finite length.

Hilbert series. In view of 3.2.1, we can now define the *Hilbert series* of a finitely generated S-module M, with S a standard graded algebra over an Artinian local ring R, as the formal power series

$$\mathrm{Hilb}_M(t) := \sum_{n \geq 0} \ell(M_n) t^n. \tag{3.1}$$

As rings will be our primary objective in these notes, rather than modules, we will be mainly interested in the properties of $\mathrm{Hilb}_S(t)$. However, it is more convenient to work in the larger module setup for inductive proofs to go through. The key result on Hilbert series is:

Theorem 3.2.2. *Let S be a standard graded algebra over an Artinian local ring R. The Hilbert series of any finitely generated S-module M is rational. In fact, for some $d = d(M) \in \mathbb{N}$, the power series $(1-t)^d \cdot \mathrm{Hilb}_M(t)$ is a polynomial with integer coefficients.*

Proof. We will prove the last assertion by induction on the minimal number r of linear R-algebra generators of S. If $r = 0$, then $S = R$, so that M is a finitely generated module over an Artinian ring, whence has finite length. It follows that $M_n = 0$ for $n \gg 0$ and we are done in this case. So assume $r > 0$ and let x be one of the linear forms generating S as an R-algebra. Multiplication by x induces maps $M_n \to M_{n+1}$ for all n. Let K_n and L_{n+1} be the respective kernel and cokernel of these maps (with $L_0 := M_0$). Define two new graded S-modules $K := \oplus_n K_n$ and $L := \oplus_n L_n$. It follows that $K \subseteq M$ and $M/xM \cong L$, proving that both modules are finitely generated over S. By construction, $xK = xL = 0$, so that both K and L are actually modules over S/xS, and hence we may apply our induction hypothesis to them. Since we have an exact sequence (see page 65 for the notion of an exact sequence)

$$0 \to K_n \to M_n \xrightarrow{x} M_{n+1} \to L_{n+1} \to 0$$

we get $\ell(K_n) - \ell(M_n) + \ell(M_{n+1}) - \ell(L_{n+1}) = 0$ by Exercise 3.4.2. Multiplying this equality with t^{n+1} and adding all terms together, we get an identity

$$t\,\mathrm{Hilb}_K(t) - t\,\mathrm{Hilb}_M(t) + \mathrm{Hilb}_M(t) - \mathrm{Hilb}_L(t) = 0.$$

Using the induction hypothesis for K and L then yields the desired result. □

Corollary 3.2.3. *For every finitely generated graded module M over a standard graded algebra over an Artinian local ring, there exists a polynomial $P_M(t) \in \mathbb{Z}[t]$, such that $\ell(M_n) = P_M(n)$ for all n sufficiently large.*

Proof. By Theorem 3.2.2 we can write $\mathrm{Hilb}_M(t) = q(t)/(1-t)^d$ for some polynomial $q(t) \in \mathbb{Z}[t]$. Using the Taylor expansion of $(1-t)^{-d}$ and then comparing coefficients at both sides, the result follows readily (see Exercise 3.4.8). Note that we have equality for all $n > \deg(q)$. □

Associated graded ring. For a given Noetherian ring $(R, \mathfrak{m})$, define its *associated graded ring* as

$$\mathrm{Gr}(R) := \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

Note that this is a standard graded algebra over the residue field $R/\mathfrak{m}$ of R (as always $\mathfrak{m}^0$ stands for the unit ideal). Applying Corollary 3.2.3 to $M = S = \mathrm{Gr}(R)$ we can find a polynomial $P_R(t)$ such that

$$P_R(n) = \ell(\mathfrak{m}^n/\mathfrak{m}^{n+1}) \tag{3.2}$$

for all $n \gg 0$. For various reasons, one often works with the 'iterate' of this function:

Hilbert-Samuel polynomial. We define the *Hilbert-Samuel function* of R as the function $n \mapsto \ell(R/\mathfrak{m}^{n+1})$. By induction, one easily shows that

$$\ell(R/\mathfrak{m}^{n+1}) = \sum_{k=0}^{n} \ell(\mathfrak{m}^k/\mathfrak{m}^{k+1}). \tag{3.3}$$

It follows from (3.2) that there then exists a polynomial $\chi_R(t)$ with integer coefficients, called the *Hilbert-Samuel polynomial*, such that

$$\ell(R/\mathfrak{m}^{n+1}) = \chi_R(n) \tag{3.4}$$

for all $n \gg 0$.

3.3 Local dimension theory

In this section, $(R, \mathfrak{m})$ denotes a local ring, which is most of the time also Noetherian. The Krull dimension of R will be denoted $\dim(R)$. We introduce two more variants, and show that they agree on Noetherian local rings.

Definition 3.3.1 (Geometric dimension). We define the *geometric dimension* of R, denoted $\operatorname{geodim}(R)$, as the least number of elements generating an $\mathfrak{m}$-primary ideal (see 1.4.9 for the definition of $\mathfrak{m}$-primary ideal). We let $\operatorname{Hilbdim}(R)$ denote the degree of the Hilbert-Samuel polynomial $\chi_R(t)$ of R given by (3.4).

As $\dim(R)$ equals the dimension of the topological space $V := |\operatorname{Spec}(R)|$, it is essentially a topological invariant. On the other hand, $\operatorname{geodim}(R)$ is the least number of hypersurfaces[1] $H_1, \dots, H_d \subseteq V$ such that $H_1 \cap \dots \cap H_d$ is a singleton (necessarily equal to the closed point x corresponding to the maximal ideal $\mathfrak{m}$), and hence is a geometric invariant. Note that the definition of geometric dimension makes sense for any local ring R (unlike the definition of Hilbert dimension which assumes the rationality of the Hilbert series), and that it is finite if and only if R has finite embedding dimension. Finally, $\operatorname{Hilbdim}(R)$ is by definition a combinatorial invariant. It follows that both geometric dimension and Hilbert dimension are finite for Noetherian local rings, but this is less obvious for Krull dimension. Nonetheless, all three seemingly unrelated invariants are always equal for Noetherian local rings (whence in particular Krull dimension is always finite):

Theorem 3.3.2. *If R is a Noetherian local ring, then*

[1] In these notes, a *hypersurface* in an affine variety V is any closed subset of the form $\mathrm{V}(I)$ with I a proper principal ideal (this does not mean that its ideal of definition is principal!) Be aware that some authors have a far more restrictive usage for this term.

$$\dim(R) = \operatorname{geodim}(R) = \operatorname{Hilbdim}(R).$$

Proof. It is not hard to verify this equality whenever one of them is zero: R has Krull dimension zero if and only if its maximal ideal is nilpotent (in other words, (0) is $\mathfrak{m}$-primary) if and only if its Hilbert-Samuel function is constant.

So we may assume that all three invariants are non-zero. First we show by induction on δ that

$$t := \operatorname{Hilbdim}(R) \leq \delta := \operatorname{geodim}(R). \tag{3.5}$$

Let $I := (a_1, \ldots, a_\delta)R$ be an $\mathfrak{m}$-primary ideal, and put $S := R/a_1R$. It is not hard so see that then necessarily $\operatorname{geodim}(S) = \delta - 1$, so that by induction, $\operatorname{Hilbdim}(S) \leq \delta - 1$. We have, for n sufficiently large,

$$\begin{aligned}\chi_S(n) = \ell(S/\mathfrak{m}^{n+1}S) = \ell(R/a_1R + \mathfrak{m}^{n+1})& \\ = \ell(R/\mathfrak{m}^{n+1}) - \ell(R/(\mathfrak{m}^{n+1} : a_1))& \\ \geq \ell(R/\mathfrak{m}^{n+1}) - \ell(R/\mathfrak{m}^{n}) = \chi_R(n) - \chi_R(n-1)&\end{aligned}$$

(where we used (5.9) below in the second line). Note that $\chi_R(n) - \chi_R(n-1)$ has degree $t-1$ (verify this!), and hence $\chi_S(n)$, a polynomial dominating the latter difference, must have degree at least $t-1$. Putting everything together, we therefore get $t-1 \leq \deg(\chi_S) \leq \delta - 1$, as we wanted to show.

For the remainder of the proof, we induct on the Krull dimension $d := \dim(R)$, and so we assume that the theorem is proven for rings of smaller Krull dimension. Let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{m}$ be a chain of prime ideals in R of maximal length. Choose x outside all minimal prime ideals but inside $\mathfrak{p}_1$. By prime avoidance (see [7, Proposition 1.11] or the more general version [18, Lemma 3.3]), such an element must exist. Put $S := R/xR$. Since $\mathfrak{p}_iS$ are distinct prime ideals, for $i > 0$, we get $\dim(S) = d-1$. Hence by induction, $\operatorname{geodim}(S) = d-1$, so that there exists an $\mathfrak{m}S$-primary ideal $I \subseteq S$ generated by $d-1$ elements. Let $J := I \cap R$. Any lifting of the $d-1$ generators of I in R together with x therefore generate J. Moreover, J is clearly $\mathfrak{m}$-primary, so that we showed $\operatorname{geodim}(R) \leq d - 1 + 1 = d$.

Let $\bar{R} := R/\mathfrak{p}_0$ and $\bar{S} := S/\mathfrak{p}_0S$. Tensoring the exact sequence

$$0 \to \bar{R} \xrightarrow{x} \bar{R} \to \bar{S} \to 0$$

with $R/\mathfrak{m}^{n+1}$, we get an exact sequence

$$0 \to H_n \to \bar{R}/\mathfrak{m}^{n+1}\bar{R} \xrightarrow{x} \bar{R}/\mathfrak{m}^{n+1}\bar{R} \to \bar{S}/\mathfrak{m}^{n+1}\bar{S} \to 0.$$

Hence, the two outer modules have the same length, so that $\chi_{\bar{S}}(n) = \ell(H_n)$ for sufficiently large n. On the other hand, using 5.6.15, we have an exact sequence

$$0 \to H_n \to \bar{R}/\mathfrak{m}^{n+1}\bar{R} \to \bar{R}/(\mathfrak{m}^{n+1}\bar{R} : x) \to 0$$

from which it follows that $\chi_{\bar{S}}(n) = \chi_{\bar{R}}(n) - \varphi(n)$, where $\varphi(n)$ denotes the length of the last module in the previous exact sequence (showing incidentally that $\varphi(n)$ too is a polynomial for $n \gg 0$). To estimate $\varphi(n)$, we use the Artin-Rees Lemma (see [41, Theorem 8.5] or [7, Proposition 10.9]).[2] By that theorem, there exists some c such that

$$\mathfrak{m}^{n+1}\bar{R} \cap x\bar{R} \subseteq \mathfrak{m}^{n+1-c}x\bar{R}$$

for all $n > c$. Hence if $s \in (\mathfrak{m}^{n+1}\bar{R} : x)$, that is to say, if $sx \in \mathfrak{m}^{n+1}\bar{R}$, then $sx \in \mathfrak{m}^{n+1-c}x\bar{R}$. Since $\bar{R}$ is a domain, this yields $s \in \mathfrak{m}^{n+1-c}\bar{R}$, and hence we have inclusions $\mathfrak{m}^{n+1}\bar{R} \subseteq (\mathfrak{m}^{n+1}\bar{R} : x) \subseteq \mathfrak{m}^{n+1-c}\bar{R}$ for all $n > c$. Therefore, for $n \gg 0$, we get inequalities

$$\chi_{\bar{R}}(n-c) \leq \varphi(n) \leq \chi_{\bar{R}}(n).$$

This shows that the (polynomial representing) φ has the same leading term as $\chi_{\bar{R}}$, and hence their difference, which is $\chi_{\bar{S}}$, has degree strictly less. Clearly, $\chi_{\bar{R}}(n) \leq \chi_R(n)$ and hence $\text{Hilbdim}(\bar{R}) \leq \text{Hilbdim}(R)$. Since $\bar{S}$ has dimension $d-1$ by choice of x, induction yields $\text{Hilbdim}(\bar{S}) = d-1$. Putting everything together, we get $\text{Hilbdim}(R) \geq d$. In summary, we proved the inequalities

$$\text{geodim}(R) \leq d \leq \text{Hilbdim}(R)$$

and hence we are done by (3.5). □

From this important theorem, various properties of dimension can now be deduced. We start with a loose end: the dimension of affine n-space (as stated in Theorem 2.2.1), or equivalently, the dimension of a polynomial ring.

Corollary 3.3.3. *If K is a field and A is either the polynomial ring or the power series ring over K in n variables ξ, then* $\dim(A) = n$.

Proof. The chain of prime ideals

$$(0) \subsetneq \xi_1 A \subsetneq (\xi_1, \xi_2)A \subsetneq \cdots \subsetneq \mathfrak{m} := (\xi_1, \ldots, \xi_n)A$$

shows that $\mathfrak{m}$ has height at least n (and, in fact, equal to n). Hence $\dim(A)$ and $\dim(A_\mathfrak{m})$ are at least n. In the power series case (so that A is local), $\mathfrak{m}$ witnesses the estimate $\text{geodim}(A) \leq n$. Hence we are done in the power series case by Theorem 3.3.2.

Let me only prove the polynomial case when K is algebraically closed (the general case is treated in Exercise 3.4.6). By Theorem 2.2.2, any maximal ideal is of the form $\mathfrak{m}_\mathbf{u}$ for some $\mathbf{u} \in K^n$. Hence $A_{\mathfrak{m}_\mathbf{u}} \cong A_\mathfrak{m}$ by a linear change of coordinates. Therefore, it suffices in view of 3.1.3 to show that $A_\mathfrak{m}$ has dimension n. However, again $\mathfrak{m}A_\mathfrak{m}$ witnesses that $\text{geodim}(A_\mathfrak{m}) \leq n$, and we are done once more by Theorem 3.3.2. □

The next application is another famous theorem due to Krull:

Theorem 3.3.4 (Hauptidealensatz/Principal Ideal Theorem). *Any proper ideal in a Noetherian ring generated by h elements has height at most h.*

[2] Unfortunately, the weak variant of Artin-Rees that we will prove in Theorem 11.2.1 below, is not sufficiently strong for the present argument to work.

Proof. Let $I \subseteq B$ be an ideal generated by h elements, let $\mathfrak{p}$ be a minimal prime of I, and put $R := B_{\mathfrak{p}}$. Since IR is then $\mathfrak{p}R$-primary, $\operatorname{geodim}(R) \leq h$. Hence $\mathfrak{p}$ has height at most h by Theorem 3.3.2 and 3.1.3. Since this holds for all minimal primes of I, the height of I is at most h. □

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. By Theorem 3.3.2, there exists a d-tuple $\mathbf{x}$ generating an $\mathfrak{m}$-primary ideal. We give a name to such a tuple:

Definition 3.3.5 (System of parameters). Any tuple of length equal to the dimension of R and generating an $\mathfrak{m}$-primary ideal will be called a *system of parameters* of R (sometimes abbreviated as *s.o.p*); the ideal it generates is then called a *parameter ideal*.

In other words, a parameter ideal is an $\mathfrak{m}$-primary ideal requiring the least possible number of generators, namely $d = \dim(R)$. The next result will enable us to construct systems of parameters. To this end, we define the *dimension* of an ideal $I \subseteq B$ as the dimension of its residue ring B/I. In particular, any d-dimensional prime ideal in a d-dimensional Noetherian local ring is a minimal prime ideal, whence there are only finitely many such ideals.

Corollary 3.3.6. *If R is a d-dimensional Noetherian local ring and x a non-unit in R, then $d-1 \leq R/xR \leq d$. The lower bound is attained if and only if x lies outside all d-dimensional prime ideals of R.*

Proof. The second inequality is obvious (from the point of view of Krull dimension). Towards a contradiction, suppose $S := R/xR$ has dimension strictly less than $d-1$. By Theorem 3.3.2 there exists a system of parameters $(x_1, \dots, x_e)$ in S with $e < d-1$. However, any liftings of the x_i to R together with x then generate an $\mathfrak{m}$-primary ideal, contradicting Theorem 3.3.2. It is now not hard to see that x lies in a d-dimensional prime ideal if and only if S admits a chain of prime ideals of length d, from which the last assertion follows. □

If R has dimension d, then element outside any d-dimensional prime is called a *parameter*. Since there are only finitely many d-dimensional prime ideals, parameters exist as soon as $d > 0$. We can now reformulate (see Exercise 3.4.10): $(x_1, \dots, x_d)$ is a system of parameters if and only if each x_i is a parameter in $R/(x_1, \dots, x_{i-1})R$.

Finite extensions. Recall that a homomorphism $R \to S$ is called *finite* if S is finitely generated as an R-module. Similarly, a morphism of affine schemes $Y \to X$ is called *finite* if the induced homomorphism on the coordinate rings is finite. Any surjective ring homomorphism $R \to R/I$ is finite.

3.3.7 *A finite morphism $Y \to X$ of affine schemes is surjective if the corresponding homomorphism of coordinate rings is injective.*

Indeed, assume $R \to S$ is a finite and injective homomorphism, and let $\mathfrak{p}$ be a prime ideal of R. Let $\mathfrak{n}$ be a maximal ideal in $S_{\mathfrak{p}} := R_{\mathfrak{p}} \otimes_R S$, and put $\mathfrak{m} := \mathfrak{n} \cap R_{\mathfrak{p}}$.

Since $R_\mathfrak{p}/\mathfrak{m} \subseteq S_\mathfrak{p}/\mathfrak{n}$ is again finite, and the latter ring is a field, so is the former by Lemma 2.2.7. Hence $\mathfrak{m}$ is a maximal ideal, necessarily equal to $\mathfrak{p}R_\mathfrak{p}$. If we put $\mathfrak{q} := \mathfrak{n} \cap S$, then an easy calculation shows $\mathfrak{p} = \mathfrak{q} \cap R$ (verify this!). By 2.3.3, this means that the morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective. □

Theorem 3.3.8. *Let $R \subseteq S$ be a finite homomorphism of Noetherian rings. If R has dimension d, then so does S.*

Proof. Put $d := \dim(R)$ and $e := \dim(S)$. To see the inequality $e \leq d$, choose a maximal ideal $\mathfrak{n}$ in S of height e, and put $\mathfrak{m} := \mathfrak{n} \cap R$. Since $R_\mathfrak{m}$ has dimension at most d, there exists an $\mathfrak{m}R_\mathfrak{m}$-primary ideal $I \subseteq R_\mathfrak{m}$ generated by at most d elements by Theorem 3.3.2. Since $S_\mathfrak{n}/IS_\mathfrak{n}$ is then a finitely generated $R_\mathfrak{m}/I$-module, it is Artinian. Hence $IS_\mathfrak{n}$ is $\mathfrak{n}S_\mathfrak{n}$-primary, showing that $\mathrm{geodim}(S_\mathfrak{n}) \leq d$. Since the left hand side is equal to e by Theorem 3.3.2, we showed $e \leq d$.

We prove the converse inequality by induction on d (where the case $d = 0$ is clearly trivial). Choose a d-dimensional prime ideal $\mathfrak{p} \subseteq R$. Using 3.3.7, we can find a prime ideal $\mathfrak{q} \subseteq S$ *lying above* $\mathfrak{p}$, that is to say, $\mathfrak{p} = \mathfrak{q} \cap R$. Put $\bar{R} := R/\mathfrak{p}$ and $\bar{S} := S/\mathfrak{q}$. In particular, $\bar{R} \subseteq \bar{S}$ is again finite and injective. By the same argument, we can take a $d-1$-dimensional prime ideal $\mathfrak{P} \subseteq \bar{R}$, and a prime ideal $\mathfrak{Q} \subseteq \bar{S}$ lying above it. By the induction hypothesis applied to the finite extension $\bar{R}/\mathfrak{P} \subseteq \bar{S}/\mathfrak{Q}$, we get $d - 1 = \dim(\bar{R}/\mathfrak{P}) \leq \dim(\bar{S}/\mathfrak{Q})$. However, since any non-zero element in a domain is a parameter (see Corollary 3.3.6), the dimension of $\bar{S}/\mathfrak{Q}$ is strictly less than the dimension of $\bar{S}$, which itself is less than or equal to e. Hence $d - 1 \leq e - 1$, as we wanted to show. □

Corollary 3.3.9. *If $V \to K^d$ is a Noether normalization of an affine variety V, then V has dimension d.*

Proof. By definition of Noether normalization, we have a finite, injective homomorphism $K[\zeta] \subseteq K[V]$ with $\zeta = (\zeta_1, \dots, \zeta_d)$. By Corollary 3.3.3, the first ring has dimension d, whence so does the second by Theorem 3.3.8. This in turn means that V has dimension d. □

3.4 Exercises

Ex 3.4.1

Prove the inequality in 3.1.1. In fact, this is often an equality, for instance if R is a polynomial ring over a field, but this is already a much less trivial result. Verify it when R is a polynomial ring over a field in a single indeterminate.

Ex 3.4.2

Show that length is additive *in the sense that if $0 \to K \to M \to N \to 0$ is a short exact sequence of A-modules, then $\ell(M) = \ell(K) + \ell(N)$.*

Ex 3.4.3

Prove 3.1.5. More generally, show that if R is an Artinian local ring with residue field k, then the length of R is equal to its vector space dimension over k. For the latter, you need to know that k is a subfield of R, and this is proven in Theorem 6.4.2 and Remark 6.4.3, but you can just assume for the moment that this is the case.

Ex 3.4.4

Let S be a standard graded R-algebra. Show that S is Noetherian if R is.

Ex 3.4.5

Show the first assertion in 3.1.3: the height of a prime ideal $\mathfrak{p} \subseteq R$ is equal to the dimension of $R_{\mathfrak{p}}$.

Ex 3.4.6

Show that any maximal ideal in $K[\xi_1, \dots, \xi_n]$ is generated by at most n elements, even if K is not algebraically closed. Use this to complete the proof of Corollary 3.3.3.

Ex 3.4.7

Generalize Corollary 3.3.3 by replacing the field by any Artinian local ring. Moreover, in the power series case, formulate a result with the base ring any Noetherian local ring. Such a result also holds in the polynomial case, but the proof requires some more powerful tools such as flatness, discussed in §5; for a proof, see for instance [41, Theorem 15.4].

Ex 3.4.8

Work out the details of the proof of Corollary 3.2.3.

*Ex 3.4.9

Develop the theory of Hilbert-Samuel polynomials also for finitely generated R-modules M and for $\mathfrak{m}$-primary ideals I, by using the graded algebra

$$\mathrm{Gr}_I(R) := \oplus_n I^n / I^{n+1}$$

and the graded module

$$\mathrm{Gr}_I(M) := \oplus_n I^n M / I^{n+1} M$$

.

Ex 3.4.10

Show that $(x_1, \dots, x_d)$ is a system of parameters in R if and only if x_i is a parameter in $R/(x_1, \dots, x_{i-1})R$ for every $i = 1, \dots, d$.

Ex 3.4.11

Show that if $\mathbf{x}$ is a tuple of length e in a Noetherian local ring R such that $\mathbf{x}R$ has height e, then $\mathbf{x}$ can be extended to a system of parameters of R. Using the same technique, also show that if $\mathfrak{p}$ is a prime ideal of height h, then there exists a system of parameters $(y_1, \dots, y_d)$ such that $\mathfrak{p}$ is a minimal prime of $(y_1, \dots, y_h)R$.

Ex 3.4.12

Prove the following more precise form of 3.3.7: a finite morphism $Y = \mathrm{Spec}(S) \to X = \mathrm{Spec}(R)$ is surjective if and only if the kernel of the corresponding ring homomorphism $R \to S$ is nilpotent. In fact, the only if direction is true for any morphism.

***Ex 3.4.13**

For non-Noetherian rings, Krull dimension and geometric dimension need not agree; here's an example. Let R be the power series ring $K[[\xi]]$ in $d > 0$ variables ξ, and let $R_\natural$ be its ultrapower. Show that $\mathrm{geodim}(R_\natural) = d = \mathrm{Hilbdim}(R_\natural)$ but $\dim(R_\natural) > d$. To establish the latter inequality, show that the ideal of infinitesimals of $R_\natural$ is a prime ideal. In fact, $R_\natural$ has infinite Krull dimension, but proving this requires some more work.

***Ex 3.4.14**

Show using Noether Normalization and Exercise 3.4.13 that any affine domain C is equidimensional*, in the sense that every maximal ideal of C has the same height.*

Additional exercises.

Ex 3.4.15

Show that a finite injective homomorphism $A \subseteq B$ satisfies the going-up theorem*, meaning that given any inclusion of prime ideals $\mathfrak{p} \subseteq \mathfrak{q} \subseteq A$ and any prime ideal $\mathfrak{P} \subseteq B$ lying over $\mathfrak{p}$, we can find a prime ideal $\mathfrak{Q} \subseteq B$ containing $\mathfrak{P}$ and lying over $\mathfrak{q}$.*

Chapter 4
Singularity theory

We gave a formal definition of a singular point in Definition 2.6.4. In this chapter, we investigate the algebraic theory behind this phenomenon. In particular, we will identify a certain type of singularity, the Cohen-Macaulay singularity, which plays an important role in the later chapters.

4.1 Regular local rings

According to our 'algebraization paradigm', geometric properties of points are reflected by their local rings. Before we make this translation, we first explore a little the classical notion, using plane curves as example.

Multiple points on a plane curve. A *plane curve* C is an irreducible affine variety given by a non-constant, irreducible polynomial $f(\xi,\zeta) \in A := K[\xi,\zeta]$, for K some algebraically closed field, that is to say, $C = \mathrm{V}(f)$. By Corollary 3.3.6, a plane curve has dimension one. So we arrive at the more general concept of a *curve* as a one-dimensional, irreducible scheme. The degree t of f is also called the *degree* of the plane curve C. If $t = 1$, then C is just a line. So from now on, we will moreover assume $t > 1$. An easy form of Bezout's theorem states:

4.1.1 *Any line intersects the plane curve C of degree t in at most t distinct points, and there exist lines which have exactly t distinct intersection points with C.*

The proof is elementary: the general equation of a line L is $a\xi + b\zeta + c = 0$ and hence the intersection $|C \cap L|$ is given by the radical of the ideal $(a\xi + b\zeta + c, f)A$ (or, viewed as an affine scheme $C \cap L$ by the ideal itself; see page 30). In terms of equations, assuming $b = 1$ for the sake of simplicity, this means that the (ξ-values of the) intersection points are given by the equation $f(\xi, -a\xi - c) = 0$, a polynomial of degree t or less, which therefore has at most t solutions. Choosing a, b, c sufficiently general, we can moreover guarantee that this polynomial has t distinct roots. We can now state when a point P on C is singular, but to not confuse with our formal definition 37, we use a different terminology:

Definition 4.1.2. A point P on a plane curve C of degree t is called *multiple*, if every line through P intersects C in less than t distinct points. More precisely, we say that P is an *n-tuple* point on C, or $\mathrm{mult}_C(P) = n$, if n is the number of points absorbed at P in each intersection with a line, that is to say,

$$\mathrm{mult}_C(P) := \min_{L \text{ line through } P} (t - \mathrm{card}(|C \cap L|) + 1).$$

Here $|C \cap L|$ denotes the (naive) intersection as sets, not as schemes. A point which is not multiple, i.e., a 1-tuple point, is called a *simple* point.

Let us look at two examples of multiple points:

An example of a node. Let $f := \xi^2 - \zeta^2 - 3\zeta^3$ and let P be the origin. Hence a line L_a through P has equation $\zeta = a\xi$ for some $a \in K$ (for sake of simplicity, we ignore the ζ-axis; the reader should check that this makes no difference in what follows). Substituting this in the equation, the intersection points with C are given by the equations $\zeta = a\xi$ and $\xi^2 - (a\xi)^2 - 3(a\xi)^3 = 0$. The second equation reduces to $\xi = 0$ or $\xi = (1-a^2)/3a^3$, thus giving only two intersection points, contrary to the expected value of three. In conclusion, P is a double point. One can check that it is the only multiple point on C (check this for instance for the point with coordinates $(2,1)$).

Moreover, note that the two diagonals $L_{\pm 1}$ intersect C in exactly one point, that is to say, the lines $y = \pm x$ have even *higher contact* with C; they are often refered to as the *tangent lines* of C at P. To formally define a tangent line, one needs to introduce the intersection number $i(L,C;P)$ of a line L with C at P, and then call L a *tangent line* if $i(L,C;P) > \mathrm{mult}_C(P)$. One way of doing this is by defining the *intersection number* $i(L,C;P)$ as the length of R/LR, where $R := (A/fA)_{\mathfrak{m}}$ is the local ring of P at C and where we identify the line L with its defining linear equation. One checks that $i(L_a,C;P)$ equals two for $a \neq \pm 1$, and three for $a = \pm 1$.

To calculate the tangent space $T_{C,P}$ as defined on page 36, let $\mathfrak{m} := (\xi,\zeta)A$ be the maximal ideal corresponding to the origin. Since $\mathfrak{m}R$ is generated by two elements, the embedding dimension of R is two, whence so is the dimension of the tangent space $T_{C,P}$ by 2.6.3. Hence, since the tangent space has higher dimension than the scheme, P is singular on C.

An example of a cusp. For our next example, let $f := \xi^4 - \zeta^3$, a curve of degree four, and let P be the origin as before. The intersection with L_a is given by the equation $\xi^4 - (a\xi)^3 = 0$, which yields two intersection points: namely P and (a^3,a^4). Hence P is a triple point of C. Moreover, there is now only one value of a which leads to a higher contact, namely $a = 0$, showing that the ξ-axis is the only tangent line (double-check that the ζ-axis does not have higher contact). A multiple point with a unique tangent line is called a *cusp*. A similar calculation as before shows that $T_{C,P}$ is again two-dimensional, whence P is singular. Let us now prove this in general:

Proposition 4.1.3. *A point on a plane curve is a multiple point if and only if it is singular.*

Proof. Let f be the equation, of degree t, defining the curve C, and let P be a point on C. After a change of coordinates, we may assume P is the origin, defined by the maximal ideal $\mathfrak{m} := (\xi, \zeta)A$. If P is non-singular, then the embedding dimension of $\mathcal{O}_{C,P} = (A/fA)_{\mathfrak{m}}$ is one. Hence either ξ or ζ generates $\mathfrak{m}R$. So, after interchanging ξ and ζ if necessary, we can write ζ as a fraction $(\xi g + f\tilde{g})/h$ in $A_{\mathfrak{m}}$, for some $g, \tilde{g}, h \in A$ with $h \notin \mathfrak{m}$. Hence the intersection with L_a is given by $\zeta = a\xi$ and

$$a\xi = \frac{\xi g(\xi, a\xi) + f(\xi, a\xi)\tilde{g}(\xi, a\xi)}{h(\xi, a\xi)}.$$

Since f has no constant term, we may divide out ξ, so that the last equation becomes

$$ah(\xi, a\xi) = g(\xi, a\xi) + \tilde{f}(\xi) \tag{4.1}$$

for some $\tilde{f} \in K[\xi]$. If P would be a multiple point of C, then $\xi = 0$ should still be a solution of (4.1). However, this can only happen if $a = (g(0,0) + \tilde{f}(0))/h(0,0)$ (note that $h(0,0) \neq 0$ by assumption). In other words, a general line has only one intersection point at P, and hence P is a simple point. Note that it has exactly one tangent line, given by the above exceptional value of a.

Conversely, assume P is simple, and write $f = u\xi + v\zeta + \tilde{f}$ with $u, v \in K$ and $\tilde{f} \in \mathfrak{m}^2$. By assumption, the equation $u\xi + va\xi + \tilde{f}(\xi, a\xi) = 0$ should have in general $t-1$ solutiuons different from $\xi = 0$. For this to be true, at least one of u or v must be non-zero. So assume, without loss of generality, that $u \neq 0$, and then multiplying with its inverse, we may even assume $u = 1$. It follows that $\xi = -v\zeta - \tilde{f}$ in R, showing that $\mathfrak{m}R = \zeta R$ by Nakayama's Lemma, and therefore that R has embedding dimension one. ☐

By the above argument, in order for P to be simple, $A_{\mathfrak{m}}/fA_{\mathfrak{m}}$ has to have embedding dimension one, which by Nakayama's lemma is equivalent with f being a minimal generator of $\mathfrak{m}A_{\mathfrak{m}}$, that is to say, $f \in \mathfrak{m}A_{\mathfrak{m}} - \mathfrak{m}^2A_{\mathfrak{m}}$. In Exercise 4.3.4 you will prove the following generalization:

4.1.4 *A point P is an n-tuple point on a plane curve $C := \mathrm{V}(f)$ if and only if n is the maximum of all k such that $f \in \mathfrak{m}^k A_{\mathfrak{m}}$, where $\mathfrak{m} := \mathfrak{m}_P$ is the maximal ideal of P.*

Geometrically, a closed point x is singular on an affine variety, or more generally, on an affine scheme X, if the dimension of its tangent space is larger than the local dimension of X at x. In particular, singularity is a local property, completely captured by the local ring of the point. Since the dimension of the tangent space is equal to embedding dimension of the local ring by 2.6.3, we can now formulate non-singularity entirely algebraically:

Definition 4.1.5 (Regular local ring). We call a Noetherian local ring $(R, \mathfrak{m})$ *regular* if and only if its dimension is equal to its embedding dimension.

In view of Theorem 3.3.2, regularity is equivalent with the maximal ideal being generated by the least possible number of elements. In particular, some system of parameters generates the maximal, and any such system is called a *regular system of parameters*. Geometrically, a point x on a scheme X is *regular*, or *non-singular*, if $\mathcal{O}_{X,x}$ is regular. An Artinian local ring is regular if and only if it is a field. By Corollary 3.3.3, a power series ring over a field is regular. Using that same theorem in conjunction with the Nullstellensatz (Theorem 2.2.2), we also get a similar result over an algebraically closed field K (for a more general version, see Exercise 4.3.6):

4.1.6 *Each closed point of affine* n*-space* $\mathbb{A}^n_K$ *is regular.*

To formulate a stronger result, let us call a ring B *regular* if each localization at a maximal ideal is regular. Similarly, we call a scheme X *regular* if all of its closed points are regular. Hence we may reformulate 4.1.6 as: $\mathbb{A}^n_K$ is regular. This begs the question: what about the non-closed points of $\mathbb{A}^n_K$? As it turns out, they too are regular, and in fact, this is a general property of regular rings:

4.1.7 *Any localization of a regular ring is again regular.*

To prove this, however, one needs a different characterization, homological in nature, of regular rings due to Serre (it was only after he proved his theorem that the above result became available). We will not provide all details, but 4.1.7 will be proved in Corollary 5.5.8 below. Another property is more readily available: geometric intuition predicts that at an intersection point of two distinct components, the scheme ought to be singular. Put differently, a variety should be irreducible in 'the neighbourhood of' a non-singular point. This translates into the following property of the local ring of the point:

4.1.8 *A regular local ring is a domain.*

To prove this, we need another characterization of regular local rings:

Theorem 4.1.9. *Let* $(R,\mathfrak{m})$ *be a* d*-dimensional Noetherian local ring with residue field* k*, and let* $S := \mathrm{Gr}(R)$ *be its associated graded ring. Then* R *is regular if and only if* S *is isomorphic to a polynomial ring over* k *in* d *variables.*

Proof. Let $A := k[\xi]$ with $\xi := (\xi_1,\dots,\xi_d)$, viewed as a standard graded k-algebra in the obvious way. If $A \cong S$, then $A_1 \cong S_1$ has k-vector space dimension d. Since $S_1 = \mathfrak{m}/\mathfrak{m}^2$, Nakayama's lemma shows that R has embedding dimension d, whence is regular. To prove the converse, assume R is regular, and we need to show that $S \cong A$. By assumption, $\mathfrak{m}$ is generated by d elements, $x_1,\dots,x_d$. Define a homomorphism $\varphi\colon k[\xi] \to S$ of graded k-algebras by the rule $\xi_i \mapsto x_i$. Since $\mathfrak{m} = (x_1,\dots,x_d)R$, the homomorphism φ is surjective (verify this!). Let I be its kernel. Hence $A/I \cong S$. Now, A has dimension d by Corollary 3.3.3. I claim that S has dimension at least d. However, if $I \neq 0$, then by Corollary 3.3.6, the dimension of A/I is strictly less than d. Hence $I = 0$, as we wanted to show (and S has actually dimension equal to d).

To prove the claim, it suffices to show that the maximal ideal $\mathfrak{n} := S_+$ has height d. Since $\mathfrak{n}^{n+1} = \oplus_{k>n} S_k$, we get $S/\mathfrak{n}^{n+1} \cong S_0 \oplus \dots \oplus S_n$, and its length is equal to $\ell(R/\mathfrak{m}^{n+1})$ by (3.3). Since $S/\mathfrak{n}^{n+1} \cong S_\mathfrak{n}/\mathfrak{n}^{n+1}S_\mathfrak{n}$ (check this!), we see that R and $S_\mathfrak{n}$ have the same Hilbert-Samuel polynomial, whence the same dimension by Theorem 3.3.2, as we wanted to show. □

Incidentally, in the last part of the proof, we did not use our hypothesis on the regularity of the ring, so that we showed one inequality in the next result; the converse will not be needed here and can be found in for instance [41, Theorem 13.9].

4.1.10 *The dimension of a Noetherian local ring is equal to the dimension of its associated graded ring.*

Proof of 4.1.8. Given two non-zero elements $a, b \in R$, we need to show that their product is non-zero too. By Theorem 1.4.11, there exist $k, l \in \mathbb{N}$ such that $a \in \mathfrak{m}^k \setminus \mathfrak{m}^{k+1}$ and $b \in \mathfrak{m}^l \setminus \mathfrak{m}^{l+1}$. Hence a and b induce two non-zero elements $\bar{a} \in S_k$ and $\bar{b} \in S_l$ respectively. Since S is a domain by Theorem 4.1.9, their product $\bar{a}\bar{b} \in S_{k+l}$ is non-zero, whence a fortiori so is ab. □

Why we need projective space. Above, we have seen examples of plane curves having a multiple point. Of course, some curves are regular. The simplest example is obviously a line. Another is given by the so-called *elliptic curves*, defined by an equation

$$\zeta^2 = \xi(\xi - 1)(\xi - u)$$

with $u \neq 0, 1$. You can use the criterion from Exercise 4.3.3 to show that every point on an elliptic curve is simple, provided the characteristic of K is not 2, whence regular by Proposition 4.1.3 (see also Exercise 4.3.9). Another example of a regular curve is the one defined by the equation $\xi\zeta^2 = 1$ (again easily verified by means of Exercise 4.3.3). However, in this latter case, we are overlooking the 'points at infinity'. More precisely, recall that $\mathbb{P}^2_K$ is obtained by glueing together three copies of $\mathbb{A}^2_K$ (see page 35), each corresponding by inverting one of the 'projective' variables. So we may view $\mathbb{A}^2_K$, with coordinates (ξ, ζ) as the copy corresponding to inverting the last variable, and embed it in $\mathbb{P}^2_K$. Given a plane curve $C = \mathrm{V}(f)$ (or rather, the affine scheme $\mathrm{Spec}(B)$ with $B := A/fA$ determined by it), let $\bar{C}$ be the closure of C inside $\mathbb{P}^2_K$. We can endow $\bar{C}$ with the structure of a projective variety as follows: let $\tilde{f}$ be the *homogenization* of f, that is to say, if f has degree t, then

$$\tilde{f}(\xi, \zeta, \eta) := \eta^t f(\xi/\eta, \zeta/\eta). \tag{4.2}$$

I claim that the underlying space of $\tilde{C} := \mathrm{Proj}(\tilde{B})$ is equal to $\bar{C}$, where $\tilde{A} := K[\xi, \zeta, \eta]$ and $\tilde{B} := \tilde{A}/\tilde{f}\tilde{A}$. Since $\tilde{A}_{(\eta)} \cong A$ by 2.5.3, we get $\tilde{B}_{(\eta)} \cong B$ by (4.2), showing that

$$\mathbb{A}^2_K \cap \tilde{C} = \tilde{\mathrm{D}}(\eta) \cap \tilde{C} = C.$$

Our claim now follows, since the closure of $\mathbb{A}^2_K$ is just $\mathbb{P}^2_K$. We call $\tilde{C}$ the *projectification* or *completion* of C.

Returning to our question on singularities: any point of $\tilde{C} \setminus C$ will be called a *point at infinity* of C. To check whether such a point is non-singular, we have to 're-coordinatize', that is to say, look at one of the two other copies of $\mathbb{A}^2_K \subseteq \mathbb{P}^2_K$. Let us do this on the example with equation $f := \xi\zeta^2 - 1$. Following the recipe in (4.2), we get $\tilde{f} = \xi\zeta^2 - \eta^3$. On the copy $\tilde{\mathrm{D}}(\xi) = \mathbb{A}^2_K$, the intersection with $\tilde{C}$ is the affine scheme given by $\zeta^2 - \eta^3$, the equation of a cusp with a singular point at $\zeta = \eta = 0$ (note that it is straightforward to undo the homogenization (4.2): just replace the pertinent variable, here ξ, by 1). Hence $\tilde{C}$ is not regular. In Exercise 4.3.10, you will show that in contrast, the projectification of any elliptic curve remains regular.

In the above discussion, we used curves merely as an illustration: a similar treatment can be given for higher dimensional affine schemes as well (see Exercise 4.3.11): any closed affine subscheme $X \subseteq \mathbb{A}^n_K$ can be projectified to a projective scheme $\tilde{X} \subseteq \mathbb{P}^n_K$. So, even if an affine scheme itself is regular, it might not be as 'good' as we believe it to be, as we do not see its points at infinity. For that we need to go to its projectification.

4.2 Cohen-Macaulay rings

Algebraic geometry has developed for a large part in an attempt to gain a better understanding of singularities, and if possible, to classify them. As it turns out, certain singularities have nicer properties than others. Our goal is to identify such a class of singularities, or equivalently, by passing to their local ring, such a class of Noetherian local rings, which are more amenable to algebraic methods: the 'Cohen-Macaulay' singularities. In order to do this, we must first study an invariant called 'depth'.

Regular sequences. Recall that an element in a ring R is called a *non-zero divisor* if multiplication with this element is injective; more generally, an element x is a *non-zero divisor* on an R-module M if multiplication by x is injective on M. Recall that a prime ideal in a Noetherian ring R is called an *associated prime* of R (respectively, of a finitely generated R-module M), if it is of the form $\mathrm{Ann}_R(x)$ for some $x \in R$ (respectively, of the form $\mathrm{Ann}_R(\mu)$ for some $\mu \in M$). Moreover, R (respectively, M) admits only finitely many associated prime ideals, among which are all the minimal prime ideals, and an element is a non-zero divisor if and only if it is not contained in any associated prime ideal (for all this, see for instance [41, §6]).

A non-zero divisor of R which is not a unit is called a *regular element* in R, or *R-regular* (do not confuse with the notion of a regular local ring!). Similarly, we say that x is *M-regular* if it is a non-zero divisor on M and $xM \neq M$ (be aware that some authors might use a slightly different definition for these notions). More generally, a sequence $(x_1,\dots,x_d)$ is called a *regular sequence* in R, or *R-regular*, (respectively, *M-regular*) if each x_i is regular in $R/(x_1,\dots,x_{i-1})R$ (respectively, in $M/(x_1,\dots,x_{i-1})M$) for $i = 1,\dots,d$. Here, and elsewhere, we do not distinguish notationally between an element in a ring R and its image in any residue ring R/I, or for that matter, in any R-algebra S. If $(x_1,\dots,x_d)$ is an R-regular sequence, then by assumption $(x_1,\dots,x_d)R$ is a proper ideal of R. In particular, if R is local, then all

x_i belong to the maximal ideal. To be a regular sequence in a local ring is quite a strong property:

4.2.1 *In a Noetherian local ring R, any regular sequence can be enlarged to a system of parameters. In particular, a regular sequence can have length at most $\dim(R)$. In fact, if $\mathbf{x}$ is a regular sequence of length e, then $\mathbf{x}R$ has height e.*

To see this, we only need to show by induction on the length of the sequence that a regular element generates a height one prime ideal and is a parameter. However, since a regular element x does not belong to any associated prime, whence in particular not to any minimal prime, the ideal xR has height one by Theorem 3.3.4. Since x then neither belongs to any prime ideal of maximal dimension, it is a parameter. Using this in conjunction with Corollary 3.3.6, we get:

4.2.2 *If $\mathbf{x}$ is a regular sequence of length e in a d-dimensional Noetherian local ring R, then $R/\mathbf{x}R$ has dimension $d-e$.*

Cohen-Macaulay local rings. A d-dimensional Noetherian local ring is called *Cohen-Macaulay* if it admits a regular sequence of length d. Trivially, any Artinian local ring is Cohen-Macaulay. The next result justifies calling the Cohen-Macaulay property a type of singularity.

Proposition 4.2.3. *Any regular local ring is Cohen-Macaulay.*

Proof. Let us induct on the dimension d of the regular local ring R. The case $d=0$ is trivial since R is then a field. By assumption, the maximal ideal $\mathfrak{m}$ is generated by d elements $x_1,\dots,x_d$. I will show by induction on d that $(x_1,\dots,x_d)$ is in fact a regular sequence. Since R is a domain by 4.1.8, the element x_1 is regular. Put $R_1 := R/x_1R$. It is a Noetherian local ring of dimension $d-1$ by Corollary 3.3.6, and its maximal ideal $\mathfrak{m}R_1$ is generated by at most $d-1$ elements. Hence R_1 is again regular. By induction, $(x_2,\dots,x_d)$ is a regular sequence in R_1, from which it follows that $(x_1,\dots,x_d)$ is a regular sequence in R. □

Depth. As we will see, being Cohen-Macaulay is a natural property, and many non-regular local rings are still Cohen-Macaulay. Since the notion hinges upon the length of a regular sequence, let us give this a name: the maximal length of a regular sequence in a Noetherian local ring R is called the *depth* of R, and is denoted $\operatorname{depth}(R)$. More generally, the *depth* of an ideal I is the maximal length of a regular sequence lying in I. We proved $\operatorname{depth}(R) \leq \dim(R)$ with equality precisely when R is Cohen-Macaulay. Immediately from our discussion on associated primes, we get:

4.2.4 *A Noetherian local ring has depth zero if and only if its maximal ideal is an associated prime.*

In particular, the one-dimensional local ring $R/(\xi^2,\xi\zeta)R$ is not Cohen-Macaulay, where $R := A_{\mathfrak{m}}$ is the local ring of the origin in $\mathbb{A}^2_K$.

4.2.5 *A one-dimensional Noetherian local domain is Cohen-Macaulay. In particular, any closed point on a (plane) curve is Cohen-Macaulay.*

As the reader might have surmised, we call a point x on a scheme X *Cohen-Macaulay* if $\mathcal{O}_{X,x}$ is Cohen-Macaulay. For an example of a non-Cohen-Macaulay local domain, necessarily of dimension at least two, see Exercise 4.3.14.

If R is Cohen-Macaulay, and $\mathbf{x}$ is a regular sequence of length $d := \dim(R)$, then $\mathbf{x}$ is automatically a system of parameters by 4.2.1. This raises the following question: what about arbitrary systems of parameters?

Theorem 4.2.6. *In a Cohen-Macaulay local ring, every system of parameters is a regular sequence. In particular, any regular sequence is* permutable*, meaning that an arbitrary permuation is again regular.*

Proof. The second statement is immediate from the first since in a system of parameters, order plays no role. However, we need it to prove the first assertion. And before we can prove this, we need to establish yet another special case of the first assertion: taking powers of the elements in a regular sequence gives again a regular sequence, and for this to hold, we do not even need the ring to be Cohen-Macaulay. Although both results have relatively elementary proofs, the combinatorics are a little involved, and so we will only present the argument for $d = 2$. Hence assume (x,y) is a regular sequence in some Noetherian local ring S. We claim that both (x^k, y^l) and (y^l, x^k) are S-regular sequences, for any $k,l \geq 1$. We first show that (x^k, y) is S-regular, for all $k \geq 1$. By induction, we only need to treat the case $k = 2$. Clearly, x^2 is S-regular, so we need to show that y is S/x^2S-regular. Hence suppose $by \in x^2S$, say $by = ax^2$. Since y is S/xS-regular, $b \in xS$, say $b = cx$. Hence, $cxy = ax^2$, and using that x is S-regular, $cy = ax$. Using again that y is S/xS-regular then yields $c \in xS$, which proves that $b = cx \in x^2S$, as we wanted to show.

Next, we show that (y,x) is S-regular. To show that y is S-regular, let $by = 0$. By our previous result, (x^n, y) is a regular sequence for every n, which means that y is S/x^nS-regular. Applied to $by = 0$, we get $b \in x^nS$. Since this holds for all n, we get $b \in \mathfrak{I}_S = 0$ by Theorem 1.4.11. So remains to show that x is S/yS-regular. Suppose $ax \in yS$, say $ax = by$. Since y is S/xS-regular, $b \in xS$, say, $b = cx$. From $ax = cxy$ and the fact that x is S-regular, we get $a = cy$, as we needed to show. Finally, to prove that (x^k, y^l) and (y^l, x^k) are S-regular, observe that the following sequences are S-regular: (x^k, y) by the first property, (y, x^k) by the second, (y^l, x^k) by the first, and finally (x^k, y^l) by the second.

So, with these two properties proven for $d = 2$, and assuming them for arbitrary d, let us turn to the proof of the theorem. Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, and let $(x_1,\dots,x_d)$ be a regular sequence. We prove by induction on d that any system of parameters $(y_1,\dots,y_d)$ is a regular sequence. There is nothing to show if $d = 0$, so assume $d > 0$. Put $I := (x_1,\dots,x_{d-1})R$. Since x_d is by assumption R/I-regular, $\mathfrak{m}(R/I)$ is not an associated prime. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ be prime ideals in R such that their image in R/I are precisely the associated primes of the latter ring. Since $J := (y_1,\dots,y_d)R$ is $\mathfrak{m}$-primary, it cannot be contained in any of the $\mathfrak{p}_i$, whence by prime avoidance, we can find $y \in J$ notin $\mathfrak{m}J$ and not in any $\mathfrak{p}_i$. In particular, $y = \sum u_i y_i$ with at least one u_i a unit in R. After renumbering, we may assume that u_d is a unit. It follows that $(y_1,\dots,y_{d-1},y)$ is again a system of parameters. Moreover, y is R/I-regular, showing that $(x_1,\dots,x_{d-1},y)$ is a regular sequence. Since we established already that any permutation is again a regular sequence, $(y,x_1,\dots,x_{d-1})$ is R-regular. Hence $(x_1,\dots,x_{d-1})$ is R/yR-regular. Since R/yR has dimension $d-1$ by Corollary 3.3.6, it is therefore Cohen-Macaulay. Hence $(y_1,\dots,y_{d-1})$, being a system of parameters in this ring, is by induction a regular sequence. In other words, $(y,y_1,\dots,y_{d-1})$ is a regular sequence, whence so is the permuted sequence $(y_1,\dots,y_{d-1},y)$. Finally, we show that y_d is R/J'-regular with $J' := (y_1,\dots,y_{d-1})R$, which then completes the proof that $(y_1,\dots,y_d)$ is a regular sequence. So assume $ay_d \in J'$. Since $y \equiv u_d y_d \mod J'$, we get $au_d y \in J'$. Since we already showed that y is R/J'-regular, we get $u_d a \in J'$, and since u_d is a unit, we finally get $a \in J'$, proving our claim. □

Corollary 4.2.7. *Let R be a Noetherian local ring, and let $\mathbf{x}$ be a regular sequence of length e. Then R is Cohen-Macaulay if and only if $R/\mathbf{x}R$ is.*

Proof. Let $d := \dim(R)$. By 4.2.2, the residue ring $R/\mathbf{x}R$ has dimension $d-e$. If it is Cohen-Macaulay, then there exists a regular sequence $\mathbf{y}$ of that length, and then $(\mathbf{x},\mathbf{y})$ (where we still write $\mathbf{y}$ for some lifting of that tuple to R) is a regular sequence of length d, showing that R is Cohen-Macaulay. Conversely, if R is Cohen-Macaulay, let $\mathbf{y}$ be a system of parameters in $R/\mathbf{x}R$. It follows that $(\mathbf{x},\mathbf{y})$ is a system of parameters in R, whence is a regular sequence by Theorem 4.2.6. Hence $\mathbf{y}$ is a regular sequence in $R/\mathbf{x}R$ of maximal length, proving that $R/\mathbf{x}R$ is Cohen-Macaulay. □

Corollary 4.2.8. *A Cohen-Macaulay local ring has no embedded primes, that is to say, any associated prime is minimal.*

Proof. Let R be a Cohen-Macaulay local ring and $\mathfrak{p}$ an associated prime. If $\mathfrak{p}$ has positive height, we can find $x \in \mathfrak{p}$ such that xR has height one. By Exercise 3.4.11, we can extend x to a system of parameters of R, which is then a regular sequence by Theorem 4.2.6. In particular, x is R-regular, contradicting that it belongs to an associated prime. □

In fact, Corollary 4.2.7 holds in far more greater generality: without assuming that R is Cohen-Macaulay, we have that the depth of R is equal to the depth of $R/\mathbf{x}R$ plus e. However, to prove this, one needs a different characterization of depth (using Ext functors), which we will not discuss in these notes. Another property that we can now prove is that any localization of a Cohen-Macaulay local ring is again Cohen-Macaulay (recall that we also still have to resolve this issue with regards to being regular).

Corollary 4.2.9. *If R is a Cohen-Macaulay local ring, then so is any localization $R_\mathfrak{p}$ at a prime ideal $\mathfrak{p} \subseteq R$.*

Proof. Let h be the height of $\mathfrak{p}$. Let us show by induction on h that $\mathfrak{p}$ contains a regular sequence of length h (that is to say, $\mathfrak{p}$ has depth h). It is not hard to check that the image of this sequence is then a regular sequence in $R_\mathfrak{p}$, showing that the latter is Cohen-Macaulay. Obviously, we may take $h > 0$. Since $\mathfrak{p}$ cannot be contained in an associated prime of R by Corollary 4.2.8, it contains an R-regular element x. Put $S := R/xR$, which is again Cohen-Macaulay by Corollary 4.2.7. As $\mathfrak{p}S$ has height $h-1$ (check this), it contains an S-regular sequence $\mathbf{y}$ of length $h-1$. But then $(x,\mathbf{y})$ is an R-regular sequence inside $\mathfrak{p}$, as we wanted to show. □

We can now say that a Noetherian ring A is *Cohen-Macaulay* if every localization at a maximal ideal is Cohen-Macaulay, and this is then equivalent by the last result with every localization being Cohen-Macaulay. Similarly, a scheme X is Cohen-Macaulay, if every local ring $\mathcal{O}_{X,x}$ at a (closed) point $x \in X$ is Cohen-Macaulay. In particular, any reduced curve is Cohen-Macaulay.

4.3 Exercises

Ex 4.3.1

Verify all the claims made on page 54 about the given node and cusp.

***Ex 4.3.2**

Prove the following more general version of Bezout's theorem: if $C := \mathrm{V}(f)$ and $D := \mathrm{V}(g)$ are two distinct plane curves of degree t and u respectively, then their scheme-theoretic intersection, given by the (Artinian) K-algebra $A/(f,g)A$ has K-vector space dimension tu. To do this, carry out effectively the proof of Noether Normalization, to get a handle on this vector space dimension.
To see how this implies the usual statement of Bezout's theorem, namely that the set-theoretic intersection $|C \cap D|$ has cardinality at most tu, show that any Artinian ring of length l has at most l maximal ideals.

Ex 4.3.3

From the proof of Proposition 4.1.3, you can extract the following criterion for f to have a simple point at the origin: its linear part should not vanish. Use this to prove that a point P on a plane curve $C := \mathrm{V}(f)$ is a multiple point if and only if $\partial f/\partial \xi$ and $\partial f/\partial \zeta$ both vanish on P. Conclude that a plane curve has at most finitely many multiple points, and find an upperbound for their number (you will need some elimination theory for this, as given, for instance, in [18, pp. 308-309]).

Ex 4.3.4

Extend the argument in the proof of Proposition 4.1.3 to prove 4.1.4.

Ex 4.3.5

Show that if R is a regular local ring, then so is the power series ring $R[[\xi]]$ in finitely many indeterminates. Prove that the ring of convergent power series over $\mathbb{C}$ (a formal power series is called convergent *if it converges on a small open disk around the origin) is regular.*

Ex 4.3.6

Use Exercise 3.4.6 to show that we may drop the condition in 4.1.6 that K is algebraically closed.

Ex 4.3.7

From the proof of 4.1.8, it is clear that any local ring whose associated graded ring is a domain, is itself a domain. Show that the coordinate ring of a cusp gives a counterexample to the converse.

Ex 4.3.8

Show that a one-dimensional Noetherian local ring R is regular if and only if it is a discrete valuation ring, that is to say, if and only if it admits a valuation $v\colon R\setminus\{0\}\to\mathbb{Z}$.

Ex 4.3.9

Using the criterion from Exercise 4.3.3, show that a plane curve with equation $\zeta^n = f(\xi)$ with f a polynomial without double roots, defines a regular plane curve if the characteristic of K does not divide n. In particular, elliptic curves are regular in all characteristics other than 2 (and in fact, also in characteristic 2, but one needs to define them by means of a different cubic polynomial). Moreover, show that if f has a double root, then the corresponding plane curve has a singularity.

Ex 4.3.10

Use the homogenization of the equation of an elliptic curve and Exercise 4.3.3 to show that the projectification of an elliptic curve is regular if the characteristic is not 2.

*Ex 4.3.11

Show that the discussion on page 57 generalizes to arbitrary affine schemes : if $X := \operatorname{Spec}(R) \subseteq \mathbb{A}^n_K$ is a closed affine subscheme, then the closure of $|X|$ in $\mathbb{P}^n_K$ can be endowed with the structure of a projectice scheme $\tilde{X} := \operatorname{Spec}(\tilde{R})$, such that $X = \tilde{X}\cap\mathbb{A}^n_K$ (as schemes). To this end, generalize the notion of 'homogenization' as described in (4.2) to arbitrary ideals.

Ex 4.3.12

Show that a prime ideal $\mathfrak{p}$ in a Noetherian ring B is associated if and only if there exists an injective B-algebra homomorphism $B/\mathfrak{p}\to B$.

*Ex 4.3.13

Show that a regular ring A is a finite direct sum of regular domains as follows. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ be the minimal primes of A. Show that A is the direct sum of the $A/\mathfrak{p}_i$, and each $A/\mathfrak{p}_i$ is regular.

Ex 4.3.14

Let $B := K[\eta_1,\dots,\eta_4]$, and let $\mathfrak{p}$ be the kernel of the K-algebra homomorphism

$$B\to K[\xi,\zeta]\colon\ \eta_1\mapsto\xi^4,\eta_2\mapsto\xi^3\zeta,\eta_3\mapsto\xi\zeta^3,\eta_4\mapsto\zeta^4$$

and let R be the localization of $B/\mathfrak{p}$ at the maximal ideal corresponding to the origin. Clearly, R is a domain, so that η_4 is a regular element. Show that the annihilator of η_3^3 in R/η_4R is equal to the maximal ideal of that ring, showing that the depth of R/η_4R is zero. Conclude that R is not Cohen-Macaulay.

*Ex 4.3.15

We call a tuple $\mathbf{x} := (x_1, \dots, x_n)$ *in a ring* A quasi-regular *if for any* k *and any homogeneous form of degree* k *in* $A[\xi]$ *with* $\xi := (\xi_1, \dots, \xi_n)$*, if* $F(\mathbf{x}) \in I^{k+1}$ *then all coefficients of* F *lie in* $I := (x_1, \dots, x_n)A$*. Show that a regular sequence is quasi-regular. To this end, first show that if* y *is a zero-divisor modulo* I*, then it is also a zero-divisor modulo any* I^k*, then show the assertion by induction on* n*.*
Show that $\mathbf{x}$ *is quasi-regular if and only if the associated graded ring* $\mathrm{Gr}_I(A) := \bigoplus_n I^n/I^{n+1}$ *of* I *is isomorphic to* $(A/I)[\xi]$*.*

*Ex 4.3.16

Give a complete proof of Theorem 4.2.6 in every dimension. To this end, you must prove that powers and permutations preserve regular sequences (the latter is also proven in Exercise 4.3.17).

*Ex 4.3.17

Show that in a Noetherian local ring R*, a sequence* $(x_1, \dots, x_d)$ *is regular if and only if it is quasi-regular, by induction on* d *as follows. Only the converse requires proof, and to this end, first show that* x_1 *is* R*-regular by proving by induction on* k *that* $x_1 z = 0$ *implies* $z \in I^k$*, where* $I := (x_1, \dots, x_d)R$*, and then using Krull's Intersection Theorem (Theorem 1.4.11). Conclude by showing that* $(x_2, \dots, x_d)$ *is* R/x_1R*-quasi-regular.*
In particular, a regular sequence in a Noetherian local ring is permutable.

Ex 4.3.18

Use Corollary 4.2.8 to prove the 'unmixedness' theorem: if I *is an ideal of height* e *in a Cohen-Macaulay local ring* R*, and if* I *is generated by* e *elements, then* I *has no embedded primes, that is to say, any associated prime of* R/I *is minimal. Also show the converse: if a Noetherian local ring has the above unmixedness property, then it is Cohen-Macaulay.*

Chapter 5
Flatness

In this chapter we will study a very important and useful property, called 'flatness'. It is best studied by homological means, so we start off with developing some homological algebra.

5.1 Homological algebra

The main tool of homological algebra is the 'homology of a complex', so let's define this notion first.

Complexes. Let A be a ring. By a *complex* we mean a (possibly infinite) sequence of A-module homomorphisms $M_i \xrightarrow{d_i} M_{i-1}$, for $i \in \mathbb{Z}$, such that the composition of any two consecutive maps is zero. We often simply will say that

$$\dots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} M_{i-2} \xrightarrow{d_{i-2}} \dots \qquad (M_\bullet)$$

is a complex. The d_i are called the the *boundary maps* of the complex, and often are omitted from the notation. Of special interest are those complexes in which all modules from a certain point on, either on the left or on the right, are zero (which forces the corresponding maps to be zero as well). Such a complex will be called *bounded* from the left or right respectively. In that case, one often renumbers so that the first non-zero module is labeled with $i = 0$. If $M_\bullet$ is bounded from the left, one also might reverse the numbering, indicate this notationally by writing $M^\bullet$, and refer to this situation as a *co-complex* (and more generally, add for emphasis the prefix 'co-' to any object associated to it).

Homology. Since the composition $d_{i+1} \circ d_i$ is zero, we have in particular an inclusion $\mathrm{Im}(d_{i+1}) \subseteq \mathrm{Ker}(d_i)$. To measure in how far this fails to be an equality, we define the *homology* $\mathrm{H}_\bullet(M_\bullet)$ of $M_\bullet$ as the collection of modules

$$\mathrm{H}_i(M_\bullet) := \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$

If all homology modules are zero, $M_\bullet$ is called *exact*. More generally, we say that $M_\bullet$ is *exact at i* (or at M_i) if $\mathrm{H}_i(M_\bullet) = 0$. Note that $M_1 \xrightarrow{d_1} M_0 \to 0$ is exact (at zero) if and only if d_1 is surjective, and $0 \to M_0 \xrightarrow{d_0} M_{-1}$ is exact if and only if d_0 is injective. An exact complex is often also called an *exact sequence*. In particular, this terminology is compatible with the nomenclature for short exact sequence. If $M_\bullet$ is bounded from the right (indexed so that the last non-zero module is M_0), then the *cokernel* of $M_\bullet$ is the cokernel of $d_1 : M_1 \to M_0$. Put differently, the cokernel is simply the zero-th homology module $\mathrm{H}_0(M_\bullet)$. We say that $M_\bullet$ is *acyclic*, if all $\mathrm{H}_i(M_\bullet) = 0$ for $i > 0$. In that case, the *augmented* complex obtained by adding the cokernel of $M_\bullet$ to the right is then an exact sequence.

5.2 Flatness

We have arrived at the main notion of this chapter. Let A be a ring and M an A-module. Recall that $\cdot \otimes_A M$, that is to say, tensoring with respect to M, is a right exact functor, meaning that given an exact sequence

$$0 \to N_2 \to N_1 \to N_0 \to 0 \tag{5.1}$$

we get an exact sequence

$$N_2 \otimes_A M \to N_1 \otimes_A M \to N_0 \otimes_A M \to 0. \tag{5.2}$$

See [7, Proposition 2.18], where one also can find a good introduction to tensor products. We now call a module M *flat* if any short exact sequence (5.1) remains exact after tensoring, that is to say, we may add an additional zero on the left of (5.2). Put differently, M is flat if and only if $N' \otimes_A M \to N \otimes_A M$ is injective whenever $N' \to N$ is an injective homomorphism of A-modules. By breaking down a long exact sequence into short exact sequences (see Exercise 5.7.1), we immediately get:

5.2.1 *If M is flat, then any exact complex $N_\bullet$ remains exact after tensoring with M.*

The easiest examples of flat modules are the free modules:

5.2.2 *Any free module, and more generally, any projective module, is flat.*

Assume first that M is a free A-module, say of the form, $M \cong A^{(I)}$, where I is a possibly infinite index set (recall that an element of $A^{(I)}$ is a sequence $\mathbf{a} := (a_i \mid i \in I)$ such that all but finitely many a_i are zero; the 'unit' vectors $\mathbf{e}_i$ form a basis of $A^{(I)}$, where all entries in $\mathbf{e}_i$ are zero except the i-th, which equals one; and, any free A-module is isomorphic to some $A^{(I)}$). For any A-module H, we have $H \otimes_A M \cong H^{(I)}$. Since direct sums preserve injectivity, we now easily conclude that M is flat. The same argument applies if M is merely *projective*, meaning that it is a direct summand of a free module, say $M \oplus M' \cong F$ with F free. This completes the proof of the assertion. In particular, $A[\xi]$, being free over A, is flat as an A-module. The same is true for power series rings, at least over Noetherian rings, but the proof is a bit more involved (see Exercise 5.7.11). Flatness is preserved under base change in the following sense (the proof is left as Exercise 5.7.3):

5.2.3 *If M is a flat A-module, then M/IM is a flat A/I-module for each ideal $I \subseteq A$. More generally, if $A \to B$ is any homomorphism, then $M \otimes_A B$ is a flat B-module.*

5.2.4 *Any localization of a flat A-module is again flat. In particular, for every prime ideal $\mathfrak{p} \subseteq A$, the localization $A_\mathfrak{p}$ is flat as an A-module.*

The last assertion follows from the first and the fact that A, being free, is flat as an A-module by 5.2.2. The first assertion is not hard and is left as Exercise 5.7.3. Our next goal is to develop a homological tool to aid us in our study of flatness.

Tor modules. Let M be an A-module. A *projective resolution* of M is a complex $P_\bullet$, bounded from the right, in which all the modules P_i are projective, and such that the *augmented complex*

$$P_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$$

is exact. Put differently, a projective resolution of M is is an acyclic complex $P_\bullet$ of projective modules whose cokernel is equal to M. Tensoring this augmented complex with a second A-module N, yields a (possibly non-exact) complex

$$P_i \otimes_A N \to P_{i-1} \otimes_A N \to \cdots \to P_0 \otimes_A N \to M \otimes_A N \to 0.$$

The homology of the non-augmented part $P_\bullet \otimes N$ (that is to say, without the final module $M \otimes N$), is denoted

$$\mathrm{Tor}_i^A(M,N) := \mathrm{H}_i(P_\bullet \otimes_A N).$$

As the notation indicates, this does not depend on the choice of projective resolution $P_\bullet$. Moreover, we have for each i an isomorphism $\mathrm{Tor}_i^A(M,N) \cong \mathrm{Tor}_i^A(N,M)$. We will not prove these properties here (the proofs are not that hard anyway, see for instance [18, Appendix 3] or [41, Appendix B]). Since tensoring is right exact, a quick calculation shows that

$$\mathrm{Tor}_0^A(M,N) \cong M \otimes_A N.$$

The next result is a general fact of ‘derived functors’ (Tor is indeed the *derived functor* of the tensor product as discussed for instance in [41, Appendix B]; for a proof of the next result, see Exercise 5.7.22).

5.2.5 *If*

$$0 \to N' \to N \to N'' \to 0$$

is a short exact sequence of A-modules, then we get for every A-module M a long exact sequence

$$\dots \to \mathrm{Tor}^A_{i+1}(M,N'') \xrightarrow{\delta_{i+1}} \mathrm{Tor}^A_i(M,N') \to \mathrm{Tor}^A_i(M,N) \to \mathrm{Tor}^A_i(M,N'') \xrightarrow{\delta_i} \mathrm{Tor}^A_{i-1}(M,N') \to \dots$$

where the δ_i are the so-called connecting homomorphisms, *and the remaining maps are induced by the original maps.*

Tor-criterion for flatness. We can now formulate a homological criterion for flatness. More flatness criteria will be discussed in §5.6 below.

Theorem 5.2.6. *For an A-module M, the following are equivalent*

1. *M is flat;*
2. $\mathrm{Tor}^A_i(M,N) = 0$ *for all $i > 0$ and all A-modules N;*
3. $\mathrm{Tor}^A_1(M,A/I) = 0$ *for all finitely generated ideals $I \subseteq A$.*

Proof. Let $P_\bullet$ be a projective resolution of N. If M is flat, then $P_\bullet \otimes_A M$ is again exact by 5.2.1, and hence its homology $\mathrm{Tor}^A_i(N,M) = \mathrm{H}_i(P_\bullet \otimes_A N)$ vanishes. Conversely, if (2) holds, then tensoring the exact sequence $0 \to N' \to N \to N/N' \to 0$ with M yields in view of 5.2.5 an exact sequence

$$0 = \mathrm{Tor}^A_1(M,N/N') \to M \otimes_A N' \to M \otimes_A N$$

showing that the latter map is injective.

Remains to show (3) $\Rightarrow$ (1), which for simplicity I will only do in the case A is Noetherian; the general case is treated in Exercise 5.7.6. We must show that if $N' \subseteq N$ is an injective homomorphism of A-modules, then $M \otimes_A N' \to M \otimes_A N$ is again injective, and we already observed that this follows once we showed that $\mathrm{Tor}^A_1(M,N/N') = 0$. I claim that it suffices to show this for N finitely generated: indeed, if N is arbitrary and $t := m_1 \otimes n_1 + \dots + m_s \otimes n_s$ is an element in $M \otimes N'$ which is sent to zero in $M \otimes N$, then by definition of tensor product, there exists a finitely generated submodule $N_1 \subseteq N$ containing all n_i such that $t = 0$ as an element of $M \otimes N_1$. In particular, t is an element of $M \otimes N'_1$, where $N'_1 := N' \cap N_1$, whose image in $M \otimes N_1$ is zero. Assuming momentarily that the finitely generated case is already proven, t is therefore zero in $M \otimes N'_1$, whence a fortiori in $M \otimes N'$.

So we may assume that N is finitely generated. We prove by induction on r, the number of generators of N/N', that $\mathrm{Tor}^A_1(M,N/N') = 0$. If $r = 1$, then N/N' is of the from A/I with $I \subseteq A$ an ideal, and the result holds by assumption. For $r > 1$, let $t \in N$ be such that its image in N/N' is a minimal generator. Put $H := N' + At$, so that N/H is generated by $r-1$ elements, and H/N' is cyclic. Tensoring the short exact sequence

$$0 \to H/N' \to N/N' \to N/H \to 0$$

yields by 5.2.5 an exact sequence

$$\mathrm{Tor}_1^A(M,H/N') \to \mathrm{Tor}_1^A(M,N/N') \to \mathrm{Tor}_1^A(M,N/H).$$

By induction, the two outer modules vanish, whence so does the inner. □

For Noetherian rings we can even restrict the test in (3) to prime ideals (but see also Theorem 5.6.7 below, which reduces the test to a single ideal):

Corollary 5.2.7. *Let A be a Noetherian ring and M an A-module. If $\mathrm{Tor}_1^A(M,A/\mathfrak{p})$ vanishes for all prime ideals $\mathfrak{p} \subseteq A$, then M is flat. More generally, if, for some $i \geq 1$, every $\mathrm{Tor}_i^A(M,A/\mathfrak{p})$ vanishes for $\mathfrak{p}$ running over the prime ideals in A, then $\mathrm{Tor}_i^A(M,N)$ vanishes for all (finitely generated) A-modules N.*

Proof. The first assertion follows from the last by (3). The last assertion, for finitely generated modules, follows from the fact that every such module N admits a *prime filtration*, that is to say, a finite ascending chain of submodules

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_e = N \tag{5.3}$$

such that each successive quotient N_j/N_{j-1} is isomorphic to the (cyclic) A-module $A/\mathfrak{p}_j$ for some prime ideal $\mathfrak{p}_j \subseteq A$, for $j = 1,\ldots,e$ (see Exercise 5.7.8). By induction on j, one then derives from the long exact sequence (5.2.5) that $\mathrm{Tor}_i^A(M,N_j) = 0$, whence in particular $\mathrm{Tor}_i^A(M,N) = 0$. To prove the same result for N arbitrary (which we will not be needing in the sequel), use an argument similar to the one in the proof of Theorem 5.2.6 (see Exercise 5.7.6). □

Corollary 5.2.8. *Let*

$$0 \to M_1 \to F \to M \to 0$$

be an exact sequence of A-modules. If F is flat, then

$$\mathrm{Tor}_i^A(M,N) \cong \mathrm{Tor}_{i-1}^A(M_1,N)$$

for all $i \geq 2$ and all A-modules N.

Proof. From the long exact sequence of Tor (see 5.2.5), we get exact sequences

$$0 = \mathrm{Tor}_i^A(F,N) \to \mathrm{Tor}_i^A(M,N) \to \mathrm{Tor}_{i-1}^A(M_1,N) \to \mathrm{Tor}_{i-1}^A(F,N) = 0$$

where the two outer most modules vanish because of Theorem 5.2.6. □

Note that in case F is actually projective in the above sequence, then M_1 is called a (first) *syzygy* of M. Therefore, the previous result is particularly useful when working with syzygies (for a typical application, see the proof of 5.5.1.)

5.3 Faithful flatness

We call an A-module M *faithful*, if $\mathfrak{m}M \neq M$ for all (maximal) ideals $\mathfrak{m}$ of A.[1] By Nakayama's Lemma, we immediately get:

5.3.1 *Any finitely generated module over a local ring is faithful.*

Of particular interest are the faithful modules which are moreover flat, called *faithfully flat* modules (see Exercise 5.7.23 for a homological characterization). It is not hard to see that any free or projective module is faithfully flat. On the other hand, no proper localization of A is faithfully flat.

5.3.2 *If M is a faithfully flat A-module, then $M \otimes_A N$ is non-zero, for every non-zero A-module N. Moreover, if $A \to B$ is an arbitrary homomorphism, then $M \otimes_A B$ is a faithfully flat B-module.*

Indeed, for the first assertion, let $N \neq 0$ and choose a non-zero element $n \in N$. Since $I := \mathrm{Ann}_A(n)$ is then a proper ideal, it is contained in some maximal ideal $\mathfrak{m} \subseteq A$. Note that $An \cong A/I$. Tensoring the induced inclusion $A/I \hookrightarrow N$ with M gives by assumption an injection $M/IM \hookrightarrow M \otimes_A N$. The first of these modules is non-zero, since $IM \subseteq \mathfrak{m}M \neq M$, whence so is the second, as we wanted to show. To prove the second assertion, $M \otimes_A B$ is flat over B by 5.2.3. Let $\mathfrak{n}$ be a maximal ideal of B, and let $\mathfrak{p} := \mathfrak{n} \cap A$ be its contraction to A. In particular, $M/\mathfrak{p}M$ is flat over $A/\mathfrak{p}$, and an easy calculation then shows that it is faithfully flat. Therefore, by the first assertion, $M/\mathfrak{p}M \otimes_{A/\mathfrak{p}} B/\mathfrak{n}$ is non-zero. As the latter is just $(M \otimes_A B)/\mathfrak{n}(M \otimes_A B)$, we showed that $M \otimes_A B$ is also faithful.

In most of our applications, the A-module has the additional structure of an A-algebra. In particular, we call a ring homomorphism $A \to B$ (*faithfully*) *flat* if B is (faithfully) flat as an A-module. Since by definition a *local homomorphism* of local rings $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a ring homomorphism with the additional property that $\mathfrak{m} \subseteq \mathfrak{n}$, we get immediately:

5.3.3 *Any local homomorphism which is flat, is faithfully flat.* □

Proposition 5.3.4. *If $A \to B$ is faithfully flat, then for every ideal $I \subseteq A$, we have $I = IB \cap A$, and hence in particular, $A \to B$ is injective.*

Proof. For I equal to the zero ideal, this just says that $A \to B$ is injective. Suppose this last statement is false, and let $a \in A$ be a non-zero element in the kernel of $A \to B$, that is to say, $a = 0$ in B. However, by 5.3.2, the module $aA \otimes_A B$ is non-zero, say, containing the non-zero element x. Hence x is of the form $ra \otimes b$ for some $r \in A$ and $b \in B$, and therefore equal to $r \otimes ab = r \otimes 0 = 0$, contradiction.

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[1] The reader be warned that this is a less conventional terminology: 'faithful' often is taken to mean that the annihilator of the module is zero. However, in view of the (well-established) term 'faithfully flat', our usage seems more reasonable: faithfully flat now simply means faithful and flat.

To prove the general case, note that B/IB is a flat A/I-module by 5.2.3. It is clearly also faithful, so that applying our first argument to the natural homomorphism $A/I \to B/IB$ yields that it must be injective, which precisely means that $I = IB \cap A$. □

A ring homomorphism $A \to B$ such that $I = IB \cap A$ for all ideals $I \subseteq A$ is called *cyclical pure*. Hence faithfully flat homomorphisms are cyclically pure (for an other example see 8.5.4 below). We can paraphrase this as 'faithful flatness preserves the ideal structure of a ring', that is to say, in terms of Grassmanians (see page 30), we have:

5.3.5 *If $A \to B$ is faithfully flat, or more generally, cyclically pure, then the induced map* $\mathrm{Grass}(A) \to \mathrm{Grass}(B)\colon I \mapsto IB$ *on the Grassmanians is injective.* □

Since a ring A is Noetherian if and only if its Grassmanian $\mathrm{Grass}(A)$ is well-ordered (i.e., has the descending chain condition; recall that the order on $\mathrm{Grass}(A)$ is given by reverse inclusion), we get immediately the following Noetherianity criterion from 5.3.5:

Corollary 5.3.6. *Let $A \to B$ be a faithfully flat, or more generally, a cyclically pure homomorphism. If B is Noetherian, then so is A.* □

A similar argument shows:

5.3.7 *If $R \to S$ is a faithfully flat homomorphism of local rings, and if $I \subseteq R$ is minimally generated by e elements, then so is IS.*

Clearly, IS is generated by at most e elements. By way of contradiction, suppose it is generated by strictly fewer elements. By Nakayama's lemma, we may choose these generators already in I. So there exists an ideal $J \subseteq I$, generated by less than e elements, such that $JS = IS$. However, by cyclic purity (Proposition 5.3.4), we have $J = JS \cap R = IS \cap R = I$, contradicting that I requires at least e generators. □

If $A \to B$ is a flat or faithfully flat homomorphism, then we also will call the corresponding morphism $Y := \mathrm{Spec}(B) \to X := \mathrm{Spec}(A)$ *flat* or *faithfully flat* respectively. In Exercise 5.7.14, you are asked to prove that:

5.3.8 *A morphism $f\colon Y \to X$ of affine schemes is flat if and only if for every (closed) point $y \in Y$, the induced homomorphism $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat.*

Theorem 5.3.9. *A morphism $Y \to X$ of affine schemes is faithfully flat if and only if it is flat and surjective.*

Proof. Let $A \to B$ be the corresponding homomorphism. Assume $A \to B$ is faithfully flat, and let $\mathfrak{p} \subseteq A$ be a prime ideal. Surjectivity of the morphism amounts to showing that there is at least one prime ideal of B lying over $\mathfrak{p}$. Now, by 5.3.2, the base change $A_\mathfrak{p} \to B_\mathfrak{p}$ is again faithfully flat, and hence in particular $\mathfrak{p}B_\mathfrak{p} \neq B_\mathfrak{p}$. In other words, the fiber ring $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is non-empty, which is what we wanted to prove (indeed, take any maximal ideal $\mathfrak{n}$ of $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ and let $\mathfrak{q} := \mathfrak{n} \cap B$; then verify that $\mathfrak{q} \cap A = \mathfrak{p}$.)

Conversely, assume $Y \to X$ is flat and surjective, and let $\mathfrak{m}$ be a maximal ideal of A. Let $\mathfrak{q} \subseteq B$ be an ideal lying over $\mathfrak{m}$. Hence $\mathfrak{m}B \subseteq \mathfrak{q} \neq B$, showing that B is faithful over A. □

5.4 Flatness and regular sequences

The first fundamental fact regarding regular sequences and flat homomorphisms is:

Proposition 5.4.1. *If $A \to B$ is a flat homomorphism and $\mathbf{x}$ is an A-regular sequence, then $\mathbf{x}$ is also B-regular.*

Proof. We induct on the length n of $\mathbf{x} := (x_1, \ldots, x_n)$. Assume first $n = 1$. Multiplication by x_1, that is to say, the homomorphism $A \xrightarrow{x_1} A$, is injective, whence remains so after tensoring with B by 5.2.3. It is not hard to see that the resulting homomorphism is again multiplication $B \xrightarrow{x_1} B$, showing that x_1 is B-regular. For $n > 1$, the base change $A/x_1A \to B/x_1B$ is flat, so that by induction $(x_2, \ldots, x_n)$ is B/x_1B-regular. Hence we are done, since x_1 is B-regular by the previous argument. □

Tor modules behave well under deformation by a regular sequence in the following sense.

Proposition 5.4.2. *Let $\mathbf{x}$ be a regular sequence in a ring A, and let M and N be two A-modules. If $\mathbf{x}$ is M-regular and $\mathbf{x}N = 0$, then we have for each i an isomorphism*

$$\operatorname{Tor}_i^A(M,N) \cong \operatorname{Tor}_i^{A/\mathbf{x}A}(M/\mathbf{x}M,N).$$

Proof. By induction on the length of the sequence, we may assume that we have a single A-regular and M-regular element x. Put $B := A/xA$. From the short exact sequence

$$0 \to A \xrightarrow{x} A \to B \to 0$$

we get after tensoring with M, a long exact sequence of Tor-modules as in 5.2.5. Since $\operatorname{Tor}_i^A(A,M)$ vanishes for all i, so must each $\operatorname{Tor}_i^A(M,B)$ in this long exact sequence for $i > 1$. Furthermore, the initial part of this long exact sequence is

$$0 \to \operatorname{Tor}_1^A(M,B) \to M \xrightarrow{x} M \to M/xM \to 0$$

proving that $\operatorname{Tor}_1^A(M,B)$ too vanishes as x is M-regular. Now, let $P_\bullet$ be a projective resolution of M. The homology of $\bar{P}_\bullet := P_\bullet \otimes_A B$ is by definition $\operatorname{Tor}_i^A(M,B)$, and since we showed that this is zero, $\bar{P}_\bullet$ is exact, whence a projective resolution of M/xM. Hence we can calculate $\operatorname{Tor}_i^B(M/xM,N)$ as the homology of $\bar{P}_\bullet \otimes_B N$ (note that by assumption, N is a B-module). However, the latter complex is equal to $P_\bullet \otimes_A N$ (which we can use to calculate $\operatorname{Tor}_i^A(M,N)$), and hence both complexes have the same homology, as we wanted to show. □

5.5 Projective dimension

If an A-module M has a projective resolution $P_\bullet$ which is also bounded from the left, that is to say, is of the form

$$0 \to P_e \to \cdots \to P_0 \to M \to 0$$

then we say that M has *finite projective dimension*. The smallest length e of such an exact sequence is called the *projective dimension* of M and is denoted $\operatorname{projdim}(M)$; if M does not have a finite projective resolution, then we set $\operatorname{projdim}(M) := \infty$. Clearly, the projective dimension of a module is zero if and only if it is projective. The connection with Tor is immediate by virtue of the latter's definition as the homology of the tensor product with a projective resolution:

5.5.1 *If M is an A-module of projective dimension e, then $\operatorname{Tor}_i^A(M,N) = 0$ for all $i > e$ and all A-modules N. Moreover, if*

$$0 \to H \to P_e \to P_{e-1} \to \cdots \to P_0 \to M \to 0$$

is exact, with all P_e projective, then H is flat (and in fact projective).

Only the second assertion requires explanation. By Corollary 5.2.8, the vanishing of $\operatorname{Tor}_{e+1}^A(M,N)$ is equivalent with the vanishing of $\operatorname{Tor}_1^A(H,N)$. Hence H is a flat A-module by Theorem 5.2.6. To prove that it is actually projective, one needs Ext-functors, which we will not treat.

If x is an A-regular element, then A/xA has projective dimension one, as is clear from the exact sequence

$$0 \to A \xrightarrow{x} A \to A/xA \to 0. \tag{5.4}$$

In fact, this is also true for regular sequences of any length, but to prove this we need a new tool:

Minimal resolutions. A complex $M_\bullet$ over a local ring $(R,\mathfrak{m})$ is called *minimal* if the kernel of each boundary $d_i\colon M_i \to M_{i-1}$ lies inside $\mathfrak{m}M_i$. The next result is easily derived from Nakayama's lemma and induction (see Exercise 5.7.9):

5.5.2 *Every finitely generated module over a Noetherian local ring admits a minimal free resolution, consisting of finitely generated free modules.*

Corollary 5.5.3. *Over a Noetherian local ring, a finitely generated module is flat if and only if it is projective if and only if it is free.*

Proof. The converse implications are all trivial. So remains to show that if G is a finitely generated flat R-module, then it is free. By 5.5.2 (or Nakayama's lemma), we can find a finitely generated free A-module F, and a surjective map $F \to G$ whose kernel H lies inside $\mathfrak{m}F$. In other words, $F/\mathfrak{m}F \cong G/\mathfrak{m}G$. On the other hand, tensoring the exact sequence $0 \to H \to F \to G \to 0$ with $k := R/\mathfrak{m}$ yields by 5.2.5 an exact sequence

$$0 = \mathrm{Tor}_1^R(G,k) \to H/\mathfrak{m}H \to F/\mathfrak{m}F \to G/\mathfrak{m}G \to 0$$

where we used the flatness of G to obtain the vanishing of the first module. Since the last arrow is an isomorphism, $H/\mathfrak{m}H = 0$, which by Nakayama's lemma implies $H = 0$, that is to say, $F = G$ is free. □

Minimal resolutions are essentially unique:

Proposition 5.5.4. *Let $(R,\mathfrak{m})$ be a Noetherian local ring with residue field k. Let M be a finitely generated R-module, and let*

$$\dots F_i \to F_{i-1} \to \cdots \to F_0 \to M \to 0 \qquad (F_\bullet)$$

be a minimal free resolution. For each $i \geq 0$, the i-th Betti number *of M, that is to say, the k-vector space dimension of* $\mathrm{Tor}_i^R(M,k)$*, is equal to the rank of F_i.*

Moreover, the projective dimension of M is equal to the supremum of all i for which $\mathrm{Tor}_i^R(M,k) \neq 0$*, and hence is less than or equal to* $\mathrm{projdim}(k)$.

Proof. By definition, $\mathrm{Tor}_i^R(M,k)$ is the homology of $F_\bullet \otimes_R k$. Since $F_\bullet$ is minimal, the boundaries in $F_\bullet \otimes_R k$ are all zero, so that $\mathrm{H}_i(F_\bullet \otimes_R k) = F_i \otimes_R k$. This shows that the Betti numbers of M coincide with the ranks of the free modules in $F_\bullet$ (and hence the latter are uniquely determined). The second assertion follows immediately from this and from 5.5.1. □

Put differently, the previous result yields a criterion for a finitely generated module to have finite projective dimension, namely that some Betti number be zero. We can now prove (5.4) for any regular sequence:

Corollary 5.5.5. *If $\mathbf{x}$ is a regular sequence in a Noetherian local ring R, then $R/\mathbf{x}R$ has finite projective dimension.*

Proof. We prove by induction on the length l of the sequence that $R/\mathbf{x}R$ has projective dimension at most l, where the case $l = 1$ is (5.4). Write $\mathbf{x} = (\mathbf{y}, x)$ with $\mathbf{y}$ a regular sequence of length $l-1$. The short exact sequence

$$0 \to R/\mathbf{y}R \xrightarrow{x} R/\mathbf{y}R \to R/\mathbf{x}R \to 0$$

when tensored with the residue field k yields by 5.2.5 a long exact sequence

$$\mathrm{Tor}_i^R(R/\mathbf{y}R,k) \to \mathrm{Tor}_i^R(R/\mathbf{x}R,k) \to \mathrm{Tor}_{i-1}^R(R/\mathbf{y}R,k)$$

For $i-1 \geq l$, both outer modules are zero by induction and Proposition 5.5.4, whence so is the inner module. Using Proposition 5.5.4 once more, we see that $R/\mathbf{x}R$ therefore has projective dimension at most l. □

In fact, the projective dimension of $R/\mathbf{x}R$ is exactly l. Moreover, this result remains true if the ring is not local, nor even Noetherian. This more general result is proven by means of a complex called the *Koszul complex*, whose homology actually measures the failure of a sequence being regular. For all this, see for instance [41, §16] or [18, §17].

Theorem 5.5.6 (Serre). *A d-dimensional Noetherian local ring R is regular if and only if its residue field k has finite projective dimension (equal to d). If this is the case, then any module has projective dimension at most d.*

Proof (partim). Regarding the first statement, we will only prove the direct implication. Since a regular local ring R is Cohen-Macaulay by Proposition 4.2.3, its maximal ideal is generated by a regular sequence $\mathbf{x}$. Hence $k = R/\mathbf{x}R$ has finite projective dimension by Corollary 5.5.5. To prove the converse, some additional tools (like Ext-functors) are required, and we refer the reader to the literature (see for instance [41, Theorem 19.2] or [18, Theorem 19.12]).

The second assertion for finitely generated modules now follows immediately from the first and Proposition 5.5.4. To also prove this for non-finitely generated modules, again Ext-functors are needed (see for instance [41, §19 Lemma 2] or [18, Theorem A3.18]). □

Although we did not give a complete proof, we did prove most of what we will use, with the most notable exception Corollary 5.5.8 below. We can even formulate a global version, which was first proven by Hilbert in the case A is a polynomial ring over a field.

Theorem 5.5.7. *Over a d-dimensional regular ring A, any finitely generated A-module M has projective dimension at most d.*

Proof. Choose an exact sequence

$$0 \to H \to A^{n_d} \to \dots A^{n_1} \to A^{n_0} \to M \to 0$$

for some n_i and some finitely generated module H, the *d-th syzygy* of M, given as the kernel of the homomorphism $A^{n_d} \to A^{n_{d-1}}$. Since $A_\mathfrak{m}$ is flat over A, for $\mathfrak{m}$ a maximal ideal of A, we get an exact sequence

$$0 \to H_\mathfrak{m} \to A_\mathfrak{m}^{n_d} \to \dots A_\mathfrak{m}^{n_1} \to A_\mathfrak{m}^{n_0} \to M_\mathfrak{m} \to 0.$$

By Theorem 5.5.6, the $A_\mathfrak{m}$-module $M_\mathfrak{m}$ has finite projective dimension, and hence, $H_\mathfrak{m}$ is flat by 5.5.1. Therefore, H is projective by Exercise 5.7.16. □

Corollary 5.5.8. *If A is a regular ring, then so is any of its localizations.*

Proof. A moment's reflection yields that we only need to prove this when A is already local, and $\mathfrak{p}$ is some (non-maximal) prime ideal. By Theorem 5.5.6, the residue ring $A/\mathfrak{p}$ admits a finite free resolution. Since localization is flat, tensoring this resolution with $A_\mathfrak{p}$ gives a finite free resolution of $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ viewed as an $A_\mathfrak{p}$-module. Hence $A_\mathfrak{p}$ is regular by Theorem 5.5.6 (this is the one spot where we use the unproven converse from that theorem). □

5.6 Flatness criteria

Because flatness will play such a crucial role in our later work, we want several ways of detecting it. In this section, we will see five such criteria.

Equational criterion for flatness Our first criterion is very useful in applications (see for instance Theorem 7.4.3), and works without any hypothesis on the ring or module. To give a streamlined presentation, let us introduce the following terminology: given an A-module N, and tuples $\mathbf{b}_i$ in A^n, by an *N-linear combination* of the $\mathbf{b}_i$, we mean a tuple in N^n of the form $n_1\mathbf{b}_1 + \cdots + n_s\mathbf{b}_s$ where $n_i \in N$. Of course, if N has the structure of an A-algebra, this is just the usual terminology. Given a (finite) homogeneous linear system of equations

$$L_1(t) = \cdots = L_s(t) = 0 \tag{$\mathscr{L}$}$$

over A in the n variables t, we denote the A-submodule of N^n consisting of all solutions of $(\mathscr{L})$ in N by $\mathrm{Sol}_N(\mathscr{L})$, and we let $f_{\mathscr{L}}\colon N^n \to N^s$ be the map given by substitution $\mathbf{x} \mapsto (L_1(\mathbf{x}), \dots, L_s(\mathbf{x}))$. In particular, we have an exact sequence

$$0 \to \mathrm{Sol}_N(\mathscr{L}) \to N^n \xrightarrow{f_{\mathscr{L}}} N^s. \tag{$\dagger_{\mathscr{L}/N}$}$$

Theorem 5.6.1. *A module M over a ring A is flat if and only if every solution in M of a homogeneous linear equation in finitely many variables over A is an M-linear combination of solutions in A. Moreover, instead of a single linear equation, we may take any finite system of linear equations in the above criterion.*

Proof. We will only prove the first assertion, and leave the second for the exercises (Exercise 5.7.10). Let $L = 0$ be a homogeneous linear equation in n variables with coefficients in A. If M is flat, then the exact sequence $(\dagger_{L/A})$ remains exact after tensoring with M, that is to say,

$$0 \to \mathrm{Sol}_A(L) \otimes_A M \to M^n \xrightarrow{f_L} M,$$

and hence by comparison with $(\dagger_{L/M})$, we get

$$\mathrm{Sol}_M(L) = \mathrm{Sol}_A(L) \otimes_A M.$$

From this it follows easily that any tuple in $\mathrm{Sol}_M(L)$ is an M-linear combination of tuples in $\mathrm{Sol}_A(L)$, proving the direct implication.

Conversely, assume the condition on the solution sets of linear forms holds. To prove that M is flat, we will verify condition (3) in Theorem 5.2.6. To this end, let $I := (a_1, \dots, a_k)A$ be a finitely generated ideal of A. Tensor the exact sequence $0 \to I \to A \to A/I \to 0$ with M to get by 5.2.5 an exact sequence

$$0 = \mathrm{Tor}_1^A(A, M) \to \mathrm{Tor}_1^A(A/I, M) \to I \otimes_A M \to M. \tag{5.6}$$

Suppose y is an element in $I \otimes M$ that is mapped to zero in M. Writing $y = a_1 \otimes m_1 + \cdots + a_k \otimes m_k$ for some $m_i \in M$, we get $a_1 m_1 + \cdots + a_k m_k = 0$. Hence by assumption, there exist solutions $\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(s)} \in A^k$ of the linear equation $a_1 t_1 + \cdots + a_k t_k = 0$, such that

$$(m_1, \ldots, m_k) = n_1 \mathbf{b}^{(1)} + \cdots + n_s \mathbf{b}^{(s)}$$

for some $n_i \in M$. Letting $b_i^{(j)}$ be the i-th entry of $\mathbf{b}^{(j)}$, we see that

$$y = \sum_{i=1}^{k} a_i \otimes m_i = \sum_{i=1}^{k} \sum_{j=1}^{s} a_i \otimes n_j b_i^{(j)} = \sum_{j=1}^{s} (\sum_{i=1}^{k} a_i b_i^{(j)}) \otimes n_j = \sum_{j=1}^{s} 0 \otimes n_j = 0.$$

Hence $I \otimes_A M \to M$ is injective, so that $\mathrm{Tor}_1^A(A/I, M)$ must be zero by (5.6). Since this holds for all finitely generated ideals $I \subseteq A$, we proved that M is flat by Theorem 5.2.6(3). □

It is instructive to view the previous result from the following perspective. To a homogeneous linear equation $L = 0$, we associated an exact sequence $(\dagger_{L/N})$. The image of f_L is of the form IN where I is the ideal generated by the coefficients of the linear form defining L. In case $N = B$ is an A-algebra, this leads to the following extended exact sequence

$$0 \to \mathrm{Sol}_B(L) \to B^n \xrightarrow{f_L} B \to B/IB \to 0. \qquad (\ddagger_{IB})$$

This justifies calling $\mathrm{Sol}_B(L)$ the *module of syzygies* of IB (one checks that it only depends on the ideal I). Therefore, we may paraphraze the equational flatness criterion for algebras as follows:

5.6.2 *A ring homomorphism $A \to B$ is flat if and only if taking syzygies commutes with extension in the sense that the module of syzygies of IB is the extension to B of the module of syzygies of I.*

Here is one application of the equational flatness criterion.

Corollary 5.6.3. *The canonical embedding of a Noetherian ring inside its ultrapower is faithfully flat.*

Proof. Let A be a ring and $A_\natural$ an ultrapower of A. Recall that $A \to A_\natural$ is given by sending an element $a \in A$ to the ultraproduct $\mathrm{ulim}_{w\to\infty} a$ of the constant sequence. If $\mathfrak{m} \subseteq A$ is a maximal ideal, then $\mathfrak{m}A_\natural$ is its ultraproduct (since $\mathfrak{m}$ is finitely generated) whence again maximal, showing that $A_\natural$ is faithful. To show it is also flat, we use the equational criterion. Let $L = 0$ be a homogeneous linear equation with coefficients in A. Let $\mathbf{a} \in A_\natural^n$ be a solution of $L = 0$ in $A_\natural$. Write $\mathbf{a}$ as an ultraproduct of tuples $\mathbf{a}_w \in A^n$. By Łos' Theorem (Theorem 1.3.1), almost each $\mathbf{a}_w \in \mathrm{Sol}_A(L)$. Hence $\mathbf{a}$ lies in the ultrapower of $\mathrm{Sol}_A(L)$. By Noetherianity, $\mathrm{Sol}_A(L)$ is finitely generated, and hence, its ultrapower is simply the $A_\natural$-module generated by $\mathrm{Sol}_A(L)$ (see Exercise 1.5.8), so that we are done by Theorem 5.6.1. □

Coherency criterion

We can turn this into a criterion for coherency. Recall that a ring A is called *coherent*, if the solution set of any homogeneous linear equation over A is finitely generated. Clearly, Noetherian rings are coherent. We have:

Theorem 5.6.4. *A ring A is coherent if and only if the canonical embedding into one of its ultrapowers is flat.*

Proof. The direct implication is proven by the same argument that proves Corollary 5.6.3, since we really only used that A is coherent in that argument. Conversely, suppose $A \to A_\natural$ is flat. Towards a contradiction, assume L is a linear form (in n indeterminates) over A whose solution set $\mathrm{Sol}_A(L)$ is infinitely generated. In particular, we can choose a sequence $\mathbf{a}_w$ in $\mathrm{Sol}_A(L)$ which is contained in no finitely generated submodule of $\mathrm{Sol}_A(L)$ (see Exercise 5.7.25). The ultraproduct $\mathbf{a}_\natural \in A_\natural^n$ of this sequence lies in $\mathrm{Sol}_{A_\natural}(L)$ by Łos' Theorem. Hence, by Theorem 5.6.1, there exists a finitely generated submodule $H \subseteq \mathrm{Sol}_A(L)$ such that $\mathbf{a}_\natural \in H \cdot A_\natural$. Therefore, almost all $\mathbf{a}_j$ lie in H by Łos' Theorem, contradiction. □

Quotient criterion for flatness. The next criterion is derived from our Tor-criterion (Theorem 5.2.6):

Theorem 5.6.5. *Let $A \to B$ be a flat homomorphism, and let $I \subseteq B$ be an ideal. The induced homomorphism $A \to B/I$ is flat if and only if $\mathfrak{a}B \cap I = \mathfrak{a}I$ for all finitely generated ideals $\mathfrak{a} \subseteq A$.*

Moreover, if A is Noetherian, we only need to check the above criterion for $\mathfrak{a}$ a prime ideal of A.

Proof. From the exact sequence $0 \to I \to B \to B/I \to 0$ we get after tensoring with $A/\mathfrak{a}$ an exact sequence

$$0 = \mathrm{Tor}_1^A(B, A/\mathfrak{a}) \to \mathrm{Tor}_1^A(B/I, A/\mathfrak{a}) \to I/\mathfrak{a}I \to B/\mathfrak{a}B$$

where we used the flatness of B for the vanishing of the first module. The kernel of $I/\mathfrak{a}I \to B/\mathfrak{a}B$ is easily seen to be $(\mathfrak{a}B \cap I)/\mathfrak{a}I$. Hence $\mathrm{Tor}_1^A(B/I, A/\mathfrak{a})$ vanishes if and only if $\mathfrak{a}B \cap I = \mathfrak{a}I$. This proves by Theorem 5.2.6 the stated equivalence in the first assertion; the second assertion follows by the same argument, this time using Corollary 5.2.7. □

To put this criterion to use, we need another definition (for further applications, see Theorem 11.2.1 and Exercise 11.3.11 below). The (A-)*content* of a polynomial $f \in A[\xi]$ (or a power series $f \in A[[\xi]]$) is by definition the ideal generated by its coefficients.

Corollary 5.6.6. *Let A be a Noetherian ring, let ξ be a finite tuple of indeterminates, and let B denote either $A[\xi]$ or $A[[\xi]]$. If $f \in B$ has content one, then B/fB is flat over A.*

Proof. By 5.2.2 or Exercise 5.7.11, the natural map $A \to B$ is flat. To verify the second criterion in Theorem 5.6.5, let $\mathfrak{p} \subseteq A$ be a prime ideal. The forward inclusion in the to be proven equality $\mathfrak{p}fB = \mathfrak{p}B \cap fB$ is immediate. To prove the other, take $g \in \mathfrak{p}B \cap fB$. In particular, $g = fh$ for some $h \in B$. Since $\mathfrak{p} \subseteq A$ is a prime ideal, so is $\mathfrak{p}B$ (this is a property of polynomial or power series rings, not of flatness!). Since f has content one, $f \notin \mathfrak{p}B$ whence $h \in \mathfrak{p}B$. This yields $g \in \mathfrak{p}fB$, as we needed to prove. □

Local criterion for flatness. For finitely generated modules, we have the following criterion:

Theorem 5.6.7 (Local flatness theorem–finitely generated case). *Let R be a Noetherian local ring with residue field k. If M is a finitely generated R-module whose first Betti number vanishes, that is to say, if $\operatorname{Tor}_1^R(M,k) = 0$, then M is flat.*

Proof. Take a minimal free resolution

$$\cdots \to F_1 \to F_0 \to M \to 0.$$

of M. By Proposition 5.5.4, the rank of F_1 is zero, so that $M \cong F_0$ is free whence flat. □

There is a much stronger version of this result, where we may replace the condition that M is finitely generated over R by the condition that M is finitely generated over a Noetherian local R-algebra S. Since we will not really need this result, we refer the reader either to the literature (see for instance [41, Theorem 22.3] or [18, Theorem 6.8]), or to Project 5.8.

Cohen-Macaulay criterion for flatness. To formulate our next criterion, we need a definition.

Definition 5.6.8 (Big Cohen-Macaulay modules). Let R be a Noetherian local ring, and let M be an arbitrary R-module. We call M a *big Cohen-Macaulay module*, if there exists a system of parameters on R which is M-regular. If moreover every system of parameters is M-regular, then we call M a *balanced big Cohen-Macaulay*.

It has become tradition to add the somehow redundant adjective 'big' to emphasize that the module is not necessarily finitely generated. It is one of the greatest open problems in homological algebra to show that every Noetherian local ring has at least one big Cohen-Macaulay module, and this is known to be the case for any Noetherian local ring containing a field (see §9.4 and §10.4).[2] A Cohen-Macaulay local ring is clearly a balanced big Cohen-Macaulay module over itself, so the problem of the existence of these modules is only important for deriving results over Noetherian local rings with 'worse than Cohen-Macaulay' singularities.

Once one has a big Cohen-Macaulay module, one can always construct, using completion (for which, see Chapter 6), a balanced big Cohen-Macaulay module

[2] A related question is even open in these cases: does there exist a 'small' Cohen-Macaulay module, i.e., a finitely generated one, if the ring is moreover complete? For the notion of a complete local ring, see §6.2; there are counterexamples to the existence of a small Cohen-Macaulay module if the ring is not complete.

from it (see for instance [13, Corollary 8.5.3]). Here is a criterion for a big Cohen-Macaulay module to be balanced taken from [6, Lemma 4.8]; its proof is a simple modification of the proof of Theorem 4.2.6 and is worked out in Exercise 5.7.12 (recall that a regular sequence is called *permutable* if any permutation is again regular).

Proposition 5.6.9. *A big Cohen-Macaulay module M over a Noetherian local ring is balanced, if every M-regular sequence is permutable.*

If R is a Cohen-Macaulay local ring, and M a flat R-module, then M is a balanced big Cohen-Macaulay module, since every system of parameters in R is R-regular by Theorem 4.2.6, whence M-regular by Proposition 5.4.1. We have the following converse:

Theorem 5.6.10. *If M is a balanced big Cohen-Macaulay module over a regular local ring, then it is flat. More generally, over an arbitrary local Cohen-Macaulay ring, if M is a balanced big Cohen-Macaulay module of finite projective dimension, then it is flat.*

Proof. The first assertion is indeed a special case of the second by Theorem 5.5.6. For simplicity, we will just prove the first, and refer to Exercise 5.7.13 for the second. So let M be a balanced big Cohen-Macaulay module over the d-dimensional regular local ring R. Since a finitely generated R-module N has finite projective dimension by the (proven part of) Theorem 5.5.6, all $\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$ by 5.5.1. Let e be maximal such that $\operatorname{Tor}_e^R(M,N) \neq 0$ for some finitely generated R-module N. If $e = 0$, then we are done by Theorem 5.2.6. So, by way of contradiction, assume $e \geq 1$. By Corollary 5.2.7, there exists a prime ideal $\mathfrak{p} \subseteq R$ such that $\operatorname{Tor}_e^R(M, R/\mathfrak{p}) \neq 0$. Let h be the height of $\mathfrak{p}$. By Exercise 3.4.11, we can choose a system of parameters $(x_1,\dots,x_d)$ in R such that $\mathfrak{p}$ is a minimal prime of $I := (x_1,\dots,x_h)R$. Since (the image of) $\mathfrak{p}$ is then an associated prime of R/I, we can find by Exercise 4.3.12 a short exact sequence

$$0 \to R/\mathfrak{p} \to R/I \to C \to 0$$

for some finitely generated R-module C. The relevant part of the long exact Tor sequence from 5.2.5, obtained by tensoring the above exact sequence with M, is

$$\operatorname{Tor}_{e+1}^R(M,C) \to \operatorname{Tor}_e^R(M,R/\mathfrak{p}) \to \operatorname{Tor}_e^R(M,R/I). \tag{5.8}$$

The first module in (5.8) is zero by the maximality of e. The last module is zero too since it is isomorphic to $\operatorname{Tor}_e^{R/I}(M/IM, R/I) = 0$ by Proposition 5.4.2 and the fact that $(x_1,\dots,x_d)$ is by assumption M-regular. Hence the middle module in (5.8) is also zero, contradiction. □

We derive the following criterion for Cohen-Macaulayness:

Corollary 5.6.11. *If X is an irreducible affine scheme of finite type over an algebraically closed field K, and $\phi\colon X \to \mathbb{A}_K^d$ is a Noether normalization, that is to say, a finite and surjective morphism, then X is Cohen-Macaulay if and only if ϕ is flat.*

Proof. Suppose $X = \mathrm{Spec}(B)$, so that ϕ corresponds to a finite and injective homomorphism $A \to B$, with $A := K[\xi_1, \ldots, \xi_d]$ (see our discussion on page 22) and B a d-dimensional affine domain. Let $\mathfrak{n}$ be a maximal ideal of B, and let $\mathfrak{m} := \mathfrak{n} \cap A$ be its contraction to A. Since $A/\mathfrak{m} \to B/\mathfrak{n}$ is finite and injective, and since the second ring is a field, so is the former by Lemma 2.2.7. Hence $\mathfrak{m}$ is a maximal ideal of A, and $A_\mathfrak{m}$ is regular by 4.1.6. By Exercise 3.4.14, the height of $\mathfrak{n}$ is d. Choose an ideal $I := (x_1, \ldots, x_d)A$ whose image in $A_\mathfrak{m}$ is a parameter ideal. Since the natural homomorphism $A/I \to B/IB$ is finite, the latter ring is Artinian since the former is (note that $A/I = A_\mathfrak{m}/IA_\mathfrak{m}$). It follows that $IB_\mathfrak{n}$ is a parameter ideal in $B_\mathfrak{n}$.

Now, if B, whence also $B_\mathfrak{n}$ is Cohen-Macaulay, then $(x_1, \ldots, x_d)$, being a system of parameters in $B_\mathfrak{n}$, is $B_\mathfrak{n}$-regular by Theorem 4.2.6. This proves that $B_\mathfrak{n}$ is balanced big Cohen-Macaulay module over $A_\mathfrak{m}$, whence is flat by Theorem 5.6.10. Hence ϕ is flat by 5.3.8.

Conversely, assume $X \to \mathbb{A}^d_K$ is flat. Therefore, $A_\mathfrak{m} \to B_\mathfrak{n}$ is flat, and hence $(x_1, \ldots, x_d)$ is $B_\mathfrak{n}$-regular by Proposition 5.4.1. Since we already showed that this sequence is a system of parameters, we see that $B_\mathfrak{n}$ is Cohen-Macaulay. Since this holds for all maximal prime ideals of B, we proved that B is Cohen-Macaulay. □

Remark 5.6.12. The above argument proves the following more general result in the local case: if $A \subseteq B$ is a finite and faithfully flat extension of local rings with A regular, then B is Cohen-Macaulay. For the converse, we can even formulate a stronger criterion.

Theorem 5.6.13. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of Noetherian local rings. If R is regular of dimension d, if S is Cohen-Macaulay of dimension e, and if $S/\mathfrak{m}S$ has dimension $e - d$, then $R \to S$ is flat.*

Proof. Let $(x_1, \ldots, x_d)$ be a system of parameters of R. Since $S/\mathfrak{m}S$ has dimension $e - d$, there exist $x_{d+1}, \ldots, x_e$ in S such that their image in $S/\mathfrak{m}S$ is a system of parameters. Hence $(x_1, \ldots, x_e)$ is a system of parameters in S, whence is S-regular by Theorem 4.2.6. In particular, $(x_1, \ldots, x_d)$ is S-regular, showing that S is a balanced big Cohen-Macaulay R-module, and therefore is flat by Theorem 5.6.10. □

The residue ring $S/\mathfrak{m}S$ is called the *closed fiber* of $R \to S$. Note that the affine scheme defined by it is indeed the fiber of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ of the unique closed point of $\mathrm{Spec}(R)$; see (2.5). Exercise 5.7.17 establishes that flatness in turn forces the dimension equality in the theorem, without any singularity assumptions on the rings. We conclude with an application of the above Cohen-Macaulay criterion:

Corollary 5.6.14. *Any hypersurface in $\mathbb{A}^n_K$ is Cohen-Macaulay.*

Proof. Recall that a hypersurface Y is an affine closed subscheme of the form $\mathrm{Spec}(A/fA)$ with $A := K[\xi_1, \ldots, \xi_n]$ and $f \in A$. Moreover, Y has dimension $n-1$ (by an application of Corollary 3.3.6), whence its Noether normalization is of the form $Y \to \mathbb{A}^{n-1}_K$. In fact, after a change of coordinates (see the proof of Theorem 2.2.5), we may assume that f is monic in ξ_n of degree d. It follows that A/fA is free over $A' := K[\xi_1, \ldots, \xi_{n-1}]$ with basis $1, \xi_n, \ldots, \xi_n^{d-1}$. Hence A/fA is flat over A' by 5.2.2, whence Cohen-Macaulay by Corollary 5.6.11. □

Colon criterion for flatness. Recall that $(I : a)$ denotes the *colon* ideal of all $x \in A$ such that $ax \in I$. Colon ideals are related to cyclic modules in the following way:

5.6.15 *For any ideal $I \subseteq A$ and any element $a \in A$, we have an isomorphism $a(A/I) \cong A/(I : a)$.*

Indeed, the homomorphism $A \to A/I \colon x \mapsto ax$ has image $a(A/I)$ whereas its kernel is $(I : a)$. We already saw that faithfully flat homomorphisms preserve the ideal structure of a ring (see 5.3.5). Using colon ideals, we can even give the following criterion:

Theorem 5.6.16. *A homomorphism $A \to B$ is flat if and only if*

$$(IB : a) = (I : a)B$$

for all elements $a \in A$ and all (finitely generated) ideals $I \subseteq A$.

Proof. Suppose $A \to B$ is flat. In view of 5.6.15, we have an exact sequence

$$0 \to A/(I : a) \to A/I \to A/(I + aA) \to 0 \tag{5.9}$$

which, when tensored with B gives the exact sequence

$$0 \to B/(I : a)B \to B/IB \xrightarrow{f} B/(IB + aB) \to 0.$$

However, the kernel of f is easily seen to be $a(B/IB)$, which is isomorphic to $B/(IB : a)$ by 5.6.15. Hence the inclusion $(I : a)B \subseteq (IB : a)$ must be an equality.

For the converse, we need in view of Theorem 5.2.6 to show that $\mathrm{Tor}_1^A(B, A/J) = 0$ for every finitely generated ideal $J \subseteq A$. We induct on the minimal number s of generators of J, where the case $s = 0$ trivially holds. Write $J = I + aA$ with I an ideal generated by $s - 1$ elements. Tensoring (5.9) with B, we get from 5.2.5 an exact sequence

$$0 = \mathrm{Tor}_1^A(B, A/I) \to \mathrm{Tor}_1^A(B, A/J) \xrightarrow{\delta} B/(I : a)B \to B/IB \xrightarrow{g} B/JB \to 0,$$

where the first module vanishes by induction. As above, the kernel of g is easily seen to be $B/(IB : a)$, so that our assumption on the colon ideals implies that δ is the zero map, whence $\mathrm{Tor}_1^A(B, A/J) = 0$ as we wanted to show. □

Here is a nice 'descent type' application of this criterion:

Corollary 5.6.17. *Let $A \to B \to C$ be homomorphisms whose composition is flat. If $B \to C$ is cyclically pure, then $A \to B$ is flat. In fact, it suffices that $B \to C$ is* cyclically pure with respect to ideals extended from A, *that is to say, that $JB = JC \cap B$ for all ideals $J \subseteq A$.*

Proof. Given an ideal $I \subseteq A$ and an element $a \in A$, we need to show in view of Theorem 5.6.16 that $(IB : a) = (I : a)B$. One inclusion is immediate, so take y in $(IB : a)$. By the same theorem, we have $(IC : a) = (I : a)C$, so that y lies in $(I : a)C \cap B$ whence in $(I : a)B$ by cyclical purity. □

The next criterion will be useful when dealing with non-Noetherian algebras in the next chapter. Here we call an ideal J in a ring B *finitely related*, if it is of the form $J = (I : b)$ with $I \subseteq B$ a finitely generated ideal and $b \in B$.

Theorem 5.6.18. *Let A be a Noetherian ring and B an arbitrary A-algebra. Suppose $\mathscr{P}$ is a collection of prime ideals in B such that every proper, finitely related ideal of B is contained in some prime ideal belonging to $\mathscr{P}$. If $A \to B_{\mathfrak{p}}$ is flat for every $\mathfrak{p} \in \mathscr{P}$, then $A \to B$ is flat.*

Proof. By Theorem 5.6.16, we need to show that $(IB : a) = (I : a)B$ for all $I \subseteq A$ and $a \in A$. Put $J := (I : a)$. Towards a contradiction, let x be an element in $(IB : a)$ but not in JB. Hence $(JB : x)$ is a proper, finitely related ideal, and hence contained in some $\mathfrak{p} \in \mathscr{P}$. However, $(IB_{\mathfrak{p}} : a) = JB_{\mathfrak{p}}$ by flatness and another application of Theorem 5.6.16, so that $x \in JB_{\mathfrak{p}}$, contradicting that $(JB : x) \subseteq \mathfrak{p}$. □

5.7 Exercises

Ex 5.7.1

Show that if $N_\bullet$ is an exact sequence, then there exist short exact sequences $0 \to Z_{i+1} \to N_i \to Z_i \to 0$ for some submodules $Z_i \subseteq N_i$ and all i. Use this to deduce 5.2.1.

Ex 5.7.2

Give a complete proof of 5.2.2, including the infinitely generated case.

Ex 5.7.3

Prove 5.2.3 and 5.2.4.

Ex 5.7.4

Show that if $A \to B$ is flat, and $I, J \subseteq A$ are ideals, then $IB \cap JB = (I \cap J)B$.

Ex 5.7.5

Show that if $A \to B$ is a flat homomorphism and M, N are A-modules, then

$$\mathrm{Tor}_i^A(M,N) \otimes_A B \cong \mathrm{Tor}_i^B(M \otimes_A B, N \otimes_A B)$$

for all i.

Ex 5.7.6

Show directly that for a given A-module M, if $I \otimes_A M \to M$ is injective for every finitely generated ideal I, then the same holds for every ideal. Use this to give a proof of (3) ⇒ (1) in Theorem 5.2.6 in case A is not Noetherian. Prove the infinitely generated case in Corollary 5.2.7 by using syzygies and Corollary 5.2.8, in combination with a modification of the argument in Theorem 5.2.6.

Ex 5.7.7

Show that a homomorphism $A \to B$ is cyclically pure with respect to prime ideals, meaning that $\mathfrak{p}B \cap A = \mathfrak{p}$ for all prime ideals $\mathfrak{p} \subseteq A$, if and only if the induced map of affine schemes $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *is surjective.*

Ex 5.7.8

Show using Exercise 4.3.12 that any finitely generated module N over a Noetherian ring admits a prime filtration (5.3)*. Use this to work out the details in the proof of Corollary 5.2.7.*

Ex 5.7.9

Prove 5.5.2 by constructing inductively a minimal resolution using Nakayama's lemma.

Ex 5.7.10

Generalize the proof of the first part of Theorem 5.6.1 to prove the second assertion in that theorem.

Ex 5.7.11

Mimic the proof of Corollary 5.6.3 to show that any power series ring in finitely many indeterminates over a Noetherian ring is flat.

Ex 5.7.12

Modify the argument in the last part of the proof of Theorem 4.2.6 to prove Proposition 5.6.9.

Ex 5.7.13

Make the necessary adjustments in the proof of the first assertion of Theorem 5.6.10 to derive the second.

Ex 5.7.14

Show that an A-module M is flat if and only if $M_{\mathfrak{m}}$ is flat as an $A_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m} \subseteq A$. Prove 5.3.8 (note that if X is moreover Noetherian, then this follows already from Theorem 5.6.18).

Ex 5.7.15

By 3.1.4, any Artinian ring is a finite direct sum of local rings. This no longer holds true for an arbitrary Noetherian semi-local *ring S, that is to say, a Noetherian ring with finitely many maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s$. Show that nonetheless there is always a natural homomorphism $S \to S_{\mathfrak{m}_1} \oplus \cdots \oplus S_{\mathfrak{m}_s}$, which is moreover faithfully flat.*

***Ex 5.7.16**

Show that if M is a finitely generated module over a Noetherian ring A such that $M_{\mathfrak{m}}$ is flat over $A_{\mathfrak{m}}$, for every maximal ideal $\mathfrak{m}$, then M is projective as an A-module.

*Ex 5.7.17

Show that if $A \to B$ is a flat homomorphism, then the going-down *theorem holds for $A \to B$, meaning that if $\mathfrak{p} \subsetneq \mathfrak{q}$ is a chain of prime ideals in A, and if $\mathfrak{Q}$ is a prime ideal in B lying over $\mathfrak{q}$, then there exists a prime ideal $\mathfrak{P} \subsetneq \mathfrak{Q}$ lying over $\mathfrak{p}$. Use this to prove that if $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ is a flat and local homomorphism of Noetherian local rings, then*

$$\dim(R) + \dim(S/\mathfrak{m}S) = \dim(S).$$

Ex 5.7.18

Use the Colon criterion, Theorem 5.6.16, to show that every overring without zero-divisors, or more generally, any torsion-free overring, of a discrete valuation ring is flat.

Ex 5.7.19

Show that every finitely related ideal in an ultra-ring is an ultra-ideal.

*Ex 5.7.20

Prove a version of Theorem 5.6.16 for modules, that is to say, by replacing the A-algebra B by an A-module M.

Additional exercises.

Ex 5.7.21

Show that a module P is projective (=direct summand of a free module) if and only if any map $P \to N/N'$ lifts to a map $P \to N$, where $N' \subseteq N$ are arbitrary modules.

Ex 5.7.22

Show that if

$$0 \to M'_\bullet \xrightarrow{f} M_\bullet \xrightarrow{g} M''_\bullet \to 0$$

is an exact sequence of complexes, meaning that for each i, we have an exact sequence

$$0 \to M'_i \xrightarrow{f_i} M_i \xrightarrow{g_i} M''_i \to 0,$$

such that the maps f_i and g_i commute with the maps in the various complexes, then we get a long exact sequence

$$\dots \xrightarrow{\delta_{i+1}} \mathrm{H}_i(M'_\bullet) \xrightarrow{f_i} \mathrm{H}_i(M_\bullet) \xrightarrow{g_i} \mathrm{H}_i(M''_\bullet) \xrightarrow{\delta_i} \mathrm{H}_{i-1}(M'_\bullet) \to \dots$$

where the f_i and g_i are used to denote the corresponding induced homomorphisms, and where the δ_i are the connecting homomorphisms defined as follows: for $\bar{u} \in H_i(M''_\bullet)$, choose a lifting $u \in \mathrm{Ker}(d''_i) \subseteq M''_i$ and an element $v \in M_i$ such that $g_i(v) = u$. Since $g(d_i(v)) = 0$, there exists a well-defined $w \in M'_{i-1}$ for which $f_{i-1}(w) = d_i(v)$ and $d_{i-1}(w) = 0$. Show that assigning the class of w in $H_{i-1}(M'_\bullet)$ to $\bar{u}$ gives a well-defined homomorphism δ_i, making the above sequence exact.
Use this result to now give a complete proof of 5.2.5.

Ex 5.7.23

Show that for an A-module M to be faithfully flat, it is necessary and sufficient that an arbitrary complex $N_\bullet$ is exact if and only if $N_\bullet \otimes_A M$ is exact.

Ex 5.7.24

Let $A \to B \to C$ be homomorphisms. Show that if $A \to C$ is flat, then $A \to B$ is cyclically pure. Show using Exercise 5.7.23 that if both $A \to C$ and $B \to C$ are faithfully flat, then so is $A \to B$.

Ex 5.7.25

Show that a module is finitely generated if and only if any countably generated submodule is contained in a finitely generated submodule.

Ex 5.7.26

Prove the following version of a theorem due to Chase ([14]): a ring is coherent if and only if every finitely related ideal is finitely generated. The direct implication is a simple application of the coherency condition; for the converse use Theorem 5.6.4 and the Colon Criterion for flatness, Theorem 5.6.16.
Use this to extend Theorem 5.6.18 to the case that A is only assumed to be coherent.

Ex 5.7.27

Show that an ultra-Dedekind domain R, that is to say, an ultraproduct of Dedekind domains, is coherent. In fact, prove the stronger fact that any finitely related ideal in R is generated by two elements, and then use Exercise 5.7.26.

5.8 Project: local flatness criterion via nets

Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k, and let $\mathbf{mod}_R$ be the class of all finitely generated R-modules (up to isomorphism). In [58], a subset $\mathbf{N} \subseteq \mathbf{mod}_R$ is called a *net* if it is closed under *extension* (i.e., if $0 \to H \to M \to N \to 0$ is an exact sequence in $\mathbf{mod}_R$ with $H, N \in \mathbf{N}$, then also $M \in \mathbf{N}$), and under *direct summands* (i.e., if $M \cong H \oplus N$ belongs to $\mathbf{N}$, then so do H and N). Clearly, $\mathbf{mod}_R$ itself is a net.

5.8.1 *Show that any intersection of nets is again a net. Conclude that any class $\mathbf{K} \subseteq \mathbf{mod}_R$ sits inside a smallest net, called the* net generated *by* $\mathbf{K}$.

5.8.2 *Show that the net generated by the singleton $\{k\}$ consists of all modules of finite length. Show that $\mathbf{mod}_R$ is generated as a net by all $R/\mathfrak{p}$ with $\mathfrak{p} \subseteq R$ a prime ideal.*

A net $\mathbf{N}$ is called *deformational*, if for every $M \in \mathbf{mod}_R$ and every M-regular element x, if $M/xM \in \mathbf{N}$ then $M \in \mathbf{N}$.

5.8.3 *Show that the deformational net generated by the singleton $\{k\}$ is equal to $\mathbf{mod}_R$.*

The goal is to prove the following version of the local flatness criterion:

5.8.4 *If $R \to S$ is a local homomorphism of Noetherian local rings, and Q a finitely generated S-module such that $\mathrm{Tor}_1^R(Q,k) = 0$, then Q is flat as an R-module.*

To this end, for $M \in \mathbf{mod}_R$, put $F(M) := \mathrm{Tor}_1^R(Q,M)$. In view of Theorem 5.2.6, we need to show that F is zero on $\mathbf{mod}_R$.

5.8.5 *Show that $F(M)$ carries a natural structure of an S-module, and as such is finitely generated, for any finitely generated R-module M.*

5.8.6 *Show that if F is zero on a class $\mathbf{K} \subseteq \mathbf{mod}_R$, then F is zero on the net generated by $\mathbf{K}$, and, in fact, even zero on the deformational net generated by $\mathbf{K}$. For the first assertion, use 5.2.5, and for the second, show that for any $N \in \mathbf{mod}_R$ and any $x \in \mathfrak{m}$, if $xF(N) = F(N)$ then $F(N) = 0$, using 5.8.5. Finally, conclude the proof of 5.8.4 by using 5.8.3.*

Chapter 6
Completion

6.1 Complete normed rings

Normed rings. In these notes, a *quasi-normed ring* $(A, ||\cdot||)$ will mean a ring A together with a real-valued function $A \to [0,1]\colon a \mapsto ||a||$ such that $||0|| = 0$ and such that for all $a, b \in A$ we have

1. $||a+b|| \leq \max\{||a||, ||b||\}$;
2. $||ab|| \leq ||a|| \cdot ||b||$.

We normally exclude the case that $||\cdot||$ is identical zero (the so-called *degenerated* case). Inequality (1) is called the *non-archimedean triangle inequality*, as opposed to the usual, weaker triangle inequality in the reals (note that (1) implies indeed that $||a+b|| \leq ||a|| + ||b||$). An immediate consequence of this triangle inequality is:

3. if $||a|| < ||b||$, then $||a+b|| = ||b||$,

which often is paraphrased by saying that "every triangle is isosceles". If moreover $||a|| = 0$ implies $a = 0$, then we call $(A, ||\cdot||)$ a *normed ring* (or, we simply say that $||\cdot||$ is a norm). The value $||a||$ will also be called the *norm* of a, even if $||\cdot||$ is only a quasi-norm. If in (2) we always have equality, then we call the norm *multiplicative* (be aware that some authors tacitly assume that a norm is always multiplicative; moreover, it is common to allow elements to also have norm bigger than one). Some immediate consequences of this definition (see Exercise 6.5.1):

6.1.1 *Any unit in a quasi-normed ring has norm equal to one, the elements of norm equal to zero form an ideal I_0, and so do the elements of norm strictly less than one. If $||\cdot||$ is multiplicative, then I_0 is prime. In particular, a multiplicatively normed ring is a domain.*

There is also a very canonical procedure to turn the quasi-norm into a norm:

6.1.2 *If A is a quasi-normed ring, and I_0 its ideal of elements of norm zero, then $||\cdot||$ factors through A/I_0 , making the latter into a normed ring.*

Indeed, using (3) we have $||a|| = ||a+w||$ for all $w \in I_0$, so that letting $||\bar{a}|| := ||a||$ is well-defined, where $\bar{a}$ denotes the image of a in A/I_0. The remaining properties are now easily checked. The normed ring A/I_0 is called the *Hausdorffication* or *separated quotient* of A. The name is justified by the following considerations: any quasi-normed ring inherits a topology, called the *norm topology*, simply by taking for opens the inverse images of the opens of $[0,1]$ under the norm map $A \to [0,1]$. Now, by Exercise 6.5.3, the topology on A is Hausdorff if and only if $||\cdot||$ is a norm.

Let $(A, ||\cdot||_A)$ and $(B, ||\cdot||_B)$ be two quasi-normed rings. A homomorphism $A \to B$ is called a *homomorphism of quasi-normed rings* if $||a||_B \leq ||a||_A$ for all a. We may also express this fact by saying that B is a *quasi-normed A-algebra*. If $I \subseteq A$ is an ideal, define a quasi-norm on A/I by letting $||a+I||$ be the infimum of all $||a+i||$ with $i \in I$. By Exercise 6.5.4, we have

6.1.3 *For any ideal $I \subseteq A$, the pair $(A/I, ||\cdot||)$ is a quasi-normed ring, and the residue map $A \to A/I$ is a homomorphism of quasi-normed rings.*

Cauchy sequences. Let $(A, ||\cdot||)$ be a quasi-normed ring. We will represent sequences in A as functions $\mathbf{a}\colon \mathbb{N} \to A$. Any element $a \in A$ defines a sequence, the *constant sequence* with value a defined as $\mathbf{a}(n) := a$. We will identify an element $a \in A$ with the constant sequence it defines.

We say that a sequence $\mathbf{a}$ is a *null-sequence* if for each $\varepsilon > 0$, there exists $N := N(\varepsilon)$ such that $||\mathbf{a}(n)|| \leq \varepsilon$ for all $n \geq N$. In particular, a constant sequence a is null if and only if $||a|| = 0$. The *twist* $\mathbf{a}^+$ of a sequence $\mathbf{a}$ is the sequence defined by $\mathbf{a}^+(n) := \mathbf{a}(n+1)$. We say that $\mathbf{a}$ is a *Cauchy sequence*, if $\mathbf{a} - \mathbf{a}^+$ is a null-sequence. We say that an element $b \in A$ is a *limit* of a sequence $\mathbf{a}$, if $\mathbf{a} - b$ is a null-sequence. A sequence admitting a limit is called a *converging* sequence. We have:

6.1.4 *Any converging sequence is Cauchy. If b is a limit of a sequence $\mathbf{a}$, then so is $b+w$ for any w of norm zero. In particular, if $||\cdot||$ is a norm, then a Cauchy sequence has at most one limit.*

If the converse also holds, that is to say, if any Cauchy sequence is convergent, then we say that $(A, ||\cdot||)$ is *quasi-complete*. We call $(A, ||\cdot||)$ *complete* if it is quasi-complete and $||\cdot||$ is a norm, that is to say, if any Cauchy sequence has a unique limit.

6.1.5 *If A is quasi-complete and $I \subsetneq A$ is a proper ideal, then A/I is again quasi-complete.*

This is proven in Exercise 6.5.4. In particular, we can turn any quasi-complete ring into a complete one: simply consider its Hausdorffication A/I_0. A sequence $\mathbf{b}$ is called a *subsequence* of a sequence $\mathbf{a}$ if there exists some strictly increasing function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\mathbf{a}(f(n)) = \mathbf{b}(n)$ for all n. The following is left as an exercise (Exercise 6.1.6):

6.1.6 *Any subsequence of a Cauchy sequence is a Cauchy sequence, and any limit of a sequence is also a limit of any of its subsequences. Moreover, for a Cauchy sequence to be convergent it suffices that some subsequence is convergent.*

Note that a (non-Cauchy) sequence can very well have a converging subsequence without itself being convergent.

Adic norms. Let A be any ring, and I an ideal. We can associate a quasi-norm to this situation, called the *I-adic quasi-norm* defined as $||a||_I := \exp(-n)$ where n is the supremum of all k for which $a \in I^k$. We allow for this supremum to be infinite, with the understanding that $\exp(-\infty) = 0$. By Exercise 6.5.6 this is indeed a quasi-norm, which is degenerated if and only if I is the unit ideal. Hence $||\cdot||_I$ is a norm if and only if the intersection of all I^k is zero. The only case of interest to us is when $(R, \mathfrak{m})$ is local viewed in its $\mathfrak{m}$-adic quasi-norm, which we then call the *canonical* quasi-norm of R, or when there is no confusion, *the* quasi-norm of R. By what we just said, the quasi-norm of $(R, \mathfrak{m})$ is a norm if and only if its ideal of infinitesimals, $\mathfrak{I}_R$ (see Definition 1.4.10), is equal to zero. By Exercise 6.5.6, we have:

6.1.7 *Any polynomial $f \in A[\xi]$ in a single indeterminate ξ defines a continuous function $A \to A\colon a \mapsto f(a)$ in the topology induced by an I-adic quasi-norm.*

If $A \to B$ is a homomorphism and $I \subseteq A$ an ideal, then $A \to B$ is a homomorphism of quasi-normed rings if we take the I-adic quasi-norm on A and the IB-adic quasi-norm on B.

6.2 Complete local rings

We call a local ring R *(quasi-)complete*, if it is (quasi-)complete with respect to its $\mathfrak{m}$-adic quasi-norm. By Exercise 6.5.7, we have

6.2.1 *A local ring $(R, \mathfrak{m})$ is quasi-complete if and only if every sequence $\mathbf{a}$ satisfying $\mathbf{a}(n) \equiv \mathbf{a}(n+1) \mod \mathfrak{m}^n$ for all n, has a limit, and for this it suffices that we can find a subsequence $\mathbf{b}$ of $\mathbf{a}$ and an element $b \in R$ such that $b \equiv \mathbf{b}(n) \mod \mathfrak{m}^n$.*

Fields are obviously complete local rings, and more generally, so are Artinian local rings. Any power series ring over a field (or an Artinian local ring) in finitely many indeterminates is complete. This follows by induction from the following more general result.

Proposition 6.2.2. *If R is a quasi-complete local ring, then so is $R[[\xi]]$ with ξ a single variable.*

Proof. The maximal ideal $\mathfrak{n}$ of $R[[\xi]]$ is generated by ξ and the maximal ideal $\mathfrak{m}$ of R. By (6.2.1), we need to show that a sequence $\mathbf{f}$ in $R[[\xi]]$ such that

$$\mathbf{f}(k) \equiv \mathbf{f}(k+1) \mod \mathfrak{n}^k \tag{6.1}$$

for all k, has a limit. Write each $\mathbf{f}(n) = \sum_j \mathbf{a}_j(n)\xi^j$. Expanding (6.1) and comparing coefficients, we get $\mathbf{a}_j(k) \equiv \mathbf{a}_j(k-1) \mod \mathfrak{m}^{k-j}$ for all $j \leq k$. In particular, each $\mathbf{a}_j$

is Cauchy, whence admits a limit $b_j \in R$. I claim that $g(\xi) := \sum_j b_j \xi^j$ is a limit of $\mathbf{f}$. To verify this, fix some k. By assumption, there exists for each j some $N_j(k)$ such that $b_j \equiv \mathbf{a}_j(n) \mod \mathfrak{m}^k$ for all $n \geq N_j(k)$. Let $N(k)$ be the maximum of all $N_j(k)$ with $j < k$. For $n \geq N(k)$, the terms in $g - \mathbf{f}(n)$ of degree at least k clearly lie inside $\mathfrak{n}^k$. The coefficient of the term of degree $j < k$ is $b_j - \mathbf{a}_j(n)$, which lies in $\mathfrak{m}^k$ by the choice of $N(k)$. Hence $g \equiv \mathbf{f}(n) \mod \mathfrak{n}^k$ for all $n \geq N(k)$, proving the claim. □

Immediately from 6.1.5 we get:

6.2.3 *Any homomorphic image of a quasi-complete local ring is again quasi-complete.*

In particular, the Hausdorffication of a quasi-complete local ring R, that is to say, the separated quotient $R/\mathfrak{I}_R$, is a complete local ring.

Hensel's lemma. The next result is a formal version of Newton's method for finding approximate roots.

Theorem 6.2.4. *Let $(R, \mathfrak{m})$ be a complete local ring with residue field k. Let $f \in R[\xi]$ be a monic polynomial in the single variable ξ, and let $\bar{f} \in k[\xi]$ denote its reduction modulo $\mathfrak{m}R[\xi]$. For every simple root $u \in k$ of $\bar{f} = 0$, we can find $a \in R$ such that $f(a) = 0$ and u is the image of a in k.*

Proof. Let $a_1 \in R$ be any lifting of u. Since $\bar{f}(u) = 0$, we get $f(a_1) \equiv 0 \mod \mathfrak{m}$. We will define elements $a_n \in R$ recursively such that $f(a_n) \equiv 0 \mod \mathfrak{m}^n$ and $a_n \equiv a_{n-1} \mod \mathfrak{m}^{n-1}$ for all $n > 1$. Suppose we already defined $a_1, \dots, a_n$ satisfying the above conditions. Conisder the Taylor expansion

$$f(a_n + \xi) = f(a_n) + f'(a_n)\xi + \xi^2 g_n(\xi) \tag{6.2}$$

where $g_n \in R[\xi]$ is some polynomial. Since the image of a_n in k is equal to u, and since $\bar{f}'(u) \neq 0$ by assumption, $f'(a_n)$ does not lie in $\mathfrak{m}$ whence is a unit, say, with inverse u_n. Define $a_{n+1} := a_n - u_n f(a_n)$. Substituting this in f and using (6.2), we get

$$f(a_{n+1}) \in (u_n f(a_n))^2 R \subseteq \mathfrak{m}^{2n},$$

as required.

To finish the proof, note that the sequence $\mathbf{a}$ given by $\mathbf{a}(n) := a_n$ is by construction Cauchy, and hence by assumption admits a limit $a \in R$ (whose residue is necessarily again equal to u). By continuity, $f(a)$ is equal to the limit of the $f(a_n)$ whence is zero. □

There are sharper versions of this result, where the root in the residue field need not be simple (See Exercise 6.5.13), or even involving systems of equations. Any local ring satisfying the hypothesis of the above theorem is called a *Henselian* ring. From a model-theoretic point of view, it is sometimes more convenient to work with Henselian local rings than with complete ones, since they form a first-order definable class (as is clear from the defining condition).

As with completion, there exists a 'smallest' Henselian overring. More precisely, for each Noetherian local ring R, there exists a Noetherian local R-algebra $R^\sim$, its *Henselization*, satisfying the following universal property: any local homomorphism

$R \to H$ with H a Henselian local ring, factors uniquely through an R-algebra homomorphism $R^{\sim} \to H$. The existence of such a Henselization will be proven in Project 6.6. Note that Theorem 6.2.4 and the universal property imply that $R^{\sim}$ is a subring of $\widehat{R}$, and in particular, the latter is the completion of the former.

Let $A := k[\xi]$ be a polynomial ring over a field k. For simplicity, we will denote the Henselization of the localization of A with respect to the variables also by $A^{\sim}$. It can be shown that $A^{\sim} = k[[\xi]]^{\text{alg}}$, the ring of *algebraic power series* over k, where we call a power series in $k[[\xi]]$ *algebraic* if it is algebraic over $k[\xi]$, that is to say, satisfies a non-zero polynomial equation with coefficients in $k[\xi]$ (for a discussion see [2] or 6.6.4 below).

Lifting generators. The next property of quasi-complete local rings, a generalization of Nakayama's lemma, is also quite useful.

Theorem 6.2.5. *Let $(R, \mathfrak{m})$ be a quasi-complete local ring, and let M be an R-module which is $\mathfrak{m}$-adically Hausdorff, in the sense that the intersection of all $\mathfrak{m}^n M$ is zero. If $M/\mathfrak{m}M$ has vector space dimension e over the residue field $R/\mathfrak{m}$, then M is generated as an R-module by e elements. In fact, any lifting of a set of generators of $M/\mathfrak{m}M$ generates M.*

Proof. Let $v_1, \dots, v_e \in M$ be liftings of the generators of $M/\mathfrak{m}M$ and let N be the submodule they generate. In particular, $M = N + \mathfrak{m}M$. Take an arbitrary $\mu \in M$. We can find some $a_i^{(0)} \in A$ such that $\mu = \sum_i a_i^{(0)} v_i + \mu^{(1)}$ with $\mu^{(1)} \in \mathfrak{m}M$. Applying the same to $\mu^{(1)}$, we can find $a_i^{(1)} \in \mathfrak{m}$ such that $\mu^{(1)} = \sum_i a_i^{(1)} v_i + \mu^{(2)}$ with $\mu^{(2)} \in \mathfrak{m}^2 M$. Continuing this way, we find $a_i^{(n)} \in \mathfrak{m}^n$ such that

$$\mu \equiv \sum_{i=1}^{s} (\sum_{j=0}^{n} a_i^{(j)}) v_i \mod \mathfrak{m}^{n+1} M. \tag{6.3}$$

Putting $\mathbf{b}_i(n) := \sum_{j \le n} a_i^{(j)}$, it follows that each $\mathbf{b}_i$ is a Cauchy sequence, whence has a limit $a_i \in R$. Using (6.3), one easily verifies that $\mu - \sum a_i v_i$ lies in every $\mathfrak{m}^n M$ whence is zero, showing that $\mu \in N$, and therefore $M = N$. □

6.3 Completions

We have seen in the previous section that complete local rings satisfy many good properties. In this section, we will describe how to construct complete local rings from arbitrary local rings. Let again start in a more general setup.

Quasi-completion of a quasi-norm. Let $(A, ||\cdot||)$ be a quasi-normed ring. Let $\mathscr{C}(A)$ be the collection of all Cauchy sequences. We make $\mathscr{C}(A)$ into a ring by adding and multiplying sequences coordinate wise. In this way, $\mathscr{C}(A)$ becomes an A-algebra, via the canonical map $A \to \mathscr{C}(A)$ sending an element to the constant sequence it determines. Note that this is in fact an embedding.

6.3.1 *A sequence* $\mathbf{a}$ *in* A *is Cauchy if and only if* $||\mathbf{a}(n)||$ *converges in* $\mathbb{R}$ *as* $n \to \infty$*. This latter limit is denoted* $||\mathbf{a}||$*; it induces a quasi-norm on* $\mathscr{C}(A)$*. A Cauchy sequence has norm zero if and only if it is a null-sequence.*

From now on, we view $\mathscr{C}(A)$ as a quasi-normed ring with the above norm.

Proposition 6.3.2. *The ring of Cauchy sequences* $\mathscr{C}(A)$ *of* A *is quasi-complete. Moreover,* A *is dense in* $\mathscr{C}(A)$*, and the following universal property holds: if we have a homomorphism of quasi-normed rings* $A \to B$ *with* B *complete, then* $A \to B$ *extends uniquely to a homomorphism* $\mathscr{C}(A) \to B$ *of quasi-normed rings.*

Proof. Let $\mathbf{a}$ be a Cauchy sequence in A, that is to say, an element in $\mathscr{C}(A)$. It follows that the limit of $||\mathbf{a}(n) - \mathbf{a}||$ is zero, for $n \mapsto \infty$, proving that the sequence $\mathbf{a}(n)$ converges to the element $\mathbf{a} \in \mathscr{C}(A)$. This already shows that A is dense in $\mathscr{C}(A)$. Let $\mathbf{B}$ be a Cauchy sequence in $\mathscr{C}(A)$, so that $\mathbf{B}(m)$ is a Cauchy sequence in A, for each m. Replacing $\mathbf{B}$ by a subsequence if necessary, we may assume $||\mathbf{B}(m) - \mathbf{B}(m+1)|| \leq \exp(-m)$ for all m. Moreover, by the previous observation, for each m, there exists $g(m)$ such that $||\mathbf{B}(m)(g(m)) - \mathbf{B}(m)|| \leq \exp(-m)$. Define a sequence $\mathbf{c}$ by the rule $\mathbf{c}(m) := \mathbf{B}(m)(g(m))$. Since $||\mathbf{c}(m) - \mathbf{c}(m+1)||$ is equal to

$$||\mathbf{c}(m) - \mathbf{B}(m) + \mathbf{B}(m) - \mathbf{B}(m+1) + \mathbf{B}(m+1) - \mathbf{c}(m+1)|| \leq \exp(-m)$$

we conclude that $\mathbf{c}$ is a Cauchy sequence. In particular, for a fixed n we can find $N \geq n$ such that $||\mathbf{c}(m) - \mathbf{c}|| \leq \exp(-n)$ for all $m \geq N$. To show that $\mathbf{c}$ is the limit of $\mathbf{B}$, we use the estimate

$$\begin{aligned} ||\mathbf{B}(m) - \mathbf{c}|| &= ||\mathbf{B}(m) - \mathbf{c}(m) + \mathbf{c}(m) - \mathbf{c}|| \\ &\leq \max\{\exp(-m), \exp(-n)\} = \exp(-n), \end{aligned}$$

for all $m \geq N$. This proves that $\mathbf{c}$ is the limit of $\mathbf{B}$.

To prove the last assertion, we define $\varphi\colon \mathscr{C}(A) \to B$ as follows. Let $\mathbf{a}$ be a Cauchy sequence in A. From the definition of homomorphism of quasi-normed rings, it follows that $\mathbf{a}$ is a Cauchy sequence in B. Since B is complete, $\mathbf{a}$ has a unique limit $b \in B$. The assignment $\mathbf{a} \mapsto b$ is now easily seen to be an A-algebra homomorphism of quasi-normed rings. □

In view of this result, we call $\mathscr{C}(A)$ the *quasi-completion* of A. The *completion* of A is then the Hausdorffication of $\mathscr{C}(A)$, that is to say, the ring $\mathscr{C}(A)/\mathscr{I}_0$, where $\mathscr{I}_0$ is the ideal of all null-sequences. If the quasi-norm is understood, as will be the case with the canonical quasi-norm of a local ring, we denote the completion by $\widehat{A}$. From Proposition 6.3.2, we get the following universal property of completion:

6.3.3 *If* B *is a normed* A*-algebra which is complete, then there exists a unique* A*-algebra homomorphism of normed rings* $\widehat{A} \to B$*.*

Completion of a Noetherian local ring. We now apply the previous theory to the canonical norm on a Noetherian local ring R. Its completion $\mathscr{C}(R)/\mathscr{I}_0$ is denoted $\widehat{R}$. It is easy to see that $\mathfrak{m}\mathscr{C}(R)$ cannot be the unit ideal, whence neither is $\mathfrak{m}\widehat{R}$. We will shortly show that $\widehat{R}$ is in fact a Noetherian local ring with maximal ideal $\mathfrak{m}\widehat{R}$, and in its adic norm, it is complete. Moreover, the norm inherited from the norm on $\mathscr{C}(R)$ is identical to the $\mathfrak{m}\widehat{R}$-adic norm. To prove all these claims, we resort to flatness.

Theorem 6.3.4. *The canonical homomorphism $R \to \widehat{R}$ of a Noetherian local ring into its completion is faithfully flat. Moreover, $\widehat{R}$ is a Noetherian local ring with the same residue field as R.*

Proof. Since $\mathfrak{m}\widehat{R} \neq \widehat{R}$, it suffices to show that $R \to \widehat{R}$ is flat. Let $\mathbf{x} := (x_1, \dots, x_e)$ generate the maximal ideal $\mathfrak{m}$ of R, and let $\xi := (\xi_1, \dots, \xi_e)$ be a tuple of indeterminates. Define an R-algebra homomorphism $S := R[[\xi]] \to \widehat{R}$ as follows. Let f be a power series and let f_n be is its truncation consisting of all terms up to degree n. The sequence $\mathbf{a}$ defined by $\mathbf{a}(n) := f_n(\mathbf{x})$ is easily seen to be a Cauchy sequence in R, whence has a unique limit in $\widehat{R}$, which we simply denote by $f(\mathbf{x})$. The homomorphism $S \to \widehat{R}$ is given by the rule $f \mapsto f(\mathbf{x})$. A moment's reflection shows that its kernel is $I := (\xi_1 - x_1, \dots, \xi_e - x_e)S$. I claim that $S \to \widehat{R}$ is surjective, so that $\widehat{R} = S/I$, showing already that it $\widehat{R}$ is a Noetherian local ring with the same residue field as R. To prove surjectivity, let $\mathbf{a}$ be a Cauchy sequence, that is to say, an element of $\mathscr{C}(R)$. Since any subsequence of $\mathbf{a}$ has the same image in $\widehat{R}$, we may assume that $\mathbf{a}(n) \equiv \mathbf{a}(n+1) \mod \mathfrak{m}^n$ for all n. Hence we can write

$$\mathbf{a}(n+1) = \mathbf{a}(n) + \sum_{|\nu|=n} r_\nu \mathbf{x}^\nu$$

where the sum runs over all e-tuples ν such that $|\nu| := \nu_1 + \dots + \nu_e = n$. Define

$$f(\xi) := \mathbf{a}(0) + \sum_{\nu} r_\nu \xi^\nu$$

where the sum is now over all non-zero e-tuples ν. Hence $f_n(\mathbf{x}) = \mathbf{a}(n)$ for all n (where as before f_n is the n-th degree truncation of f), showing that $f(\mathbf{x}) = \mathbf{a}$.

Since $R \to S$ is flat by Exercise 5.7.11, the flatness of $R \to \widehat{R}$ will follow from Thereom 5.6.5 once we show that $I \cap \mathfrak{a}S = \mathfrak{a}I$ for every ideal $\mathfrak{a} \subseteq R$. Let $\mathfrak{a} := (a_1, \dots, a_n)R$. Let $f \in I \cap \mathfrak{a}S$ so that we can write it in two different ways as

$$f = a_1 s_1 + \dots + a_n s_n = t_1(\xi_1 - x_1) + \dots + t_e(\xi_e - x_e) \tag{6.4}$$

for some $s_i, t_i \in S$. We want to show that $s_i \in I$. By Taylor expansion, we can write each s_i as $s_i = b_i + s_i'$ with $b_i \in R$ and $s_i' \in I$. Hence $f \equiv c \mod \mathfrak{a}I$ where $c := a_1 b_1 + \dots + a_n b_n$. However, $R \to \widehat{R}$ is injective (since $\mathfrak{I}_R = 0$), so that $I \cap R = (0)$. Since c lies in $I \cap R$ it is therefore zero, showing that $f \in \mathfrak{a}I$. □

Corollary 6.3.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with completion $\widehat{R}$. For all n we have an isomorphism $R/\mathfrak{m}^n R \cong \widehat{R}/\mathfrak{m}^n \widehat{R}$. In particular, $\widehat{R}$ is a complete Noetherian local ring, that is to say, is complete in its canonical $\mathfrak{m}\widehat{R}$-adic norm, of the same dimension as R.*

Proof. Let $R_n := R/\mathfrak{m}^n$, and let $S_n := \widehat{R}/\mathfrak{m}^n \widehat{R}$. Note that R_n is Artinian, whence complete. As $S_n/\mathfrak{m}S_n$ is equal to the residue field of R whence of R_n by Theorem 6.3.4, we get $S_n \cong R_n$ by Theorem 6.2.5. In particular, R and $\widehat{R}$ have the same Hilbert-Samuel polynomial, whence the same dimension by Theorem 3.3.2.

I claim that if $\mathbf{a}$ is a Cauchy sequence such that $\mathbf{a}(k) \in \mathfrak{m}^n$ for all $k \gg 0$, then $\mathbf{a} \in \mathfrak{m}^n\widehat{R}$. Indeed, by what we just proved, we have $\widehat{R} = R + \mathfrak{m}^n\widehat{R}$. Hence if we choose generators $\mathbf{x}$ for $\mathfrak{m}$, then we can write

$$\mathbf{a} = r + \sum_{|\nu|=n} \mathbf{x}^\nu \mathbf{b}_\nu \tag{6.5}$$

with $r \in R$ and $\mathbf{b}_\nu \in \widehat{R}$. Substituting k such that $\mathbf{a}(k) \in \mathfrak{m}^n$ in (6.5) shows that $r \in \mathfrak{m}^n\widehat{R}$. Since $\mathfrak{m}^n\widehat{R} \cap R = \mathfrak{m}^n$ by faithful flatness (or the above isomorphism), we get $\mathbf{a} \in \mathfrak{m}^n\widehat{R}$, as claimed. It follows that the $\mathfrak{m}\widehat{R}$-adic norm of an element is at most its norm as a Cauchy sequence. The converse is easy, thus proving the last assertion. □

Immediate from 6.2.3 we get:

6.3.6 *If I is an ideal in a Noetherian local ring R, then $\widehat{R}/I\widehat{R}$ is the completion of R/I.*

Another extremely useful property of completion is that it "transfers singularities" in the following sense:

Corollary 6.3.7. *A Noetherian local ring is regular or Cohen-Macaulay if and only if its completion is.*

Proof. Let $(R,\mathfrak{m})$ be a d-dimensional Noetherian local ring. The completion $\widehat{R}$ of R also has dimension d by Corollary 6.3.5. If R is regular, then $\mathfrak{m}$ is generated by d elements, whence so is $\mathfrak{m}\widehat{R}$, showing that $\widehat{R}$ is regular. Conversely, if $\widehat{R}$ is regular, so that $\mathfrak{m}\widehat{R}$ is generated by a d-tuple $\mathbf{x}$, then by Nakayama's lemma, we may choose these generators already in $\mathfrak{m}$. From $\mathbf{x}\widehat{R} = \mathfrak{m}\widehat{R}$, the cyclic purity of faithfully flat homomorphisms (Proposition 5.3.4) yields $\mathbf{x}R = \mathfrak{m}$, showing that R is regular. If R is Cohen-Macaulay and $\mathbf{x}$ is an R-regular sequence of length d, then $\mathbf{x}$ is also $\widehat{R}$-regular by faithful flatness and Proposition 5.4.1, showing that $\widehat{R}$ is also Cohen-Macaulay. Conversely, assume $\widehat{R}$ is Cohen-Macaulay, and let $\mathbf{x} := (x_1,\dots,x_d)$ be a system of parameters of R. Using Corollary 6.3.5, we get $R/\mathbf{x}R \cong \widehat{R}/\mathbf{x}\widehat{R}$, showing that $\mathbf{x}$ is also a system of parameters in $\widehat{R}$, whence $\widehat{R}$-regular. Since $R/(x_1,\dots,x_e)R \hookrightarrow \widehat{R}/(x_1,\dots,x_e)\widehat{R}$ for all e by faithful flatness and Proposition 5.3.4, it follows easily that $\mathbf{x}$ is also R-regular. □

For those that know inverse limits (also called projective limits), one can give the following alternative construction of the completion:

Proposition 6.3.8. *The completion of a Noetherian local ring $(R,\mathfrak{m})$ is equal to the inverse limit $\varprojlim R/\mathfrak{m}^n$.*

Proof. Here we view the $R_n := R/\mathfrak{m}^n$ as an inverse system via the canonical residue maps $R_m \to R_n$ for all $m \geq n$. A typical element of the inverse limit is represented by a sequence $\mathbf{a}$ in R such that $\mathbf{a}(m) + \mathfrak{m}^m$ is mapped to $\mathbf{a}(n) + \mathfrak{m}^n$ under the residue map $R_m \to R_n$ for all $m \geq n$; two sequences $\mathbf{a}$ and $\mathbf{a}'$ then give rise to the same element in the inverse limit if $\mathbf{a}(m) \equiv \mathbf{a}'(m) \mod \mathfrak{m}^m$ for all m. The first of these conditions simply translates into $\mathbf{a}(m) \equiv \mathbf{a}(n) \mod \mathfrak{m}^n$ for all $m \geq n$, showing that $\mathbf{a}$ is a Cauchy sequence; the second condition says that $\mathbf{a} - \mathbf{a}'$ is a null-sequence. Hence we have a map $\varprojlim R_n \to \mathscr{C}(R)/\mathscr{I}_0 = \widehat{R}$. The reader can check that this gives indeed an isomorphism of rings. □

6.4 Complete Noetherian local rings

Classifying Noetherian local rings is a daunting task, but under the additional completeness assumption, we can say much more, as we will now explore. This will even aid us in the study of non-complete Noetherian local rings by the faithful flatness of completion proven in Theorem 6.3.4.

Cohen's structure theorem. A local ring $(R, \mathfrak{m})$ may or may not contain a field. In the former case, we say that R has *equal characteristic*; the remaining case is refered to as *mixed characteristic*. The name is justified in Exercise 6.5.10: a ring has equal characteristic if and only if has the same characteristic as its residue field. A subfield $\kappa \subseteq R$ which under the canoncial residue map $R \to k := R/\mathfrak{m}$ maps surjectively, whence isomorphically, onto k, is called a *coefficient field*. These might not always exist, but we do have a weaker version:

Lemma 6.4.1. *Let R be an equal characteristic local ring with residue field k. Then there exists a subfield $\kappa \subseteq R$, such that k is algebraic over the image $\pi(\kappa)$ of κ under the residue map $\pi \colon R \to k$.*

Proof. The collection of subfields of R is non-empty by assumption, and is clearly closed under chains. Hence by Zorn's lemma there exists a maximal subfield $\kappa \subseteq R$. Let u be an arbitrary element in $k \setminus \pi(\kappa)$, and choose $a \in R$ with $\pi(a) = u$. In particular, $a \notin \kappa$. Put $S := \kappa[a]$, the κ-subalgebra of R generated by a, and let $\mathfrak{p} := \mathfrak{m} \cap S$. Since $S_\mathfrak{p} \subseteq R$, it cannot be a field by maximality of κ, and hence $\mathfrak{p} \neq 0$. Choose a non-zero element $b \in \mathfrak{p}$, and write it as $b = f(a)$ for some $f \in \kappa[\xi]$. If we let $f^\pi \in \pi(k)[\xi]$ be the (non-zero) polynomial obtained from f by applying π to its coefficients, then $f^\pi(u) = 0$, showing that u is algebraic over $\pi(\kappa)$. □

Theorem 6.4.2 (Equal characteristic). *Let $(R, \mathfrak{m})$ be a local ring of equal characteristic. If R is complete, then it admits a coefficient field κ. If R has moreover finite embedding dimension e, then R is Noetherian, and in fact isomorphic to a homomorphic image of a power series ring in e variables over k.*

Proof. To prove the existence of a coefficient field in positive characteristic, one normally resorts to the theory of etale extensions (as the proof in [41, Theorem 28.3]) or differential bases (as in [18, Theorem 16.14]); an alternative proof is given below in Remark 6.4.3. Here I will only give the proof in equal characteristic zero, that is to say, when the residue field of k has characteristic zero. By (the proof of) Proposition 6.4.1, if $\kappa \subseteq R$ is a maximal subfield, then k is algebraic over $\pi(\kappa)$, where $\pi\colon R \to k$ is the residue map. Towards a contradiction, assume there is some $u \in k \setminus \pi(\kappa)$. Let $f \in \kappa[\xi]$ be such that f^π is a minimal polynomial of u. Since we are in characteristic zero, u must be a simple root of f^π. Hence by Hensel's Lemma, Theorem 6.2.4, we can find $a \in R$ such that $f(a) = 0$ and $\pi(a) = u$. Since clearly $a \notin \kappa$, the strictly larger field $\kappa(a) \cong \kappa[\xi]/f\kappa[\xi]$ embeds into R, violating the maximality of κ.

To prove the last assertion (in either characteristic), assume the maximal ideal is finitely generated, say, $\mathfrak{m} = (x_1,\dots,x_e)R$. By Exercise 6.5.11, every element of R can be expanded as a power series in $(x_1,\dots,x_e)$ with coefficients in κ. In particular, R is a homomorphic image of the regular local ring $\kappa[[\xi_1,\dots,\xi_e]]$ (for the regularity of this latter ring, see Exercise 4.3.5). □

Remark 6.4.3 (CORRECT?). To prove the existence of coefficient fields in equal characteristic $p > 0$, we may reason as follows. First assume R is Artinian (indeed an instance of a complete local ring). We induct on the length l of R, where the case $l = 1$ is trivial. Suppose we have proven that any equal characteristic ring of length two admits an embedding of its residue field, and let R have length $l > 2$. Choose some non-zero x in the socle of R, that is to say, such that $x\mathfrak{m} = 0$, and let $\pi\colon R \to R/xR$ denote the canonical surjection. By assumption, there exists a homomorphism $\iota\colon k \to R/xR$, where k is the residue field of R. Let $S := \pi^{-1}(\iota(k)) \subseteq R$. I claim that $xR = xS \subseteq S$ is the maximal ideal of S. To verify the equality, let rx be a non-zero element with $r \in R$. In particular, r must be a unit in R, whence $\pi(r)$ in R/xR. Since k embeds in the latter, we can write $\pi(r) = \iota(c) + \pi(m)$, for some m in the maximal ideal of R. Hence, $r - m \in S$. Since $rx = (r-m)x$, we showed that $xR = xS$. Now, S/xS clearly equals $\iota(k) \cong k$, proving that xS is maximal. Since $x^2 = 0$, the length of S is two, and hence by assumption, there exists an embedding $k \to S$, which composed with the inclusion $S \subseteq R$ yields the desired embedding into R.

So remains to treat the case that R has length two. In particular, its maximal ideal is of the form xR with $x^2 = 0$. Let, as before, $\pi\colon R \to R/xR \cong k$ be the residue map. By the argument in the characteristic zero case, if $\kappa \subseteq R$ is a maximal subfield, then the extension $\pi(\kappa) \subseteq k$ is purely inseparable. Towards a contradiction, assume it is non-trivial. For any $a \in R$, some p-th power of $\pi(a)$ lies in $\pi(\kappa)$. In other words, $a^q = c + rx$ for some $c \in \kappa$, some $r \in R$, and some power q of p. Taking the p-th power of this equality, we get $a^{qp} = c^p + r^p x^p = c^p$. In other words, any element of R has some p-th power inside κ. In particular, R is integral over κ. Let $S := \kappa + xR$. It is easy to verify that this is a subring of R, and that xR is its maximal ideal (however, it is in general not generated by x, and a priori even infinitely generated in S). Nonetheless, $S \subseteq R$ is integral, and by assumption proper.

Let q be minimal among all p-th powers so that there exists $a \in R \setminus S$ such that $a^q \in S$. If $q > p$, then by minimality $b := a^{q/p}$ does not belong to S, yet b^p does. Hence $q = p$, and $a^p = c + rx$ for some $c \in \kappa$ and some $r \in R$. Let D be the κ-subalgebra of R generated by a. Since $a^{p^2} = c^p \in \kappa$, the dimension of D over κ is at most p^2. Since D is a purely inseparable algebra over κ, its dimension is therefore either p or p^2. If the latter, consider the κ-subalgebra generated by a^2. If this is equal to D, then $a = f(a^2)$, for some polynomial $f \in \kappa[\xi]$. Taking reductions then shows that $\pi(a)$ satisfies a separable equation over $\pi(\kappa)$, whence must belong to it, since $\pi(\kappa) \subseteq k$ is purely inseparable. This in turn implies $a \in S$, contradiction. Hence the subalgebra generated by a^2 has dimension strictly less than D, whence equal to p. Therefore, upon replacing a by a^2 if necessary, we may assume that D has dimension p over κ. Hence $a^p \in \kappa$, say $a^p = c$. If $c = d^p$ for some $d \in \kappa$, then $0 = a^p - d^p = (a-d)^p$, proving that $a - d \in Rx$ and hence $a \in S$. Hence $\xi^p - c$ is an irreducible polynomial over κ.

Therefore, the field $\kappa[\xi]/(\xi^p - c)\kappa[\xi]$ embeds in R, contradicting the maximality of κ.

For arbitrary R of equal characteristic p, now use that we can embed k in each $R_n := R/\mathfrak{m}^n$ by the above argument. Moreover, one checks that these embeddings are compatible with the residue maps $R_m \to R_n$ for $m \geq n$. Hence we get a homomorphism $k \to \varprojlim R_n$. This gives the required embedding, since $\varprojlim R_n$ is equal to $\widehat{R} = R$ by Proposition 6.3.8.

The analogue in mixed characteristic requires even more work, and so again we only quote the result here (see [41, Theorem 29.4] for a proof).

Theorem 6.4.4 (Mixed characteristic). *Let $(R, \mathfrak{m})$ be a complete local ring of mixed characteristic, with residue field k of characteristic $p > 0$. If R has embedding dimension e, then there exists a complete discrete valuation ring V with with maximal ideal pV and residue field k, and there exists an ideal $I \subseteq V[[\xi]]$ with $\xi = (\xi_1, \dots, \xi_{e-1})$ such that $R \cong V[[\xi]]/I$. In particular, R is Noetherian.*

The complete discrete valuation ring V from the statement is in fact uniquely determined by p and k, and called the *complete p-ring* with residue field k (see [41, Theorem 29.2 and Corollary]).

Immediately some important corollaries follow from these structure theorems.

Theorem 6.4.5. *A complete regular local ring of equal characteristic is isomorphic to a power series ring over a field.*

Proof. Let R be a d-dimensional complete regular local ring with residue field k. By definition, R has embedding dimension d, so that $R \cong k[[\xi]]/I$ by Theorem 6.4.2, with $\xi = (\xi_1, \dots, \xi_d)$ and $I \subseteq k[[\xi]]$. Since $k[[\xi]]$ has dimension d by Corollary 3.3.3, the ideal I must be zero by Corollary 3.3.6. □

There is also a structure theorem for complete regular local rings of mixed characteristic, but it is less straightforward and we will omit it.

Cohen normalization. The next result is the analogue for complete local rings of Noether normalization. Again we will only give the proof in equal characteristic.

Theorem 6.4.6. *If R is a d-dimensional Noetherian local ring, then there exists a (complete) d-dimensional regular local subring $S \subseteq R$ over which R is finite.*

Proof. Assume R has equal characteristic, and view its residue field k as a coefficient field of R (see Theorem 6.4.2). Let $\mathbf{x} := (x_1, \dots, x_d)$ be a system of parameters of R. Let $k[[\xi]] \to R$ be the k-algebra homomorphism given by $\xi_i \mapsto x_i$, where $\xi = (\xi_1, \dots, \xi_d)$, let I be the kernel of this homomorphism, and let S be its image. Hence $S \cong k[[\xi]]/I$. Since $R/\mathbf{x}R$ is Artinian by definition of system of parameters, it is a finite dimensional vector space over $S/\xi S = k$. Since S is also complete, R is a finite S-module by Theorem 6.2.5 (notice that $\mathfrak{I}_R = 0$ by Theorem 1.4.11 so that the Hausdorff condition is satisfied). In particular, by Theorem 3.3.8, both rings have the same dimension d. However, this then forces by Corollaries 3.3.3 and 3.3.6 that $I = 0$, so that S is regular (by Exercise 4.3.5). □

Complete scalar extensions. Sometimes it is desirable to have a residue field with some additional properties. We finish with discussing a technique of extending the residue field in equal characteristic (for the mixed characteristic case, we refer to [62]).

Theorem 6.4.7. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of equal characteristic with residue field k. Every extension $k \subseteq K$ of fields can be lifted to a faithfully flat extension $R \to R^{\frown}_K$, inducing the given extension on the residue fields, with $R^{\frown}_K$ a complete local ring with maximal ideal $\mathfrak{m}R^{\frown}_K$ and residue field K. In fact, $R^{\frown}_K$ is a solution to the following universal property: any complete Noetherian local R-algebra T with residue field K has a unique structure of a local $R^{\frown}_K$-algebra. In particular, $R^{\frown}_K$ is uniquely determined by R and K up to isomorphism, and is called the* complete scalar extension *of R along K.*

Proof. By Theorem 6.4.2, the completion $\widehat{R}$ of R is isomorphic to $k[[\xi]]/I$ for some ideal I and some tuple of indeterminates ξ. Put $R^{\frown}_K := K[[\xi]]/IK[[\xi]]$. By Theorem 6.3.4 and base change, S has all the required properties.

To prove the universal property, let T be any complete Noetherian local R-algebra, given by the local homomorphism $R \to T$. By the universal property of completions, we have a unique extension $k[[\xi]]/I \cong \widehat{R} \to T$, and by the universal property of tensor products, and this uniquely extends to a homomorphism $R^{\frown}_K = K[[\xi]]/IK[[\xi]] \to T$. □

Note that complete scalar extension is actually a functor, that is to say, any local homomorphism $R \to S$ of Noetherian local rings whose residue fields are subfields of K extends to a local homomorphism $R^{\frown}_K \to S^{\frown}_K$. In particular, complete scalar extension commutes with homomorphic images:

$$(R/I)^{\frown}_K \cong R^{\frown}_K/IR^{\frown}_K, \tag{6.6}$$

for all ideals $I \subseteq R$. By Exercise 6.5.12, the complete scalar extension $R^{\frown}_K$ has the same dimension as R, and one is respectively regular or Cohen-Macaulay if and only if the other is.

6.5 Exercises

Ex 6.5.1

Prove the statements in 6.1.1. Show moreover that the set I_r of all elements of norm at most r, and the set I_r^- of all elements of norm strictly less than r, are ideals, for all $r \in [0,1]$ (called norm-ideals*).*

Ex 6.5.2

Show that the canonical norm on a regular local ring is multiplicative.

Ex 6.5.3

Show that all norm-ideals (see Exercise 6.5.1) in a quasi-normed ring A are open in the norm topology. Show that A is Hausdorff if and only if $||\cdot||$ is a norm.

Ex 6.5.4

Prove the statements in 6.1.3 and 6.1.5. Prove that I is closed in the norm topology if and only if the quasi-norm on A/I is a norm.

Ex 6.5.5

Prove 6.1.6.

Ex 6.5.6

Show that the I-adic quasi-norm $||\cdot||_I$ is indeed a quasi-norm. Show that I and any of its powers define equivalent quasi-norms, in the sense that both norms are mutually bounded. Prove 6.1.7.

Ex 6.5.7

Prove 6.2.1 by finding for each Cauchy sequence an appropriate subsequence satisfying the hypothesis, and a subsequence of this satisfying the conclusion.

***Ex 6.5.8**

Show that the Jacobson radical *(:=intersection of all maximal ideals) in a quasi-complete ring is the ideal of all elements of norm strictly less than one.*

Ex 6.5.9

Formulate, and then prove a generalization of Theorem 6.2.5 which works for any ring which is quasi-complete in its I-adic quasi-norm. In fact, you can even formulate a version for any quasi-complete ring $(A, ||\cdot||)$.

Ex 6.5.10

Show that a local ring R has equal characteristic if and only if it has the same characteristic as its residue field.

Ex 6.5.11

Show that if κ is a coefficient field of a local ring $(R,\mathfrak{m})$ and $\mathfrak{m} = \mathbf{x}R$ is finitely generated, then for every $a \in R$ and each $n \in \mathbb{N}$, we can find a polynomial $f_n \in \kappa[\xi]$ such that $a \equiv f_n(\mathbf{x}) \mod \mathfrak{m}^n$. Deduce from this the assertion about power series expansions in the last paragraph of the proof of Theorem 6.4.2.

Ex 6.5.12

Show using Exercise 5.7.17 that R and its complete scalar extension $R^{\widehat{}}_K$ have the same dimension. Prove that R is regular or Cohen-Macaulay if and only if $R^{\widehat{}}_K$ is.

Additional exercises.

Ex 6.5.13

Show the following more general version of Hensel's lemma for a complete local ring R: if $f \in R[\xi]$, $c \in \mathbb{N}$ and $a \in R$ are such that $f(a)$ lies in the ideal $f'(a)^2\mathfrak{m}^c$, then there exists $b \in R$ with $f(b) = 0$ and $b \equiv a \mod \mathfrak{m}^c$.

6.6 Project: Henselizations

There are many ways to construct Henselizations (see for instance [42, 43, 46]), most of which rely on some more sophisticated notions, such as etale extensions, etc. There is, however, also a direct construction, which we will now discuss. Let $(R,\mathfrak{m})$ be a Noetherian local ring. By a *Hensel system* over R of size N, we mean a pair $(\mathscr{H},\mathbf{u})$ consisting of a system $(\mathscr{H})$ of N polynomial equations $f_1,\dots,f_N \in R[t]$ in the N unknowns $t := (t_1,\dots,t_N)$, and an approximate solution $\mathbf{u}$ modulo $\mathfrak{m}$ in R (meaning that $f_i(\mathbf{u}) \equiv 0 \mod \mathfrak{m}$ for all i), such that associated Jacobian matrix

$$\operatorname{Jac}(\mathscr{H}) := \begin{pmatrix} \partial f_1/\partial t_1 & \partial f_1/\partial t_2 & \dots & \partial f_1/\partial t_N \\ \partial f_2/\partial t_1 & \partial f_2/\partial t_2 & \dots & \partial f_2/\partial t_N \\ \vdots & \vdots & \ddots & \vdots \\ \partial f_N/\partial t_1 & \partial f_N/\partial t_2 & \dots & \partial f_N/\partial t_N \end{pmatrix} \tag{6.7}$$

evaluated at $\mathbf{u}$ is invertible over R (that is to say, its determinant is a unit in R). An N-tuple $\mathbf{x}$ in some local R-algebra S is called a *solution* of the Hensel system $(\mathscr{H},\mathbf{u})$, if it is a solution of the system $(\mathscr{H})$ and $\mathbf{x} \equiv \mathbf{u} \mod \mathfrak{m}S$. Note that a Hensel system of size $N = 1$ is just a Hensel equation together with a solution in the residue field, as in the statement of Hensel's lemma. In fact, R is Henselian (that is to say, satisfies Hensel's lemma) if and only if any Hensel system over R has a solution in R. The proof of this equivalence is not that easy (one can give for instance a proof using standard etale extensions; see [42] or [18, Exercise 7.26]). However, you can modify the proof of Theorem 6.2.4 to show that complete local rings have this property. In fact, using multivariate Taylor expansion, show the following stronger version (it is instructive to try this first for a single Hensel equation).

6.6.1 *Any Hensel system $(\mathscr{H},\mathbf{u})$ over R admits a unique solution in the completion $\widehat{R}$.*

We call an element $s \in \widehat{R}$ a *Hensel element* if there exists a Hensel system $(\mathscr{H},\mathbf{u})$ over R such that s is the first entry of the (unique) solution of this system in $\widehat{R}$. Let $R^\sim$ be the subset of $\widehat{R}$ of all Hensel elements. For given Hensel elements s and t, construct from their associated Hensel systems a new Hensel system for $s+t$ (respectively, for st), and use this to prove:

6.6.2 *The collection of all Hensel elements is a local ring $R^\sim$ with maximal ideal $\mathfrak{m}R^\sim$. Moreover, $R^\sim$ is Henselian, with completion equal to $\widehat{R}$.*

It is unfortunately less easy to prove that $R^\sim$ is also Noetherian. One way is to first show that $R^\sim \to \widehat{R}$ is faithfully flat, and then use this to deduce the Noetherianity of $R^\sim$ from that of $\widehat{R}$.

6.6.3 *Show that $R^\sim$ satisfies the universal property of Henselization: any Henselian local R-algebra S admits a unique structure of $R^\sim$-algebra.*

You could also try to prove:

6.6.4 *A power series over a field k in n indeterminates ξ is a Hensel element over the localization of $k[\xi]$ with respect to the maximal ideal generated by the ξ if and only if it is algebraic over that ring. In other words, $k[\xi]^\sim = k[[\xi]]^{alg}$.*

Part III
Fugue in U major[1]

[1] The major theme being ultraproducts. . .

Chapter 7
Uniform bounds

In this chapter, we will discuss our first application of ultraproducts: the existence of uniform bounds over polynomial rings. The method goes back to A. Robinson, but really gained momentum by the work of Schmidt and van den Dries in [52], where they brought in flatness as an essential tool. Most of our applications will be concerned with affine algebras over a field. So let us fix an ultra-field K, realized as the ultraproduct of fields K_w for $w \in \mathbb{W}$. For a concrete example, one may take $K := \mathbb{C}$ and $K_p := \mathbb{F}_p^{\text{alg}}$ by Theorem 1.4.3 (with $\mathbb{W}$ the set of prime numbers).

7.1 Ultra-hulls

Ultra-hull of a polynomial ring. In this section, we let $A := K[\xi]$, where $\xi := (\xi_1, \dots, \xi_n)$ are indeterminates. We define the *ultra-hull* (called the *non-standard hull* in the earlier papers [53, 54, 57]) of A as the ultraproduct of the $A_w := K_w[\xi]$, and denote it $U(A)$. The inclusions $K_w \subseteq A_w$ induce an inclusion $K \subseteq U(A)$. Let ξ_i also denote the ultraproduct $\operatorname{ulim}_w \xi_i$ of the constant sequence ξ_i. By Łos' Theorem, Theorem 1.3.1, the ξ_i are algebraically independent over K. Hence, we may view them as indeterminates over K in $U(A)$, thus yielding an embedding $A = K[\xi] \subseteq U(A)$. To see why this is called an ultra-hull, let us introduce the category of ultra-K-algebras: a K-algebra $B_\natural$ is called an *ultra-K-algebra* if it is the ultraproduct of K_w-algebras B_w; a morphism of ultra-K-algebras $B_\natural \to C_\natural$ is any K-algebra homomorphism obtained as the ultraproduct of K_w-algebra homomorphisms $B_w \to C_w$. It follows that any ultra-K-algebra is a K-algebra. The ultra-hull $U(A)$ is clearly an ultra-K-algebra. We have:

7.1.1 *The ultra-hull $U(A)$ satisfies the following universal property: if $B_\natural$ is an ultra-K-algebra, and $A \to B_\natural$ is any K-algebra homomorphism, then there exists a unique ultra-K-algebra homomorphism $U(A) \to B_\natural$ extending $A \to B$.*

Indeed, by assumption, $B_\natural$ is the ultraproduct of K_w-algebras B_w. Let $b_{i\natural}$ be the image of ξ_i under the the homomorphism $A \to B_\natural$, and choose $b_{iw} \in B_w$ whose ultraproduct equals $b_{i\natural}$. Define K_w-algebra homomorphisms $A_w \to B_w$ by the rule $\xi_i \mapsto b_{iw}$. The ultraproduct of these homomorphisms is then the required ultra-K-algebra homomorphism $\mathcal{U}(A) \to B_\natural$. Its uniqueness follows by an easy application of Łos' Theorem.

An intrinsic characterization of A as a subset of $\mathcal{U}(A)$ is provided by the next result (in the terminology of Chapter 12, this exhibits A as a certain *protoproduct*):

7.1.2 *An ultraproduct $f_\natural = \operatorname{ulim} f_w$ in $\mathcal{U}(A)$ belongs to A if and only if the $f_w \in A_w$ have bounded degree, meaning that there is a d such that almost all f_w have degree at most d.*

Indeed, if $f \in A$ has degree d, then we can write it as $f = \sum_\nu u_\nu \xi^\nu$ for some $u_\nu \in K$, where ν runs over all n-tuples with $|\nu| \leq d$. Choose $u_{\nu w} \in K_w$ such that their ultraproduct is u_ν, and put

$$f_w := \sum_{|\nu| \leq d} u_{\nu w} \xi^\nu. \tag{7.1}$$

An easy calculation shows that the ultraproduct of the f_w is equal to f, viewed as an element in $\mathcal{U}(A)$. Conversely, if almost each f_w has degree at most d, so that we can write it in the form (7.1), then

$$\operatorname*{ulim}_{w\to\infty} f_w = \sum_{|\nu| \leq d} (\operatorname*{ulim}_{w\to\infty} u_{\nu w}) \xi^\nu$$

is a polynomial (of degree at most d).

Ultra-hull of an affine algebra. More generally, let C be a K-*affine* ring, that is to say, a finitely generated K-algebra, say of the form $C = A/I$ for some ideal $I \subseteq A$. We define the *ultra-hull* of C to be $\mathcal{U}(A)/I\mathcal{U}(A)$, and denote it $\mathcal{U}(C)$. It is clear that the canoncial embedding $A \subseteq \mathcal{U}(A)$ induces by base change a homomorphism $C \to \mathcal{U}(C)$. Less obvious is that this is still an injective map, which we will prove in Corollary 7.2.3 below. To show that the construction of $\mathcal{U}(C)$ does not depend on the choice of presentation $C = A/I$, we verify that $\mathcal{U}(C)$ satisfies the same universal property 7.1.1 as $\mathcal{U}(A)$: any K-algebra homomorphism $C \to B_\natural$ to some ultra-K-algebra $B_\natural$ extends uniquely to a homomorphism $\mathcal{U}(C) \to B_\natural$ of ultra-K-algebras (recall that any solution to a universal property is necessarily unique). To see why the universal property holds , apply 7.1.1 to the composition $A \twoheadrightarrow A/I = C \to B_\natural$ to get a unique extension $\mathcal{U}(A) \to B_\natural$. Since any element in I is sent to zero under the composition $A \to B_\natural$, this homomorphism factors through $\mathcal{U}(A)/I\mathcal{U}(A)$, yielding the required homomorphism $\mathcal{U}(C) \to B_\natural$ of ultra-K-algebras. Uniqueness follows from the uniqueness of $\mathcal{U}(A) \to B_\natural$.

Since $I\mathcal{U}(A)$ is finitely generated, it is an ultra-ideal, that is to say, an ultraproduct of ideals $I_w \subseteq A_w$. By 1.1.6, the ultraproduct of the $C_w := A_w/I_w$ is equal to $\mathcal{U}(C) = \mathcal{U}(A)/I\mathcal{U}(A)$. If $C = A'/I'$ is a different presentation of C as a K-algebra (with A' a polynomial ring in finitely many indeterminates), and $C'_w := A'_w/I'_w$ the corresponding K_w-algebras, then their ultraproduct $\mathsf{U}(A')/I'\mathsf{U}(A')$ is another way of defining the ultra-hull of C, whence it must be isomorphic to $\mathsf{U}(C)$. Without loss of generality, we may assume $A \subseteq A'$ and hence $A_w \subseteq {A'}_w$. Since $\mathsf{U}(A)/I\mathsf{U}(A) \cong \mathsf{U}(C) \cong \mathsf{U}(A')/I'\mathsf{U}(A')$, the homomorphisms $A_w \subseteq {A'}_w$ induce homomorphisms $C_w \to {C'}_w$, and by Łos' Theorem, almost all are isomorphisms. This justifies the usage of calling the C_w *approximations* of C (in spite of the fact that they are not uniquely determined by C).

7.1.3 *The ultra-hull $\mathsf{U}(\cdot)$ is a functor from the category of K-affine rings to the category of ultra-K-algebras.*

The only thing which remains to be verified is that an arbitrary K-algebra homomorphism $C \to D$ of K-affine rings induces a homomorphism of ultra-K-algebras $\mathsf{U}(C) \to \mathsf{U}(D)$. However, this follows from the universal property applied to the composition $C \to D \to \mathsf{U}(D)$, admitting a unique extension so that the following diagram is commutative

$$\begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ \mathsf{U}(C) & \longrightarrow & \mathsf{U}(D). \end{array} \tag{7.2}$$

Ultra-hull of a local affine algebra. Recall that a K-affine local ring R is simply the localization $C_{\mathfrak{p}}$ of a K-affine algebra C at a prime ideal $\mathfrak{p}$. Let us call R *geometric*, if $\mathfrak{p}$ is a maximal ideal $\mathfrak{m}$ of C. By Proposition 2.6.1, a geometric K-affine local ring, in other words, is the local ring of a closed point on an affine scheme of finite type over K. Note that a K-affine local ring is in general not finitely generated as a K-algebra; one usually says that R is *essentially of finite type* over K. The next result will enable us to define the ultra-hull of a geometric affine local ring; we shall discuss the general case on page 112 below:

7.1.4 *Let C be a K-affine ring. If $\mathfrak{m}$ is a maximal ideal in C, then $\mathfrak{m}\mathsf{U}(C)$ is a maximal ideal in $\mathsf{U}(C)$, and $C/\mathfrak{m} \cong \mathsf{U}(C)/\mathfrak{m}\mathsf{U}(C)$.*

By our previous discussion, $\mathsf{U}(L) := \mathsf{U}(C)/\mathfrak{m}\mathsf{U}(C)$ is the ultra-hull of the field $L := C/\mathfrak{m}$. By Proposition 2.2.6, the extension $K \subseteq L$ is finite. It follows by Exercise 1.5.9 that L is an ultra-field. By the universal property L is equal to its own ultra-hull, and hence $\mathfrak{m}\mathsf{U}(C)$ is a maximal ideal. □

We can now define the ultra-hull of a K-affine local ring $R = C_{\mathfrak{m}}$ as the localization $\mathsf{U}(R) := \mathsf{U}(C)_{\mathfrak{m}\mathsf{U}(C)}$. Note that $\mathsf{U}(R)$ is again an ultra-ring: let C_w be approximations of C, and let $\mathfrak{m}_w \subseteq C_w$ be ideals whose ultraproduct is equal to $\mathfrak{m}\mathsf{U}(C)$. Since the latter is maximal, so are almost all $\mathfrak{m}_w$. For those w, set $R_w := (C_w)_{\mathfrak{m}_w}$ (and arbitrary for the remaining w). By Exercise 1.5.2, the ultraproduct of the R_w is equal to $\mathsf{U}(R)$, and for this reason we call them again *approximations* of R. We can formulate a similar universal property which is satisfied by $U(R)$, and then show that any local homomorphism $R \to S$ of K-affine local rings induces a unique homomorphism $U(R) \to U(S)$. Moreover, any two approximations agree almost everywhere (see Exercise 7.5.1). In particular, for homomorphic images we have:

7.1.5 *If $I \subseteq C$ is an ideal in a K-affine (local) ring, then $U(C/I) = U(C)/IU(C)$.*

We extend our naming practice also to elements or ideals: if $a \in C$ is an element or $I \subseteq C$ is an ideal, and $a_w \in C_w$ and $I_w \subseteq C_w$ are such that their ultraproduct equals $a \in U(C)$ and $IU(C)$ respectively, then we call the a_w and the I_w *approximations* of a and I respectively. In particular, by 7.1.4, the approximations of a maximal ideal are almost all maximal. The same holds true with 'prime' instead of 'maximal', but the proof is more involved, and we have to postpone it until Theorem 7.3.4 below.

7.2 Schmidt-van den Dries theorem

The ring $U(A)$ is highly non-Noetherian. In particular, although each $\mathfrak{m}U(A)$ is a maximal ideal for $\mathfrak{m}$ a maximal ideal of A, these are not the only maximal ideals of $U(A)$ (see Exercise 7.5.2). Nonetheless, they somehow 'cover' enough of $U(A)$ so that we can apply Theorem 5.6.18. More precisely:

7.2.1 *If almost all K_w are algebraically closed, then any proper finitely related ideal of $U(A)$ is contained in some $\mathfrak{m}U(A)$ with $\mathfrak{m} \subseteq A$ a maximal ideal.*

Indeed, this is even true for any proper ultra-ideal $I \subseteq U(A)$ (and finitely related ideals are ultra-ideals by Exercise 5.7.19). Namely, let I be the ultraproduct of ideals $I_w \subseteq A_w$. By Łos' Theorem, almost each I_w is a proper ideal whence contained in some maximal ideal $\mathfrak{m}_w$. By the Nullstellensatz 2.2.2, we can write $\mathfrak{m}_w$ as $(\xi_1 - u_{1w}, \dots, \xi_n - u_{nw})A_w$ for some $u_{iw} \in K_w$. Let $u_i \in K$ be the ultraproduct of the u_{iw}, so that the ultraproduct of the $\mathfrak{m}_w$ is equal to $(\xi_1 - u_1, \dots, \xi_n - u_n)U(A)$, and by Łos' Theorem it contains I. □

Theorem 7.2.2. *For any K-affine ring, the canonical homomorphism $C \to U(C)$ is faithfully flat, whence in particular injective.*

Proof. If we have proven this result for the ultra-hull $U(A)$ of A, then it will follow from 5.2.3 for any $C \to U(C)$, since the latter is just a base change $C = A/I \to U(A)/IU(A) = U(C)$, where $C = A/I$ is some presentation of C. The faithfulness of $U(A)$ is immediate from 7.1.4. So remains to show the flatness of $A \to U(A)$, and for this we may assume that K and all K_w are algebraically closed. Indeed, if K' is the ultraproduct of the algebraic closures of the K_w, then $A \to A' := K'[\xi]$ is flat by 5.2.3. By Exercise 7.5.3, the canonical homomorphism $U(A) \to U(A')$ is cyclically pure with respect to ideals extended from A, where $U(A')$ is the ultra-K'-hull of A. Hence if we showed that $A' \to U(A')$ is flat, then so is $A \to U(A)$ by Corollary 5.6.17. Hence we may assume all K_w are algebraically closed. By Theorem 5.6.18 in conjunction with 7.2.1, we only need to show that $R := A_\mathfrak{m} \to U(R) = U(A)_{\mathfrak{m}U(A)}$

is flat for every maximal ideal $\mathfrak{m} \subseteq A$. After a translation, we may assume $\mathfrak{m} = (\xi_1, \dots, \xi_n)A$. By Łos' Theorem, $(\xi_1, \dots, \xi_n)$ is $U(A)$-regular whence $U(R)$-regular. This proves that $U(R)$ is a big Cohen-Macaulay R-module. By Proposition 5.6.9 it is therefore a balanced big Cohen-Macaulay module, since any regular sequence in $U(R)$ is permutable by Łos' Theorem, because this is so in the Noetherian local rings $(A_w)_{\mathfrak{m}_w}$ (see Theorem 4.2.6). Hence $U(R)$ is flat over R by Theorem 5.6.10. □

Immediately from this and the cyclic purity of faithfully flat homomorphisms (Proposition 5.3.4) we get:

Corollary 7.2.3. *The canonical map $C \to U(C)$ is injective, and $IU(C) \cap C = I$ for any ideal $I \subseteq C$.* □

7.3 Transfer of structure

We will use ultra-hulls in our definition of tight closure in characteristic zero (see §9), and to this end, we need to investigate more closely the relation between an affine algebra and its approximations. We start with the following far reaching generalization of 7.1.4.

Finite extensions.

Proposition 7.3.1. *If $C \to D$ is a finite homomorphism of K-affine rings, then $U(D) \cong U(C) \otimes_C D$, and hence $U(C) \to U(D)$ is also finite.*

Proof. By Exercise 1.5.9, the tensor product $U(C) \otimes_C D$ is an ultra-K-algebra, since it is finite over $U(C)$. By the universal property of the ultra-hull of D, we therefore have a unique homomorphism $U(D) \to U(C) \otimes_C D$ of ultra-K-algebras. On the other hand, by the universal property of tensor products, we have a unique homomorphism $U(C) \otimes_C D \to U(D)$. It is no hard to see that the latter is in fact a morphism of ultra-K-algebras. By uniqueness of both homomorphisms, they must be therefore each other's inverse. □

Corollary 7.3.2. *If C is a K-affine Artinian ring, then $C \cong U(C)$.*

Proof. Since C is a direct product of local Artinian rings by 3.1.4, and since ultra-hulls are easily seen to commute with direct products, we may assume C is moreover local, with maximal ideal $\mathfrak{m}$, say. Let $L := C/\mathfrak{m}$, so that $L \cong U(L)$ by 7.1.4. Note that the vector space dimension of C over L is equal to the length of C by Exercise 3.4.3. In any case, C is a finite L-module, so that by Proposition 7.3.1 we get $U(C) = U(L) \otimes_L C = C$. □

Corollary 7.3.3. *The dimension of a K-affine ring is equal to the dimension of almost all of its approximations.*

Proof. Let C be an n-dimensional K-affine ring, with approximations C_w. The assertion is trivial for $C = A$ a polynomial ring. For the general case, let $A \subseteq C$ be a finite extension, as given by Theorem 2.2.5. The induced homomorphism $\mathsf{U}(A) \to \mathsf{U}(C) \cong \mathsf{U}(A) \otimes_A C$ is finite, by Proposition 7.3.1, and injective since $A \to \mathsf{U}(A)$ is flat by Theorem 7.2.2. By Łos' Theorem, almost all $A_w \to C_w$ are finite and injective. Hence almost all C_w have dimension n by Theorem 3.3.8. □

Prime ideals. We return to our discussion on the behavior of prime ideals under the ultra-hull, and we are ready to prove the promised generalization of 7.1.4.

Theorem 7.3.4. *A K-affine ring C is a domain if and only if $\mathsf{U}(C)$ is, if and only if almost all of its approximations are. In particular, if $\mathfrak{p}$ is a prime ideal in an arbitrary K-affine ring D, then $\mathfrak{p}\mathsf{U}(D)$ is again a prime ideal, and so are almost all of its approximations $\mathfrak{p}_w$.*

Proof. By Łos' Theorem, almost all C_w are domains if and only if $\mathsf{U}(C)$ is a domain. If this holds, then C too is a domain since it is a subring of $\mathsf{U}(C)$ by Corollary 7.2.3. Conversely, assume C is a domain, and let $A \subseteq C$ be a Noether normalization of C, that is to say a finite and injective extension. Let $A_w \subseteq C_w$ be the corresponding approximations implied by Proposition 7.3.1. Let $\mathfrak{p}_w$ be a prime ideal in C_w of maximal dimension, and let $\mathfrak{P}$ be their ultraproduct, a prime ideal in $\mathsf{U}(C)$. An easy dimension argument shows that $\mathfrak{p}_w \cap A_w = (0)$ and hence by Łos' Theorem, $\mathfrak{P} \cap \mathsf{U}(A) = (0)$. Let $\mathfrak{p} := \mathfrak{P} \cap C$. Since $\mathfrak{p} \cap A$ is contained in $\mathfrak{P} \cap \mathsf{U}(A)$, it is also zero. Hence $A \to C/\mathfrak{p}$ is again finite and injective. Since C is a domain, a dimension argument using Theorem 3.3.8 yields that $\mathfrak{p} = 0$. On the other hand, we have an isomorphism $\mathsf{U}(C) = \mathsf{U}(A) \otimes_A C$, so that by general properties of tensor products

$$\mathsf{U}(C)/\mathfrak{P} = \mathsf{U}(A)/(\mathfrak{P} \cap \mathsf{U}(A)) \otimes_{A/(\mathfrak{P}\cap A)} C/(\mathfrak{P} \cap C) = \mathsf{U}(A) \otimes_A C = \mathsf{U}(C)$$

showing that $\mathfrak{P}$ is zero, whence so are almost all $\mathfrak{p}_w$. Hence almost all C_w are domains, and hence by Łos' Theorem, so is $\mathsf{U}(C)$.

The last assertion is immediate from the first applied to $C := D/\mathfrak{p}$. □

This allows us to define the ultra-hull of an arbitrary local K-affine ring $C_\mathfrak{p}$ as the localization $\mathsf{U}(C)_{\mathfrak{p}\mathsf{U}(C)}$. To show that a local affine ring has the same dimension as almost all of its approximations, one can use either some deeper results on the dimension of an affine ring (see Exercise 7.5.6), or we proceed with some further transfer results.

Recall (see Definition 3.3.1) that the geometric dimension geodim(R) of a local ring $(R,\mathfrak{m})$ of finite embedding dimension is by definition the least number of generators needed to generate an $\mathfrak{m}$-primary ideal.

Proposition 7.3.5. *If $(R,\mathfrak{m})$ is a d-dimensional local K-affine ring, then $\mathsf{U}(R)$ has geometric dimension d.*

Proof. We induct on the dimension d, where the case $d = 0$ follows from Corollary 7.3.2. So assume $d > 0$, and let x be a parameter in R. Hence, R/xR has dimension $d-1$, so that by induction, $\mathsf{U}(R/xR)$ has geometric dimension $d-1$. Since

$U(R/xR) = U(R)/xU(R)$ by 7.1.5, we see that $U(R)$ has geometric dimension at most d. By way of contradiction, suppose its geometric dimension is at most $d-1$. In particular, there exists an $\mathfrak{m}U(R)$-primary ideal $\mathfrak{N}$ generated by $d-1$ elements. Put $\mathfrak{n} := \mathfrak{N} \cap R$, and let n be such that $\mathfrak{m}^n U(R) \subseteq \mathfrak{N}$. By faithful flatness, that is to say, by Corollary 7.2.3, we have an inclusion $\mathfrak{m}^n \subseteq \mathfrak{n}$, showing that $\mathfrak{n}$ is $\mathfrak{m}$-primary. Hence $R/\mathfrak{n} \cong U(R/\mathfrak{n}) = U(R)/\mathfrak{n}U(R)$ by Corollary 7.3.2. Hence $U(R)/\mathfrak{N}$ is a homomorphic image of $R/\mathfrak{n}$ whence equal to it by definition of $\mathfrak{n}$. In conclusion, $\mathfrak{N} = \mathfrak{n}U(R)$. By Theorem 3.3.2, the geometric dimension of R is d, so that $\mathfrak{n}$ requires at least d generators. Since $R \to U(R)$ is flat by Theorem 7.2.2, also $\mathfrak{n}U(R)$ requires at least d generators by 5.3.7, contradiction. □

Corollary 7.3.6. *The dimension of a local K-affine ring R is equal to the dimension of almost all of its approximations R_w. Moreover, if $\mathbf{x}$ is a sequence in R with approximations $\mathbf{x}_w$, then $\mathbf{x}$ is a system of parameters if and only if almost all $\mathbf{x}_w$ are.*

Proof. The second assertion follows immediately from the first and Łos' Theorem. By Proposition 7.3.5, the geometric dimension of $U(R)$ is equal to $d := \dim(R)$. Let R_w be approximations of R, so that their ultraproduct equals $U(R)$. If I is an $\mathfrak{m}U(R)$-primary ideal generated by d elements, then its approximation I_w is an $\mathfrak{m}_w$-primary ideal generated by d elements for almost all w by 1.4.9. Hence almost all R_w have geometric dimension at most d, whence dimension at most d by Theorem 3.3.2.

Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{m}$ be a chain of prime ideals in R of maximal length. By faithfull flatness (in the form of Corollary 7.2.3), this chain remains strict when extended to $U(R)$, and by Theorem 7.3.4, it consists again of prime ideals. Hence if $\mathfrak{p}_{iw} \subseteq R_w$ are approximations of $\mathfrak{p}_i$, then by Łos' Theorem, we get a strict chain of prime ideals $\mathfrak{p}_{0w} \subsetneq \cdots \subsetneq \mathfrak{p}_{dw} = \mathfrak{m}_w$ for almost all w, proving that almost all R_w have dimension at least d. □

Note that it is not true that if $\mathbf{x}_w$ are systems of parameters in the approximations, then their ultraproduct (which in general even lies outside R) does not necesarily generate an $\mathfrak{m}U(R)$-primary ideal.

Singularities. Now that we know how dimension behaves under ultra-hulls, we can investigate singularities.

Theorem 7.3.7. *A local K-affine ring is respectively regular or Cohen-Macaulay if and only if almost all its approximations are.*

Proof. Let R be a d-dimensional local K-affine ring, and let R_w be its approximations. If R is regular, then its embedding dimension is d, whence so is the embedding dimension of $U(R)$, and by Łos' Theorem, then so is the embedding dimension of R_w for almost each w, and conversely. This proves the assertion for regularity. As for the Cohen-Macaulay condition, let $\mathbf{x}$ be a system of parameters with approximation $\mathbf{x}_w$. Hence almost each $\mathbf{x}_w$ is a system of parameters in R_w by Corollary 7.3.6. If R is Cohen-Macaulay, then $\mathbf{x}$ is R-regular, hence $U(R)$-regular by flatness (see Proposition 5.4.1), whence almost each $\mathbf{x}_w$ is R_w-regular by Łos' Theorem, and conversely. □

7.4 Uniform bounds

In this last section, we are ready to deduce some applications of ultraproducts to the study of rings. The results as well as the proof method via ultraproducts are due to Schmidt and van den Dries from their seminal paper [52], and were further developped in [51, 53, 54, 64].

Linear equations. The proof of the next result is very typical for an argument based on ultraproducts, and will be the template for all future proofs.

Theorem 7.4.1. *There exists a function* $N\colon \mathbb{N}^2 \to \mathbb{N}$ *with the following property. If* k *is a field, and if* $f_0,\dots,f_s \in k[\xi]$ *are polynomials of degree at most* d *in at most* n *indeterminates* ξ *such that* $f_0 \in (f_1,\dots,f_s)k[\xi]$*, then there exist* $g_1,\dots g_s \in k[\xi]$ *of degree at most* $N(d,n)$ *such that* $f_0 = g_1f_1 + \dots + g_sf_s$*.*

Proof. By way of contradiction, suppose this result is false for some pair (d,n). This means that we can produce counterexamples requiring increasingly high degrees. Before we write these down, observe that the number s of polynomials in these counterexamples can be taken to be the same by Lemma 7.4.2 below (by adding zero polynomials if necessary). So, for each $w \in \mathbb{N}$, we can find counterexamples consisting of a field K_w, and polynomials $f_{0w},\dots,f_{sw} \in A_w := K_w[\xi]$ of degree at most d, such that f_{0w} can be written as an A_w-linear combination of the $f_{1w},\dots,f_{sw}$, but any such linear combination involves a polynomial of degree at least w. Let f_i be the ultraproduct of the f_{iw}. This is again a polynomial of degree d in A by 7.1.2. Moreover, by Łos' Theorem, $f_0 \in (f_1,\dots,f_d)U(A)$. We use the flatness of $A \to U(A)$ via its corollary in 7.2.3, to conclude that $f_0 \in (f_1,\dots,f_s)U(A) \cap A = (f_1,\dots,f_s)A$. Hence we can find polynomials $g_i \in A$ such that

$$f_0 = g_1f_1 + \dots + g_sf_s. \tag{7.3}$$

Let e be the maximum of the degrees of the g_i. By 7.1.2 again, we can choose approximations $g_{iw} \in A_w$ of g, of degree at most e. By Łos' Theorem, (7.3) yields $f_{0w} = \sum_i g_{iw}f_{iw}$, contradicting our assumption. □

Lemma 7.4.2. *Any ideal in* A *generated by polynomials of degree at most* d *requires at most* $N := \binom{d+n}{n}$ *generators.*

Proof. Note that N is equal to the number of monomials of degree at most d in n variables. Let $I := (f_1,\dots,f_s)A$ be an ideal in A with each f_i of degree at most d. Choose some (total) ordering $<$ on these monomials (e.g., the lexicographical ordering on the exponent vectors), and let $l(f)$ denote the largest monomial appearing in f with non-zero coefficient, for any $f \in A$ a polynomial of degree at most d (where we put $l(0) := -\infty$). If $l(f_i) = l(f_j)$ for some non-zero f_i, f_j with $i < j$, then $l(uf_i - vf_j) < l(f_i)$ for some non-zero elements $u, v \in K$, and we may replace the generator f_j by the new generator $uf_i - vf_j$. Doing this recursively for all i, we arrive at a situation in which all non-zero f_i have different $l(f_i)$, and hence there can be at most N of these. □

We can reformulate the result in Theorem 7.4.1 to arrive at some further generalizations. The ideal membership condition in that theorem is really about solving an (inhomogeneous) linear equation in A: the equation $f_0 = f_1t_1 + \cdots + f_st_s$, where the t_i are the unknowns of this equation (as opposed to ξ, which are indeterminates). This is the perspective taken in Exercise 7.5.5, which shows there exists a bound, only depending on the degree and the number of variables, for every system of linear equations. In the homogeneous case we can say even more:

Theorem 7.4.3. *There exists a bound $N := N(d,n)$ such that for any field k, any homogeneous system of equations in $k[\xi_1,\dots,\xi_n]$ all of whose coefficients have degree at most d, admits a finite number of solutions of degree at most N such that any other solution is a linear combination of these finitely many solutions.*

Proof. The proof once more is by contradiction. Assume the statement is false for the pair (n,d). Hence we can find for each $w \in \mathbb{N}$, a field K_w, and a system of linear homogeneous equations

$$\lambda_{1w}(t) = \cdots = \lambda_{sw}(t) = 0 \qquad (\mathscr{L}_w)$$

in the variables $t = (t_1,\dots,t_m)$ with coefficients in A_w, such that the module of solutions $\mathrm{Sol}_{A_w}(\mathscr{L}_w) \subseteq A_w^k$ requires at least one generator one of whose entries is a polynomial of degree at least w. Here, we may again take the number m of t-variables as well as the number s of equations to be the same in all counterexamples, by another use of Lemma 7.4.2. The ultraproduct of each λ_{iw} is, as before by 7.1.2, an element $\lambda_i \in A[t]$ which is a linear form in the t-variables (and has degree at most d in ξ). By the equational criterion for flatness, Theorem 5.6.1, the flatness of $A \to U(A)$, proven in Theorem 7.2.2, amounts to the existence of solutions $\mathbf{b}_1,\dots,\mathbf{b}_l \in \mathrm{Sol}_A(\mathscr{L})$ such that any solution of the homogeneous linear system $(\mathscr{L})$ of equations $\lambda_1 = \cdots = \lambda_s = 0$ in $U(A)$ lies in the $U(A)$-module generated by the $\mathbf{b}_i$. Let e be the maximum of the degrees occuring in the $\mathbf{b}_i$. In particular, we can find approximations $\mathbf{b}_{iw} \in A_w^m$ of $\mathbf{b}_i$ whose entries all have degree at most e. I claim that almost each $\mathrm{Sol}_{A_w}(\mathscr{L}_w)$ is equal to the submodule H_w generated by $\mathbf{b}_{1w},\dots,\mathbf{b}_{lw}$, which would then contradict our assumption.

To prove the claim, one inclusion is clear, so assume by way of contradiction that we can find for almost all w a solution $\mathbf{q}_w \in \mathrm{Sol}_{A_w}(\mathscr{L}_w)$ outside H_w. Let $\mathbf{q}_\natural \in U(A)^m$ be its ultraproduct (note that this time, we cannot guarantee that its entries lie in A since the degrees might be unbounded). By Łos' Theorem, $\mathbf{q}_\natural \in \mathrm{Sol}_{U(A)}(\mathscr{L})$, whence can be written as an $U(A)$-linear combination of the $\mathbf{b}_i$. Writing this out and using Łos' Theorem once more, we conclude that $\mathbf{q}_w$ lies in H_w for almost all w, contradiction. □

Primality testing.

Theorem 7.4.4. *There exists a function $N\colon \mathbb{N}^2 \to \mathbb{N}$ with the following property. If k is a field, and if $\mathfrak{p}$ is an ideal in $k[\xi_1,\dots,\xi_n]$ generated by polynomials of degree at most d, then $\mathfrak{p}$ is a prime ideal if and only if for any two polynomials f,g of degree at most $N(d,n)$ which do not belong to $\mathfrak{p}$, neither does their product.*

Proof. One direction in the criterion is obvious. Suppose the other is false for the pair (d,n), so that we can find for each $w \in \mathbb{N}$, a field K_w and a non-prime ideal $\mathfrak{a}_w \subseteq A_w$ generated by polynomials of degree at most d, such that any two polynomials of degree at most w not in $\mathfrak{a}_w$ have their product also outside $\mathfrak{a}_w$. Taking ultraproducts of the generators of the $\mathfrak{a}_w$ of degree at most d gives polynomials of degree at most d in A by 7.1.2, and by Łos' Theorem if $\mathfrak{a} \subseteq A$ is the ideal they generate, then $\mathfrak{a}U(A)$ is the ultraproduct of the $\mathfrak{a}_w$. I claim that $\mathfrak{a}$ is a prime ideal. However, this implies that almost all $\mathfrak{a}_w$ must be prime ideals by Theorem 7.3.4, contradiction.

To verify the claim, let $f, g \notin \mathfrak{a}$. We want to show that $fg \notin \mathfrak{a}$. Let e be the maximum of the degrees of f and g. Choose approximations $f_w, g_w \in A_w$ of degree at most e, of f and g respectively. By Łos' Theorem, $f_w, g_w \notin \mathfrak{a}_w$ for almost all w. For $w \geq e$, our assumption then implies that $f_w g_w \notin \mathfrak{a}_w$, whence by Łos' Theorem, their ultraproduct $fg \notin \mathfrak{a}U(A)$. A fortiori, then neither does fg belong to $\mathfrak{a}$, as we wanted to show. □

The pattern by now must become clear: prove a particular property of ideals is preserved under ultra-hulls, and use this to deduce uniform bounds. For instance you are asked in Exercise 7.5.7 to prove the following two results.

Proposition 7.4.5. *The image of a radical ideal in the ultra-hull remains radical.*

Since the radical of an ideal is the intersection of its minimal overprimes, we derive from this the following uniform bounds property:

Theorem 7.4.6. *There exists a function $N\colon \mathbb{N}^2 \to \mathbb{N}$ with the following property. If k is a field, and if I is an ideal in $k[\xi_1, \dots, \xi_n]$ generated by polynomials of degree at most d, then its radical $J := \operatorname{rad}(I)$ is generated by polynomials of degree at most $N := N(n,d)$. Moreover, $J^N \subseteq I$ and I has at most N distinct minimal overprimes, all of which are generated by polynomials of degree at most N.*

7.5 Exercises

Ex 7.5.1

Call a ring $S_\natural$ an ultra-local K-algebra*, if it is an ultraproduct of local K_w-algebras S_w; any ultraproduct of local K_w-algebra homomorphisms $S_w \to T_w$ is called a* morphism of ultra-local K-algebras*. Show that if R is a local K-affine ring, then its ultra-hull $U(R)$ is an ultra-local K-algebra. Moreover, we have the following universal property: if $R \to S_\natural$ is a local K-algebra homomorphism into an ultra-local K-algebra $S_\natural$, then there exists a unique morphism $U(R) \to S_\natural$ of ultra-local K-algebras. Prove 7.1.5 and the assertions preceding it.*

Ex 7.5.2

The maximal ideals of $U(A)$ that are not extended from A are harder to describe. To show that they at least exist, we reason as follows. For each w, choose a polynomial $f_w \in A_w$ of degree w with distinct roots (assuming K_w has at least size w), and let $f \in U(A)$ be their *ultraproduct. Let $\mathfrak{a}$ be the ideal generated by all f/h where h runs over all elements in A such that $f \in hU(A)$. Show that $\mathfrak{a}$ is not the unit ideal, and hence is contained in some maximal ideal $\mathfrak{M}$ of $U(A)$. Show that $\mathfrak{a}$ cannot be inside a maximal ideal of the form $\mathfrak{m}U(A)$ with $\mathfrak{m} \subseteq A$, showing that $\mathfrak{M}$ is not of the latter form. Give an example, assuming that the K_w are not algebraically closed, of a maximal ultra-ideal of $U(A)$ which is not extended from A.*

Ex 7.5.3

Show that if $C_\natural \to D_\natural$ is an ultraproduct of cyclically pure homomorphisms $C_w \to D_w$, then $C_\natural \to D_\natural$ is cyclically pure with respect to ultra-ideals. Deduce from this the claim in the proof of Theorem 7.2.2 about the cyclical purity of $U(A) \to U(A')$ with respect to ideals extended from A.

Ex 7.5.4

Show the 'global' counterparts of Theorem 7.3.7, that is to say, a K-affine ring is respectively regular or Cohen-Macaulay if and only if almost all of its approximations are.

Ex 7.5.5

Show that there exists a bound $N := N(d,n)$ such that for any field k, and for any (not necessarily homogeneous) linear system $(\mathscr{L})$ of equations $\lambda_1 = \cdots = \lambda_s = 0$ with $\lambda_i \in k[\xi, t]$ of ξ-degree at most d and t-degree at most one, where ξ is an n-tuple of indeterminates and t is a finite tuple of variables, if the system admits a solution in $K[\xi]$, then it admits a solutions all of whose entries have degree at most N.

Ex 7.5.6

In a K-affine domain D, we always have an equality $\dim(D/\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}) = \dim(D)$ (for a special case, see Exercise 3.4.14). Assuming this result, use it to give an alternative proof of Corollary 7.3.6 which does not rely on Proposition 7.3.5, but instead uses Corollary 7.3.3.

Ex 7.5.7

Prove Proposition 7.4.5 and derive Theorem 7.4.6 from it by the typical ultraproduct argument.

Ex 7.5.8

Use Theorem 5.6.16, the Colon Criterion, to show that there exists a bound $N := N(d,n)$ such that for any field k, any ideal $I \subseteq k[\xi]$ generated by polynomials of degree at most d, and any $a \in k[\xi]$ of degree at most d, where ξ is an n-tuple of indeterminates, the ideal $(I : a)$ is generated by polynomials of degree at most N.

Chapter 8
Tight closure in positive characteristic

In this chapter, p is a fixed prime number, and all rings are assumed to have characteristic p, unless explicitly mentioned otherwise. We review the notion of tight closure due to Hochster and Huneke (as a general reference, we will use [36]).

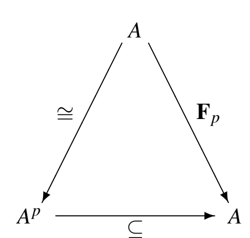
8.1 Frobenius

The major advantage of rings of positive characteristic is the presence of an algebraic endomorphism: the Frobenius. More precisely, let A be a ring of characteristic p, and let $\mathbf{F}_p$, or more accurately, $\mathbf{F}_{p,A}$, be the ring homomorphism $A \to A\colon a \mapsto a^p$, called the *Frobenius* on A. Recall that this is indeed a ring homomorphism, where the only thing to note is that the coefficients in the binomial expansion

$$\mathbf{F}_p(a+b) = \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i} = \mathbf{F}_p(a) + \mathbf{F}_p(b)$$

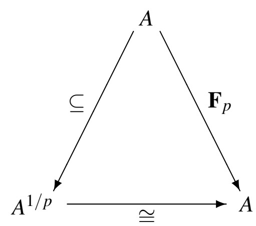
are divisible by p for all $0 < i < p$ whence zero in A, proving that $\mathbf{F}_p$ is additive.

When A is reduced, $\mathbf{F}_p$ is injective whence yields an isomorphism with its image $A^p := \mathrm{Im}(\mathbf{F}_p)$ consisting of all p-th powers of elements in A (and not to be confused with the p-th Cartesian power of A). The inclusion $A^p \subseteq A$ is isomorphic with the Frobenius on A because we have a commutative diagram



(8.1)

When A is a domain, then we can also define the ring $A^{1/p}$ as the subring of the algebraic closure of the field of fractions of A consisting of all elements b satisfying $b^p \in A$. Hence $A \subseteq A^{1/p}$ is integral. Since, $\mathbf{F}_p(A^{1/p}) = A$ and $\mathbf{F}_p$ is injective, we get $A^{1/p} \cong A$. Moreover, we have a commutative diagram



(8.2)

showing that the Frobenius on A is also isomorphic to the inclusion $A \subseteq A^{1/p}$. It is sometimes easier to work with either of these inclusions rather than with the Frobenius itself, especially to avoid notational ambiguity between source and target of the Frobenius (instances where this approach would clarify the argument are the proofs of Theorem 8.1.2 and Corollary 8.1.3 below).

Often, the inclusion $A^p \subseteq A$ is even finite, and hence so is the Frobenius itself. One can show (see Exercise 8.7.11) using Noether normalization (Theorem 2.2.5) or Cohen normalization (Theorem 6.4.6) that this is true when A is respectively an affine k-algebra or a complete Noetherian local ring with residue field k, and k is perfect, or more generally, $(k : k^p) < \infty$.

Frobenius transforms. Given an ideal $I \subseteq A$, we will denote its extension under the Frobenius by $\mathbf{F}_p(I)A$, and call it the *Frobenius transform* of I. Note that $\mathbf{F}_p(I)A \subseteq I^p$, but the inclusion is in general strict. In fact, one easily verifies that

8.1.1 *If* $I = (x_1, \dots, x_n)A$, *then* $\mathbf{F}_p(I)A = (x_1^p, \dots, x_n^p)A$.

If we repeat this process, we get the *iterated Frobenius transforms* $\mathbf{F}_p^n(I)A$ of I, generated by the p^n-th powers of elements in I, and in fact, of generators of I. In tight closure theory, the simplified notation

$$I^{[n]} := \mathbf{F}_p^n(I)A$$

is normally used, but for reasons that will become apparent once we defined tight closure as a difference closure (see page 138), we will use the 'heavier' notation. On the other hand, since we fix the characteristic, we may omit p from the notation and simply write $\mathbf{F} : A \to A$ for the Frobenius.

Kunz's theorem. The next result, due to Kunz, characterizes regular local rings in positive characteristic via the Frobenius. We will only prove the direction that we need.

Theorem 8.1.2. *Let R be a Noetherian local ring. If R is regular, then* $\mathbf{F}_p$ *is flat. Conversely, if R is reduced and* $\mathbf{F}_p$ *is flat, then R is regular.*

Proof. We only prove the direct implication; for the converse see [40, §42]. Let $\mathbf{x}$ be a system of parameters of R, whence an R-regular sequence by Proposition 4.2.3.

Since $\mathbf{F}(\mathbf{x})$ is also a system of parameters, it too is R-regular (Theorem 4.2.6). Hence, R viewed as an R-algebra via $\mathbf{F}$ is a balanced big Cohen-Macaulay module, and therefore flat by Theorem 5.6.10. □

Corollary 8.1.3. *If R is a regular local ring, $I \subseteq R$ an ideal, and $a \in R$ an arbitrary element, then $a \in I$ if and only if $\mathbf{F}(a) \in \mathbf{F}(I)R$.*

Proof. One direction is of course trivial, so assume $\mathbf{F}(a) \in \mathbf{F}(I)R$. However, since $\mathbf{F}$ is flat by Theorem 8.1.2, the contraction of the extended ideal $\mathbf{F}(I)R$ along $\mathbf{F}$ is again I by Proposition 5.3.4, and a lies in this contraction (recall that $\mathbf{F}(I)R \cap R$ stands really for $\mathbf{F}^{-1}(\mathbf{F}(I)R)$.) □

8.2 Tight closure

The definition of tight closure, although not complicated, is at first hard to grasp, and only by working with it enough, and realizing its versatility, does one get a knack of it. The idea is inspired by the ideal membership test of Corollary 8.1.3. Unfortunately, that test only works over regular local rings, so that it will be no surprise that whatever test we design, it will have to be more involved. Moreover, the proposed test will in fact fail in general, that is to say, the elements satisfying the test form an ideal which might be strictly bigger than the original ideal. But not too much bigger, so that we may view this bigger ideal as a closure of the original ideal, and as such, it is a 'tight' fit.

In the remainder of this section, A is a Noetherian ring, of characteristic p. A first obvious generalization of the ideal membership test from Corollary 8.1.3 is to allow iterates of the Frobenius: we could ask, given an ideal $I \subseteq A$, what are the elements x such that $\mathbf{F}^n(x) \in \mathbf{F}^n(I)A$ for some power n? They do form an ideal and the resulting closure operation is called the *Frobenius closure*. However, its properties are not sufficiently strong to derive all the results tight closure can.

Tight closure. The adjustment to make in the definition of Frobenius closure, although minor, might at first be a little surprising. To make the definition, we will call an element $a \in A$ a *multiplier*, if it is either a unit, or otherwise generates an ideal of positive height (necessarily one by Theorem 3.3.4). Put differently, a is a multiplier if it does not belong to any minimal prime ideal of A. In particular, the product of two multipliers is again a multiplier. In a domain, a situation we can often reduce to, a multiplier is simply a non-zero element.

The name 'multiplier' comes from the fact that we will use such elements to multiply our test condition with. However, for this to make sense, we cannot just take one iterate of the Frobenius, we must take all of them, or at least all but finitely many. So we now define: an element $x \in A$ belongs to the *tight closure* $\mathrm{cl}_A(I)$ of an ideal $I \subseteq A$, if there exists a multiplier $c \in A$ and a positive integer N such that

$$c\,\mathbf{F}^n(x) \in \mathbf{F}^n(I)A \tag{8.3}$$

for all $n \geq N$. Note that the multiplier c and the bound N may depend on x and I, but not on n. We will write $\operatorname{cl}(I)$ for $\operatorname{cl}_A(I)$ if the ring A is clear from the context. In the literature, tight closure is invariably denoted I^*, but again for reasons that will become clear in the next chapter, our notation better suits our purposes. Let us verify some elementary properties of this closure operation:

8.2.1 *The tight closure of an ideal I in a Noetherian ring A is again an ideal, it contains I, and it is equal to its own tight closure. Moreover, we can find a multiplier c and a positive integer N which works simultaneous for all elements in $\operatorname{cl}(I)$ in criterion (8.3).*

It is easy to verify that $\operatorname{cl}(I)$ is closed under multiples, and contains I. To show that it is closed under sums, whence an ideal, assume $x, x' \in A$ both lie in $\operatorname{cl}(I)$, witnessed by the equations (8.3) for some multipliers c and c', and some positive integers N and N' respectively. However, $cc'\mathbf{F}^n(x+x')$ then lies in $\mathbf{F}^n(I)A$ for all $n \geq \max\{N, N'\}$, showing that $x + x' \in \operatorname{cl}(I)$ since cc' is again a multiplier. Let $J := \operatorname{cl}(I)$ and choose generators $y_1, \dots, y_s$ of J. Let c_i and N_i be the corresponding multiplier and bound for y_i. It follows that $c := c_1 c_2 \cdots c_s$ is a multiplier such that (8.3) holds for all $n \geq N := \max\{N_1, \dots, N_s\}$ and all $x \in J$, since any such element is a linear combination of the y_i. In particular, $c\,\mathbf{F}^n(J)A \subseteq \mathbf{F}^n(I)A$ for all $n \geq N$. Hence if z lies in the tight closure of J, so that $d\mathbf{F}^n(z) \in \mathbf{F}^n(J)A$ for some multiplier d and for all $n \geq M$, then $cd\,\mathbf{F}^n(z) \in \mathbf{F}^n(I)A$ for all $n \geq \max\{M, N\}$ whence $z \in \operatorname{cl}(I)$. The last assertion now easily follows from the above analysis. In the sequel, we will therefore no longer make the bound N explicit and instead of "for all $n \geq N$" we will just write "for all $n \gg 0$".

Example 8.2.2. It is instructive to look at an example. Let K be a field of characteristic $p > 3$, and let $A := K[\xi, \zeta, \eta]/(\xi^3 - \zeta^3 - \eta^3)K[\xi, \zeta, \eta]$ be the projective coordinate ring of the *cubic Fermat curve*. Let us show that ξ^2 is in the tight closure of $I := (\zeta, \eta)A$. For a fixed e, write $2p^e = 3h + r$ for some $h \in \mathbb{N}$ and some remainder $r \in \{1, 2\}$, and let c be the multiplier ξ^3. Hence

$$c\,\mathbf{F}^e(\xi^2) = \xi^{3(h+1)+r} = \xi^r(\zeta^3 + \eta^3)^{h+1}.$$

A quick calculation shows that any monomial in the expansion of $(\zeta^3 + \eta^3)^{h+1}$ is a multiple of $\mathbf{F}^e(\zeta)$ or of $\mathbf{F}^e(\eta)$, showing that (8.3) holds for all e, and hence that $(\xi^2, \zeta, \eta)A \subseteq \operatorname{cl}(I)$.

It is often much harder to show that an element does not belong to the tight closure of an ideal. Shortly, we will see in Theorem 8.3.6 that any element outside the integral closure is also outside the tight closure. Since $(\xi^2, \zeta, \eta)A$ is integrally closed, we conclude that it is equal to $\operatorname{cl}(I)$.

We will encounter many operations similar to tight closure, and so we formally define:

Definition 8.2.3 (Closure operation). A *closure operation* on a ring A is any order-preserving, contractive, idempotent endomorphism of the Grassmanian $\operatorname{Grass}(A)$

(recall that Grass(A) is ordered by reverse inclusion, so that *contractive* means that I lies in its own closure).

For instance, taking the radical of an ideal is a closure operation, and so is *integral closure* discussed below. Tight closure too is a closure operation on A, since it clearly also preserves inclusion: if $I \subseteq I'$, then $\mathrm{cl}(I) \subseteq \mathrm{cl}(I')$. An ideal that is equal to its own tight closure is called *tightly closed*. Recall that the *colon* ideal $(I : J)$ is the ideal of all elements $a \in A$ such that $aJ \subseteq I$; here $I \subseteq A$ is an ideal, but $J \subseteq A$ can be any subset, which, however, most of the time is either a single element or an ideal. Almost immediately from the definitions, we get

8.2.4 *If I is tightly closed, then so is $(I : J)$ for any $J \subseteq A$.*

One of the most outstanding open problems in tight closure theory is its behavior under localization: do we always have

$$\mathrm{cl}_A(I)A_{\mathfrak{p}} \stackrel{?}{=} \mathrm{cl}_{A_{\mathfrak{p}}}(IA_{\mathfrak{p}}) \tag{8.4}$$

for every prime ideal $\mathfrak{p} \subseteq A$. This disturbing gap in our knowledge explains the awkward terminology in the next definition.

Definition 8.2.5. A Noetherian ring A is called *weakly F-regular* if each of its ideals is tightly closed. If all localizations of A are weakly F-regular, then A is called *F-regular*.

It is sometimes cumbersome to work with multipliers in arbitrary rings, but in domains they are just non-zero elements. Fortunately, we can always reduce to the domain case when calculating the tight closure:

Proposition 8.2.6. *Let A be a Noetherian ring, let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be its minimal primes, and put $\bar{A}_i := A/\mathfrak{p}_i$. For all ideals $I \subseteq A$ we have*

$$\mathrm{cl}_A(I) = \bigcap_{i=1}^{s} \mathrm{cl}_{\bar{A}_i}(I\bar{A}_i) \cap A. \tag{8.5}$$

Proof. The same equations which exhibit x as en element of $\mathrm{cl}_A(I)$ also show that it is in $\mathrm{cl}_{\bar{A}_i}(I\bar{A}_i)$ since any multiplier in A remains, by virtue of its definition, a multiplier in $\bar{A}_i$ (moreover, the converse also holds: by prime avoidance, we can lift any multiplier in $\bar{A}_i$ to one in A). So one inclusion in (8.5) is clear.

Conversely, suppose x lies in the intersection on the right hand side of (8.5). Let $c_i \in A$ be a multiplier in A (so that its image is a multiplier in $\bar{A}_i$), such that

$$c_i \mathbf{F}^n_{\bar{A}_i}(x) \in \mathbf{F}^n_{\bar{A}_i}(I)\bar{A}_i$$

for all $n \gg 0$. This means that each $c_i \mathbf{F}^n_A(x)$ lies in $\mathbf{F}^n_A(I)A + \mathfrak{p}_i$ for $n \gg 0$. Choose for each i, an element $t_i \in A$ inside all minimal primes except $\mathfrak{p}_i$, and let $c := c_1t_1 + \cdots + c_st_s$. A moment's reflection yields that c is again a multiplier. Moreover, since $t_i\mathfrak{p}_i \subseteq \mathfrak{n}$, where $\mathfrak{n} := \mathrm{nil}(R)$ is the nil-radical of A, we get

$$c\,\mathbf{F}^n_A(x) \in \mathbf{F}^n_A(I)A + \mathfrak{n}$$

for all $n \gg 0$. Choose m such that $\mathfrak{n}^{p^m}$ is zero, whence also the smaller ideal $\mathbf{F}_A(\mathfrak{n})$. Apply $\mathbf{F}_A^m$ to the previous equations, yielding

$$\mathbf{F}_A^m(c)\mathbf{F}_A^{m+n}(x) \in \mathbf{F}_A^{m+n}(I)A$$

for all $n \gg 0$, which means that $x \in \mathrm{cl}_A(I)$ since $\mathbf{F}_A^m(c)$ is again a multiplier. □

8.3 Five key properties of tight closure

In this section we derive five key properties of tight closure, all of which admit fairly simple proofs. It is important to keep this in mind, since these five properties will already suffice to prove in the next section some deep theorems in commutative algebra. In fact, as we will see, any closure operation with these five properties on a class of Noetherian local rings would establish these deep theorems for that particular class (and there are still classes for which this is not known to be true). Moreover, the proofs of the five properties themselves rest on a few simple facts about the Frobenius, so that this will allow us to also carry over our arguments to characteristic zero in Chapters 9 and 10.

The first property, stated here only in its weak version, is merely an observation. Namely, any equation (8.3) in a ring A extends to a similar equation in any A-algebra B. In order for the latter to calculate tight closure, the multiplier $c \in A$ should remain a multiplier in B, and so we proved:

Theorem 8.3.1 (Weak Persistence). *Let $A \to B$ be a ring homomorphism, and let $I \subseteq A$ be an ideal. If $A \to B$ is injective and B is a domain, or more generally, if $A \to B$ preserves multipliers, then* $\mathrm{cl}_A(I) \subseteq \mathrm{cl}_B(IB)$. □

The remarkable fact is that this is also true if $A \to B$ is arbitrary and A is of finite type over an excellent Noetherian local ring (see [36, Theorem 2.3]). We will not need this stronger version, the proof of which requires another important ingredient of tight closure: the notion of a test element. A multiplier $c \in A$ is called a *test element* for A, if for every $a \in \mathrm{cl}(I)$, we have $c\mathbf{F}^n(a) \in \mathbf{F}^n(I)A$ for all n. The existence of test elements is not easy, but once one has established their existence, many arguments become even more streamlined.

Theorem 8.3.2 (Regular closure). *In a regular local ring, every ideal is tightly closed. In fact, a regular ring is* F-regular.

Proof. Let R be a regular local ring. By Corollary 5.5.8, any localization of R is again regular, so that the second assertion follows from the first. To prove the first, let I be an ideal and $x \in \mathrm{cl}(I)$. Towards a contradiction, assume $x \notin I$. In particular, we must have $(I : x) \subseteq \mathfrak{m}$. Choose a non-zero element c such that (8.3) holds for all $n \gg 0$. This means that c lies in the colon ideal $(\mathbf{F}^n(I)R : \mathbf{F}^n(x))$, for all $n \gg 0$. Since $\mathbf{F}$ is flat by Theorem 8.1.2, the colon ideal is equal to $\mathbf{F}^n(I : x)R$ by Theorem 5.6.16. Since $(I : x) \subseteq \mathfrak{m}$, we get $c \in \mathbf{F}^n(\mathfrak{m})R \subseteq \mathfrak{m}^{p^n}$. Since this holds for all $n \gg 0$, we get $c = 0$ by Theorem 1.4.11, clearly a contradiction. □

Theorem 8.3.3 (Colon Capturing). *Let R be a Noetherian local domain which is a homomorphic image of a regular (or even Cohen-Macaulay) local ring, and let $(x_1,\dots,x_d)$ be a system of parameters in R. Then for each i, the colon ideal $((x_1,\dots,x_i)R : x_{i+1})$ is contained in* $\mathrm{cl}((x_1,\dots,x_i)R)$.

Proof. Let S be a local Cohen-Macaulay ring such that $R = S/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq S$ of height h. By prime avoidance, we can lift the x_i to elements in S, again denoted for simplicity by x_i, and find elements $y_1,\dots,y_h \in \mathfrak{p}$ such that $(y_1,\dots,y_h,x_1,\dots,x_d)$ is a system of parameters in S, whence an S-regular sequence (see Exercise 8.7.3). Since $\mathfrak{p}$ contains the ideal $J := (y_1,\dots,y_h)S$ of the same height (see 4.2.1), it is a minimal prime of J. Let $J = \mathfrak{g}_1 \cap \dots \mathfrak{g}_s$ be a minimal primary decomposition of J, with $\mathfrak{g}_1$ the $\mathfrak{p}$-primary component of J. In particular, some power of $\mathfrak{p}$ lies in $\mathfrak{g}_1$, and we may assume that this power is of the form p^m for some m. Choose c inside all $\mathfrak{g}_i$ with $i > 1$, but outside $\mathfrak{p}$ (note that this is possible by prime avoidance). Putting everything together, we have

$$c\,\mathfrak{p}^{p^m} \subseteq J. \tag{8.6}$$

Fix some i, let $I := (x_1,\dots,x_i)S$ and assume $zx_{i+1} \in IR$, for some $z \in S$. Lifting this to S, we get $zx_{i+1} \in I + \mathfrak{p}$. Applying the n-th power of Frobenius to this for $n > m$, we get $\mathbf{F}^n(z)\mathbf{F}^n(x_{i+1}) \in \mathbf{F}^n(I)S + \mathbf{F}^n(\mathfrak{p})S$. By (8.6), this means that $c\,\mathbf{F}^n(z)\mathbf{F}^n(x_{i+1})$ lies in $\mathbf{F}^n(I)S + \mathbf{F}^{n-m}(J)S$. Since the $\mathbf{F}^{n-m}(y_j)$ together with the $\mathbf{F}^n(x_j)$ form again an S-regular sequence, we conclude that

$$c\,\mathbf{F}^n(z) \in \mathbf{F}^n(I)S + \mathbf{F}^{n-m}(J)S \subseteq \mathbf{F}^n(I)S + J$$

whence $c\,\mathbf{F}^n(z) \in \mathbf{F}^n(I)R$ for all $n > m$. By the choice of c, it is non-zero in R, so that the latter equations show that $z \in \mathrm{cl}(IR)$. □

The condition that R is a homomorphic image of a regular local ring is satisfied either if R is a local affine algebra, by 4.1.6, or if R is complete, by Theorems 6.4.2 and 6.4.4. These are the two only cases in which we will apply the previous theorem. There is a more general version which does not require R to be a domain, but only to be *equidimensional*, meaning that all minimal primes have the same dimension (Exercise 8.7.13).

Theorem 8.3.4 (Finite extensions). *If $A \to B$ is a finite, injective homomorphism of domains, and $I \subseteq A$ be an ideal, then* $\mathrm{cl}_B(IB) \cap A = \mathrm{cl}_A(I)$.

Proof. One direction is immediate by Theorem 8.3.1. For the converse, there exists an A-module homomorphism $\varphi\colon B \to A$ such that $c := \varphi(1) \neq 0$, by Lemma 8.3.5 below. Suppose $x \in \mathrm{cl}_B(IB) \cap A$, so that for some non-zero $d \in B$, we have $d\,\mathbf{F}^n(x) \in \mathbf{F}^n(I)B$ for $n \gg 0$. Since B is finite over A, some non-zero multiple of d lies in A, and hence without loss of generality, we may assume $d \in A$. Applying φ to these equations, we get

$$cd\,\mathbf{F}^n(x) \in \mathbf{F}^n(I)A$$

showing that $x \in \mathrm{cl}_A(I)$. □

Lemma 8.3.5. *If $A \subseteq B$ is a finite extension of domains, then there exists an A-linear map $\varphi: B \to A$ with $\varphi(1) \neq 0$.*

Proof. Suppose B is generated over A by the elements $b_1, \dots, b_s$. Let K and L be the fields of fractions of A and B respectively. Since B is a domain, it lies inside the K-vector subspace $V \subseteq L$ generated by the b_i. Choose an isomorphism $\gamma: V \to K^t$ of K-vector spaces. After renumbering, we may assume that the first entry of $\gamma(1)$ is non-zero. Let $\pi: K^t \to K$ be the projection onto the first coordinate, and let $d \in A$ be the common denominator of the $\pi(\gamma(b_i))$ for $i = 1, \dots, s$. Now define an A-linear homomorphism φ by the rule $\varphi(y) = d\pi(\gamma(y))$ for $y \in B$. Since y is an A-linear combination of the b_i and since $d\pi(\gamma(b_i)) \in A$, also $\varphi(y) \in A$. Moreover, by construction, $\varphi(1) \neq 0$. □

Note that a special case of Theorem 8.3.4 is the fact that tight closure measures the extent to which an extension of domains $A \subseteq B$ fails to be cyclically pure: $IB \cap A$ is contained in the tight closure of I, for any ideal $I \subseteq A$. In particular, in view of Theorem 8.3.2, this reproves the well-known fact that if $A \subseteq B$ is an extension of domains with A regular, then $A \subseteq B$ is cyclically pure. The next and last property involves another closure operation, integral closure. It will be discussed in more detail below (§8.4), and here we just state its relationship with tight closure:

Theorem 8.3.6 (Integral closure). *For every ideal $I \subseteq A$, its tight closure is contained in its integral closure. In particular, radical ideals, and more generally integrally closed ideals, are tightly closed.*

Proof. The second assertion is an immediate consequence of the first. We verify condition (4) of Theorem 8.4.1 to show that if x belongs to the tight closure $\mathrm{cl}_A(I)$, then it also belongs to the integral closure $\bar{I}$. Let $A \to V$ be a homomorphism into a discrete valuation ring V, such that its kernel is a minimal prime of A. We need to show that $x \in IV$. However, this is clear since $x \in \mathrm{cl}_V(IV)$ by Theorem 8.3.1 (note that $A \to V$ preserves multipliers), and since $\mathrm{cl}_V(IV) = IV$, by Theorem 8.3.2 and the fact that V is regular (Exercise 4.3.8). □

It is quite surprising that there is no proof, as far as I am aware of, that a prime ideal is tightly closed without reference to integral closure.

8.4 Integral closure

The *integral closure* $\bar{I}$ of an ideal I is the collection of all elements $x \in A$ satisfying an integral equation of the form

$$x^d + a_1 x^{d-1} + \cdots + a_d = 0 \tag{8.7}$$

with $a_j \in I^j$ for all $j = 1, \dots, d$. We say that I is *integrally closed* if $I = \bar{I}$. Since clearly $\bar{I} \subseteq \mathrm{rad}(I)$, radical ideals are integrally closed. It follows from either characterization (2) or (4) below that $\bar{I}$ is an ideal.

Theorem 8.4.1. *Let A be an arbitrary Noetherian ring (not necessarily of characteristic p). For an ideal $I \subseteq A$ and an element $x \in A$, the following are equivalent*

1. *x belongs to the integral closure, $\bar{I}$;*
2. *there is a finitely generated A-module M with zero annihilator such that $xM \subseteq IM$;*
3. *there is a multiplier $c \in A$ such that $cx^n \in I^n$ for infinitely many n;*
4. *for every homomorphism $A \to V$ into a discrete valuation ring V with kernel equal to a minimal prime of A, we have $x \in IV$;*

Proof. We postpone the proof to Exercise 8.7.14, except for the equivalence of (1) with (4) (note that this is the only equivalence used so far, in the proof of Theorem 8.3.6). By Exercise 8.7.12, we may reduce to the case that A is moreover a domain.

To prove (1) $\Rightarrow$ (4), suppose $x \in \bar{I}$ and $A \subseteq V$ is an injective homomorphism into a discrete valuation ring V. Let v be the valuation on V. Suppose towards a contradiction that $x \notin IV$, and therefore $m := v(x) < n := v(IV)$. By assumption, x satisfies an integral equation (8.7). For all $i = 1, \dots, d$, we have $v(a_i x^{d-i}) \geq ni + (d-i)m > dm$. However, this is in contradiction with $v(x^d) = md$.

To prove the converse, assume $x \in IV$ for every embedding $A \subseteq V$ into a discrete valuation ring V. Let $I = (a_1, \dots, a_n)A$, and consider the homomorphism $A[\xi] \to A_x$ given by $\xi_i \mapsto a_i/x$, where $\xi := (\xi_1, \dots, \xi_n)$. Let B be its image, so that $A \subseteq B \subseteq A_x$ (one calls B the *blowing-up* of $I + xA$ at x). Let $\mathfrak{m} := (\xi_1, \dots, \xi_n)A[\xi]$. I claim that $\mathfrak{m}B = B$. Assuming the claim, we can find $f \in \mathfrak{m}$ such that $f(\mathbf{a}/x) = 1$ in A_x, where $\mathbf{a} := (a_1, \dots, a_n)$. Write $f = f_1 + \dots + f_d$ in its homogeneous parts f_j of degree j, so that

$$1 = x^{-1} f_1(\mathbf{a}) + \dots + x^{-d} f_d(\mathbf{a}).$$

Multiplying with x^d, and observing that $f_j(\mathbf{a}) \in I^j$, we see that x satisfies an integral equation (8.7), and hence $x \in \bar{I}$.

To prove the claim ex absurdum, suppose $\mathfrak{m}B$ is not the unit ideal, whence is contained in a maximal ideal $\mathfrak{n}$ of B. By Exercise 8.7.15, there exists an injective, local homomorphism $B_\mathfrak{n} \subseteq V$ with V a discrete valuation ring. Hence also $A \subseteq V$. Since $\mathfrak{m}V$ lies in the maximal ideal πV, we get $a_i \in x\pi V$ for all i. Hence $IV \subseteq x\pi V$ contradicting that $x \in IV$. □

From this we readily deduce (see Exercise 8.7.10):

Corollary 8.4.2. *A domain A is normal (=integrally closed) if and only if each principal ideal is integrally closed if and only if each principal ideal is tightly closed.*

In one of our applications below (Theorem 8.5.1), we will make use of the following nice application of the chain rule:

Proposition 8.4.3. *Let K be a field of characteristic zero, and let R be either the power series ring $K[[\xi]]$, the ring of convergent power series $K\{\xi\}$ (assuming K is a normed field), or the localization of $K[\xi]$ at the ideal generated by the indeterminates $\xi := (\xi_1, \dots, \xi_n)$. If f is a non-unit, then it lies in the integral closure of its Jacobian ideal* $\operatorname{Jac}(f) := (\partial f/\partial \xi_1, \dots, \partial f/\partial \xi_n)R$.

Proof. Recall that $K\{\xi\}$ consists of all formal power series f such that $f(\mathbf{u})$ is a convergent series for all $\mathbf{u}$ in a small enough neighborhood of the origin. Put $J := \mathrm{Jac}(f)$. In view of (4) in Theorem 8.4.1, we need to show that given an embedding $R \subseteq V$ into a discrete valuation ring V, we have $f \in JV$. Since completion is faithfully flat by Theorem 6.3.4, we may replace V by its completion, and hence already assume V is complete. By Theorem 6.4.2 therefore, V is a power series ring $\kappa[[\zeta]]$ in a single variable over a field extension κ of K. Viewing the image of f in $\kappa[[\zeta]]$ as a power series in ζ, the multi-variate chain rule yields

$$\frac{\mathrm{d}f}{\mathrm{d}\zeta} = \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i} \cdot \frac{\mathrm{d}\xi_i}{\mathrm{d}\zeta} \in JV.$$

However, since f has order $e \geq 1$ in V, its derivative $\mathrm{d}f/\mathrm{d}\zeta$ has order $e-1$, and hence $f \in (\mathrm{d}f/\mathrm{d}\zeta)V \subseteq JV$. Note that for this to be true, however, the characteristic needs to be zero. For instance, in characteristic p, the power series ξ^p would already be a counterexample to the proposition. □

Since the integral closure is contained in the radical closure, we get that some power of f lies in its Jacobian $\mathrm{Jac}(f)$. A famous theorem due to Briançon-Skoda states that in fact already the n-th power lies in the Jacobian (where n is the number of variables; we will prove this via an elegant tight closure argument in Theorem 8.5.1 below).

8.5 Applications

We will now discuss three important applications of tight closure. Perhaps surprisingly, the original statements all were in characteristic zero (with some of them in their original form plainly false in positive characteristic), and their proofs required deep and involved arguments, some even based on transcendental/analytic methods. However, they each can be reformulated so that they also make sense in positive characteristic, and then can be established by surprisingly elegant tight closure arguments. As for the proofs of their characteristic zero counterparts, they must wait until we have developed the theory in characteristic zero in Chapters 9 and 10 (or one can use the 'classical' tight closure in characteristic zero discussed in §8.6).

The Briançon-Skoda theorem. We already mentioned this famous result, proven first in [12].

Theorem 8.5.1 (Briançon-Skoda). *Let R be either the ring of formal power series $\mathbb{C}[[\xi]]$, or the ring of convergent power series $\mathbb{C}\{\xi\}$, or the localization of the polynomial ring $\mathbb{C}[\xi]$ at the ideal generated by ξ, where $\xi := (\xi_1, \dots, \xi_n)$ are some indeterminates. If f is not a unit, then $f^n \in \mathrm{Jac}(f) := (\partial f/\partial \xi_1, \dots, \partial f/\partial \xi_n)R$.*

This theorem will follow immediately from the characteristic zero analogue of the next result (with $l = 1$), in view of Proposition 8.4.3 and Exercise 4.3.5; we will do this in Theorem 9.2.5 below.

Theorem 8.5.2 (Briançon-Skoda—tight closure version). *Let A be a Noetherian ring of characteristic p, and $I \subseteq A$ an ideal generated by n elements. Then we have for all $l \geq 1$ an inclusion*

$$\overline{I^{n+l-1}} \subseteq \mathrm{cl}(I^l).$$

In particular, if A is a regular local ring, then the integral closure of I^{n+l-1} lies inside I^l for all $l \geq 1$.

Proof. For simplicity, I will only prove the case $l = 1$ (see Exercise 8.7.7 for the general case). Assume $z \in \overline{I^n}$. By (3) in Theorem 8.4.1, there exists a multiplier $c \in A$ such that $cz^k \in I^{kn}$ for all $k \gg 0$. Since $I := (f_1, \dots, f_n)A$, we have an inclusion $I^{kn} \subseteq (f_1^k, \dots, f_n^k)A$. Hence with k equal to p^m, we get $c\mathbf{F}^m(z) \in \mathbf{F}^m(I)A$ for all $m \gg 0$. In conclusion, $z \in \mathrm{cl}(I)$. The last assertion then follows from Theorem 8.3.2. □

The Hochster-Roberts theorem. We will formulate the next result without defining in detail all the concepts involved, except when we get to its algebraic formulation. A *linear algebraic group G* is an affine subscheme of the general linear group $\mathrm{GL}(K,n)$ over an algebraically closed field K (see Example 2.3.7) such that its K-rational points form a subgroup of the latter group. When G acts (as a group) on a closed subscheme $X \subseteq \mathbb{A}^n_K$ (more precisely, for each algebraically closed field L containing K, there is an action of the L-rational points of $G(L)$ on $X(L)$), we can define the *quotient space X/G*, consisting of all orbits under the action of G on X, as the affine space $\mathrm{Spec}(R^G)$, where R^G denotes the subring of G-invariant sections in $R := \Gamma(X, \mathscr{O}_X)$ (the action of G on X induces an action on the sections of X, and hence in particular on R). For this to work properly, we also need to impose a certain finiteness condition: G has to be *linearly reductive*. Although not usually its defining property, we will here take this to mean that there exists an R^G-linear map $R \to R^G$ which is the identity on R^G, called the *Reynold operator* of the action. For instance, if $K = \mathbb{C}$, then an algebraic group is linearly reductive if and only if it is the complexification of a real Lie group, where the Reynolds operator is obtained by an integration process. This is the easiest to understand if G is finite, when the integration is just a finite sum

$$\rho \colon R \to R^G \colon a \mapsto \frac{1}{|G|} \sum_{\sigma \in G} a^\sigma,$$

where a^σ denotes the effect of $\sigma \in G$ acting on $a \in R$. In fact, as indicated by the above formula, a finite group is linearly reductive over a field of positive characteristic, only if its cardinality is not divisible by the characteristic. If X is non-singular and G is linearly reductive, then we will call X/G a *quotient singularity*.[1] The celebrated Hochster-Roberts theorem now states:

Theorem 8.5.3. *Any quotient singularity is Cohen-Macaulay.*

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[1] The reader should be aware that other authors might use the term more restrictively, only allowing X to be affine space $\mathbb{A}^n_K$, or G to be finite.

To state a more general result, we need to take a closer look at the Reynolds map. A ring homomorphism $A \to B$ is called *split*, if there exists an A-linear map $\sigma\colon B \to A$ which is the identity on A (note that σ need not be multiplicative, that is to say, is not a ring homomorphism, only a module homomorphism). We call σ the *splitting* of $A \to B$. Hence the Reynold map is a splitting of the inclusion $R^G \subseteq R$. The only property of split maps that will matter is the following:

8.5.4 *A split homomorphism $A \to B$ is cyclically pure.*

See the discussion following Proposition 5.3.4 for the definition of cyclic purity. Let $a \in IB \cap A$ with $I = (f_1, \dots, f_s)A$ an ideal in A. Hence $a = f_1 b_1 + \dots + f_s b_s$ for some $b_i \in B$. Applying the splitting σ, we get by A-linearity $a = f_1\sigma(b_1) + \dots + f_s\sigma(b_s) \in I$, proving that A is cyclically pure in B. □

We can now state a far more general result, of which Theorem 8.5.3 is just a special case (see Exercise 8.7.9).

Theorem 8.5.5. *If $R \to S$ is a cyclically pure homomorphism and if S is regular, then R is Cohen-Macaulay.*

Proof. In fact, we can split the proof in two parts. Namely, we first show that R is F-regular, and then show that any F-regular ring is Cohen-Macaulay.

8.5.6 *A cyclically pure subring of a regular ring is F-regular.*

Indeed, since both cyclic purity and regularity are preserved under localization, we only need to show that every ideal in R is tightly closed. To this end, let $I \subseteq R$ and $x \in \mathrm{cl}(I)$. Hence x lies in the tight closure of IS by (weak) persistence (Theorem 8.3.1), and therefore in IS by Theorem 8.3.2. Hence by cyclic purity, $x \in I = IS \cap R$, proving that R is weakly F-regular. Note that we actually proved that a cyclically pure subring of a (weakly) F-regular ring is again (weakly) F-regular.

8.5.7 *An F-regular domain is Cohen-Macaulay.*

Without loss of generality, we may assume R is local. Assume R is F-regular and let $(x_1, \dots, x_d)$ be a system of parameters in R. To show that x_{i+1} is $R/(x_1, \dots, x_i)R$-regular, assume $zx_{i+1} \in (x_1, \dots, x_i)R$. Colon Capturing (Theorem 8.3.3) yields that z lies in the tight closure of $(x_1, \dots, x_i)R$, whence in the ideal itself since R is F-regular. □

In fact, R is then also normal (this follows easily from 8.5.6 and Corollary 8.4.2). A far more difficult result is that R is then also *pseudo-rational* (a concept that lies beyond the scope of these notes; see for instance [36, 65] for a discussion of what follows). This was first proven by Boutot in [11] for $\mathbb{C}$-affine algebras by means of deep vanishing theorems. The positive characteristic case was proven by Smith in [68] by tight closure methods, where she also showed that pseudo-rationality is in fact equivalent with the weaker notion of F-rationality (a local ring is *F-rational* if some parameter ideal is tightly closed). The general characteristic zero case was proven in [65] by means of ultraproducts (as described in §10). In fact, being F-regular is equivalent under the $\mathbb{Q}$-Gorenstein assumption with having log-terminal singularities (see [23, 59]). It should be noted that 'classical' tight closure theory in characteristic zero (see §8.6 below) is not sufficiently versatile to derive these results: so far, only our present ultraproduct method seems to work.

The Ein-Lazardsfeld-Smith theorem. If P is a point in the affine plane K^2, and $f \in K[\xi,\zeta]$, then we say that f has *multiplicity* k at P if P is a k-multiple point of the curve $\mathrm{V}(f)$ (as defined in Definition 4.1.2). The next result, although elementary in its formulation, was only proven recently in [17] using quite complicated methods (which only work over $\mathbb{C}$), but then soon after in [32] by an elegant tight closure argument (see also [55]), which proves the result over any field K.

Theorem 8.5.8. *Let $V \subseteq K^2$ be a finite subset with ideal of definition $I := \mathfrak{I}(V)$. For each k, let $J_k(V)$ be the ideal of all polynomials f having multiplicity at least k at each point $x \in V$. Then $J_{2k}(V) \subseteq I^k$, for all k.*

To formulate the more general result of which this is just a corollary, we need to introduce symbolic powers. We first do this for a prime ideal $\mathfrak{p}$: its k-th *symbolic power* is the contracted ideal $\mathfrak{p}^{(k)} := \mathfrak{p}^k R_{\mathfrak{p}} \cap R$. In general, the inclusion $\mathfrak{p}^k \subseteq \mathfrak{p}^{(k)}$ may be strict, and in fact, $\mathfrak{p}^{(k)}$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^k$. If $\mathfrak{a}$ is a radical ideal (we will not treat the more general case), then we define its k-th *symbolic power* $\mathfrak{a}^{(k)}$ as the intersection $\mathfrak{p}_1^{(k)} \cap \cdots \cap \mathfrak{p}_s^{(k)}$, where the $\mathfrak{p}_i$ are all the minimal overprimes of $\mathfrak{a}$. The connection with Theorem 8.5.8 is given by:

8.5.9 *The k-th symbolic power of the ideal of definition $I := \mathfrak{I}(V)$ of a finite subset $V \subseteq K^2$ is equal to the ideal $J_k(V)$ of all polynomials that have multiplicity at least k at any point of V.*

Indeed, for $\mathbf{x} \in V$, let $\mathfrak{m} := \mathfrak{m}_{\mathbf{x}}$ be the corresponding maximal ideal. By 4.1.4, a polynomial f has multiplicity at least k at each $\mathbf{x} \in V$, if $f \in \mathfrak{m}^k A_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ containing I. The latter condition simply means that $f \in \mathfrak{m}^{(k)}$, so that the claim follows from the definition of symbolic power. □

Hence, in view of this, Theorem 8.5.8 is an immediate consequence of the following theorem (at least in positive characteristic; for the characteristic zero case, see Theorems 9.2.6 and 10.2.4 below):

Theorem 8.5.10. *Let A be a regular domain of characteristic p. Let $\mathfrak{a} \subseteq A$ be a radical ideal and let h be the maximal height of its minimal overprimes. Then we have an inclusion $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$, for all n.*

Proof. We start with proving the following useful inclusion:

$$\mathfrak{a}^{(hp^e)} \subseteq \mathbf{F}^e(\mathfrak{a})A \tag{8.8}$$

for all e. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals of $\mathfrak{a}$. We first prove (8.8) locally at one of these minimal primes $\mathfrak{p}$. Since $A_{\mathfrak{p}}$ is regular and $\mathfrak{a}A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, we can find $f_i \in \mathfrak{a}$ such that $\mathfrak{a}A_{\mathfrak{p}} = (f_1, \dots, f_h)A_{\mathfrak{p}}$. By definition of symbolic powers, $\mathfrak{a}^{(hp^e)}A_{\mathfrak{p}} = \mathfrak{a}^{hp^e}A_{\mathfrak{p}}$. On the other hand, $\mathfrak{a}^{hp^e}A_{\mathfrak{p}}$ consists of monomials in the f_i of degree hp^e, and hence any such monomial lies in $\mathbf{F}^e(\mathfrak{a})A_{\mathfrak{p}}$. This establishes (8.8) locally at $\mathfrak{p}$. To prove this globally, take $z \in \mathfrak{a}^{(hp^e)}$. By what we just proved, there exists $s_i \notin \mathfrak{p}_i$ such that $s_i z \in \mathbf{F}^e(\mathfrak{a})A$ for each $i = 1, \dots, m$. For each i, choose an element t_i in all $\mathfrak{p}_j$ except $\mathfrak{p}_i$, and put $s := t_1 s_1 + \cdots + s_m t_m$. It follows that s multiplies z inside $\mathbf{F}^e(\mathfrak{a})A$, whence a fortiori, so does $\mathbf{F}^e(s)$. Hence

$$z \in (\mathbf{F}^e(\mathfrak{a})A : \mathbf{F}^e(s)) = \mathbf{F}^e(\mathfrak{a} : s)A$$

where we used Theorem 5.6.16 and the fact that $\mathbf{F}$ is flat on A by Theorem 8.1.2. However, s does not lie in any of the $\mathfrak{p}_i$, whence $(\mathfrak{a} : s) = \mathfrak{a}$, proving (8.8).

To prove the theorem, let $f \in \mathfrak{a}^{(hn)}$, and fix some e. We may write $p^e = an + r$ for some $a, r \in \mathbb{N}$ with $0 \leq r < n$. Since the usual powers are contained in the symbolic powers, and since $r < n$, we have inclusions

$$\mathfrak{a}^{hn} f^a \subseteq \mathfrak{a}^{hr} f^a \subseteq \mathfrak{a}^{(han+hr)} = \mathfrak{a}^{(hp^e)} \subseteq \mathbf{F}^e(\mathfrak{a})A \tag{8.9}$$

where we used (8.8) for the last inclusion. Taking n-th powers in (8.9) shows that $\mathfrak{a}^{hn^2} f^{an}$ lies in the n-th power of $\mathbf{F}^e(\mathfrak{a})A$, and this in turn lies inside $\mathbf{F}^e(\mathfrak{a}^n)A$. Choose some non-zero c in $\mathfrak{a}^{hn^2}$. Since $p^e \geq an$, we get $c\mathbf{F}^e(f) \in \mathbf{F}^e(\mathfrak{a}^n)A$ for all e. In conclusion, f lies in $\mathrm{cl}(\mathfrak{a}^n)$ whence in $\mathfrak{a}^n$ by Theorem 8.3.2. □

One might be tempted to try to prove a more general form which does not assume A to be regular, replacing $\mathfrak{a}^n$ by its tight closure. However, we used the regularity assumption not only via Theorem 8.3.2 but also via Kunz's theorem that the Frobenius is flat. Hence the above proof does not work in arbitrary rings.

8.6 Classical tight closure in characteristic zero

To prove the previous three theorems in a ring of equal characteristic zero, Hochster and Huneke also developed tight closure theory for such rings. One of the precursors to tight closure theory was the proof of the Intersection Theorem by Peskine and Szpiro in [44]. They used properties of the Frobenius together with a method to transfer results from characteristic p to characteristic zero, which was then generalized by Hochster in [26]. This same technique is also used to obtain a tight closure theory in equal characteristic zero, as we will discuss briefly in this section. However, using ultraproducts, we will bypass in Chapters 9 and 10 this rather heavy-duty machinery, to arrive much quicker at proofs in equal characteristic zero.

Let A be a Noetherian ring containing the rationals. The idea is to associate to A some rings in positive characteristic, its *reductions modulo* p, and calculate tight closure in the latter. More precisely, let $\mathfrak{a} \subseteq A$ be an ideal, and $z \in A$. We say that z lies in the *HH-tight closure* of $\mathfrak{a}$ (where "HH" stands for Hochster-Huneke), if there exists a $\mathbb{Z}$-affine subalgebra $R \subseteq A$ containing z, such that (the image of) z lies in the tight closure of $I(R/pR)$ for all primes numbers p, where $I := \mathfrak{a} \cap R$.

It is not too hard to show that this yields a closure operation on A (in the sense of Definition 8.2.3). Much harder is showing that it satisfies all the necessary properties from §8.3. For instance, to prove the analogue of Theorem 8.3.2, want needs some results on generic flatness, and some deep theorems on Artin Approximation (see for instance [36, Appendix 1] or [31]; for a brief discussion of Artin Approximation, see §10.1 below; for an example of the technique, see Project 10.6 below). In contrast, using ultraproducts, one can avoid all these complications in the affine case (Chapter 9), or get by with a more elementary version of Artin Approximation in the general case (Chapter 10).

8.7 Exercises

Ex 8.7.1

Let A be the coordinate ring of the hypersurface in K^3 given by the equation $\xi^2 - \zeta^3 - \eta^7 = 0$. Show that ξ lies in the tight closure of $(\zeta, \eta)A$.
A far more difficult result is to show that this is not true if we replace η^7 by η^5 in the above equation. In fact this new coordinate ring is F-regular, but this is a deep fact, following from it being log-terminal (see also the discussion following Theorem 8.5.5).

Ex 8.7.2

Show that any regular ring of prime characteristic is F-regular.

Ex 8.7.3

Prove the existence of the y_i in the proof of Theorem 8.3.3.

Ex 8.7.4

Work out the details of the following alternative proof of Colon Capturing for a local domain R admitting Noether Normalization with parameters*, meaning that for any system of parameters $(x_1, \dots, x_d)$ in R, there exists a regular local subring $S \subseteq R$ containing the x_i such that $S \subseteq R$ is finite and $(x_1, \dots, x_d)S$ is the maximal ideal of S. Suppose $z \in ((x_1, \dots, x_i)R : x_{i+1})$ and let A be the S-subalgebra of R generated by z. Show that A is a hypersurface ring and hence is Cohen-Macaulay, by modifying the proof of Corollary 5.6.14. By Lemma 8.3.5, there exists an R-linear map $\varphi\colon R \to A$ with $c := \varphi(1) \neq 0$. Apply the n-th iterate of Frobenius to the relation $zx_{i+1} \in (x_1, \dots, x_i)R$ and then apply φ to get ideal membership relations in A. Use that $\mathbf{F}^n(x_i)$ is a regular sequence in A to derive from these relations that z lies in the tight closure of $(x_1, \dots, x_i)A$, and finish with an application of weak persistence (Theorem 8.3.1).*
Show using Theorem 2.2.5 that any affine local domain admits Noether Normalization with parameters (see for instance [18, Theorem 13.3]). Prove similarly, using the argument in Theorem 6.4.6, that so does any complete Noetherian local domain.

Ex 8.7.5

Prove, using tight closure, that a Noether normalization $A \subseteq B$ of an affine algebra B over a field of positive characteristic is cyclically pure. Use this, together with Corollary 5.6.11, to give an example of a finite cyclically pure homomorphism of local rings which is not flat.

Ex 8.7.6

Show that if $z \in \bar{I}$ satisfies an integral equation (8.7) of degree d, then $I^{d-1}z^k \in I^k$ for all k.

Ex 8.7.7

Prove the general version of Theorem 8.5.2.

Ex 8.7.8

Give an alternative proof that $\xi^2 \in \mathrm{cl}(I)$ in Example 8.2.2 using the Briançon-Skoda Theorem instead.

Ex 8.7.9

Derive Theorem 8.5.3 from Theorem 8.5.5 using 8.5.4.

Additional exercises.

Ex 8.7.10

Prove Corollary 8.4.2.

Ex 8.7.11

Prove that if A is an affine k-algebra, or a complete Noetherian local ring with residue field k, and if k is perfect, or more generally, if $(k : k^p) < \infty$, then $\mathbf{F}_p \colon A \to A$ is finite.

Ex 8.7.12

Show that x lies in the integral closure of an ideal I if and only if it lies in the integral closure of each $I(A/\mathfrak{p})$, for $\mathfrak{p}$ a minimal prime of A.

Ex 8.7.13

Prove Theorem 8.3.3 under the weaker assumption that R is an equidimensional homomorphic image of a Cohen-Macaulay local ring.

Ex 8.7.14

To show the equivalence of (1) with (2) in Theorem 8.4.1, use in one direction the ideal $J := x^{d-1}A + x^{d-2}I + \cdots + I^d$, and in the other use a 'determinantal trick'. Use the ideal J to also prove (1) $\Rightarrow$ (3), and finish the proof of Theorem 8.4.1 by showing (3) $\Rightarrow$ (4). See also Exercise 8.7.6.

Ex 8.7.15

Let $(R, \mathfrak{m})$ be a Noetherian local domain. We want to show that there exists a discrete valuation ring V and a local injective homomorphism $R \to V$. Let $(x_1, \dots, x_n)$ be a generating tuple of $\mathfrak{m}$ and let R' be the R-algebra generated by the fractions x_i/x_1 with $i = 1, \dots, n$ (one often refers to B as a blowing-up *of R at $\mathfrak{m}$). Show that $\mathfrak{m}B$ is principal, and using Krull's Principal Ideal Theorem (Theorem 3.3.4), that there exists a height one prime ideal $\mathfrak{p}$ in B containing $\mathfrak{m}B$. Let V be the integral closure of $B_{\mathfrak{p}}$. Show that V is a discrete valuation ring, and that the natural embedding $R \to V$ is local.*

Ex 8.7.16

In this exercise, we will explore some of the concepts of invariant theory briefly mentioned at the beginning of our discussion on the Hochster-Roberts Theorem. Let K be an algebraically closed field, let $X = \operatorname{Spec}(R) \subseteq \mathbb{A}^n_K$ be an irreducible, reduced closed subscheme, and let G be a linearly reductive algebraic group acting on X. In particular, the K-rational points $G(K)$ of G form an (abstract) group acting on the variety $X(K) \subseteq K^n$ consisting of the K-rational points of X (see page 27). For a given section $p \colon X(K) \to K$, and an element $g \in G(K)$, define a new section p^g given by the rule $p^g(\mathbf{u}) = p(g \cdot \mathbf{u})$. Show that we may identify R with the sections on

$X(K)$, and the above then defines an action of $G(K)$ on R. Let R^G be the subring of invariants of R under this action, that is to say, all $a \in R$ such that $a^g = a$ for all $g \in G(K)$ (notationally, one often confuses the algebraic group G with its K-rational points $G(K)$). Without proof, we state that R^G is again K-affine, that is to say, a finitely generated K-algebra. Let $Y := \operatorname{Spec}(R^G)$. Show, using Exercise 5.7.7 and the Reynolds operator, that the induced map $X \to Y$ is surjective. Show furthermore that the induced surjective map of K-rational points $X(K) \to Y(K)$ factors through the orbit space $X(K)/G(K)$. It requires some more work though to show that this actually induces an isomorphism $X(K)/G(K) \cong Y(K)$.

Chapter 9
Tight closure in characteristic zero. Affine case

We will develop a tight closure theory in characteristic zero which is different from the Hochster-Huneke approach discussed briefly in §8.6. In this chapter we treat the affine case, that is to say, we develop the theory for algebras of finite type over an uncountable algebraically closed field K of characteristic zero; the general local case will be discussed in Chapter 10. Recall that under the Continuum Hypothesis, any uncountable algebraically closed field K of characteristic zero is a *Lefschetz field*, that is to say an ultraproduct of fields of positive characteristic, by Theorem 1.4.3 and Remark 1.4.4. In particular, without any set-theoretic assumption, $\mathbb{C}$, the field of complex numbers, is a Lefschetz field. The idea now is to use the ultra-Frobenius, that is to say, the ultraproduct of the Frobenii (see Definition 1.4.14), in the same manner in the definition of tight closure as in positive characteristic. However, the ultra-Frobenius does not act on the affine algebra but rather on its ultra-hull, so that we have to introduce a more general setup. It is instructive to do this first in an axiomatic manner, and then specialize to the situation at hand.

9.1 Difference hulls

A ring C together with an endomorphism σ on C is called a *difference ring*, and for emphasis, we denote this as a pair (C,σ). If (C,σ) and $(C'\sigma')$ are difference rings, and $\varphi\colon C \to C'$ a ring homomorphism, then we call φ a *morphism of difference rings* if it commutes with the endomorphisms, that is to say, if $\varphi(\sigma(a)) = \sigma'(\varphi(a))$ for all $a \in C$. The example par excellence of a difference ring is any ring of positive characteristic endowed with his Frobenius. We will now reformulate tight closure from this perspective, but anticipating already the fact that the ultra-Frobenius acts only on a certain overring of the affine algebra, to wit, its ultra-hull defined in §7.1. Since we also want the theory to be compatible with ring homomorphisms ('Persistence'), we need to work categorically. Let $\mathfrak{C}$ be a category of Noetherian rings closed under homomorphic images (at this point we do not need to make any characteristic assumption). Often, the category will also be closed under localization, and we will tacitly assume this as well. In summary, $\mathfrak{C}$ is a collection of Noetherian rings so that for any A in $\mathfrak{C}$ any localization $S^{-1}A$ and any residue ring A/I belongs again to $\mathfrak{C}$ (and the canonical maps $A \to S^{-1}A$ and $A \to A/I$ are morphisms in $\mathfrak{C}$).

Definition 9.1.1 (Difference hull). A *difference hull* on $\mathfrak{C}$ is a functor $D(\cdot)$ from $\mathfrak{C}$ to the category of difference rings, and a natural transformation η from the identity functor to $D(\cdot)$ (that is to say, for each A in $\mathfrak{C}$, we have a difference ring $D(A)$ with endomorphism σ_A and a ring homomorphism $\eta_A \colon A \to D(A)$, and for each morphism $A \to B$ in $\mathfrak{C}$, we get an induced morphism of difference rings $D(A) \to D(B)$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & D(A) \\ \downarrow & & \downarrow \\ B & \xrightarrow{\eta_B} & D(B) \end{array} \tag{9.1}$$

commutes), with the following three additional properties:

1. each $\eta_A \colon A \to D(A)$ is faithfully flat;
2. the endomorphism σ_A of $D(A)$ preserves $D(A)$-regular sequences;
3. for any ideal $I \subseteq A$, we have $\sigma_A(I) \subseteq I^2 D(A)$.

Since η_A is in particular injective (Proposition 5.3.4), we will henceforth view A as a subring of $D(A)$ and omit, as usual, η_A from our notation.

Difference closure. Given a difference hull $D(\cdot)$ on some category $\mathfrak{C}$, we define the *difference closure* $\mathrm{cl}^D(I)$ of an ideal $I \subseteq A$ of a member A of $\mathfrak{C}$ as follows: an element $z \in A$ belongs to $\mathrm{cl}^D(I)$ if there exists a multiplier $c \in A$ and a number $N \in \mathbb{N}$ such that

$$c\sigma^n(z) \in \sigma^n(I)D(A) \tag{9.2}$$

for all $n \geq N$. Here, $\sigma^n(I)D(A)$ denotes the ideal in $D(A)$ generated by all $\sigma^n(y)$ with $y \in I$, where σ is the endomorphism of the difference ring $D(A)$. It is crucial here that the multiplier c already belongs to A, although the membership relations in (9.2) are inside the bigger ring $D(A)$. We leave it as an exercise to show that the difference closure is indeed a closure operation in the sense of Definition 8.2.3 (see Exercise 9.5.1). An ideal that is equal to its difference closure will be called *difference closed*.

Example 9.1.2 (Frobenius hull). It is clear that our definition is inspired by the membership test (8.3) for tight closure, and indeed, this is just a special case. Namely, for a fixed prime number p, let $\mathfrak{C}_p$ be the category of all Noetherian rings of characteristic p and let $D(\cdot)$ be the functor assigning to a ring A the difference ring $(A, \mathbf{F}_A)$. It is easy to see that this makes $D(\cdot)$ a difference hull in the above sense, and the difference closure with respect to this hull is just the tight closure of the ideal; we will refer to this construction as the *Frobenius hull*.

In the next section, we will view tight closure in characteristic zero as a difference closure too. For the remainder of this section, we fix a category $\mathfrak{C}$ endowed with a difference hull $D(\cdot)$, and study the corresponding difference closure on the members of $\mathfrak{C}$. For a given member A of $\mathfrak{C}$, we let σ_A, or just σ, be the endomorphism of $D(\cdot)$. In fact, we are mostly interested in the restriction of σ to A, and we also denote this homomorphism by σ (of course, this restriction is no longer an endomorphism).

Five key properties of difference closure. To derive the necessary properties of this closure operation, namely the the analogues of the five key properties of §8.3, we again depart from a flatness result, the analogue of Kunz's theorem (Theorem 8.1.2).

Proposition 9.1.3. *If A is a regular local ring in $\mathfrak{C}$, then $\sigma\colon A \to D(A)$ is faithfully flat.*

Proof. By Theorem 5.6.10, it suffices to show that $D(A)$ is a balanced big Cohen-Macaulay algebra under σ. To this end, let $(x_1,\dots,x_d)$ be an A-regular sequence. Since $A \subseteq D(A)$ is by assumption faithfully flat, $(x_1,\dots,x_d)$ is $D(A)$-regular by Proposition 5.4.1. By Condition (2) of Definition 9.1.1, the sequence $(\sigma(x_1),\dots,\sigma(x_d))$ is also $D(A)$-regular, as we wanted to show. □

Corollary 9.1.4. *Any ideal of a regular ring in $\mathfrak{C}$ is difference closed.*

Proof. Suppose first that $(R,\mathfrak{m})$ is a regular local ring in $\mathfrak{C}$, and z lies in the difference closure of an ideal $I \subseteq R$. Hence, with c and N as in (9.2), the multiplier c lies in $(\sigma^n(I)D(R) : \sigma^n(z))$ for $n \geq N$, and hence by flatness (Proposition 9.1.3) and the Colon Criterion (Theorem 5.6.16), it lies in $\sigma^n(I : z)D(R)$. If z does not belong to I, then $(I : z) \subseteq \mathfrak{m}$, and hence c belongs to $\sigma^n(\mathfrak{m})D(R)$ which in turn lies inside $\mathfrak{m}^{2^n}D(R)$ by Condition (3) of Definition 9.1.1. By faithful flatness, c therefore lies in $\mathfrak{m}^{2^n}$, for every $n \geq N$, contradicting, in view of Krull's Intersection Theorem 1.4.11, that it is a multiplier whence non-zero.

For the general case, assume z lies in the tight closure of an ideal I in a regular ring A in $\mathfrak{C}$. By weak persistence and the local case, $z \in IA_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of A. It follows that $(I : z)$ cannot be a proper ideal, whence $z \in I$. □

Remark 9.1.5. Let us call a difference hull *simple* if instead of Condition 9.1.1(3) we have the stronger condition that $\sigma(I)$ is contained in all powers of $ID(A)$, for $I \subseteq A$. In that case, we can define a variant of the difference closure, called *simple difference closure*, by requiring condition (9.2) to hold only for $n = 1$, that is to say, a single test suffices. Inspecting the above proof, one sees that for a simple difference hull, any ideal I in a regular ring is equal to its simple difference closure. We leave it to the reader (see Exercise 9.5.6) to show that simple difference closure satisfies all the properties below of its non-simple counterpart.

Weak persistence holds for the same reasons as it does for tight closure, so for the record we state:

9.1.6 *If $A \to B$ is an injective morphism in $\mathfrak{C}$ with A and B domains, then* $\mathrm{cl}^{\mathsf{D}}(I) \subseteq \mathrm{cl}^{\mathsf{D}}(IB)$.

Proposition 9.1.7 (Colon Capturing). *Let R be a Noetherian local domain which is a homomorphic image of a Cohen-Macaulay local ring in $\mathfrak{C}$, and let $(x_1,\dots,x_d)$ be a system of parameters in R. Then for each i, the colon ideal $((x_1,\dots,x_i)R : x_{i+1})$ is contained in* $\mathrm{cl}^{\mathsf{D}}((x_1,\dots,x_i)R)$.

Proof. Let S be a local Cohen-Macaulay ring in $\mathfrak{C}$ such that $R = S/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq S$, and assume the x_i already belong to S. As in the proof of Theorem 8.3.3, we can find an S-regular sequence $(y_1,\dots,y_h,x_1,\dots,x_d)$ with $y_1,\dots,y_h \in \mathfrak{p}$, an element $c \notin \mathfrak{p}$, and a number $m \in \mathbb{N}$ such that

$$c\,\mathfrak{p}^{2^m} \subseteq J := (y_1,\dots,y_h)S. \tag{9.3}$$

Let τ denote the endomorphism of $\mathsf{D}(S)$. By assumption, the canonical epimorphism $S \to R$ induces a morphism of difference rings $\mathsf{D}(S) \to \mathsf{D}(R)$. In particular, $\mathfrak{p}\mathsf{D}(R) = 0$.

Fix some i, let $I := (x_1,\dots,x_i)S$ and assume $zx_{i+1} \in IR$ some $z \in S$. Hence $zx_{i+1} \in I + \mathfrak{p}$. Applying τ^n to this for $n > m$, we get $\tau^n(z)\tau^n(x_{i+1}) \in \tau^n(I)\mathsf{D}(S) + \tau^n(\mathfrak{p})\mathsf{D}(S)$. By (9.3) and 9.1.1(3), this means that $c\tau^n(z)\tau^n(x_{i+1})$ lies in $\tau^n(I)\mathsf{D}(S) + \tau^{n-m}(J)\mathsf{D}(S)$. Since the $\tau^{n-m}(y_j)$ together with the $\tau^n(x_j)$ form again an S-regular sequence by a stronger version of 9.1.1(2) proven in Exercise 9.5.2, we conclude that

$$c\tau^n(z) \in \tau^n(I)\mathsf{D}(S) + \tau^{n-m}(J)\mathsf{D}(S) \subseteq \tau^n(I)\mathsf{D}(S) + J\mathsf{D}(S).$$

Therefore, under the induced morphism $\mathsf{D}(S) \to \mathsf{D}(R)$, we get

$$c\sigma^n(z) \in \sigma^n(I)\mathsf{D}(R)$$

for all $n > m$, showing that $z \in \mathrm{cl}^{\mathsf{D}}(IR)$. □

To prove the remaining two properties (the analogues of Theorems 8.3.4 and 8.3.6 respectively), some additional assumptions are needed. To compare with integral closure, we have to make a rather technical assumption on the underlying category $\mathfrak{C}$. We say that $\mathfrak{C}$ has the *Néron* property if for any homomorphism $A \to V$ with A in $\mathfrak{C}$ and V a discrete valuation ring (not necessarily belonging to $\mathfrak{C}$), there exists a faithfully flat extension $V \to W$ and a morphism $A \to R$ in $\mathfrak{C}$ with $R \in \mathfrak{C}$ a regular local ring such that the following diagram commutes

$$\begin{array}{ccc} A & \longrightarrow & V \\ \downarrow & & \downarrow \\ R & \longrightarrow & W. \end{array} \tag{9.4}$$

Clearly the Frobenius hull in prime characteristic trivially satisfies this property since we then may take $R = V = W$.

Proposition 9.1.8. *If $\mathfrak{C}$ is a difference hull satisfying the Néron property, then the difference closure of any ideal is contained in its integral closure.*

Proof. Let $I \subseteq A$ be an ideal of a ring A in $\mathfrak{C}$, and let $z \in A$ be in the difference closure of I. In order to show that z lies in the integral closure of I, we use criterion (4) in Theorem 8.4.1. To this end, let $A \to V$ be a homomorphism into a discrete valuation ring V whose kernel is a minimal prime of A. We need to show that $z \in IV$. Since $\mathfrak{C}$ has the Néron property, we can find a faithfully flat extension $V \to W$ and a morphism $A \to R$ in $\mathfrak{C}$ with R a regular local ring, yielding a commutative diagram (9.4). By assumption, there exists a multiplier $c \in A$ and a number N such that (9.2) holds in $D(A)$. Since c does not lie in the kernel of $A \to V$, its image in R must, a fortiori, be non-zero. Hence the same ideal membership relations viewed in $D(R)$ show that z lies in the difference closure of IR. By Corollary 9.1.4, this implies that z already lies in IR whence in IW. By faithful flatness and Proposition 5.3.4, we get $z \in IV$, as we wanted to show. ◻

Let us say that the difference hull $D(\cdot)$ *commutes with finite homomorphisms* if for each finite homomorphism $A \to B$ in $\mathfrak{C}$, the canonical homomorphism $D(A) \otimes_A B \to D(B)$ is an isomorphism of $D(A)$-algebras. Once more, this property holds trivially for the Frobenius hull.

Proposition 9.1.9. *If $D(\cdot)$ commutes with finite homomorphisms, and if $A \subseteq B$ is a finite extension of domains, then $\mathrm{cl}^D(I) = \mathrm{cl}^D(IB) \cap A$ for any ideal $I \subseteq A$.*

Proof. As in the proof of Theorem 8.3.4, we have an A-linear map $\varphi\colon B \to A$ with $\varphi(1) \neq 0$. By base change, this yields a $D(A)$-linear map $D(A) \otimes_A B \to D(A)$, whence a $D(A)$-linear map $D(B) \to D(A)$. The remainder of the argument is now as in the proof of Theorem 8.3.4, and is left to the reader. ◻

9.2 Tight closure

Our axiomatic treatment in terms of difference closure now only requires us to identify the appropriate difference hull. For the remainder of this chapter, K denotes a fixed algebraically closed Lefschetz field, and $\mathfrak{C}_K$ is the category of K-affine algebras (that is to say, the algebras essentially of finite type over K). By definition, we can realize K as an ultraproduct of fields K_p of characteristic p, where for simplicity we index these fields by their characteristic although this is not necessary. We remind the reader that $K = \mathbb{C}$ is an example of a Lefschetz field (Theorem 1.4.3). As difference hull, we now take the ultra-hull as defined in §7.1, viewing it as a difference ring by means of its ultra-Frobenius (see Definition 1.4.14).

Theorem 9.2.1. *The category $\mathfrak{C}_K$ has the Néron property, and the ultra-hull constitutes a simple difference hull which commutes with finite homomorphisms.*

Proof. We defer the proof of the Néron property to Proposition 9.2.2 below. The ultra-hull is functorial by 7.1.3. Property (1) in Definition 9.1.1 holds by Theorem 7.2.2, and the two remaining properties (2) and (3) hold trivially. By Łos' Theorem, the ultra-hull is a simple difference hull as defined in Remark 9.1.5; and it commutes with finite homomorphisms by Proposition 7.3.1. □

Proposition 9.2.2. *The category $\mathfrak{C}_K$ has the Néron property.*

Proof. Assume $A \to V$ is a homomorphism from a K-affine ring A into a discrete valuation ring V. Replacing A by its image in V, we may view A as a subring of V. By Theorem 6.4.5, the completion of V is isomorphic to $L[[t]]$ for some field L extending K and for t a single indeterminate. Let $\bar{L}$ be the algebraic closure of L and put $W := \bar{L}[[t]]$. By Theorem 6.3.4 and base change, the natural homomorphism $V \to W$ is faithfully flat (see also Theorem 6.4.7). The image of A in W has the same (uncountable) cardinality as K, whence is already contained in a subring of the form $k[[t]]$ with k an algebraically closed subfield of $\bar{L}$ of the same cardinality as K. By Theorem 1.4.5, we have an isomorphism $k \cong K$, and so we may assume that the composition $A \to W$ factors through $K[[t]]$. Let B' be the A-subalgebra of W generated by t, and let B be its localization at $tW \cap B'$, so that B is a local V_0-affine ring, where V_0 is the localization of $K[t]$ at the ideal generated by t. By Néron p-desingularization (see for instance [2, §4]), the embedding $B \subseteq K[[t]]$ factors through a regular local V_0-algebra R. Since R is then also a K-affine local ring, it satisfies all the required properties. □

The difference closure obtained from this choice of difference hull on $\mathfrak{C}_K$ will simply be called again *tight closure* (in the paper [57] it was called *non-standard tight closure*). For ease of reference, we repeat its definition here: an element z in a K-affine ring A belongs to the tight closure of an ideal $I \subseteq A$ if there exists a multiplier $c \in A$ such that

$$c\,\mathbf{F}^n_\infty(z) \in \mathbf{F}^n_\infty(I)U(A) \tag{9.5}$$

for all $n \gg 0$. We will denote the tight closure of I by $\mathrm{cl}_A(I)$ or simply $\mathrm{cl}(I)$, and we adopt the corresponding terminology from positive characteristic. Immediately from Theorem 9.2.1 and the results in the previous section we get:

Theorem 9.2.3. *Tight closure on K-affine rings satisfies the five key properties:*

1. *if $A \to B$ is an extension of K-affine domains, or more generally, a homomorphism of K-affine rings preserving multipliers, then $\mathrm{cl}_A(I) \subseteq \mathrm{cl}_B(IB)$ for every ideal $I \subseteq A$;*
2. *if A is a K-affine regular ring, then any ideal in A is tightly closed, and in fact, A is F-regular;*
3. *if R is a K-affine local ring and $(x_1, \dots, x_d)$ a system of parameters in R, then $((x_1, \dots, x_i)R : x_{i+1}) \subseteq \mathrm{cl}((x_1, \dots, x_i)R)$ for all i;*
4. *the tight closure of an ideal is contained in its integral closure;*
5. *if $A \subseteq B$ is a finite extension of K-affine domains, then $\mathrm{cl}_A(I) = \mathrm{cl}_B(IB) \cap A$.*

□

Of all five properties, only (4) relies on a deeper theorem, to wit Néron p-desingularization (which, nonetheless, is a much weaker form of Artin Approximation than needed for the HH-tight closure as discussed in §8.6). Is there a more elementary argument, at least for proving that tight closure is inside the radical of an ideal? On the other hand, property (5) is not such a very impressive fact in characteristic zero by Exercise 9.5.9 (see also the discussion following Theorem 9.4.1 below).

Since the ultra-hull is a simple difference hull, we can also define *simple tight closure* by requiring that (9.5) only holds for $n = 1$ (this was termed *non-standard closure* in [57]). For more on this closure, see Exercise 9.5.6. As already remarked, the five key properties form the foundation for deriving several deep theorems, as we now will show.

Theorem 9.2.4 (Hochster-Roberts—affine case). *If $R \to S$ is a cyclically pure homomorphism of K-affine local rings and if S is regular, then R is Cohen-Macaulay.*

The argument is exactly as in positive characteristic: one shows first that R is weakly F-regular, and then that any weakly F-regular ring is Cohen-Macaulay because we have Colon Capturing (in fact, one can prove an analogue of this result in any difference hull, see Exercise 9.5.5). Note that by our discussion on page 129, we have now completed the proof of Theorem 8.5.3 (to prove the result, we may always extend the base field to a Lefschetz field). The next result, however, cannot be proven—it seems—within the framework of difference hulls, although its proof is still elementary.

Theorem 9.2.5 (Briançon-Skoda—affine case). *Let A be a K-affine ring, and let $I \subseteq A$ be an ideal generated by n elements. If I has positive height, then we have for all $l \geq 1$ an inclusion*

$$\overline{I^{n+l-1}} \subseteq \mathrm{cl}(I^l).$$

In particular, if A is a K-affine regular local ring, then the integral closure of I^{n+l-1} lies inside I^l for all $l \geq 1$.

Proof. Again we only proof the case $l = 1$. Let z be in the integral closure of I^n, and let A_p, z_p and I_p be approximations of A, z and I respectively. The integral equation (similar to (8.7)), say, of degree d, witnessing that z lies in the integral closure of I^n, shows by Łos' Theorem that almost each z_p satisfies a similar integral equation of degree d, and hence, in particular, z_p belongs to the integral closure of I_p^n. By Exercise 8.7.6, for those p we have

$$I_p^{n(d-1)} z_p^k \in I_p^{kn}$$

for all k. As in the proof of Theorem 8.5.2, this implies that $I_p^{n(d-1)} \mathbf{F}_p^e(z_p)$ is contained in $\mathbf{F}_p^e(I_p) A_p$ for all e. Taking ultraproducts then yields

$$I^{n(d-1)} \mathbf{F}_\infty^e(z) \subseteq \mathbf{F}_\infty^e(I) U(A).$$

Since I has positive height, we can find by prime avoidance a multiplier $c \in I^{n(d-1)}$. In particular, $c\mathbf{F}^e_\infty(z) \in \mathbf{F}^e_\infty(I)U(A)$ for all e, whence $z \in \mathrm{cl}(I)$, as we wanted to show. The last assertion then follows from Theorem 9.2.3. □

We would of course prefer a version in which no assumption on I needs to be made. This indeed exists, but requires an intermediary closure operation, *ultra-closure* (see §9.3 below and Exercise 9.5.16). Using the previous result, we have now proven the polynomial case in the Briançon-Skoda theorem (Theorem 8.5.1). The last of our applications, the Ein-Lazardsfeld-Smith Theorem, can neither be carried out in the purely axiomatic setting of difference closure, but relies on some additional properties of the ultra-hull.

Theorem 9.2.6. *Let A be a K-affine regular domain, and let $\mathfrak{a} \subseteq A$ be a radical ideal, given as the intersection of finitely many prime ideals of height at most h. Then for all n, we have an inclusion $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$.*

Proof. Let $z \in \mathfrak{a}^{(hn)}$, and let A_p, z_p and $\mathfrak{a}_p$ be approximations of A, z and $\mathfrak{a}$ respectively. By Theorem 7.3.7 (or rather Exercise 7.5.4), almost all A_p are regular, and by Corollary 7.3.3 and Theorem 7.3.4, almost each $\mathfrak{a}_p$ is the intersection of finitely many prime ideals of height at most h. As in the proof of Theorem 8.5.10, for those p we therefore have $\mathfrak{a}_p^{hn^2}\mathbf{F}^e_p(z_p) \subseteq \mathbf{F}^e_p(\mathfrak{a}_p^n)A_p$ for all e. Taking ultraproducts then yields $\mathfrak{a}^{hn^2}\mathbf{F}^e_\infty(z) \subseteq \mathbf{F}^e_\infty(\mathfrak{a}^n)U(A)$, showing that z lies in $\mathrm{cl}(\mathfrak{a}^n)$ whence in $\mathfrak{a}^n$ by Theorem 9.2.3. □

9.3 Ultra-closure

In the two last proofs, we derived some membership relations in the approximations of an affine algebra and then took ultraproducts to get the same relations in its ultra-hull. However, each time the relations in the approximations already established tight closure membership in those rings. This suggests the following definition. Let A be a K-affine algebra, $I \subseteq A$ an ideal and $z \in A$. We say that z lies in the *ultra-closure* $\text{ultra-cl}(I)$ of I (called the *generic tight closure* in [57, 59]), if z_p lies in the tight closure of I_p for almost all p, where A_p, z_p and I_p are approximations of A, z and I respectively. Put differently

$$\text{ultra-cl}(I) = (\operatorname*{ulim}_{p\to\infty} \mathrm{cl}_{A_p}(I_p)) \cap A,$$

where we view the ultraproduct of the tight closures as an ideal in $U(A)$.

With little effort (Exercise 9.5.15) one shows:

Proposition 9.3.1. *Ultra-closure is a closure operation satisfying the five key properties listed in Theorem 9.2.3.*

To relate ultra-closure with tight closure, some additional knowledge of the theory of test elements (see the discussion following Theorem 8.3.1) is needed. Since we did not discuss these in detail, I quote the following result without proof.

Proposition 9.3.2 ([57, Proposition 8.4]). *Given a K-affine algebra A, there exists a multiplier $c \in A$ with approximation $c_p \in A_p$ such that c_p is a test element in A_p for almost all p.* □

Theorem 9.3.3. *The ultra-closure of an ideal is contained in its tight closure (and also in its simple tight closure).*

Proof. Let $z \in \text{ultra-cl}(I)$, with I an ideal in a K-affine algebra A. Let A_p, z_p and I_p be approximations of A, z and I respectively. By definition, z_p lies in the tight closure of I_p for almost all p. Let c be a multiplier as in Proposition 9.3.2, with approximations c_p. For almost all p for which c_p is a test element, we get $c_p\mathbf{F}_p^e(z_p) \in \mathbf{F}_p^e(I_p)A_p$ for all $e \geq 0$. Taking ultraproducts then yields $c\mathbf{F}_\infty^e(z) \in \mathbf{F}_\infty^e(I)U(A)$ for all e, showing that z lies in the (simple) tight closure of I. □

Without proof, we state the following comparison between our theory and the classical theory due to Hochster and Huneke (see §8.6); for a proof see [57, Theorem 10.4].

Proposition 9.3.4. *The HH-tight closure of an ideal is contained in its ultra-closure, whence in its tight closure.* □

9.4 Big Cohen-Macaulay algebras

Although the material in this section is strictly speaking not part of tight closure theory, the development of the latter was germane to the discovery by Hochster and Huneke of Theorem 9.4.1 below.

Big Cohen-Macaulay algebras in prime characteristic. Recall that the *absolute integral closure* A^+ of a domain A with field of fractions F, is the integral closure of A inside an algebraic closure of F. Since algebraic closure is unique up to isomorphism, so is absolute integral closure. Nonetheless it is not functorial, and we only have the following quasi-functorial property: given a homomorphism $A \to B$ of domains, there exists a (not necessarily unique) homomorphism $A^+ \to B^+$ making the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A^+ & \longrightarrow & B^+ \end{array} \tag{9.6}$$

commute (see Exercise 9.5.10).

Theorem 9.4.1 ([29]). *For every excellent local domain R in characteristic p, the absolute integral closure R^+ is a balanced big Cohen-Macaulay algebra.*

The condition that a Noetherian local ring is 'excellent' is for instance satisfied when R is either K-affine or complete (see [41, §32]). The proof of the above result is beyond the scope of these notes (see for instance [36, Chapters 7& 8]) although we will present a 'dishonest' proof shortly. It is quite a remarkable fact that the same result is completely false in characteristic zero: in fact any extension of a normal domain is split, and hence provides a counterexample as soon as R is not Cohen-Macaulay (see Exercise 9.5.9). One can use the absolute integral closure to define a closure operation in an excellent local domain R of prime characteristic as follows. For an ideal I, let the *plus-closure* of I be the ideal $I^+ := IR^+ \cap R$. One can show (see Exercise 9.5.12) that I^+ is a closure operation in the sense of Definition 8.2.3, satisfying the five key properties listed in Theorem 9.2.3. Moreover, unlike tight closure, it is not hard to show that it commutes with localization.

Proposition 9.4.2. *In an excellent local domain R of prime characteristic, the plus-closure of an ideal $I \subseteq R$ is contained in its tight closure.*

Proof. Let $z \in I^+$. By definition, there exists a finite extension $R \subseteq S \subseteq R^+$ such that $z \in IS$ (note that R^+ is the direct limit of all finite extensions of R by local domains). Hence $z \in \mathrm{cl}(I)$ by Theorem 8.3.4. □

It is conjectured that plus closure always equals tight closure (and hence in particular this would answer the localization problem for tight closure in the affirmative). Smith has verified this conjecture for a special, but important class of ideals:

Theorem 9.4.3 ([67]). *Any ideal generated by part of a system of parameters in an excellent local domain of prime characteristic has the same plus closure as tight closure.*

Proof of Theorem 9.4.1 assuming Theorem 9.4.3.

The proof we will present here is dishonest in the sense that Smith made heavy use of Theorem 9.4.1 to derive her result. However, here is how the converse direction goes. Let $(x_1,\dots,x_d)$ be a system of parameters in an excellent local domain R of characteristic p, and suppose $zx_{i+1} \in IR^+$ for some $z \in R^+$ and $I := (x_1,\dots,x_i)R$. Hence there already exists a finite extension $R \subseteq S \subseteq R^+$ containing z such that $zx_{i+1} \in IS$. Since $R \subseteq S$ is finite, $(x_1,\dots,x_d)$ is also a system of parameters in S by Theorem 3.3.8. By Colon Capturing (Theorem 8.3.3), we get $z \in \mathrm{cl}(IS)$. By Theorem 9.4.3, this implies that z lies in the plus closure of IS, whence in IS^+. However, it is not hard to see that $R^+ = S^+$, proving that $(x_1,\dots,x_d)$ is R^+-regular. □

9.4.4 *If R is an excellent regular local ring of prime characteristic, then R^+ is faithfully flat over R.*

This follows immediately from Theorem 9.4.1 and the Cohen-Macaulay criterion for flatness (Theorem 5.6.10). Interestingly enough, it also provides an alternative strategy to prove Theorem 9.4.1.

Proposition 9.4.5. *Let k be a field of positive characteristic. Suppose we can show that any k-affine (respectively, complete) regular local ring has a faithfully flat absolute integral closure, then the absolute integral closure of any k-affine (respectively, complete Noetherian) local domain is a balanced big Cohen-Macaulay algebra.*

Proof. I will only treat the affine case and leave the complete case to Exercise 9.5.18. Let R be a k-affine local domain, and let $\mathbf{x}$ be a system of parameters in R. By Noether Normalization with parameters (see the second part of Exercise 8.7.4), we can find a k-affine regular local subring $S \subseteq R$, such that $S \subseteq R$ is finite and $\mathbf{x}S$ is the maximal ideal of S. By assumption, S^+ is faithfully flat over S, and hence $(x_1, \dots, x_d)$ is S^+-regular. Finiteness yields $S^+ = R^+$, and so we are done. □

Big Cohen-Macaulay algebras in characteristic zero. As already mentioned, if R is a K-affine local domain of characteristic zero, then R^+ will in general not be a big Cohen-Macaulay algebra. However, we can still associate to any such R (in a quasi-functorial way) a canonically defined balanced big Cohen-Macaulay algebra as follows. Let R_p be an approximation of R. By Theorem 7.3.4, almost all R_p are domains. Let $\mathcal{B}(R)$ be the ultraproduct of the R_p^+; this is independent form the choice of approximation (see Exercise 9.5.19). By Łos' Theorem, there is a canonical homomorphism $R \to \mathcal{B}(R)$.

Theorem 9.4.6. *If R is a K-affine local domain, then $\mathcal{B}(R)$ is a balanced big Cohen-Macaulay algebra over R.*

Proof. Since almost each approximation R_p is a K_p-affine (whence excellent) local domain, R_p^+ is a balanced big Cohen-Macaulay R_p-algebra by Theorem 9.4.1. Let $\mathbf{x}$ be a system of parameters of R, with approximation $\mathbf{x}_p$. By Corollary 7.3.6, almost each $\mathbf{x}_p$ is a system of parameters in R_p, whence R_p^+-regular. By Łos' Theorem, $\mathbf{x}$ is therefore $\mathcal{B}(R)$-regular, as we wanted to show. □

Hochster and Huneke ([30]) arrive differently at balanced big Cohen-Macaulay algebras in characteristic zero, via their lifting method discussed in §8.6. However, their construction, apart from being rather involved, is far less canonical. In contrast, although it appears that $\mathcal{B}(R)$ depends on R, we have in fact:

9.4.7 *For each d, there exists a ring B_d such that for any K-affine local domain R, we have $\mathcal{B}(R) \cong B_d$ if and only if R has dimension d. In other words, B_d is a balanced big Cohen-Macaulay algebra for R if and only if R has dimension d.*

Indeed, by Noether Normalization (with parameters, see Exercise 8.7.4), R is finite over the localization of $K[\xi]$ at the ideal generated by the indeterminates $\xi := (\xi_1, \dots, \xi_d)$. By Łos' Theorem, the approximation R_p is finite over the corresponding localization of $K_p[\xi]$. If B_p is the absolute integral closure of this localization, then $B_p = R_p^+$. Hence the ultraproduct of the B_p only depends on d and is isomorphic to $\mathcal{B}(R)$. □

In analogy with plus closure, we define the *B-closure* $\operatorname{cl}^{\mathcal{B}}(I)$ of an ideal I in a K-affine local domain R as the ideal $I\mathcal{B}(R) \cap R$. As in positive characteristic, it is a closure operation satisfying the five key properties of Theorem 9.2.3 (see Exercise 9.5.12). Using Proposition 9.4.2 and Łos' Theorem, together with Theorem 9.3.3 we get:

9.4.8 *For any ideal I in a K-affine local domain R, we have inclusions* $\mathrm{cl}^{\mathsf{B}}(I) \subseteq \text{ultra-cl}(I) \subseteq \mathrm{cl}(I)$. □

Like tight closure theory, the existence of balanced big Cohen-Macaulay algebras does have many important applications. To illustrate this, we give an alternative proof of the Hochster-Roberts theorem, as well as a proof of the Monomial Conjecture (as far as I am aware of, no tight closure argument proves the latter). We will treat only the affine characteristic zero case here, but the same argument applies in positive characteristic, and, once we have developed the theory in Chapter 10, for arbitrary equicharacteristic Noetherian local rings.

Alternative proof of Theorem 9.2.4. Let $R \to S$ be a cyclically pure homomorphism of K-affine local domains with S regular, and let $\mathbf{x} := (x_1, \dots, x_d)$ be a system of parameters in R. To show that this is R-regular, assume $zx_{i+1} \in I := (x_1, \dots, x_i)R$. Since $\mathbf{x}$ is $\mathsf{B}(R)$-regular by Theorem 9.4.6, we get $z \in I\mathsf{B}(R)$. By quasi-functoriality (after applying Łos' Theorem to (9.6)) we get a homomorphism $\mathsf{B}(R) \to \mathsf{B}(S)$ making the diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathsf{B}(R) & \longrightarrow & \mathsf{B}(S) \end{array} \tag{9.7}$$

commute. In particular, $z \in I\mathsf{B}(S)$. Since S is regular, $S \to \mathsf{B}(S)$ is flat by the Cohen-Macaulay criterion for flatness (Theorem 5.6.10) and Theorem 9.4.6. Hence z belongs to IS by Proposition 5.3.4 whence to I by cyclical purity. □

As promised, we conclude with an application of the existence of big Cohen-Macaulay algebras to one of the Homological Conjectures (for further discussion, especially the still open mixed characteristic case, see Chapter 13). Let us call a tuple $(x_1, \dots, x_d)$ in a ring R *monomial*, if for all k, we have

$$(x_1 \cdots x_d)^{k-1} \notin (x_1^k, \dots, x_d^k)R. \tag{9.8}$$

We say that the *Monomial Conjecture* holds for a Noetherian local ring R, if R satisfies the hypothesis in the next result:

Theorem 9.4.9 (Monomial Conjecture). *If R is a local K-affine algebra, then any system of parameters is monomial.*

Proof. Let $(x_1, \dots, x_d)$ be a system of parameters, let x be the product of the x_i and suppose $x^{k-1} \in I_k := (x_1^k, \dots, x_d^k)R$ for some k. Let $\mathfrak{p}$ be a d-dimensional prime ideal.

Since $(x_1,\dots,x_d)$ is then also a system of parameters in $R/\mathfrak{p}$, and $x^{k-1} \in I_k(R/\mathfrak{p})$, we may after replacing R by $R/\mathfrak{p}$ assume that R is a domain. Hence $(x_1,\dots,x_d)$ is $B(R)$-regular by Theorem 9.4.6. However, it is easy to see that for a regular sequence we can never have $x^{k-1} \in I_k B(R)$ (see Exercise 9.5.13). □

Remark 9.4.10. By an argument on local cohomology, one can show that given any system of parameters $(x_1,\dots,x_d)$ in a Noetherian local ring R, there exists some t such that $(x_1^t,\dots,x_d^t)$ is monomial. Hence the real issue as far as the Monomial Conjecture is concerned is the fact that one can always take $t = 1$.

9.5 Exercises

Ex 9.5.1

Given a difference hull $D(\cdot)$ on a category $\mathfrak{C}$, and given an ideal $I \subseteq A$ in a ring A in $\mathfrak{C}$, show that $\mathrm{cl}^D(I)$ is an ideal in A containing I, and $\mathrm{cl}^D(\mathrm{cl}^D(I)) = \mathrm{cl}^D(I)$, that is to say, $\mathrm{cl}^D(I)$ is difference closed. Conclude that $\mathrm{cl}^D(\cdot)$ is a closure operation in the sense of Definition 8.2.3.

Ex 9.5.2

In this exercise, you are asked to prove that if $D(R)$ is a difference hull of a local ring R in $\mathfrak{C}$ with endomorphism σ, and if $(x_1,\dots,x_h)$ is an R-regular sequence, then $(\sigma^{e_1}(x_1),\dots,\sigma^{e_h}(x_h))$ is $D(R)$-regular, for any $e_i \geq 0$. First prove the case when $e_1 \geq e_2 \geq \dots \geq e_h$ by induction on the length h, and using that an arbitrary element $a \in R$ divides $\sigma(a)$ by Condition 9.1.1(3). To prove the general case, show that in an arbitrary ring A, if $(a_1b_1,\dots,a_sb_s)$ is a permutable A-regular sequence (meaning that any permutation is A-regular), then so is $(a_1,\dots,a_s)$.

Ex 9.5.3

Show that any Lefschetz ring is a difference ring.

Ex 9.5.4

Complete the proof of Proposition 9.1.9.

Ex 9.5.5

Let $D(\cdot)$ be a difference hull on $\mathfrak{C}$. Show that if $R \to S$ is a cyclically pure homomorphism of local rings in $\mathfrak{C}$, if R is a homomorphic image of a Cohen-Macaulay ring in $\mathfrak{C}$, and if S is regular, then R is Cohen-Macaulay.

Ex 9.5.6

Show that simple tight closure is a closure operation satisfying the five key properties of Theorem 9.2.3. In fact, you can prove the same for simple difference closure, with the necessary assumptions on the difference hull. See also Exercise 9.5.7 for more variants.

Ex 9.5.7

Proposition 8.2.6 essentially reduces the study of tight closure in arbitrary rings to domains. Unfortunately, for both difference closure and ultra-closure, I cannot yet prove this in general, the problem being that the endomorphism/ultra-Frobenius does not preserve multipliers. To circumvent this problem, the following variant—which I only explain for difference closure—is probably the 'correct' definition. Say that z lies in the stable difference closure *of an ideal $I \subseteq A$, if there exists a multiplier $c \in A$ and some $N \in \mathbb{N}$, such that $\sigma^N(c)\sigma^n(z) \in \sigma^n(I)\mathsf{D}(A)$ for all $n \geq N$. Prove that stable difference closure is a closure operation in the sense of Definition 8.2.3, verify that the analogue of Proposition 8.2.6 holds, and show that it satisfies the five key properties. Define the analogue stable variant for simple tight closure, and prove the same properties. Show that stable tight closure is always contained in stable simple tight closure.*

Ex 9.5.8

Give an alternative proof of the flatness of $\mathbf{F}_\infty$ on a K-affine regular local ring, by means of the equational criterion for flatness (Theorem 5.6.1), Theorem 7.3.7, and Łos' Theorem.

*Ex 9.5.9

Let A be a Noetherian normal domain(=integrally closed in its field of fractions L) containing the rationals, and let $A \subseteq B$ be a finite extension. Show that $A \to B$ is split (see the discussion following Theorem 8.5.3) as follows. Argue that after taking a homomorphic image, we may assume that B is a domain, with field of fractions L. We then we may replace B and L in such way that L is a Galois extension of K, say of degree d. Show that the trace map $L \to K$ (=the sum of all conjugates), followed by division by d, is a splitting of $A \subseteq B$. Use this to show that if R is K-affine local domain which is normal but not Cohen-Macaulay, then R^+ is not a big Cohen-Macaulay algebra.

*Ex 9.5.10

Show the existence of a map $A^+ \to B^+$ making diagram (9.6) commute. To this end, factor $A \to B$ as a surjection followed by an inclusion, and then treat each of these two cases separately.

Ex 9.5.11

Show that for any K-affine local domain R, the canonical map $R \to \mathsf{B}(R)$ factors through the ultra-hull $\mathsf{U}(R)$. Argue that $\mathsf{B}(R)$ is no longer integral over R if R is non-Artinian. Show that if $R \subseteq S$ is a finite extension of affine local domains, then $\mathsf{B}(R) = \mathsf{B}(S)$.

Ex 9.5.12

Show that plus closure and B-closure are closure operations in the sense of Definition 8.2.3, satisfying the five key properties listed in Theorem 9.2.3. In fact, quasi-functoriality (in the sense of (9.6)) yields persistence under arbitrary homomorphisms of local domains.

Ex 9.5.13

Show that a permutable regular sequence $\mathbf{x}$ in an arbitrary ring A is monomial. In particular, any local Cohen-Macaulay ring satifies the Monomial Conjecture.

*Ex 9.5.14

Let $Z := \mathbb{Z}[\xi]$ with $\xi := (\xi_1, \dots, \xi_n)$. Let us say that a tuple $\mathbf{x}$ in a ring A is strongly monomial, *if $I \neq J$ implies $IA \neq JA$ for any two monomial ideals $I, J \subseteq Z$ (that is to say, ideals generated by monomials), where we view A as a Z-algebra via the homomorphism $Z \to A\colon \xi_i \mapsto x_i$. Show that a regular sequence, and more generally, a quasi-regular sequence, is always strongly monomial (use Exercise 4.3.15). This proves in particular the claim in Exercise 9.5.13 for any regular sequence. Modify the argument in the proof of Theorem 9.4.9 to deduce that a system of parameters in a K-affine local ring, or in a Noetherian local ring of prime characteristic, is strongly monomial.*

Additional exercises.

Ex 9.5.15

Prove Proposition 9.3.1.

Ex 9.5.16

Show that if A is a K-affine ring and $I \subseteq A$ an ideal generated by at most n elements, then $\overline{I^{n+l-1}} \subseteq \text{ultra-cl}(I^l)$ for all l.

Ex 9.5.17

Show that if R is a K-affine local domain and $\mathfrak{p}$ a prime ideal in R, then

$$\mathsf{B}(R_\mathfrak{p}) \cong \mathsf{B}(R) \otimes_{\mathsf{U}(R)} \mathsf{U}(R_\mathfrak{p}).$$

Use this to prove that if $I \subseteq R$ and $\mathfrak{p}$ a minimal prime of I, then $\mathsf{B}(IR_\mathfrak{p}) = \mathsf{B}(I)R_\mathfrak{p}$. I do not know whether B-closure commutes in general with localization.

Ex 9.5.18

Prove the complete case in Proposition 9.4.5 using the Cohen structure theorems of Chapter 6.

Ex 9.5.19

Our goal is to give an alternative description of $\mathsf{B}(A)$ for A a K-affine local domain, showing that its construction is canonical. Let $\mathbb{N}_\natural$ be the ultrapower of the set of natural numbers, and let t be an indeterminate. For an element $f \in \mathsf{U}(A[t])$, define its ultra-degree *$\alpha \in \mathbb{N}_\natural$ (with respect to t) to be the ultraproduct of the t-degrees α_p of the f_p, where f_p is an approximation of f. Call an element $f \in \mathsf{U}(A[t])$* ultra-monic *if there exists $\alpha \in \mathbb{N}_\natural$ such that $f - t^\alpha$ has ultra-degree strictly less than α (see page 12 for the ultra-exponent notation). By a* root *of $g \in \mathsf{U}(A[t])$ in a Lefschetz field L containing K we mean an element $a \in L$ such that $g \in (t-a)\mathsf{U}(A_L[t])$, where $A_L := A \otimes_K L$ and its ultra-hull is taken in the category $\mathfrak{C}_L$.*
Show that there exists an algebraically closed Lefschetz field L containing K such that $\mathsf{B}(A)$ is isomorphic to the ring of all $a \in L$ that are a root of some ultra-monic polynomial in $\mathsf{U}(A[t])$.

Ex 9.5.20

Part of the descent theory of Hochster and Huneke for defining their HH-tight closure in characteristic zero (see §8.6), is the following special case: given a complete Noetherian local ring R containing a field K, a system of parameters $\mathbf{x}$ in R, and a finite subset $\Sigma \subseteq R$, we can find a K-affine local subring $S \subseteq R$ containing $\mathbf{x}$ and Σ, such that $\mathbf{x}$ is part of a system of parameters of S (see for instance [36, App. 1, Theorem 5.1]; this is also explained in more detail in Exercise 10.6.3 below). Use this to deduce the Monomial Conjecture (and even the stronger version discussed in Exercise 9.5.14) for any Noetherian local ring R of equal characteristic zero as follows. Assume that we have a counterexample to (9.8) for some k. Argue, using Theorem 6.4.7, that we may assume that R is complete with an algebraically closed Lefschetz residue field. Use the previous property to obtain a counterexample inside a K-affine local ring, and then finish with Theorem 9.4.9. A more direct proof can be given using the construction from §10.4 below.

Chapter 10
Tight closure in characteristic zero. Local case

The goal of this chapter is to extend the tight closure theory from the previous chapter to include all Noetherian rings containing a field. However, the theory becomes more involved, especially if one wants to maintain full functoriality. We opt in these notes to forego this cumbersome route (directing the interested reader to the joint paper [6] with Ashenbrenner), and only develop the theory minimally as to still obtain the desired applications. In particular, we will only focus on the local case.

From our axiomatic point of view, we need to define a difference hull on the category of Noetherian local rings containing $\mathbb{Q}$. The main obstacle is how to define an ultra-hull-like object, on which we then have automatically an action of the ultra-Frobenius. By Cohen's structure theorems, the problem can be reduced to constructing a difference hull for the power series ring $R := K[[\xi]]$ in a finite number of indeterminates ξ over an algebraically closed Lefschetz field K. A candidate presents itself naturally: let $\mathsf{U}(R)$ be the ultraproduct of the $K_p[[\xi]]$, where the K_p are algebraically closed fields of characteristic p whose ultraproduct is K. However, unlike in the polynomial case, there is no obvious homomorphism from R to $\mathsf{U}(R)$, and in fact, the very existence of such a homomorphism implies already some form of Artin Approximation. It turns out, however, that we can embed R in an ultrapower of $\mathsf{U}(R)$, and this is all we need, since the latter is still a Lefschetz ring. So we start with a discussion of this construction.

10.1 Artin Approximation

Constructing algebra homomorphisms. In this section, we study the following problem: Given two A-algebras S and T, when is there an A-algebra homomorphism $S \to T$? We will only provide a solution to the weaker version in which we are allowed to replace T by one of its ultrapowers. Since we want to apply this problem when T is equal to $\mathsf{U}(R)$, we will merely have replaced one type of ultraproduct with another.

Theorem 10.1.1. *For a Noetherian ring A, and A-algebras S and T, the following are equivalent:*

1. *every system of polynomial equations with coefficients from A which is solvable in S, is solvable in T;*
2. *for each finitely generated A-subalgebra C of S, there exists an A-algebra homomorphism $\varphi_C\colon C \to T$;*
3. *there exists an A-algebra homomorphism $\eta\colon S \to T_\natural$, where $T_\natural$ is some ultrapower of T.*

Proof. Suppose that (1) holds, and let $C \subseteq S$ be an A-affine subalgebra. Hence C is isomorphic to $A[\xi]/I$ with ξ a finite tuple of indeterminates and I some ideal in $A[\xi]$. Let $\mathbf{x}$ be the image of ξ in S, so that $\mathbf{x}$ is a solution of the system of equations $f_1 = \cdots = f_s = 0$, where $I = (f_1,\dots,f_s)A[\xi]$. By assumption, there exists therefore a solution $\mathbf{y}$ of this system of equations in T. Hence the A-algebra homomorphism $A[\xi] \to T$ given by sending ξ to $\mathbf{y}$ factors through an A-algebra homomorphism $\varphi_C\colon C \to T$, proving implication (1) $\Rightarrow$ (2).

Assume next that (2) holds. Let $\mathbb{W}$ be the collection of all A-affine subalgebras of S (there is nothing to show if S itself is A-affine, so we may assume $\mathbb{W}$ is in particular infinite). For each finite subset $\Sigma \subseteq S$ let $\langle\Sigma\rangle$ be the subset of $\mathbb{W}$ consisting of all A-affine subalgebras $C \subseteq S$ containing Σ. Any finite intersection of sets of the form $\langle\Sigma\rangle$ is again of that form. Hence we can find an ultrafilter on $\mathbb{W}$ containing each $\langle\Sigma\rangle$, where Σ runs over all finite subsets of S. Let $T_\natural$ be the ultrapower of T with respect to this ultrafilter. For each A-affine subalgebra $C \subseteq S$, let $\widetilde{\varphi}_C\colon S \to T$ be the map which coincides with φ_C on C and which is identically zero outside C. (This is of course no longer a homomorphism.) Define $\eta\colon S \to T_\natural$ to be the restriction to S of the ultraproduct of the $\widetilde{\varphi}_C$. In other words,

$$\eta(x) := \operatorname*{ulim}_{C\to\infty} \widetilde{\varphi}_C(x)$$

for any $x \in S$. It remains to verify that η is an A-algebra homomorphism. For $x, y \in S$, we have for each $C \in \langle\{x,y\}\rangle$ that

$$\widetilde{\varphi}_C(x+y) = \varphi_C(x+y) = \varphi_C(x) + \varphi_C(y) = \widetilde{\varphi}_C(x) + \widetilde{\varphi}_C(y),$$

since $\widetilde{\varphi_C}$ and φ_C agree on elements in C. Since this holds for almost all C, Łos' Theorem yields $\eta(x+y) = \eta(x) + \eta(y)$. By a similar argument, one also shows that $\eta(xy) = \eta(x)\eta(y)$ and $\eta(ax) = a\eta(x)$ for $a \in A$, proving that η is an A-algebra homomorphism.

Finally, suppose that $\eta\colon S \to T_\natural$ is an A-algebra homomorphism, for some ultrapower $T_\natural$ of T. Let $f_1 = \cdots = f_s = 0$ be a system of polynomial equations with coefficients in A, and let $\mathbf{x}$ be a solution in S. Since η is an A-algebra homomorphism, $\eta(\mathbf{x})$ is a solution of this system of equations in $T_\natural$. Hence by Łos' Theorem, this system must have a solution in T, proving (3) $\Rightarrow$ (1). □

Artin Approximation. We already got acquainted with Artin Approximation in our discussion of HH-tight closure, or in the guise of Néron p-desingularization as used in Proposition 9.2.2. The time has come, however, to present a more detailed discussion. Let S be a Noetherian local ring. We say that S satisfies the *Artin Approximation property* if any system of polynomial equations with coefficients in S which is solvable in $\widehat{S}$ is already solvable in S (for some equivalent conditions, see Exercise 10.5.3). So immediately from Theorem 10.1.1, or rather by the embedding version of Exercise 10.5.2, we get:

10.1.2 *A Noetherian local ring S has the Artin Approximation property if and only if its completion embeds in some ultrapower of S.*

Not any Noetherian local ring can have the Artin Approximation property:

Proposition 10.1.3. *A Noetherian local ring $(S,\mathfrak{m})$ with the Artin Approximation property is Henselian.*

Proof. Recall that this means that S satisfies Hensel's Lemma: any simple root $\bar{a}$ in $R/\mathfrak{m}$ of a monic polynomial $f \in S[t]$ lifts to a root in the ring itself. By Theorem 6.2.4, we can find such a root in $\widehat{S}$, and therefore by Artin Approximation, we then also must have a root in S itself (see Exercise 10.5.3 for how to ensure that it is a lifting of $\bar{a}$). □

Artin conjectured in [2] that the converse also holds if S is moreover excellent (it can be shown that any ring having the Artin Approximation property must be excellent). Although one has now arrived at a positive solution by means of very deep tools ([45, 69, 70]), the ride has been quite bumpy, with many false proofs appearing during the intermediate decades. Luckily, we only need this in the following special case due to Artin himself, admitting a fairly simple proof (which nonetheless is beyond the scope of these notes; see page 92 for the notion of Henselization).

Theorem 10.1.4 ([2, Theorem 1.10]). *The Henselization $k[\xi]^{\sim}$, with k a field and ξ a finite tuple of indeterminates, admits the Artin Approximation property.* □

Embedding power series rings. From now on, unless stated otherwise, K denotes an arbitrary ultra-field, given as the ultraproduct of fields K_m (for simplicity we assume $m \in \mathbb{N}$). We fix a tuple of indeterminates $\xi := (\xi_1,\dots,\xi_n)$, define $A := K[\xi]$ and $R := K[[\xi]]$, and let $\mathfrak{m} := (\xi_1,\dots,\xi_n)\mathbb{Z}[\xi]$. Similarly, for each m, we let $A_m := K_m[\xi]$ and $R_m := K_m[[\xi]]$, and in accordance with our notation from §7.1, we denote their respective ultraproducts by $\mathsf{U}(A)$ and $\mathsf{U}(R)$. By Łos' Theorem, we get a homomorphism $\mathsf{U}(A) \to \mathsf{U}(R)$ so that $\mathsf{U}(R)$ is in particular an A-algebra, but unlike the affine case, it is no longer clear how to make $\mathsf{U}(R)$ into an R-algebra. Note that $\mathsf{U}(R)$ is only quasi-complete (see the proof of Theorem 11.1.4), so that limits are not unique. In particular, although the truncations $f_n \in A$ of a power series $f \in R$ form a Cauchy sequence in $\mathsf{U}(R)$, there is no obvious choice for their limit.

Theorem 10.1.5. *There exists an ultrapower $\mathsf{L}(R)$ of $\mathsf{U}(R)$ and a faithfully flat A-algebra homomorphism $\eta_R\colon R \to \mathsf{L}(R)$.*

Proof. We start with proving the existence of an A-algebra homomorphism η_R from R to some ultrapower of $U(R)$. To this end, we need to show in view of Theorem 10.1.1 that any polynomial system of equations $(\mathscr{L})$ over A which is solvable in R, is also solvable in $U(R)$. By Theorem 10.1.4, the system has a solution $\mathbf{y}$ in $A^{\sim}$. Since the complete local rings R_m are Henselian by Theorem 6.2.4, so is $U(R)$ by Łos' Theorem. By the universal property of Henselization, the canonical homomorphism $A \to U(R)$ extends to a (unique) A-algebra homomorphism $A^{\sim} \to U(R)$. Hence the image in $U(R)$ of $\mathbf{y}$ is a solution of $(\mathscr{L})$ in $U(R)$, as we wanted to show.

Let $L(R)$ be the ultrapower of $U(R)$ given by Theorem 10.1.1 with corresponding A-algebra homomorphism $\eta : R \to L(R)$. Since $\eta(\xi) = \xi$, the maximal ideal of $L(R)$ is generated by ξ, and so η is local. By the Cohen-Macaulay criterion for flatness (Theorem 5.6.10), it suffices to show that $L(R)$ is a balanced big Cohen-Macaulay algebra. Since ξ is an R_m-regular sequence, so is its ultraproduct $\eta(\xi) = \xi$ in $L(R)$. This proves that $L(R)$ is a big Cohen-Macaulay algebra, and we can now use Proposition 5.6.9 and Łos' Theorem, to conclude that it is balanced, and hence that $\eta : R \to L(R)$ is faithfully flat. □

Being an ultrapower of an ultraproduct, $U(R)$ itself is an ultra-ring. More precisely (see Exercise 10.5.4):

10.1.6 *There exists an index set $\mathbb{W}$ and an $\mathbb{N}$-valued function assigning to each $w \in \mathbb{W}$ an index $m(w)$, such that*

$$L(R) = \operatorname*{ulim}_{w \to \infty} R_{m(w)}.$$

Strong Artin Approximation. We say that a local ring $(S, \mathfrak{n})$ has the *strong Artin Approximation property* if the following holds: given a system $(\mathscr{L})$ of polynomial equations $f_1 = \dots = f_s = 0$ with coefficients in S, if $(\mathscr{L})$ has an approximate solution in S modulo $\mathfrak{n}^m$ for all m, then $(\mathscr{L})$ has a (true) solution in S. Here by an *approximate* solution of $(\mathscr{L})$ modulo an ideal $\mathfrak{a} \subseteq S$, we mean a tuple $\mathbf{x}$ in S such that the congruences $f_1(\mathbf{x}) \equiv \dots \equiv f_s(\mathbf{x}) \equiv 0 \mod \mathfrak{a}$ hold.

We start with the following observation regarding the connection between R and its Lefschetz hull $L(R)$ (this will be explored in more detail in §11.1 where we will call the separated quotient the *cataproduct* of the R_m).

Proposition 10.1.7. *The separated quotient $U(R)/\mathfrak{I}_{U(R)}$ of $U(R)$ is isomorphic to R.*

Proof. We start by defining a homomorphism $U(R) \to R$ as follows. Given $f \in U(R)$, choose approximations $f_m \in R_m$ and expand each as a power series

$$f_m = \sum_{\nu \in \mathbb{N}^n} a_{\nu,m} \xi^{\nu}$$

for some $a_{\nu,m} \in K_m$. Let $a_\nu \in K$ be the ultraproduct of the $a_{\nu,m}$ and define

$$\tilde{f} := \sum_{\nu \in \mathbb{N}^n} a_\nu \xi^\nu \in R.$$

One checks that the map $f \mapsto \tilde{f}$ is well-defined (that is to say, independent of the choice of approximation), and is a ring homomorphism. It is not hard to see that it is moreover surjective. So remains to show that its kernel equals the ideal of infinitesimals $\mathfrak{I}_{U(R)}$. Suppose $\tilde{f} = 0$, whence all $a_\nu = 0$. For fixed d, almost all $a_{\nu,m} = 0$ whenever $|\nu| < d$. Hence $f_m \in \mathfrak{m}^d R_m$ for almost all m, and therefore $f \in \mathfrak{m}^d U(R)$ by Łos' Theorem. Since this holds for all d, we see that $f \in \mathfrak{I}_{U(R)}$. Conversely, any infinitesimal is easily seen to lie in the kernel by simply reversing this argument. □

In [10], a paper the methods of which are germane in the development of the present theory, the following ultraproduct argument was used to derive a strong Artin Approximation result.

Theorem 10.1.8. *The ring $R := K[[\xi]]$, for K an arbitrary algebraically closed ultra-field and ξ a finite tuple of indeterminates, has the strong Artin Approximation.*

Proof. Let $(\mathscr{L})$ be a system of equations over R, and for each m, let $\mathbf{x}_m$ be an approximate solution of $(\mathscr{L})$ modulo $\mathfrak{m}^m R$. Let $R_\natural$ be the ultrapower of R, and let $\mathbf{x}$ be the ultraproduct of the $\mathbf{x}_m$. By Łos' Theorem, $\mathbf{x}$ is an approximate solution of $(\mathscr{L})$ modulo any $\mathfrak{m}^m R_\natural$, whence modulo $\mathfrak{I}_{R_\natural}$, the ideal of infinitesimals of $R_\natural$ (see Definition 1.4.10). By Proposition 10.1.7 (or rather by a variant admitting a similar argument), the separated quotient $R_\natural/\mathfrak{I}_{R_\natural}$ is isomorphic to $K_\natural[[\xi]]$, where $K_\natural$ is the ultrapower of K. The image of $\mathbf{x}$ in $K_\natural[[\xi]]$ is therefore a solution of the system $(\mathscr{L})$. Let $k \subseteq K$ be a countable algebraically closed subfield such that $(\mathscr{L})$ is already defined over k, and let $L \subseteq K_\natural$ be the algebraic closure of the field generated over K by all the coefficients of the entries in the image of $\mathbf{x}$ in $K_\natural[[\xi]]$. Since L has the same cardinality as K, they are isomorphic as fields by Theorem 1.4.5, and in fact, by a simple modification of its proof, these fields are isomorphic over their common countable subfield k. In particular, the image of $\mathbf{x}$ under the induced $k[[\xi]]$-algebra isomorphism of $L[[\xi]]$ with $K[[\xi]]$, gives the desired solution of $(\mathscr{L})$ in $R = K[[\xi]]$. □

Any version in which the same conclusion as in the strong Artin Approximation property can be reached just from the solvability modulo a single power $\mathfrak{n}^N$ of the maximal ideal $\mathfrak{n}$, where N only depends on (some numerical invariants of) the system of equations, is called the *uniform strong Artin Approximation* property. In [10], the uniform strong Artin Approximation for certain Henselizations was derived from the Artin Approximation property of those rings via ultraproducts. To get a uniform version in more general situations, additional restrictions have to be imposed on the equations (see [2, Theorem 6.1] or [10, Theorem 3.2]) and substantially more work is required [15, 16]. We will here present a version which requires the equations to have polynomial coefficients as well.

Theorem 10.1.9 (Uniform strong Artin Approximation). *There exists a function $N\colon \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let k be a field, put $A := k[\xi]$ with ξ an*

n-tuple of indeterminates, and let $\mathfrak{m}$ be the ideal generated by these indeterminates. Let $(\mathscr{L})$ be a polynomial system of equations with coefficients from A, in the n unknowns t, such that each polynomial in $(\mathscr{L})$ has total degree (with respect to both ξ and t) at most d. If $(\mathscr{L})$ admits an approximate solution in A modulo $\mathfrak{m}^{N(n,d)}A$, then it admits a true solution in $k[[\xi]]$.

Proof. Towards a contradiction, assume such a bound does not exist for the pair (d,n), so that for each $m \in \mathbb{N}$ we can find a counterexample consisting of a field K_m, and of polynomials f_{im} for $i = 1,\dots,s$ over this field of total degree at most d in the indeterminates ξ and t, such that viewed as a system of equations $(\mathscr{L}_m)$ in the unknowns t, it has an approximate solution $\mathbf{x}_m$ in $A_m := K_m[\xi]$ modulo $\mathfrak{m}^m A_m$ but no actual solution in $R_m := K_m[[\xi]]$. Note that by Lemma 7.4.2 we may assume that the number of equations s is independent from m. Let K, $U(A)$ and $U(R)$ be the ultraproduct of the K_m, A_m and R_m respectively, and let f_i and $\mathbf{x}$ be the ultraproduct of the f_{im} and $\mathbf{x}_m$ respectively. By 7.1.2, the f_i are polynomials over K, and by Łos' Theorem, $f_i(\mathbf{x}) \equiv 0 \mod \mathfrak{I}_{U(R)}$. By Proposition 10.1.7, we have an epimorphism $U(R) \to R$. In particular, the image of $\mathbf{x}$ in R is a solution of the system $(\mathscr{L})$ given by $f_1 = \cdots = f_s = 0$.

Since we have an A-algebra homomorphism $R \to L(R)$ by Theorem 10.1.5, the image of $\mathbf{x}$ in $L(R)$ remains a solution of the system $(\mathscr{L})$, and hence by Łos' Theorem, we can find for almost each w, a solution of $(\mathscr{L}_{m(w)})$ in $R_{m(w)}$, contradicting our assumption on the systems $(\mathscr{L}_m)$. □

Note that the above proof only uses the existence of a homomorphism from R to some ultrapower of $U(R)$, showing that mere existence is already a highly non-trivial result, and hence it should not come as a surprise that we needed at least some form of Artin Approximation to prove the latter. Of course, by combining this with Theorem 10.1.4, we may even conclude that $(\mathscr{L})$ has a solution in $A^{\sim}$, thus recovering the original result [2, Theorem 6.1] (see also [10, Theorem 3.2]). If instead we use the filtered version of Theorem 10.1.5, to be discussed briefly after Proposition 10.3.2 below, we get filtered versions of this uniform strong Artin Approximation property, as explained in [6] (for a special case, see Exercise 10.5.8).

We conclude with the non-linear analogue of Theorem 7.4.3 (or rather of the version given in Exercise 7.5.5). We cannot simply expect the same conclusion as in the linear case to hold: there is not bound on the degree of polynomial solutions in terms of the degrees of the system of equations (a counterexample is discussed in [54, Theorem 9.1]). However, we can recover bounds when we allow for power series solutions. Of course degree makes no sense in this context, and so we define the following substitute. By Project 6.6, a power series y lies in the Henselization $A^{\sim}$ if there exists an N-tuple $\mathbf{y}$ in R with first coordinate equal to y, and a *Hensel system* $(\mathscr{H})$, consisting of N polynomials $f_1,\dots,f_N \in A[t]$ in the N unknowns t such that the Jacobian matrix $\operatorname{Jac}(\mathscr{H})$ evaluated at $\mathbf{x}$ is invertible in R. We say that y has *etale complexity* as most d, if we can find such a Hensel system of size $N \leq d$ with all f_i of total degree at most d (in ξ and t).

Theorem 10.1.10. *There exists a function $N\colon \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let k be a field and put $A := k[\xi]$ with ξ an n-tuple of indeterminates. Let $(\mathscr{L})$*

be a system of polynomial equations in $A[t]$ in the n unknowns t, such that each polynomial in $(\mathscr{L})$ has total degree (with respect to ξ and t) at most d. If $(\mathscr{L})$ is solvable in $k[[\xi]]$, then it has a solution in $A^{\sim}$ of etale complexity at most $N(d,n)$.

Proof. Suppose no such bound on the etale complexity exists for the pair (d,n), yielding for each m a counterexample consisting of a field K_m, and a system of polynomial equations $(\mathscr{L}_m)$ over K_m of total degree at most d with a solution $\mathbf{y}_m$ in the power series ring R_m, such that, however, any solution in $A_m^{\sim}$ has etale complexity at least m (notation as before). Let $(\mathscr{L})$ be the ultraproduct of the $(\mathscr{L}_m)$, a system of polynomial equations over K by 7.1.2 (and an application of Lemma 7.4.2), and let $\mathbf{y}$ be the ultraproduct of the $\mathbf{y}_m$, a solution of $(\mathscr{L})$ in $U(R)$ by Łos' Theorem. By Proposition 10.1.7, under the canonical epimorphism $U(R) \to R$, we get a solution of $(\mathscr{L})$ in R, whence in $A^{\sim}$ by Theorem 10.1.4. Let $(\mathscr{H})$ be a Hensel system for this solution $\mathbf{x}$ viewed as a tuple in $A^{\sim}$ (note that one can always combine Hensel systems for each entry of a tuple to a Hensel system for the whole tuple), and let d be its total degree. Since the ultraproduct $H_\natural$ of the $A_m^{\sim}$ is a Henselian local ring containing A, the universal property of Henselizations yields an A-algebra homomorphism $A^{\sim} \to H_\natural$. Viewing therefore $\mathbf{x}$ as a solution of $(\mathscr{L})$ in $H_\natural$, we can find approximations $\mathbf{x}_m$ in $A_m^{\sim}$ which are solutions of $(\mathscr{L}_m)$ for almost all m. If we let $(\mathscr{H}_m)$ be an approximation of $(\mathscr{H})$, then by Łos' Theorem, for almost all m, it is a Hensel system for $\mathbf{x}_m$ of degree at most d, thus contradicting our assumption. □

10.2 Tight closure

For the remainder of this chapter, we specify the previous theory to the case that K is an algebraically closed Lefschetz field, given as the ultraproduct of the algebraically closed fields K_p of characteristic p.

Lefschetz hulls. In particular, $L(R)$ is a Lefschetz ring, given as the ultraproduct of the power series rings $R_{p(w)} := K_{p(w)}[[\xi]]$, where $p(w)$ is equal to the underlying characteristic. The ultraproduct $\mathbf{F}_\infty$ of the $\mathbf{F}_{p(w)}$ acts on $L(R)$, making it a difference ring. This immediately extends to homomorphic images:

Corollary 10.2.1. *The assignment $R/I \mapsto L(R/I) := L(R)/IL(R)$ constitutes a difference hull on the category of all homomorphic images of R.* □

Note that any complete Noetherian local ring with residue field K and embedding dimension at most n is a homomorphic image of R by Theorem 6.4.2. However, a local homomorphism between two such rings is not necessarily an epimorphism, so that the previous statement is much weaker than obtaining a difference hull on the category of complete Noetherian local ring with residue field K. We will address this issue further in §10.3 below.

We can easily extend the previous construction to include any Noetherian local ring S of equal characteristic zero. Our definition though will depend on some choices. We start by taking K sufficiently large so that it contains the residue field

k of S as a subfield. Let $\widehat{S_K}$ be the complete scalar extension of S along K as given by Theorem 6.4.7. By Cohen's Theorem (Theorem 6.4.2), we may write $\widehat{S_K}$ as $R/\mathfrak{a}$ for some ideal $\mathfrak{a} \subseteq R$ (assuming that the number n of indeterminates ξ is at least the embedding dimension of S). We now define $L(S) := L(\widehat{S_K}) = L(R)/\mathfrak{a}L(R)$. Since $S \to \widehat{S_K}$ is faithfully flat by Theorem 6.4.7, this assignment is a difference hull on the category of all homomorphic images of S by Corollary 10.2.1 and Exercise 9.5.3, called a *Lefschetz hull* of S (for another type of Lefschetz hull, see page 163 below).

Tight closure. The *tight closure* of an ideal $I \subseteq S$ is by definition the difference closure of I with respect to a (choice of) Lefschetz hull, and is again denoted $\mathrm{cl}_S(I)$ or simply $\mathrm{cl}(I)$ (although technically speaking, we should also include the Lefschetz hull in the notation). In other words, $z \in \mathrm{cl}(I)$ if and only if there exists a multiplier $c \in S$ such that

$$c\,\mathbf{F}^e_\infty(z) \in \mathbf{F}^e_\infty(I)L(S) \tag{10.1}$$

for all $e \gg 0$ (again we suppress the embedding $\eta_S\colon S \to L(S)$ in our notation).

By our axiomatic treatment of difference closure, we therefore immediately obtain the five key properties of Theorem 9.2.3 for this category. However, this is a severely limited category, and the only two properties that do not rely on any functoriality with respect to general homomorphisms are:

10.2.2 *Any regular local ring of equal characteristic zero is F-regular, and any complete local domain S (or more generally, any equidimensional homomorphic image of a Cohen-Macaulay local ring) of equal characteristic zero admits Colon Capturing: for any system of parameters $(x_1,\dots,x_d)$ in S, we have $((x_1,\dots,x_i)S : x_{i+1}) \subseteq \mathrm{cl}((x_1,\dots,x_i)S)$ for all i.*

Inspecting the proofs of Theorems 9.2.5 and 9.2.6, we see that these carry over immediately to the present case, and hence we can now state:

Theorem 10.2.3 (Briançon-Skoda—local case). *Let S be a Noetherian local ring of equal characteristic zero, and let $I \subseteq S$ be an ideal generated by n elements. If I has positive height, then we have for all $l \geq 1$ an inclusion*

$$\overline{I^{n+l-1}} \subseteq \mathrm{cl}(I^l).$$

In particular, if S is moreover regular, then the integral closure of I^{n+l-1} lies inside I^l for all $l \geq 1$. □

Theorem 10.2.4. *Let S be a regular local ring of equal characteristic zero, and let $\mathfrak{a} \subseteq S$ be the intersection of finitely many prime ideals of height at most h. Then for all n, we have an inclusion $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$.*

In particular, we also proved the original version of the Briançon-Skoda theorem (Theorem 8.5.1).

10.3 Functoriality

Unfortunately, the last of our three applications, the Hochster-Roberts Theorem, requires functoriality beyond the one provided by Corollary 10.2.1. In Project 10.6 we will describe an alternative strategy to prove the Hochster-Roberts theorem in the general case. Here, we discuss briefly how to extend some form of functoriality to the whole category of all Noetherian local rings of equal characteristic zero, which suffices to derive the theorem. As we will see shortly, functoriality requires a 'filtered' version of Theorem 10.1.1. To show that this version holds for power series rings over K, we require the following more sophisticated Artin Approximation result due to Rotthaus (its proof is still relatively simple in comparison with those of the general Artin Conjecture needed in the Hochster-Huneke version). As before, $R := K[[\xi]]$.

Theorem 10.3.1 ([49]). *The Henselization $R[\zeta]^\sim$ of the localization of $R[\zeta]$ at the maximal ideal generated by all the indeterminates admits the Artin Approximation property.*

We extend the terminology used in §7.1: given an ultra-ring $C_\natural$, realized as the ultraproduct of rings C_w, then by an *ultra-$C_\natural$-algebra $D_\natural$*, we mean an ultraproduct $D_\natural$ of C_w-algebras D_w. If almost each C_w is local and D_w is a local C_w-algebra (meaning that the canonical homomorphism $C_w \to D_w$ is a local homomorphism), then we call $D_\natural$ an *ultra-local $C_\natural$-algebra*. Similarly, a *morphism of ultra-(local) $C_\natural$-algebras* is by definition an ultraproduct of (local) C_w-algebra homomorphisms.

For our purposes, we only will need the following quasi-functorial version of the Lefschetz hull.

Proposition 10.3.2. *Let S be a Noetherian local ring of equal characteristic zero with a given choice of Lefschetz hull $\eta_S\colon S \to \mathsf{L}(S)$. For every Noetherian local S-algebra T whose residue field embeds in K, there exists a choice of Lefschetz hull $\eta_T\colon T \to \mathsf{L}(T)$ on T, having in addition the structure of an ultra-local $\mathsf{L}(S)$-algebra.*

Proof. By taking an isomorphic copy of the S-algebra T, we may assume that the induced homomorphism on the residue fields is an inclusion of subfields of K. In that case, one easily checks that the complete scalar extension $\widehat{S_K} \to \widehat{T_K}$ of the canonical homomorphism $S \to T$ is in fact a K-algebra homomorphism. Taking n sufficiently large, $\widehat{S_K}$ and $\widehat{T_K}$ are homomorphic images of R, and the K-algebra homomorphism $\widehat{S_K} \to \widehat{T_K}$ lifts to a K-algebra endomorphism φ of R by an application of Theorem 6.4.2. So without loss of generality, we may assume $S = T = R$. Let $\mathbf{x} := (x_1, \dots, x_n)$ be the image of ξ under φ, so that in particular, each x_i is a power series without constant term. Note that the K-algebra local homomorphism φ is completely determined by this tuple, namely $\varphi(f) = f(\mathbf{x})$ for any $f \in R$ (see Exercise 10.5.5). Let $R' := R[[\zeta]]$, where ζ is another n-tuple of indeterminates, and put $R'_p := R_p[[\zeta]]$. Note that φ is isomorphic to the composition $R \subseteq R' \twoheadrightarrow R'/J \cong K[[\zeta]]$, where the first map is just inclusion, and where J is the ideal generated by all $\xi_i - x_i$. Since Lefschetz hulls commute with homomorphic images, we reduced the problem to finding a Lefschetz hull $\eta_{R'}\colon R' \to \mathsf{L}(R')$, together with a morphism $\mathsf{L}(R) \to \mathsf{L}(R')$ of ultra-local K-algebras extending the inclusion $R \subseteq R'$.

By Theorem 10.1.5, there exists some ultrapower of $\mathsf{U}(R)$ which is faithfully flat over R. Since we will have to further modify this ultrapower, we denote it by $Z_\natural$. Recall that it is in fact an ultraproduct of the R_p by 10.1.6. Let $Z'_\natural$ denote the corresponding ultraproduct of the $R'_{p(w)}$. In particular, we get a morphism $Z_\natural \to Z'_\natural$ of ultra-local K-algebras. Moreover, $Z'_\natural$ is an R-algebra via the composition $R \to Z_\natural \to Z'_\natural$, whence also an $R[\zeta]$-algebra, since in $Z'_\natural$, the indeterminates ζ remain algebraically independent over R. We will obtain $\mathsf{L}(R')$ as a (further) ultrapower of $Z'_\natural$ from an application of Theorem 10.1.1, which at the same time then also provides the desired R-algebra homomorphism $R' \to \mathsf{L}(R')$. So, given a polynomial system of equations $(\mathscr{L})$ with coefficients in R having a solution in R', we need to find a solution in $Z'_\natural$. By Theorem 10.3.1, we can find a solution in $R[\zeta]^\sim$, since R' is the completion of the latter ring. By the universal property of Henselizations, we get a local $R[\zeta]$-algebra homomorphism $R[\zeta]^\sim \to Z'_\natural$, and hence via this homomorphism, we get a solution for $(\mathscr{L})$ in $Z'_\natural$, as we wanted to show. Let $R' \to \mathsf{L}(R')$ be the homomorphism given by Theorem 10.1.1, which is then faithfully flat by Theorem 10.1.5. Let $\mathsf{L}(R)$ be the corresponding ultrapower of $Z_\natural$, so that $R \to \mathsf{L}(R)$ too is faithfully flat. Moreover, the homomorphism $Z_\natural \to Z'_\natural$ then yields, after taking ultrapowers, a morphism of ultra-local K-algebras $\mathsf{L}(R) \to \mathsf{L}(R')$. We leave it to the reader to verify that it extends the inclusion $R \subseteq R'$, and admits all the desired properties. □

In [6], a much stronger form of functoriality is obtained, by making the ad hoc argument in the previous proof more canonical. In particular, we construct $\eta_R\colon R \to \mathsf{L}(R)$ in such way that it maps each of the subrings $K[[\xi_1,\dots,\xi_i]]$ to the corresponding subring of $\mathsf{L}(R)$ of all elements depending only on the indeterminates $\xi_1,\dots,\xi_i$, that is to say, the ultraproduct of the $K_{p(w)}[[\xi_1,\dots,\xi_i]]$ (our treatment of the inclusion $R \subseteq R'$ in the previous proof is a special instance of this). However, this is not a trivial matter, and caution has to be exercised as to how much we can preserve. For instance, in [6, §4.33], we show that 'unnested' subrings cannot be preserved, that is to say, there cannot exist an η_R which maps any subring $K[[\xi_{i_1},\dots,\xi_{i_s}]]$ into the corresponding subring of all elements depending only on the indeterminates $\xi_{i_1},\dots,\xi_{i_s}$ (the concrete counterexample requires $n=6$, and it would be of interest to get already a counterexample for $n=2$).

Proposition 10.3.2 is sufficiently strong to get the following form of weak persistence: if $S \to T$ is a local homomorphism of Noetherian local domains of equal characteristic zero, then we can define tight closure operations $\mathrm{cl}_S(\cdot)$ and $\mathrm{cl}_T(\cdot)$ on S and T respectively, such that $\mathrm{cl}_S(I) \subseteq \mathrm{cl}_T(IT)$ for all $I \subseteq S$ (see the argument in the next proof).

Theorem 10.3.3 (Hochster-Roberts). *If $S \to T$ is a cyclically pure homomorphism of Noetherian local rings of equal characteristic, and if T is regular, then S is Cohen-Macaulay.*

Proof. We already dealt with the positive characteristic case, so assume the characteristic is zero. By Exercise 10.5.10, we may assume S and T are complete, and by Proposition 10.3.2, we may assume that $L(T)$ is an ultra-$L(S)$-algebra (by taking K sufficiently large). Let $(x_1,\dots,x_d)$ be a system of parameters in S, and assume $zx_{i+1} \in I := (x_1,\dots,x_i)S$. By Colon Capturing (10.2.2), we get $z \in \mathrm{cl}(I)$, so that (10.1) holds for all $e \gg 0$. However, we may now view these relations also in $L(T)$ via the S-algebra homomorphism $L(S) \to L(T)$, showing that $z \in \mathrm{cl}(IT)$. By 10.2.2 therefore, $z \in IT$ whence by cyclic purity, $z \in I$, as we wanted to show. □

We can now also tie up another loose end, the last of our five key properties, namely the connection with integral closure (recall that 9.2.3(5) is not really an issue in characteristic zero by Exercise 9.5.9):

Theorem 10.3.4. *The tight closure of an ideal lies inside its integral closure.*

Proof. Let $I \subseteq S$ be an ideal in a Noetherian local ring $(S,\mathfrak{n})$ of equal characteristic zero, and let $z \in \mathrm{cl}(I)$. By Exercise 10.5.11, we may reduce to the case that I is $\mathfrak{n}$-primary. In view of 8.4.1(4), we need to show that $z \in IV$, for every homomorphism $S \to V$ into a discrete valuation ring V with kernel a minimal prime ideal of S. There is nothing to show if $\mathfrak{n}V = V$ whence $IV = V$, so that we may assume $S \to V$ is local. Moreover, by a similar cardinality argument as in Proposition 9.2.2, we may replace V by a sub-discrete valuation ring whose residue field embeds in K. By Proposition 10.3.2, there exists a Lefschetz hull $L(V)$ on V which is an ultra-local $L(S)$-algebra. In particular, z lies in the tight closure of IV with respect to this choice of Lefschetz hull, and so we are done by an application of 10.2.2 to the regular ring V. □

10.4 Big Cohen-Macaulay algebras

As in the affine case, we can also associate to each Noetherian local domain of equal characteristic zero a balanced big Cohen-Macaulay algebra. However, to avoid some complications caused by the fact that the completion of a domain need not be a domain, I will only discuss this in case S is a complete Noetherian local domain with residue field K (for the general case, see [6, §7]). But even in this case, the Lefschetz hull defined above does not have the desired properties: we do not know whether the approximations of S are again domains. So we discuss first a different construction of a Lefschetz hull.

Relative hulls. Fix some Noetherian local ring $(S,\mathfrak{n})$ with residue field k contained in K, and let $L(S)$ be a Lefschetz hull for S with approximations S_w. We want to construct a Lefschetz hull on the category of S-affine algebras, extending the Lefschetz hull defined on page 159. Let us first consider the polynomial ring $B := S[\zeta]$ in finitely many indeterminates ζ. Let $L_S(B)$ be defined as the ultraproduct of the $B_w := S_w[\zeta]$, so that $L_S(B)$ is an ultra-$L(S)$-algebra. The homomorphism $S \to L_S(B)$ extends naturally to a homomorphism $B \to L_S(B)$, since the ζ remain algebraically independent over $L(S)$. We call $L_S(B)$ the *relative Lefschetz hull of B* (with respect to the Lefschetz hull $S \to L(S)$). Similarly, if $C = B/I$ is an arbitrary S-affine algebra, then we define $L_S(C)$ as the residue ring $L_S(B)/IL_S(B)$, and we call this the *relative Lefschetz hull* of C (with respect to the choice of Lefschetz hull $L(S)$). By base change the homomorphism $B \to L_S(B)$ induces a homomorphism $C \to L_S(C)$. Moreover, $L_S(C)$ is an ultra-$L(S)$-algebra, since I is finitely generated.

It is instructive to calculate $L_S(B)/\mathfrak{n}L_S(B) = L_S(B/\mathfrak{n}B) = L_S(k[\zeta])$, where k is the residue field of S. Since $\mathfrak{n}\widehat{S_K}$ is the maximal ideal in $\widehat{S_K}$, we get $L(S)/\mathfrak{n}L(S) = L(k) = L(K)$, and this field is just an ultrapower of $K = U(K)$. Hence $B_w/\mathfrak{n}_w B_w = K_{p(w)}[\zeta]$, and we see that $L_S(B)/\mathfrak{n}L_S(B)$ is an ultrapower of $U(K[\zeta])$. Next, suppose T is a local S-affine algebra, say of the form $B_{\mathfrak{p}}/IB_{\mathfrak{p}}$, with $\mathfrak{p} \subseteq B$ a prime ideal containing I. Moreover, since we assume that $S \to T$ is local, $\mathfrak{n}B \subseteq \mathfrak{p}$. In order to define the relative Lefschetz hull $L_S(T)$ of T as the localization of $L_S(B/IB)$ with respect to $\mathfrak{p}L_S(B/IB)$, we need:

10.4.1 *If $\mathfrak{p}$ is a prime ideal in B containing $\mathfrak{n}B$, then $\mathfrak{p}L_S(B)$ is prime.*

We need to show that $L_S(B/\mathfrak{p})$ is a domain. Since $B/\mathfrak{p}$ is a homomorphic image of $B/\mathfrak{n}B$, it suffices to show that $\mathfrak{p}$ extends to a prime ideal in $L_S(B/\mathfrak{n}B)$. By Theorem 7.3.4, the extension of $\mathfrak{p}$ to $U(K[\zeta])$ remains prime. Since $L_S(B/\mathfrak{n}B)$ is an ultrapower of $U(K[\zeta])$, the extension of $\mathfrak{p}$ to the former is again prime by Łos' Theorem. □

To prove that these are well-defined objects, that is to say, independent of the choice of presentation $C = B/I$ (or its localization), we prove (see Exercise 10.5.12) a similar universal property as for ultra-hull:

10.4.2 *Any S-algebra homomorphism $C \to D_\natural$ with $D_\natural$ an ultra-$L(S)$-algebra, extends uniquely to a morphism $L_S(C) \to D_\natural$ of ultra-$L(S)$-algebras. Similarly, any local $L(S)$-algebra homomorphism $T \to D_\natural$ with $D_\natural$ an ultra-local $L(S)$-algebra, extends uniquely to a morphism $L_S(T) \to D_\natural$ of ultra-local $L(S)$-algebras.*

Proposition 10.4.3. *On the category of S-affine algebras, $L_S(\cdot)$ is a difference hull.*

Proof. Let T be a local S-affine algebra (for the global case see Exercise 10.5.13). Clearly, the ultra-Frobenius $\mathbf{F}_\infty$ acts on each $L_S(T)$, making the latter into a difference ring. So remains to show that the canonical map $T \to L_S(T)$ is faithfully flat. By Cohen's structure theorem, $\widehat{S_K}$ is a homomorphic image of $R := K[[\xi]]$. A moment's reflection shows that $L_S(T) = L_R(\widehat{T_K})$, so that by an application of Theorem 6.4.7, we may reduce to the case that $S = R$. By another application of Cohen's structure theorem, T is a homomorphic image of a localization of $R[\zeta]$, and hence without loss of generality, we may assume that T is moreover regular. Flatness of $T \to L_R(T)$ then follows from the Cohen-Macaulay criterion of flatness in the same way as in the proof of Theorem 7.2.2 (see Exercise 10.5.13). □

Big Cohen-Macaulay algebras. For the remainder of this section, S is a complete Noetherian local domain with residue field K. By Theorem 6.4.6, we have a finite extension $R \subseteq S$ (for an appropriate choice of n and $R := K[[\xi]]$ as before). The

Lefschetz hull we will use for S to construct a balanced big Cohen-Macaulay algebra is the relative hull $\mathfrak{L}_R(S)$ (with respect to a fixed Lefschetz hull for R). Let S_w be the approximations of S with respect to this choice of Lefschetz hull, that is to say, S_w are the complete local $K_{p(w)}$-algebras whose ultraproduct is $\mathfrak{L}_R(S)$. By the above discussion, $\mathfrak{L}_R(S)$ is a domain, whence so are almost all S_w. Let $\mathfrak{B}(S)$ be the ultraproduct of the S_w^+, so that $\mathfrak{B}(S)$ is in particular an ultra-$\mathfrak{L}_R(S)$-algebra whence an S-algebra. In Exercise 10.5.14, you are asked to prove:

Theorem 10.4.4. *For each complete Noetherian local domain S with residue field K, the S-algebra $\mathfrak{B}(S)$ is a balanced big Cohen-Macaulay algebra.* □

Theorem 10.4.5 (Monomial Conjecture). *The Monomial Conjecture holds for any Noetherian local ring S of equal characteristic, that is to say, any system of parameters is monomial.*

Proof. I will only explain the equal characteristic zero case; the positive characteristic case is analogous, using instead Theorem 9.4.1. Towards a contradiction, suppose $(x_1,\dots,x_d)$ is a counterexample, that is to say, a system of parameters which fails (9.8) for some k. After taking a complete scalar extension (which preserves the system of parameters), we may assume that S is complete with residue field K. After killing a prime ideal of maximal dimension (which again preserves the system of parameters), we then may assume moreover that S is a domain. The counterexample then also holds in $\mathfrak{B}(S)$, contradicting that $(x_1,\dots,x_d)$ is $\mathfrak{B}(S)$-regular by Theorem 10.4.4. □

As before, we can also define the *B-closure* of an ideal $I \subseteq S$ by the rule $\mathrm{cl}^{\mathfrak{B}}(I) := I\mathfrak{B}(S) \cap S$ and prove that it satisfies the five key properties (see Exercise 9.5.12).

10.5 Exercises

Ex 10.5.1

One can make the choice of ultrapower in Theorem 10.1.1 independent from the particular choices of A-algebra homomorphisms $\varphi_C\colon C \to T$ as follows. Let $\mathbb{W}'$ be the set of all A-algebra homomorphisms $C \to T$ whose domain C is an A-affine subalgebra of S. Define an appropriate ultrafilter on this set, let $T_\natural$ be the ultrapower of T with respect to this ultrafilter, and modify the argument in the proof of the theorem accordingly.

Ex 10.5.2

To obtain embeddings rather than just homomorphisms, prove that the following are equivalent for algebras S and T over a Noetherian ring A:

1. *every finite system of polynomial equations and inequalities with coefficients from A which is solvable in S, is solvable in T;*
2. *given an A-affine subalgebra $C \subseteq S$ and finitely many non-zero elements $c_1,\dots,c_n$ of C there exists an A-algebra homomorphism $C \to T$ sending each c_i to a non-zero element of T;*

3. *there exists an embedding $S \to T_\natural$ of A-algebras into an ultrapower $T_\natural$ of T.*

Ex 10.5.3

Let $(R,\mathfrak{m})$ be a Noetherian local ring. Show that given finitely many congruence relations $f_i \equiv 0 \mod \mathfrak{m}^{c_i}$ with $f_i \in R[t]$ can be turned in to a system of equations, such that the congruences are solvable in $\widehat{R}$ or R if and only if the equations are. Prove the same for a system of equations and negations of equations. Conclude that to admit Artin Approximation is equivalent with either of the following two apparently stronger conditions:

1. *any system of polynomial equations and negations of equations over R which is solvable in $\widehat{R}$ is already solvable in R;*
2. *given some c and a system of equations over R with a solution $\widehat{\mathbf{x}}$ in $\widehat{R}$, we can find a solution $\mathbf{x}$ in R such that $\mathbf{x} \equiv \widehat{\mathbf{x}} \mod \mathfrak{m}^c\widehat{R}$, that is to say, a solution in $\widehat{R}$ can be 'approximated' arbitrarily close by solutions in R.*

The last condition also explains the name of this property.

Ex 10.5.4

Prove 10.1.6.

Ex 10.5.5

Show that a K-algebra endomorphism of $R := K[[\xi]]$ is completely determined by the image of ξ. More generally, if S is a complete local K-algebra, then there is a one-one correspondence between local K-algebra homomorphisms $R \to S$, and tuples in S with entries in the maximal ideal. This is no longer true if S is only quasi-complete, and hence explains why we needed the more elaborate theory using Theorem 10.1.1.

Ex 10.5.6

Show that $(R,\mathfrak{m})$ has the strong Artin Approximation property if and only if the product of all $R/\mathfrak{m}^k$ embeds in some ultrapower of R. Use this to then prove that R has the strong Artin Approximation property if and only if R has the Artin Approximation property and $\widehat{R}$ has the strong Artin Approximation property.

Ex 10.5.7

Show the following more general 'approximating' version of Theorem 10.1.9 by modifying its proof accordingly (see (2) in Exercise 10.5.3): There exists a function $N\colon \mathbb{N}^3 \to \mathbb{N}$ with the following property. Let k be a field and let $(\mathscr{L})$ be a polynomial system of equations in the n unknowns t with coefficients in $k[\xi]$, such that the total degree (with respect to ξ and t) is at most d. If $(\mathscr{L})$ has an approximate solution $\mathbf{x}$ in $R := k[[\xi]]$ modulo $\mathfrak{m}^{N(n,d,c)}R$, then there exists a solution $\mathbf{y}$ in R such that $\mathbf{x} \equiv \mathbf{y}$ modulo $\mathfrak{m}^cR$.

Ex 10.5.8

Prove the following one-nested generalization of [10, Theorem 4.3] (the latter only treats the case $s = 1$): There exists a function $N\colon \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let k be a field and let $(\mathscr{L})$ be a polynomial system of equations in the n unknowns t with coefficients in $A := k[\xi]$ with ξ an n-tuple of indeterminates, such that the total degree (with respect to ξ and t) is at most d. If $(\mathscr{L})$ has an approximate solution $(x_1,\dots,x_n)$ in A

modulo $\mathfrak{m}^{N(n,d,c)}A$ *with* $x_1,\dots,x_l$ *depending only on* $\xi_1,\dots,\xi_s$*, then there exists a solution* $(y_1,\dots,y_n)$ *in* $k[[\xi]]$ *with* $y_1,\dots,y_l$ *depending only on* $\xi_1,\dots,\xi_s$*. Start as always with assuming towards a contradiction that there exist counterexamples* $(\mathscr{L}_m)$ *over* $A_m := K_m[[\xi]]$ *of degree at most d with an approximate solution modulo* $\mathfrak{m}^m A_m$ *whose first l entries belong to* $A'_m := K_m[\xi_1,\dots,\xi_s]$*, but having no solution in* $R_m := K_m[[\xi]]$ *whose first l entries belong to* $R'_m := K_m[[\xi_1,\dots,\xi_s]]$*. Use Proposition 10.3.2 to get a commutative diagram of corresponding Lefschetz hulls*

$$\begin{array}{ccc} R' & \longrightarrow & R \\ \downarrow & & \downarrow \\ L(R') & \longrightarrow & L(R) \end{array} \qquad (10.2)$$

where $R := K[[\xi]]$ *and* $R' = K[[\xi_1,\dots,\xi_s]]$*, and where K is the ultraproduct of the* K_m*. Use the existence of these embeddings in the same way as in the proof of Theorem 10.1.9 to derive the desired contradiction.*

Ex 10.5.9

Give a proof of Corollary 10.2.1.

Ex 10.5.10

Show that the completion of a cyclically pure homomorphism is again cyclically pure.

*Ex 10.5.11

Show that the integral closure $\bar{I}$ *of an ideal I in a local ring* $(S,\mathfrak{m})$ *is equal to the intersection of the integral closures of the* $\mathfrak{m}$*-primary ideals* $I+\mathfrak{m}^n$*. Show how this allows us to reduce to the* $\mathfrak{m}$*-primary case in the proof of Theorem 10.3.4.*

Ex 10.5.12

Show the universal property 10.4.2 of the relative hull.

Ex 10.5.13

Fill in the details of the proof of Proposition 10.4.3.

Ex 10.5.14

To prove Theorem 10.4.4, you first need to show that S has the same dimension as almost all of its approximations S_w*, by an argument similar to the one in Corollary 7.3.3. In particular, almost each approximation of a system of parameters is again a system of parameters. Now apply Theorem 9.4.1.*

Additional exercises

Ex 10.5.15

Show that condition (1) in Theorem 10.1.1 is equivalent with the model-theoretic assertion that T is a model of the positive existential theory of S in the language $\mathscr{L}(A)$ of rings with constant symbols for the elements in A. Similarly, condition (1) in Exercise 10.5.2 is equivalent with T being a model of the (full) existential $\mathscr{L}(A)$-theory of S.

Ex 10.5.16

We can even relax the hypothesis of Theorem 10.1.10 so that the system of equations $(\mathscr{L})$ has only to be of the form $f_1 = \cdots = f_s = 0$ with each $f_i \in A^{\sim}[t]$ of t-degree at most d, and each coefficient of f_i of etale complexity at most d. Namely, given such a more general system, replace each coefficient with a new indeterminate, and add a new Hensel system for that coefficient (with first variable corresponding to the new indeterminate). For this you also will need the uniqueness of a Hensel solution, proved in 6.6.1.

Ex 10.5.17

Generalize the construction of the relative hull on page 163 as follows. Let $S_\natural$ be the ultraproduct of rings S_w, let $B_w := S_w[\zeta]$, and define the relative $S_\natural$-hull *of $B := S_\natural[\zeta]$ as the the ultraproduct of the B_w, denoted $\mathrm{L}_{S_\natural}(B)$. Argue that the relative hull $\mathrm{L}_S(B)$ as defined page 163 is just the relative $S_\natural$-hull of B.*
Show that $\mathrm{L}_{S_\natural}(B)$ satisfies the following universal property: any $S_\natural$-algebra homomorphism $B \to D_\natural$ into an ultra-$S_\natural$-algebra $D_\natural$, extends uniquely to a homomorphism $\mathrm{L}_{S_\natural}(B) \to D_\natural$ of ultra-$S_\natural$-algebras. Define similarly the relative $S_\natural$-hull of an $S_\natural$-affine algebra C (recall that this means by definition—see page 21—that $C \cong B/I$ with I finitely generated), and prove again a universal property. Do the same in case $S_\natural$ is local and T is a local $S_\natural$-affine algebra.

10.6 Project: proof of Hochster-Roberts Theorem

Our goal is to give a different proof of Theorem 10.3.3. By an argument similar to that in the text, we may reduce the problem to complete local domains. Hence let S be an arbitrary complete Noetherian local domain containing the algebraically closed Lefschetz field K. Define a closure operation on S as follows: an element $z \in S$ lies in the *inductive tight closure* $\mathrm{cl}^{ind}(\mathfrak{a})$ of an ideal $\mathfrak{a} \subseteq S$, if there exists a local K-affine subalgebra $C \subseteq S$ containing z, and an ideal $I \subseteq C$, such that $\mathfrak{a} = IS$ and $z \in \mathrm{cl}_C(I)$ (where we take tight closure $\mathrm{cl}_C(\cdot)$ in C in the sense of Chapter 9). Show that weak persistence holds:

10.6.1 *If $S \to T$ is an injective local K-algebra homomorphism of complete Noetherian local domains, then $\mathrm{cl}^{ind}(\mathfrak{a}) \subseteq \mathrm{cl}^{ind}(\mathfrak{a}T)$ for all $\mathfrak{a} \subseteq S$.*

Call S *inductively F-regular*, if every ideal in S is equal to its own inductive tight closure. To prove the Hochster-Roberts Theorem, we again split the proof into two parts. The easy part is:

10.6.2 *If $S \to T$ is cyclically pure, and T is inductively F-regular, then so is S.*

To prove the analogue of 10.2.2, we need to understand how arbitrary K-algebras are approximated by K-affine algebras. You may take the following theorem for granted, but see below for how to prove it.

10.6.3 *Let S be a complete Noetherian local domain containing K, and let C be a local K-affine subalgebra of S. Then the embedding $C \subseteq S$ factors through a local K-affine domain D, satisfying the following additional conditions*

1. *if S is regular, then we may take D to be regular too;*
2. *if $x_1, \dots, x_d \in C$ are a system of parameters in S, then D can be chosen in such way that $(x_1, \dots, x_d)$ is part of a system of parameters in D.*

The dimension of D will in general be larger than d, the dimension of S. Note also that we are not requiring that the canonical map $D \to S$ has to be injective. After reduction to the case $S = R := K[[\xi]]$, assertion (1) follows from the Artin-Rotthaus theorem ([3])—a stronger form of p-desingularization, of which also Theorem 10.1.4 is an immediate consequence. To prove (2), apply Theorem 6.4.6 to S to get a finite extension $R \subseteq S$ sending ξ_i to x_i, then apply [3] to obtain a finite extension $D' \subseteq D$ of K-affine algebras with D' flat over A (and regular), and a factorization $C \to D \to S$. Using 10.6.3, derive the analogues of 10.2.2:

10.6.4 *Any regular local ring containing K is inductively F-regular.*

10.6.5 *If S is a complete Noetherian local domain containing K, then Colon Capturing holds in S: if $(x_1, \dots, x_d)$ is a system of parameters, then $(I_k : x_{k+1}) \subseteq \mathrm{cl}^{ind}(I_k)$ for every k, where $I_k := (x_1, \dots, x_k)S$.*

To conlcude, combine all these results to give an alternative proof of the Hochster-Roberts theorem.

Chapter 11
Cataproducts

11.1 Cataproducts

Recall from 1.4.7 that the ultraproduct of local rings of bounded embedding dimension is again a local ring of finite embedding dimension. In this chapter, we will be mainly concerned with the following subclass.

Definition 11.1.1 (Ultra-Noetherian ring). We call a local ring $R_\natural$ *ultra-Noetherian* if it is the ultraproduct of Noetherian local rings of bounded embedding dimension, that is to say, of Noetherian local rings R_w such that the embedding dimension of R_w is at most e, for some e independent of w.

The Noetherian local rings R_w will be called *approximations* of $R_\natural$ (note the more liberal use of this term than in the previous chapters, which, however, should not cause any confusion). It is important to keep in mind that approximations are not uniquely determined by $R_\natural$.

We introduced the *geometric dimension* of a Noetherian local ring in our study of Krull dimension, see Theorem 3.3.2. This notion carries over naturally to any local ring $(S,\mathfrak{n})$ of finite embedding dimension, namely, $\operatorname{geodim}(S)$ is the least number d of elements $x_1,\dots,x_d$ in its maximal ideal such that $S/(x_1,\dots,x_d)S$ is Artinian, that is to say, such that $(x_1,\dots,x_d)S$ is $\mathfrak{n}$-primary. Any tuple $(x_1,\dots,x_d)$ with this property is then called a *system of parameters* of R.[1] Any element of R which belongs to some system of parameters will be called a *parameter*. We immediately get:

11.1.2 *The geometric dimension of a local ring is at most its embedding dimension, whence in particular is finite for any ultra-Noetherian local ring.*

By Exercise 11.3.1, the geometric dimension of an ultra-Noetherian local ring is larger than or equal to the (geometric) dimension of its Noetherian approximations, and this inequality can be strict (for an example see Exercise 11.3.3). To study this phenomenon as well as further properties of ultra-Noetherian local rings, we first introduce a new kind of product:

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[1] In [56, 61, 63] such a tuple was called *generic*.

Cataproducts. In 1.4.13 we saw that most ultra-Noetherian rings are not Noetherian (in model-theoretic terms this means that the class of Noetherian local rings of fixed embedding dimension is not first order definable; see Exercise 1.5.17). However, there is a Noetherian local ring closely associated to any ultra-Noetherian local ring. Fix an ultra-Noetherian local ring

$$R_\natural := \operatorname*{ulim}_{w\to\infty} R_w,$$

and define the *cataproduct* of the R_w as the separated quotient of $R_\natural$, that is to say,

$$R_\sharp := R_\natural/\mathfrak{I}_{R_\natural}.$$

If all R_w are equal to a fixed Noetherian local ring $(R,\mathfrak{m})$, then we call $R_\sharp$ the *catapower* of R. In this case, the natural (diagonal) embedding $R \to R_\natural$ induces a natural homomorphism $R \to R_\sharp$. Since $\mathfrak{m}R_\natural$ is the maximal ideal of $R_\natural$, likewise, $\mathfrak{m}R_\sharp$ is the maximal ideal of $R_\sharp$. The relationship between the rings R_w and their cataproduct $R_\sharp$ is much less strong than in the ultraproduct case, as the following example illustrates.

11.1.3 *The catapower of a Noetherian local ring $(R,\mathfrak{m})$ is isomorphic to the cataproduct of the Artinian local rings $R/\mathfrak{m}^n$.*

Indeed, if $R_\natural$ and $S_\natural$ denote the ultrapower of R and the ultraproduct of the $R/\mathfrak{m}^n$ respectively, then we get a surjective homomorphism $R_\natural \to S_\natural$. However, any element in the kernel of this homomorphism is an infinitesimal, so that the induced homomorphism $R_\sharp \to S_\sharp$ is an isomorphism. □

Nonetheless, as before, we will still refer to the R_w as *approximations* of $R_\sharp$, and given an element $x \in R_\sharp$, we call any choice of elements $x_w \in R_w$ whose ultraproduct is a lifting of x to $R_\natural$, an *approximation* of x.

Theorem 11.1.4. *The cataproduct of Noetherian local rings of bounded embedding dimension is complete and Noetherian.*

Proof. In almost all our applications,[2] the ultrafilter lives on a countable index set $\mathbb{W}$, but nowhere did we exclude larger cardinalities. For simplicity, however, I will assume countability, and treat the general case in a separate remark below. Hence, we may assume $\mathbb{W} = \mathbb{N}$. Let $(R_\natural,\mathfrak{m})$ be the ultraproduct of Noetherian local rings R_w of embedding dimension at most e. It follows that $R_\natural$ too has embedding dimension at most e. Let us first show that $R_\natural$ is quasi-complete (note that it is not Hausdorff in general, because $\mathfrak{I}_{R_\natural} \neq 0$). To this end, we only need to consider by 6.2.1 a Cauchy sequence $\mathbf{a}$ in $R_\natural$ such that $\mathbf{a}(n) \equiv \mathbf{a}(n+1) \mod \mathfrak{m}^n R_\natural$. Choose approximations $\mathbf{a}_w(n) \in R_w$ such that

$$\mathbf{a}(n) = \operatorname*{ulim}_{w\to\infty} \mathbf{a}_w(n)$$

for each $n \in \mathbb{N}$. By Łos' Theorem, we have for a fixed n that

[2] A notable exception is the construction of a Lefschetz hull given in Theorem 10.1.5.

$$\mathbf{a}_w(n) \equiv \mathbf{a}_w(n+1) \mod \mathfrak{m}_w^n \tag{11.1}$$

for almost all w, say, for all w in D_n. I claim that we can modify the $\mathbf{a}_w(n)$ in such way that (11.1) holds for all n and all w. More precisely, for each n there exists an approximation $\tilde{\mathbf{a}}_w(n)$ of $\mathbf{a}(n)$, such that

$$\tilde{\mathbf{a}}_w(n) \equiv \tilde{\mathbf{a}}_w(n+1) \mod \mathfrak{m}_w^n \tag{11.2}$$

for all n and w. We will construct the $\tilde{\mathbf{a}}_w(n)$ recursively from the $\mathbf{a}_w(n)$. When $n = 0$, no modification is required (since by assumption $\mathfrak{m}_w^0 = R_w$), and hence we set $\tilde{\mathbf{a}}_w(0) := \mathbf{a}_w(0)$ and $\tilde{\mathbf{a}}_w(1) := \mathbf{a}_w(1)$. So assume we have defined already the $\tilde{\mathbf{a}}_w(j)$ for $j \leq n$ such that (11.2) holds for all w. Now, for those w for which (11.1) fails for some $j \leq n$, that is to say, for $w \notin (D_0 \cup \cdots \cup D_n)$, let $\tilde{\mathbf{a}}_w(n+1)$ be equal to $\tilde{\mathbf{a}}_w(n)$; for the remaining w, that is to say, for almost all w, we make no changes: $\tilde{\mathbf{a}}_w(n+1) := \mathbf{a}_w(n+1)$. It is now easily seen that (11.2) holds for all w, and $\tilde{\mathbf{a}}_w(n)$ is another approximation of $\mathbf{a}(n)$, for all n, thus establishing our claim.

So we may assume (11.1) holds for all j and w. Define $b := \operatorname{ulim} \mathbf{a}_w(w)$. Since $\mathbf{a}_w(w) \equiv \mathbf{a}_w(n) \mod \mathfrak{m}_w^n$ for all $w \geq n$, Łos' Theorem yields $b \equiv \mathbf{a}(n) \mod \mathfrak{m}^n R_\natural$, showing that b is a limit of $\mathbf{a}$.

Since the cataproduct $R_\sharp$ of the R_w is a homomorphic image of $R_\natural$, it is again quasi-complete by 6.1.5. By construction, $R_\sharp$ is Hausdorff and therefore even complete. Since $R_\sharp$ has finite embedding dimension, it is therefore Noetherian by Theorem 6.4.2 (or, in mixed characteristic, by Theorem 6.4.4). □

Remark 11.1.5. In order for the above argument to work for arbitrary index sets $\mathbb{W}$, we need to make one additional assumption on the ultrafilter $\mathscr{U}$: it needs to be *countably incomplete*, meaning that there exists a function $f\colon \mathbb{W} \to \mathbb{N}$ such that for each n, almost all $f(w)$ are greater than or equal to n. Of course, if $\mathbb{W} = \mathbb{N}$ such a function exists, namely the identity will already work. Countably incomplete ultrafilters exist on any infinite set. In fact, it is a strong set-theoretic condition to assume that not every ultrafilter is countable incomplete! Now, the only place where we need this assumption is to build the limit element b. This time we should take it to be the ultraproduct of the $\mathbf{a}_w(f(w))$. The reader can verify that this one modification makes the proof work for any index set.

Proposition 11.1.6. *Let $R_\natural$ be an ultra-Noetherian local ring with separated quotient $R_\sharp$. For any ideal $I \subseteq R_\natural$, its $\mathfrak{m}$-adic closure is equal to $I + \mathfrak{I}_{R_\natural}$. In particular, the separated quotient of $R_\natural/I$ is $R_\sharp/IR_\sharp$.*

Proof. It suffices to show the first assertion. Clearly, $I + \mathfrak{I}_{R_\natural}$ is contained in the $\mathfrak{m}$-adic closure of I. To prove the other inclusion, assume a lies in the $\mathfrak{m}$-adic closure of I. Hence its image in $R_\sharp$ lies in the $\mathfrak{m}$-adic closure of $IR_\sharp$, and this is just $IR_\sharp$ by Theorem 1.4.11, since $R_\sharp$ is Noetherian by Theorem 11.1.4. Therefore, a lies in $IR_\sharp \cap R_\natural = I + \mathfrak{I}_{R_\natural}$. □

From a model-theoretic point of view, Łos' Theorem explicates which properties are preserved in ultraproducts, to wit, any first-order property. Since cataproducts are residue rings, they, therefore, inherit any positive first-order property from their components (Exercise 11.3.12). However, we do not want to derive properties of the cataproduct via a syntactical analysis, but instead use an algebraic approach. The first issue to address is the way dimension behaves under cataproducts. We already mentioned that the geometric dimension of an ultra-Noetherian ring can exceed that of its components (see Exercise 11.3.3). The same phenomenon occurs for cataproducts because we have:

11.1.7 *For an ultra-Noetherian local ring $(R_\natural, \mathfrak{m})$ its geometric dimension is equal to the dimension of its separated quotient $R_\sharp$.*

Let $\mathbf{x} := (x_1, \ldots, x_d)$ be a system of parameters in $R_\natural$ (recall that this means that $(x_1, \ldots, x_d)R_\natural$ is an $\mathfrak{m}$-primary ideal, with d the geometric dimension of $R_\natural$). So $S_\natural := R_\natural/\mathbf{x}R_\natural$ is an Artinian local ring, whence must be equal to its separated quotient $S_\sharp$ (see Exercise 11.3.5). By Proposition 11.1.6, we have $S_\sharp = R_\sharp/\mathbf{x}R_\sharp$, showing that $R_\sharp$ has geometric dimension at most d. Since $R_\sharp$ is Noetherian by Theorem 11.1.4, it has dimension at most d by Theorem 3.3.2. Moreover, we may reverse the argument, for if $S_\sharp$ is Artinian, then necessarily $S_\natural = S_\sharp$ (again by Exercise 11.3.5). □

To investigate when the dimension of a cataproduct is equal to the dimension of almost all of its approximations, we need to introduce a new invariant.

Definition 11.1.8 (Parameter degree). Given a local ring $(R, \mathfrak{m})$ of finite embedding dimension, its *parameter degree*, denoted $\operatorname{pardeg}(R)$, is by definition the least possible length of the residue rings $R/\mathbf{x}R$, where $\mathbf{x}$ runs over all systems of parameters.

Note that by definition of geometric dimension, the parameter degree of R is always finite. Closely related to this invariant, is the *parameter degree* $\operatorname{pdeg}_R(x)$ of an element $x \in R$, defined as follows: if x is a unit, then we set $\operatorname{pdeg}_R(x)$ equal to zero, and if x is not a parameter, then we set $\operatorname{pdeg}_R(x)$ equal to ∞; in the remaining case, we let $\operatorname{pdeg}_R(x)$ be the parameter degree of R/xR. In Exercise 11.3.6, you are asked to prove:

11.1.9 *Let R be a d-dimensional Noetherian local ring, or more generally, a local ring of geometric dimension d, and let $x \in R$. Then the parameter degree of x is equal to the minimal length of any residue ring of the form $R/(xR + I)$, where I runs over all ideals generated by $d-1$ non-units.*

In [63, Proposition 2.2 and Theorem 3.4] we prove the following generalization of Theorem 11.1.4: the completion of a local ring R of finite embedding dimension is Noetherian, and has dimension equal to the geometric dimension d of R; moreover, both rings have the same Hilbert polynomial whence their Hilbert dimension is also d by Theorem 3.3.2. We define the *multipliciy* of R to be the leading coefficient of its Hilbert polynomial times $d!$ (this coincides with the classical definition in the Noetherian case). The multiplicity of R is always at most its parameter degree, and provided R is Noetherian with infinite residue field, both are equal if and only if R is Cohen-Macaulay (see [60, Lemma 3.3] for the Noetherian case, and [63, Lemma 6.10] for a generalization).

Theorem 11.1.10. *Let R_w be d-dimensional Noetherian local rings of embedding dimension at most e. Their cataproduct $R_\natural$ has dimension d if and only if almost all R_w have bounded parameter degree (that is to say,* $\operatorname{pardeg}(R_w) \leq r$ *for some r and for almost all w).*

Proof. Assume almost all R_w have parameter degree at most r, so that their exists a d-tuple $\mathbf{x}_w$ in R_w such that $S_w := R_w/\mathbf{x}_w R_w$ has length at most r. Hence the cataproduct $S_\natural$ has length at most r by Exercise 11.3.5. By Proposition 11.1.6, the cataproduct $S_\natural$ is isomorphic to $R_\natural/\mathbf{x}R_\natural$, where $\mathbf{x}$ is the ultraproduct of the $\mathbf{x}_w$. Hence $R_\natural$, being Noetherian by Theorem 11.1.4, has dimension at most d by Theorem 3.3.2, whence equal to d by Exercise 11.3.1.

Conversely, suppose $R_\natural$ has dimension d, and let $\mathbf{x}$ be a system of parameters of $R_\natural$. Let r be the length of $R_\natural/\mathbf{x}R_\natural$. Let $\mathbf{x}_w$ be approximations of $\mathbf{x}$. By Exercise 11.3.5, almost all $R_w/\mathbf{x}_w R_w$ have length at most r. It follows that almost each $\mathbf{x}_w$ is a system of parameters, and hence that R_w has parameter degree at most r. ☐

Catapowers. We can apply this to catapowers. In the next result, the first statement is immediate from Theorem 11.1.4 and Proposition 11.1.6; the second follows immediately from Theorem 11.1.10.

Corollary 11.1.11. *Let R be a Noetherian local ring with catapower $R_\natural$. For any ideal $I \subseteq R$, the catapower of R/I is $R_\natural/IR_\natural$. Moreover, R and $R_\natural$ have the same dimension.* ☐

Corollary 11.1.12. *The catapower of a regular local ring is again regular (of the same dimension).*

Proof. Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring. If $d = 0$, then R is a field, and $R_\natural$ is equal to the ultrapower $R_\natural$ whence a field. So we may assume $d > 0$. Let x be a minimal generator of $\mathfrak{m}$. Hence R/xR is regular of dimension $d-1$, so that by induction, its catapower is also regular of dimension $d-1$. But this catapower is just $R_\natural/xR_\natural$ by Corollary 11.1.11. It follows that $\mathfrak{m}R_\natural$ is generated by at most d elements. Since $R_\natural$ has dimension d by Corollary 11.1.11, it is regular. ☐

Flatness of catapowers. To further explore the connection between a ring and its catapower, we require a flatness result.

Theorem 11.1.13. *For each Noetherian local ring R, the induced homomorphism $R \to R_\natural$ into its catapower $R_\natural$ is faithfully flat.*

Proof. Since $R \to R_\natural$ is local, we only need to verify flatness. Moreover, since $R_\natural$ is complete by Theorem 11.1.4, we get $(\widehat{R})_\natural = R_\natural$ by a double application of 11.1.3, whence an induced homomorphism $\widehat{R} \to R_\natural$. As $R \to \widehat{R}$ is flat by Theorem 6.3.4, we only need to show that $\widehat{R} \to R_\natural$ is flat, and hence we may already assume that R is complete.

Suppose first that R is moreover regular. By Corollary 11.1.12, so is then $R_\natural$. In particular, the generators of $\mathfrak{m}$ are $R_\natural$-regular, so that $R_\natural$ is flat over R by Theorem 5.6.10. For R arbitrary, note that $R = S/I$ for some regular local ring S and some ideal $I \subseteq S$ by Theorems 6.4.2 and 6.4.4. By our previous argument the ultrapower $S_\sharp$ of S is flat, whence so is $R = S/I \to S_\sharp/IS_\sharp = R_\sharp$ by 5.2.3 (where we used Corollary 11.1.11 for the last equality). □

The reader who is willing to use some heavier commutative algebra can prove the following stronger fact:

Corollary 11.1.14. *If R is an excellent local ring, then the natural map $R \to R_\sharp$ is regular.*

Proof. For the notion of excellence and regular maps, see [41, §32]. By Theorem 11.1.13, the map $R \to R_\sharp$ is flat. It is also unramified, in the sense that $\mathfrak{m}R_\sharp$ is the maximal ideal of $R_\sharp$. If R is a field k, then $R_\sharp$ is just its ultrapower $k_\natural$. Using Maclane's criterion for separability, one shows that the extension $k \to k_\natural$ is separable (Exercise 1.5.16). For R arbitrary, this shows in view of Corollary 11.1.11 that $R \to R_\sharp$ induces a separable extension of residue fields. Hence $R \to R_\sharp$ is formally smooth by [41, Theorem 28.10], whence regular by [1]. □

We can now generalize the fact that catapowers preserve regularity (Corollary 11.1.12) to:

Corollary 11.1.15. *If R is an excellent local ring, then R is regular, normal, reduced or Cohen-Macaulay, if and only if $R_\sharp$ is.*

Proof. Immediate from Corollary 11.1.14 and the fact that regular maps preserve these properties in either direction (see [41, Theorem 32.2]). □

Corollary 11.1.16. *If R is a complete Noetherian local domain, then so is its catapower $R_\sharp$.*

Proof. Let S be the normalization of R (that is to say, the integral closure of R inside its field of fractions). By [41, §33], the extension $R \subseteq S$ is finite, and S is also a complete Noetherian local ring. I claim that the induced homomorphism of catapowers $R_\sharp \to S_\sharp$ is again finite and injective. Since $S_\sharp$ is normal by Corollary 11.1.15, it is a domain, whence so it its subring $R_\sharp$.

So remains to prove the claim. By the weak Artin-Rees Lemma applied to the finite R-module S (see Exercise 11.3.4), we can find for each m a bound $e(m)$ such that $\mathfrak{m}^{e(m)}S \cap R \subseteq \mathfrak{m}^m$ for all m. Let $\mathfrak{n}$ be the maximal ideal of S. Since $S/\mathfrak{m}S$ is finite over $R/\mathfrak{m}$ by base change, it is Artinian, and hence $\mathfrak{n}^l \subseteq \mathfrak{m}S$ for some l. Together with the weak Artin-Rees bound, this yields

$$\mathfrak{n}^{le(m)} \cap R \subseteq \mathfrak{m}^m \tag{11.3}$$

for all m.

Let $S_\natural$ be the ultrapower of S, so that $S_\natural$ is a finite $R_\natural$-module. The inclusion $\mathfrak{I}_{R_\natural} \subseteq \mathfrak{I}_{S_\natural} \cap R_\natural$ is clear, and we need to prove the converse, for then $R_\sharp \to S_\sharp$ will be injective. So let $z \in R_\natural$ be be such that it is an infinitesimal in $S_\natural$, and let $z_w \in R$ be approximations of z. Fix some m. Since $z \in \mathfrak{n}^{le(m)}S_\natural$, by Łos' Theorem $z_w \in \mathfrak{n}^{le(m)}$ for almost all w, whence $z_w \in \mathfrak{m}^m$ by (11.3). By another application of Łos' Theorem, we get $z \in \mathfrak{m}^m R_\natural$, and since this holds for all m, we get $z \in \mathfrak{I}_{R_\natural}$, as we wanted to show. □

Theorem 11.1.17. *Let R be a Noetherian local ring of equal characteristic, with residue field k, and let $R_\sharp$ and $k_\sharp$ be their respective catapowers. Then $R_\sharp$ is isomorphic to the complete scalar extension $\widehat{R_{k_\sharp}}$ over $k_\sharp$.*

Proof. Since a ring and its completion have the same complete scalar extensions, we may assume R is complete. By Cohen's structure theorem, R is a homomorphic image of a power series ring $k[[\xi]]$, with ξ an n-tuple of indeterminates. Since complete scalar extensions (by (6.6)) as well as catapowers (Corollary 11.1.11) commute with homomorphic images, we may assume $R = k[[\xi]]$. So remains to show that $R_\sharp \cong k_\sharp[[\xi]]$. However, this is clear by Theorem 6.4.5, since $R_\sharp$ is regular by Corollary 11.1.12, with residue field $k_\sharp$, having dimension n by Corollary 11.1.11. □

11.2 Uniform behavior

In Chapter 7 we amply illustrated how ultraproducts can be used to prove several uniformity results. This section contains more results derived by this technique.

Weak Artin-Rees. The Artin-Rees lemma is an important tool in commutative algebra, especially when using 'topological' arguments. Its proof is not that hard, but we have not given it in these notes. However, there is a weaker form of Artin-Rees, which is often really the only property one uses (a notable exception is the proof of Theorem 3.3.2) and which we can now prove easily by non-standard methods.

Theorem 11.2.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, and let $\mathfrak{a} \subseteq R$ be an ideal. For each m, there exists $e := e(\mathfrak{a},m)$ such that*

$$\mathfrak{a} \cap \mathfrak{m}^e \subseteq \mathfrak{m}^m\mathfrak{a}.$$

Proof. Suppose not, so that for some m, none of the intersections $\mathfrak{a}\cap\mathfrak{m}^n$ is contained in $\mathfrak{m}^m\mathfrak{a}$. Hence we can find elements $a_n \in \mathfrak{a}\cap\mathfrak{m}^n$ outside $\mathfrak{m}^m\mathfrak{a}$. Let $R_\natural$ and $R_\sharp$ be the respective ultrapower and catapower of R. The canonical homomorphisms $R \to R_\natural$ and $R \to R_\sharp$ are both flat by Corollary 5.6.3 and Theorem 11.1.13 respectively. Since $R_\sharp = R_\natural/\mathfrak{I}_R$, the intersection criterion, Theorem 5.6.5, yields $\mathfrak{a}R_\natural \cap \mathfrak{I}_R = \mathfrak{a}\mathfrak{I}_R$. Let a be the ultraproduct of the a_n, so that by Łos' Theorem, $a \in \mathfrak{a}R_\natural \cap \mathfrak{I}_R = \mathfrak{a}\mathfrak{I}_R$. The latter ideal is in particular contained in $\mathfrak{a}\mathfrak{m}^m R_\natural$, and hence by Łos' Theorem once more, $a_n \in \mathfrak{m}^m\mathfrak{a}$ for almost all n, contradiction. □

Uniform arithmetic in a complete Noetherian local ring. In what follows, our invariants are allowed to take values in $\overline{\mathbb{N}} := \mathbb{N}\cup\{\infty\}$. To an n-ary $\mathbb{N}$-valued function $F\colon \mathbb{N}^n \to \mathbb{N}$, we associate its *extension at infinity*, defined as the map $\bar{F}\colon \overline{\mathbb{N}}^n \to \overline{\mathbb{N}}$ sending any tuple outside $\mathbb{N}^n$ to ∞. Any such extended map will be called a *numerical function*. By the *order* $\operatorname{ord}_R(x)$ of an element x in a local ring $(R,\mathfrak{m})$ we mean the supremum of all m such that $x \in \mathfrak{m}^m$ (so that in particular $\operatorname{ord}_R(x) = \infty$ if and only if $x \in \mathfrak{I}_R$; in the terminology of page 91, the order of x is the negative logarithm of its adic norm).

Theorem 11.2.2. *A complete Noetherian local ring R is a domain if and only if there exists a binary function F such that*

$$\operatorname{ord}_R(xy) \leq \bar{F}(\operatorname{ord}_R(x), \operatorname{ord}_R(y)) \tag{11.4}$$

for all $x, y \in R$.

Proof. Assume first that (11.4) holds for some F. If x and y are non-zero, then their order is finite by Theorem 1.4.11. Hence $\bar{F}(\operatorname{ord}(x), \operatorname{ord}(y))$ is finite by definition of F. In particular, xy must be non-zero, showing that R is a domain.

Conversely, assume towards a contradiction that no such function F can be defined on a pair $(a,b) \in \mathbb{N}^2$. This implies that there exist for each n, elements x_n and y_n in R of order at most a and b respectively, but such that their product $x_n y_n$ has order at least n. Let $R_\natural$ and $R_\sharp$ be the ultrapower and catapower of R respectively, and let x and y be the ultraproducts of x_n and y_n respectively. It follows from Łos' Theorem that $\operatorname{ord}_{R_\natural}(x) \leq a$ and $\operatorname{ord}_{R_\natural}(y) \leq b$, and hence in particular, x and y are non-zero in $R_\sharp$. By Corollary 11.1.16, the catapower $R_\sharp$ is again a domain. In particular, xy is a non-zero element in $R_\sharp$, and hence has finite order, say, c, by Theorem 1.4.11. However, then also $\operatorname{ord}_{R_\natural}(xy) = c$ whence $\operatorname{ord}_R(x_n y_n) = c$ for almost all n by Łos' Theorem, contradiction. □

Remark 11.2.3. Theorem 11.2.2 is classically proven by a valuation argument. By [71, Theorem 3.4] and [35, Proposition 2.2], we may take F linear, or rather, of the form $F(a,b) := c\max\{a,b\}$, for some $c \in \mathbb{N}$ (one usually expresses this by saying that R has *c-bounded multiplication*.

Theorem 11.2.4. *A d-dimensional Noetherian local ring $(R, \mathfrak{m})$ is Cohen-Macaulay if and only if there exists a binary function G such that*

$$\operatorname{ord}_{R/I}(xy) \leq \bar{G}(\operatorname{pdeg}_{R/I}(x), \operatorname{ord}_{R/I}(y)) \tag{11.5}$$

for all $x, y \in R$ and all ideals $I \subseteq R$ generated by part of a system of parameters of length $d-1$.

Proof. Suppose a function G satisfying (11.5) exists, and let $(z_1, \dots, z_d)$ be a system of parameters in R. Fix some i and let $y \in (J : z_{i+1})$ with $J := (z_1, \dots, z_i)R$. We need to show that $y \in J$. For each m, let $I_m := J + (z_{i+2}^m, \dots, z_d^m)R$, and put $x := z_{i+1}$. Since $xy \in J \subseteq I_m$, the left hand side in (11.5) for $I = I_m$ is infinite, whence so must the right hand side be. However, x is a parameter in R/I_m, and therefore has finite parameter degree. Hence, the second argument of $\bar{G}$ must be infinite, that is to say, $\operatorname{ord}_{R/I_m}(y) = \infty$. In other words, $y \in I_m$, and since this holds for all m, we get $y \in J$ by Theorem 1.4.11, as we wanted to show.

Conversely, towards a contradiction, suppose R is Cohen-Macaulay but no such function G can be defined on the pair $(a,b) \in \mathbb{N}^2$. This means that there exist elements $x_n, y_n \in R$ and a $d-1$-tuple $\mathbf{z}_n$ which is part of a system of parameters in R, such that $\mathrm{pdeg}_{S_n}(x_n) \leq a$ and $\mathrm{ord}_{S_n}(y_n) \leq b$, but $x_n y_n$ has order at least n in $S_n := R/\mathbf{z}_n R$. Let $R_\natural$ and $R_\sharp$ be the respective ultrapower and catapower of R. Since R is Cohen-Macaulay, so is $R_\sharp$ by Corollary 11.1.15 (or Exercise 11.3.7). Let x, y and $\mathbf{z}$ be the ultraproduct of the x_n, y_n and $\mathbf{z}_n$ respectively. By Proposition 11.1.6, the cataproduct of the S_n is equal to $S_\sharp := R_\sharp/\mathbf{z}R_\sharp$. Since each S_n has dimension one, and parameter degree at most a by assumption on x_n, the dimension of $S_\sharp$ is again one by Theorem 11.1.10. Since $R_\sharp$ has dimension d by 11.1.7, the $d-1$-tuple $\mathbf{z}$ is part of a system of parameters in $R_\sharp$, whence is $R_\sharp$-regular by Theorem 4.2.6. This in turn implies that $S_\sharp = R_\sharp/\mathbf{z}R_\sharp$ is Cohen-Macaulay. Moreover, by Łos' Theorem, y has order b in $R_\natural/\mathbf{z}R_\natural$ whence also in $S_\sharp$, and x has parameter degree a in $S_\sharp$. In particular, x is a parameter in $S_\sharp$ whence $S_\sharp$-regular. On the other hand, by Łos' Theorem, xy is an infinitesimal in $R_\natural/\mathbf{z}R_\natural$, whence zero in $S_\sharp$. Since x is $S_\sharp$-regular, y is zero in $S_\sharp$, contradicting that its order in that ring is b. □

11.3 Exercises

Ex 11.3.1

Let R be an ultraproduct of d-dimensional Noetherian local rings of embedding dimension at most e, and let δ be its geometric dimension. Show that $d \leq \delta \leq e$.

Ex 11.3.2

Let $R_\natural$ be an ultra-Noetherian ring and $R_\sharp$ its separated quotient. Show that $x \in R_\natural$ is a parameter if and only if its image in $R_\sharp$ is a parameter if and only if it is not contained in any prime ideal of $R_\natural$ obtained as the pre-image of a maximal dimensional prime ideal of $R_\sharp$.

Ex 11.3.3

Let $R_n := K[[\xi]]/\xi^n K[\xi]$ with ξ a single indeterminate over the field K. Show that their ultraproduct $R_\natural$ has geometric dimension at least one.

Ex 11.3.4

Given finitely generated modules $N \subseteq M$ over a Noetherian local ring $(R,\mathfrak{m})$, apply Exercise 5.7.8 to the module M/N and use Theorem 11.2.1 to show that for each m, there exists $e := e(N,M,m)$ such that $N \cap \mathfrak{m}^e M \subseteq \mathfrak{m}^m N$.

Ex 11.3.5

Show that the separated quotient of a local ring of finite embedding dimension is Artinian if and only if the ring itself is Artinian. More generally, show that the ultraproduct of local rings R_w is Artinian of length l if and only if the cataproduct is Artinian of length l if and only if almost all R_w are Artinian of length l (see also Exercise 1.5.10).

Ex 11.3.6

Prove 11.1.9.

Ex 11.3.7

Prove without using Corollary 11.1.15, but relying only on Theorem 11.1.13, that a Noetherian local ring is Cohen-Macaulay if and only if its catapower is.

Ex 11.3.8

Show that a Noetherian local ring R admits a function satisfying (11.4) if and only if its completion does. Use this to show that one can weaken the assumption on R in Theorem 11.2.2 from being a 'complete domain' to being analytically irreducible *(meaning that its completion is a domain).*

Ex 11.3.9

Show that if R_w are domains admitting the same function F satisfying (11.4), then so does their cataproduct, and hence the cataproduct is in particular a domain. Show by a counterexample that the cataproduct of complete Noetherian local domains of fixed dimension and parmeter degree is not necesarily a domain.

*Ex 11.3.10

Show that a complete Noetherian local ring is a domain if and only if there is a unary function H such that $\operatorname{pdeg}_R(x) \leq \bar{H}(\operatorname{ord}_R(x))$ for all $x \in R$.

Additional exercises.

Ex 11.3.11

Prove the following converse of Theorem 11.2.1: if R is a coherent local ring (see Theorem 5.6.4) such that for each finitely generated ideal $\mathfrak{a} \subseteq R$ and each m, there exists some $e := e(\mathfrak{a}, m)$ such that $\mathfrak{a} \cap \mathfrak{m}^e \subseteq \mathfrak{a}^m \mathfrak{m}$, then R is Noetherian. You will also need the flatness criterion from Theorem 5.6.5 and the Noetherianity criterion from Corollary 5.3.6.

Ex 11.3.12

Let φ be a positive sentence in the language of rings, that is to say, equivalent with a quantification of a disjunction of systems of polynomial equations. Show that if φ holds for almost all R_w, then it holds for their cataproduct $R_\sharp$.

Ex 11.3.13

Given a Noetherian local ring R, show that R is regular if and only if $\operatorname{ord}_R(x) = \operatorname{pdeg}_R(x)$ for all $x \in R$.

Chapter 12
Protoproducts

In Chapter 7, we used ultraproducts to derive uniform bounds for various algebraic operations, where the bounds are given in terms of the degrees of the polynomials involved. This was done by constructing a faithfully flat embedding of the polynomial ring A into an ultraproduct $\mathsf{U}(A)$ of polynomial rings, called its ultra-hull. Moreover, A is characterized as the subring of $\mathsf{U}(A)$ of all elements of finite degree. In this chapter, we want to put these uniformity results in a more general context, by replacing the degree on A by what we will call a *proto-grading*. However, as the notion of ultra-hull is no longer available, we must replace the latter by the ultrapower $A_\natural$. Moreover, there now may be elements of finite proto-grading in $A_\natural$ outside A, leading to the notion of the *protopower* $A_\flat$ of A, sitting in between A and its ultrapower, and these embeddings may or may not be (faithfully) flat. The existence of uniform bounds in terms of the proto-grading follow from good properties of this protopower. This can be extended to several rings simultaneously by using protoproducts instead.

12.1 Protoproducts

Whereas as cataproducts are homomorphic images of ultraproducts, protoproducts will be subrings. To define them, we need to formalize the notion of the degree of a polynomial.

Proto-graded rings. By a *pre-proto-grading* $\Gamma_\bullet(A)$ on a ring A, we mean an ascending chain of subsets

$$\Gamma_0(A) \subseteq \Gamma_1(A) \subseteq \Gamma_2(A) \subseteq \dots$$

and a unary function $F\colon \mathbb{N} \to \mathbb{N}$ such that

$$\Gamma_n(A) + \Gamma_m(A) \subseteq \Gamma_{F(n+m)}(A) \quad \text{and} \quad \Gamma_n(A) \cdot \Gamma_m(A) \subseteq \Gamma_{F(n+m)}(A)$$

for all $n, m \in \mathbb{N}$, and such that for any unit $u \in A$, if u lies in $\Gamma_n(A)$, then its inverse lies in $\Gamma_{F(n)}(A)$. The following terminology will prove to be quite convenient when discussing proto-gradings: we will say that an element $x \in A$ *has proto-grade at most* n, if it lies in $\Gamma_n(A)$. The minimal value n such that x has proto-grade at most n will occasionally be called the proto-grade of x, allowing the case that x lies in no $\Gamma_n(A)$, in which case it is said to have *infinite proto-grade*. In this terminology, the above conditions read: for any x of proto-grade at most n and any y of proto-grade at most m, their sum, $x+y$, and product, xy, both have proto-grade at most $F(n+m)$; and if, moreover, x is a unit, then its inverse has proto-grade at most $F(n)$. A less accurate but more telling paraphrase of these conditions is that the arithmetic of the ring (addition, product, and inverse) is uniformily bounded with respect to proto-grade. If we want to emphasize the function F, we call $\Gamma_\bullet(A)$ a pre-proto-F-grading. We call $\Gamma_\bullet(A)$ a *proto-F-grading*, or simply a *proto-grading*, if moreover the union of all $\Gamma_n(A)$ is equal to A, that is to say, if there are no elements of infinite proto-grade. In Exercise 12.5.1 we prove:

12.1.1 *If $\Gamma_\bullet(A)$ is a pre-proto-F-grading on A, then the collection of all elements of finite proto-grade form a subring A' of A, called the* proto-graded subring *associated to the pre-proto-grading, and $\Gamma_n(A') := \Gamma_n(A)$ defines a proto-F-grading on this subring.*

A *proto-graded* ring (A, Γ) is a ring endowed with a proto-grading $\Gamma_\bullet(A)$. Two proto-gradings $\Gamma_\bullet(A)$ and $\Theta_\bullet(A)$ are *equivalent* if there exists a unary function G such that $\Gamma_n(A) \subseteq \Theta_{G(n)}(A)$ and $\Theta_n(A) \subseteq \Gamma_{G(n)}(A)$ for all n. For all intent and purposes, as we shall see, we may replace any proto-grading by an equivalent one. For instance, since any proto-grading is equivalent with a proto-grading $\Gamma_\bullet(A)$ such that $0, \pm 1 \in \Gamma_0(A)$, there is no harm in assuming this already from the start.

Any polynomial ring $A := Z[\xi]$ over an arbitrary ring Z is proto-F-graded by letting $\Gamma_n(A)$ consist of all elements of degree at most n, where F is now just the identity function. Note that in particular $Z \subseteq \Gamma_0(A)$, or put differently, all coefficients have proto-grade zero. We will shortly generalize this, and refer to this situation as the *Z-affine proto-grading*, or simply *affine proto-grading* on A. The following are some other immediate examples.

If $A = \oplus_n A_n$ is an $\mathbb{N}$-graded ring, then letting $\Gamma_n(A)$ be equal to the direct sum $A_0 \oplus A_1 \oplus \cdots \oplus A_n$, makes A into a proto-graded ring. A different generalization, proven in Exercise 12.5.2, is:

12.1.2 *Let (A, Γ) be a reduced proto-graded ring, let ξ be a (finite) tuple of indeterminates, and put $B := A[\xi]$. Let $\Gamma_n(B)$ be the set of all polynomials of degree at most n each coefficient of which has proto-grade at most n. Then $\Gamma_\bullet(B)$ is a proto-grading on B, called the* extended degree proto-grading.

In particular, the affine proto-grading on $A[\xi]$ is the extended degree proto-grading where A is given the *trivial proto-grading* in which any element has proto-grade zero. Note that the protopower of a trivially proto-graded ring is simply its ultrapower.

For a given subring A of $\mathbb{R}$, let $\Gamma_n(A)$ be the set of all elements $a \in A$ of absolute value at most n. In order for this to be a proto-grading, we need to exclude the possibility that there exist units in A converging to zero. For instance, if A is not dense at zero (that is to say, if there exists $\varepsilon > 0$ such that $A \cap [-\varepsilon, \varepsilon] = \{0\}$), then the absolute value yields a proto-grading on A. In particular, this defines a proto-grading on the ring of integers $\mathbb{Z}$. The degree proto-grading on $\mathbb{Z}[\xi]$ extending this absolute value proto-grading, as defined in 12.1.2, will be called the *Kronecker proto-grading* on $\mathbb{Z}[\xi]$, and will be studied further in §12.3.

The category of proto-graded rings. A *morphism* of proto-graded rings $(A,\Gamma) \to (B,\Theta)$ is a ring homomorphism $A \to B$ for which there exists a unary function $G \colon \mathbb{N} \to \mathbb{N}$ such that $\Gamma_n(A) \subseteq \Theta_{G(n)}(B)$, for all n. In particular, if Γ and Γ' are equivalent proto-gradings on A, then the identity on A induces an isomorphism of proto-graded rings. These definitions give rise to the category of proto-graded rings.

Let (A,Γ) be a proto-F-graded ring, and let $\varphi \colon A \to B$ be a ring homomorphism. We define the *push-forward* of Γ by the rule $\Gamma_n(B) := \varphi(\Gamma_n(A))$ for all n. In general, $\Gamma_\bullet(B)$ is only a pre-proto-F-grading, since elements outside the image of the homomorphism have infinite proto-grade. In particular, the push-forward is a proto-grading if φ is surjective, that is to say, if B is of the form A/I for some ideal $I \subseteq A$. We call this proto-grading the *residual proto-grading* on A/I. As for localizations, we can show:

12.1.3 *Let (A,Γ) be a proto-graded rings and let $S \subseteq A$ be a multiplicative subset. There exists a natural proto-grading on the localization $S^{-1}A$, such that the natural map $A \to S^{-1}A$ is a morphism of proto-gradings.*

Indeed, suppose Γ is a proto-F-grading. On $B := S^{-1}A$, define a proto-grading by the rule that x/s has proto-grade at most n, for $x \in A$ and $s \in S$, if both x and s have proto-grade at most n. This yields a proto-G-grading on B, where $G = F \circ F$ (see Exercise 12.5.3 for details).

In view of 12.1.2, we can now extend a given proto-grading on a ring A to any A-affine algebra B. Namely, write B as $A[\xi]/I$ and give B the residual proto-grading of the extended degree proto-grading on $A[\xi]$ (and similarly, for local A-affine algebras, using 12.1.3). In Exercise 12.5.4, we show that any two presentations of B as an A-affine algebra yield equivalent proto-gradings. In case the base ring Z is trivially proto-graded, then we refer to the thus obtained proto-grading on a (local) Z-affine algebra B as the *Z-affine proto-grading*, or simply, the *affine proto-grading* on B.

Protopowers. Let (A,Γ) be a proto-graded ring, and let $A_\natural$ be some ultrapower of A. We define a pre-proto-grading on $A_\natural$ by letting $\Gamma_n(A_\natural)$ be the ultrapower of $\Gamma_n(A)$ (viewed as a subset of $A_\natural$). The *protopower* $A_\flat$ of A is defined as the proto-graded subring associated to this pre-proto-grading, that is to say,

$$A_\flat := \bigcup_n \Gamma_n(A_\natural).$$

By 12.1.1, the protopower is again a ring, and Γ induces a proto-grading on $A_\flat$. The following characterization of $A_\flat$ easily follows from Łos' Theorem:

12.1.4 *An element in the ultraproduct $A_\natural$ lies in the protopower $A_\flat$ if and only if for some n, almost all of its approximations have proto-grade at most n.*

We may express this more simply by saying that $x \in A_\natural$ belongs to $A_\flat$ if and only if some (equivalently, any) approximation of x has *bounded proto-grade*. Some care has to be exercised when working with residual proto-gradings.

12.1.5 *Let (A,Γ) be a proto-graded ring, and let $I \subseteq A$ be a finitely generated ideal. With respect to the residual proto-grading on A/I, the protopower of A/I is equal to $A_\flat/(IA_\natural \cap A_\flat)$.*

Indeed, let $A_\flat$ and $A_\natural$ be the respective protopower and ultrapower of A. Since the ultrapower of A/I is equal to $A_\natural/IA_\natural$ by Exercise 1.5.7, an element $x \in A_\natural$ viewed as an element of $A_\natural/IA_\natural$ has an approximation x_w of bounded proto-grade in A/I if and only if almost all $x_w \in \Gamma_n(A)$ for some n. This in turn is equivalent with $x \in A_\flat$. Hence the protopower of A/I is equal to the image of $A_\flat$ in $A_\natural/IA_\natural$, and this is just $A_\flat/(IA_\natural \cap A_\flat)$. Note, however, that in general $A_\flat \to A_\natural$ will not be cyclically pure, and therefore $IA_\natural \cap A_\flat$ can be strictly larger than $IA_\flat$. □

We have now completed the *chromatic scale* of a proto-graded Noetherian local ring A: there exist natural local A-algebra homomorphisms

$$A \to A_\flat \hookrightarrow A_\natural \twoheadrightarrow A_\sharp. \tag{12.1}$$

Protoproducts. To define protoproducts, we need to make an assumption on the sequence of proto-graded rings A_w. We say that the A_w are *uniformly proto-graded* if there exists a unary function F, such that the proto-grading on A_w is equivalent with a proto-F-grading $\Gamma_\bullet(A_w)$ for (almost) all w. If this is the case, let $A_\natural$ be the ultraproduct of the A_w, and for each n, let $\Gamma_n(A_\natural)$ be the subset of $A_\natural$ given as the ultraproduct of the $\Gamma_n(A_w)$. By Łos' Theorem, $\Gamma_\bullet(A_\natural)$ is a pre-proto-F-grading on $A_\natural$. The associated proto-graded subring $A_\flat$ is called the *protoproduct* of the A_w. One checks that this definition does not depend on the choice of the unary function F, or the particular equivalent proto-F-grading. Of course, a protopower is just a special instance of a protoproduct where all the rings are equal to a single proto-graded ring. The protoproduct of trivially proto-graded rings is just their ultrapower, so that protoproducts generalize the notion of ultraproduct.

Lemma 12.1.6. *The protoproduct of uniformly proto-graded local rings is a local ring.*

Proof. Let $(R_m, \mathfrak{m}_w)$ be proto-F-graded local rings, and let $(R, \mathfrak{m})$ be their ultraproduct. I claim that $\mathfrak{m} \cap R_\flat$ is the unique maximal ideal of the protoproduct $R_\flat$. To this end, we have to show that if $x \in R_\flat$ does not belong to $\mathfrak{m}$, then it is invertible in $R_\flat$. Let x_w be an approximation of x. In particular, almost all x_w are units, and have proto-grade at most n, for some n independent of w. Hence their respective inverses y_w have proto-grade at most $F(n)$. The ultraproduct y of the y_w lies therefore also in $R_\flat$. By Łos' Theorem, $xy = 1$ holds in R, whence also in the subring $R_\flat$, as we wanted to show. □

Protoproducts commute with the formation of a polynomial ring in the following sense:

Proposition 12.1.7. *Let A_w be uniformly proto-graded rings, which we assume to be either reduced or Noetherian, let ξ be a finite tuple of indeterminates, and view each $B_w := A_w[\xi]$ with its extended degree proto-grading. If $A_\flat$ and $B_\flat$ are the respective protoproducts of A_w and B_w, then $B_\flat = A_\flat[\xi]$.*

Proof. Note that it follows from Exercise 12.5.2 that all B_w are also uniformly proto-graded, so that it makes sense to talk about their protoproduct. Let $A_\natural \subseteq B_\natural$ be the ultraproducts of the A_w and B_w respectively. By definition of the extended degree proto-grading, an element f in the ultraproduct $B_\natural$ of the B_w has an approximation f_w of bounded proto-grade if and only if, for some n, almost all f_w have degree at most n with coefficients of proto-grade at most n in A_w. Hence such an f belongs to $A_\natural[\xi]$ by an argument similar to the one used for 7.1.2. Moreover, by Łos' Theorem, the coefficients of f are then all in $\Gamma_n(A_\flat)$, showing that $f \in A_\flat[\xi]$. Reversing the argument yields the converse inclusion. □

For instance, the protopower of $\mathbb{Z}[\xi]$ with respect to its Kronecker proto-grading is $\mathbb{Z}[\xi]$ itself, whereas with respect to its affine proto-grading we get $\mathbb{Z}_\natural[\xi]$, where $\mathbb{Z}_\natural$ is the ultrapower of $\mathbb{Z}$. It is instructive to revisit our construction of an ultra-hull in this new formalism: let K be the ultraproduct of fields K_w, each of which we view with its trivial proto-grading. In particular, the protoproduct of the K_w is just K, and hence by the above result, the protoproduct of the $A_w := K_w[\xi]$ in their affine proto-grading, is $A := K[\xi]$, whereas the ultraproduct of the A_w is the ultra-hull $U(A)$ of A.

12.2 Uniform bounds

In Chapter 7, the main tool for deriving uniform bounds was the faithful flatness of the ultra-hull. In the more general setup of proto-gradings, this is no longer a property holding automatically, but rather an hypothesis, and so we have to investigate when it is satisfied. Ideally, we would derive uniform bounds for a class of uniformly proto-graded rings (with the bounds only depending on some numerical invariants of the ring and the data), and for this we will need good properties of protoproducts. An example of this method will be discussed in §13.2 in the next chapter. In this chapter, however, we content ourselves with bounds that work for a single proto-graded ring, for which it suffices to work with protopowers.

In what follows A is a proto-graded ring, with proto-grading $\Gamma_\bullet(A)$, protopower $A_\flat$ and ultrapower $A_\natural$. We need to study the properties of the inclusions $A \subseteq A_\flat \subseteq A_\natural$. If A is Noetherian, then $A \to A_\natural$ is faithfully flat by Corollary 5.6.3, and hence $A \subseteq A_\flat$ is cyclically pure (in fact, this remains true without the Noetherianity assumption; see Exercise 12.5.8). It is natural to ask under which conditions will $A \subseteq A_\flat$ be faithfully flat, and we will see criteria for that below (see for instance Exercise 12.5.9).

However, keeping the example of an ultra-hull in the above discussion in mind, the more important question is the nature of the embedding $A_\flat \subseteq A_\natural$ (recall that this corresponds in the ultra-hull case precisely to the embedding $A \subseteq U(A)$). Unlike the ultra-hull case, this embedding may fail to be faithfully flat, an a priori obstruction for deriving uniform bounds à la Chapter 7, and so we make the following definition.

Definition 12.2.1. A proto-grading on a ring A is called respectively *flat*, *faithful*, *faithfully flat* or *cyclically pure*, if the natural embedding $A_\flat \subseteq A_\natural$ has the corresponding property.

To better formulate the next results, we introduce the following terminology. Let A be a proto-graded ring, with $A_\flat$ and $A_\natural$ its respective protopower and ultrapower. An ideal $I \subseteq A$ is said to have *proto-grade at most n*, if it can be generated by n elements of proto-grade at most n (note the bound on the number of generators!). In particular, for any $n \geq 1$, an element has proto-grade at most n if and only if the ideal it generates has proto-grade at most n. The usefulness of this concept is exhibited by the following result:

12.2.2 *Let $I_w \subseteq A$ be ideals of proto-grade at most n, for some n independent from w. Then there exists a finitely generated ideal $I \subseteq A_\flat$ such that $IA_\natural$ is equal to the ultraproduct $I_\natural$ of the I_w.*

Indeed, if $f_{1w}, \dots, f_{nw}$ are generators of I_w of proto-grade at most n, and if $f_1, \dots, f_n \in A_\natural$ are they respective ultraproducts, belonging therefore to the subring $A_\flat$, then we may take $I := (f_1, \dots, f_n)A_\flat$ (see Exercise 1.5.7). □

A note of caution: the ideal I is not uniquely determined by the I_w, and neither is it necesarily equal to $I_\natural \cap A_\flat$. We will return to this issue in Definition 12.4.1 below.

Noetherian proto-gradings. Trivial proto-gradings are automatically faithfully flat, for then protopower and ultrapower agree. However, to derive meaningful bounds, some finiteness assumptions are required, the most natural of which is that the protopower should also be Noetherian. Let us therefore call a proto-grading *Noetherian* if its protopower $A_\flat$ is a Noetherian ring. If the proto-grading on A is Noetherian, then A itself must also be Noetherian by Corollary 5.3.6, since $A \to A_\flat$ is cyclically pure by Exercise 12.5.8. The trivial proto-grading shows that the converse fails in general. So of real interest to us will be the proto-gradings which are at the same time Noetherian and faithfully flat. The example par excellence, of course, is the ultra-hull as discussed above. For technical purposes, we also need the following definition: a proto-grading on A is *coherent* if $A_\flat$ is a coherent ring (see page 78).

Proposition 12.2.3. *If A has a Noetherian proto-grading, then for each n there exists a bound n' such that any ideal generated by elements of proto-grade at most n has itself proto-grade at most n' (and whence in particular can be generated by at most n' elements).*

Proof. Suppose no such bound exists for n, so that we can find counterexamples I_w which are ideals of proto-grade at least w but generated by elements of proto-grade at most n. Let $I_\natural$ be the ultraproduct of the I_w. By Exercise 1.5.7, there exists an ideal

$I \subseteq A_\flat$ such that $I_\natural = IA_\natural$. Since $A_\flat$ is Noetherian, we can write $I = (f_1, \dots, f_s)A_\flat$. Choose $n \geq s$ so that all f_i have proto-grade at most n, and let f_{iw} be approximations , also of proto-grade at most n. By the same exercise, we get $I_w = (f_{1w}, \dots, f_{sw})A$ for almost all w, showing that I_w has proto-grade at most n, contradiction. □

The existence of certain uniform bounds also characterizes Noetherianity via the following generalization of a theorem due to Seidenberg in [66].

Theorem 12.2.4. *For a proto-graded ring A, the following are equivalent:*

1. *the proto-grading is Noetherian;*
2. *there exists for each function $\lambda\colon \mathbb{N} \to \mathbb{N}$, a bound $n := n(\lambda)$ with the property that given a sequence of elements f_i of proto-grade at most $\lambda(i)$ for all $i \in \mathbb{N}$, then for some $i \leq n$, we can write $f_i = q_0 f_0 + \dots + q_{i-1} f_{i-1}$ with all q_j of proto-grade at most n.*

Proof. By way of contradiction, assume that $A_\flat$ is Noetherian, but that for some λ, no such bound exists. Therefore, we can find for each w, a counterexample consisting of the following data: elements $f_{iw} \in A$ of proto-grade at most $\lambda(i)$, for $i \leq w$, such that no f_{iw} can be written as a linear combination of the $f_{0w}, \dots, f_{i-1w}$ with coefficients of proto-grade at most w. For $i > w$, set f_{iw} equal to zero, and, for each i, let f_i be the ultraproduct of the f_{iw}, so that by construction, $f_i \in A_\flat$. Since $A_\flat$ is Noetherian, the ideal generated by all the f_i is equal to $(f_0, \dots, f_{m-1})A_\flat$ for some m. In particular, there exist $q_i \in A_\flat$ such that $f_m = q_0 f_0 + \dots + q_{m-1} f_{m-1}$. Choose $n \geq m$ such that all q_i for $i < m$ have proto-grade at most n. Hence, by Łos' Theorem, f_{nw} is a linear combination of $f_{0w}, \dots, f_{n-1,w}$ with coefficients of proto-grade at most n, for almost all w, contradicting our assumption for $w > n$.

Conversely, assume that (2) holds but that there exists an infinite strictly ascending chain of ideals $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \dots$ in $A_\flat$. Choose for each i, an element f_i in $\mathfrak{a}_i$ but not in $\mathfrak{a}_{i-1}$. Let I be the ideal in $A_\flat$ generated by these f_i. For each i, choose $\lambda(i)$ so that f_i has proto-grade at most $\lambda(i)$. Let f_{iw} be an approximation of f_i of proto-grade at most $\lambda(i)$. By assumption, there is a bound $n := n(\lambda)$ such that for some $i \leq n$ and some q_{jw} of proto-grade at most n, we have

$$f_{iw} = q_{0w} f_{0w} + \dots + q_{i-1,w} f_{i-1,w}.$$

Let $q_j \in A_\flat$ be the ultraproduct of the q_{jw}. Since there are only finitely many possibilities for $i \leq n$, there is one such which holds for almost all w. For this i, we have therefore by Łos' Theorem that

$$f_i = q_0 f_0 + \dots + q_{i-1} f_{i-1} \tag{12.2}$$

in $A_\natural$. Since all elements in (12.2) belong to the subring $A_\flat$, this equation itself holds in this subring, showing that $f_i \in \mathfrak{a}_{i-1}$, contradiction. □

Applying this to a constant function yields:

Corollary 12.2.5. *If A has a Noetherian proto-grading, then for any n there exists $n' \geq n$ with the property that any ideal generated by elements of proto-grade at most n is generated already by n' of these generators, that is to say, is an ideal of proto-grade at most n'.* □

We already mentioned that the affine proto-grading on a polynomial ring $K[\xi]$ over a field K is Noetherian, and in this case, one can give a more explicit bound in the previous Corollary 12.2.5: namely we may take n' equal to the number of monomials of degree at most n in the ξ (Lemma 7.4.2). Nonetheless, I am not aware of such an explicit characterization for other functions λ in Theorem 12.2.4, that is to say, for the result that: *for each field K, any function λ admits a bound n, such that if f_i are polynomials in ξ over K of degree at most $\lambda(i)$, then for some $i \leq n$, the polynomial f_i is a linear combination of the previous f_j with coefficients themselves polynomials of degree at most n*. In Project 12.6, we will show that the ring of algebraic power series $K[[\xi]]^{\mathrm{alg}}$ over a field K admits a faithfully flat, Noetherian proto-grading, called the *etale proto-grading*. Here, even the bound given by Corollary 12.2.5 seems no longer to admit a straightforward argument.

Corollary 12.2.6. *Let A have a Noetherian proto-grading and let ξ be a finite tuple of indeterminates. For each function $\lambda\colon \mathbb{N} \to \mathbb{N}$, there exists a bound $n := n(\lambda)$ with the property that given a sequence of polynomials f_i of degree at most $\lambda(i)$ as well as each coefficient of proto-grade at most $\lambda(i)$, for all $i \in \mathbb{N}$, then for some $i \leq n$, we can write $f_i = q_0 f_0 + \cdots + q_{i-1} f_{i-1}$ with all q_j polynomials of degree at most n having coefficients of proto-grade at most n.*

Proof. Let $A_\flat$ be the protopower of A, which by assumption is Noetherian. Since A itself is in particular Noetherian, the extended degree proto-grading on $B := A[\xi]$ is well-defined by Exercise 12.5.2. By Proposition 12.1.7, the protopower of B is $A_\flat[\xi]$, again a Noetherian ring. Therefore the extended degree proto-grading is Noetherian and the bound now follows from Theorem 12.2.4. □

Another example of a Noetherian proto-grading to which we may apply the previous corollary is the Kronecker proto-grading on $\mathbb{Z}$ (since the protopower is trivial), yielding:

Corollary 12.2.7. *Given a tuple of indeterminates ξ, for each function $\lambda\colon \mathbb{N} \to \mathbb{N}$, there exists a bound $n := n(\lambda)$ with the property that if f_i are polynomials of degree at most $\lambda(i)$ with integer coefficients of absolute value at most $\lambda(i)$, for all $i \in \mathbb{N}$, then for some $i \leq n$, we can write $f_i = q_0 f_0 + \cdots + q_{i-1} f_{i-1}$ with all q_j polynomials of degree at most n having integer coefficients of absolute value at most n.* □

Faithful proto-gradings. We start with discussing faithful proto-gradings. By Corollary 12.1.6, any proto-grading on a local ring is faithful. We can characterize faithful proto-gradings by a uniformity result:

Theorem 12.2.8. *A proto-grading on a ring A is faithful if and only if for each pair (n,s) there exists a bound $m := m(n,s)$ with the property that if $(f_1,\ldots,f_s)A$ is the* *unit ideal, with each f_i of proto-grade at most n, then there exist g_i of proto-grade at most m such that $1 = f_1g_1 + \cdots + f_sg_s$.*

If the proto-grading is faithful and Noetherian, then m can be taken to be independent from s.

Proof. The last assertion follows from the first and Proposition 12.2.3. Suppose first that proto-grading is faithful, so that the natural embedding $A_\flat \to A_\natural$ is faithful, but towards a contradiction, suppose no bound exists for the pair (n,s). Hence for each w, we have a counterexample consisting of s elements f_{iw} of proto-grade at most n generating the unit ideal, but any linear combination of the f_{iw} equal to 1 requires at least one of the coefficients to have proto-grade at least w. Let f_i be the ultraproduct of the f_{iw}. By construction, $f_i \in A_\flat$, and by Łos' Theorem, $(f_1,\dots,f_s)A_\natural = A_\natural$. Since $A_\flat \to A_\natural$ is faithful, this implies that necessarily $(f_1,\dots,f_s)A_\flat = A_\flat$. Choose $g_i \in A_\flat$ such that $f_1g_1 + \cdots + f_sg_s = 1$, and let m large enough so that all g_i have of proto-grade at most m. For each i, choose an approximation g_{iw} of g_i of proto-grade at most m. Since by Łos' Theorem,

$$f_{1w}g_{1w} + \cdots + f_{sw}g_{sw} = 1 \qquad (12.3)$$

for almost all w, we get the desired contradiction for any of those $w > m$.

Conversely, suppose a bound as above exists for all pairs (n,s), and let I be an ideal in $A_\flat$ such that $IA_\natural = A_\natural$. We want to show that $I = A_\flat$. By assumption, there exist $f_1,\dots,f_s \in I$ and $h_1,\dots,h_s \in A_\natural$ such that $f_1h_1 + \cdots + f_sh_s = 1$. Choose n large enough so that all f_i have proto-grade at most n. Let f_{iw} be an approximation of f_i of proto-grade at most n. By Łos' Theorem, $(f_{1w},\dots,f_{sw})A = A$ for almost all w, and hence by assumption, we can find g_{iw} of proto-grade at most m satisfying (12.3), for some m independent of w. By construction, the ultraproduct g_i of g_{iw} belongs to $A_\flat$, and by Łos' Theorem, $f_1g_1 + \cdots + f_sg_s = 1$ in $A_\natural$, whence already in the subring $A_\flat$, as we wanted to show. □

For affine proto-gradings (recall that these are essentially given by the degree), being faithful is already a very restrictive assumption as the next result shows (recall that a ring A is called *von Neumann regular* if for each non-zero x there exists a non-zero $y \in A$ such that $xy^2 = y$, and that this is equivalent with A being *absolutely flat*, meaning that any A-module is flat):

Theorem 12.2.9. *For a reduced ring A, the following are equivalent:*

1. *A is von Neumann regular;*
2. *the A-affine proto-grading on $A[\xi]$ is faithful, for all finite tuples ξ of indeterminates;*
3. *every A-affine proto-grading is faithful.*

Proof. The equivalence of (1) and (2) is an immediate consequence of Theorem 12.2.8 and the characterization of von Neumann regularity given in the proof of [50, Proposition 5]. The equivalence of (2) and (3) follows by base change. □

In view of Proposition 12.1.7, condition (2) is equivalent with the canonical map $A_\natural[\xi] \to B_\natural$ being faithful, where $B_\natural$ is the ultrapower of $A[\xi]$. For a non-reduced example where this property holds, let A be an Artinian local ring: the embedding $A_\natural[\xi] \to B_\natural$ is then in fact faithfully flat by [64, Theorem 1.2] (note that $A_\natural$ is again an Artinian local ring). In our current terminology, any affine proto-grading over an Artinian local ring is Noetherian and faithfully flat (see Exercise 12.5.15).

A reduced Noetherian ring is von Neumann regular if and only if it is a direct sum of fields. In particular, the affine proto-grading on $A := \mathbb{Z}[\xi]$ is not faithful. This is exemplified by the following ideal: let $\omega := \mathrm{ulim}_{n\to\infty} n \in \mathbb{N}_\natural$, and let $I := (1-2\xi, 2^\omega)A_\flat$ (recall that $A_\flat = \mathbb{Z}_\natural[\xi]$ by Proposition 12.1.7). Since each $(1-2\xi, 2^n)A$ is the unit ideal, so is $IA_\natural$ by Łos' Theorem. However, in order to write 1 as linear combination of $1-2\xi$ and 2^n, we require a polynomial of degree at least n, namely,

$$(1-2\xi)(\sum_{i<n}(2\xi)^i) + 2^n(\xi^n) = 1,$$

and so I is proper ideal in $A_\flat$.

If we replace the faithfulness assumption in Theorem 12.2.8 by the stronger assumption that the proto-grading is cyclically pure, then virtually an identical proof yields (see Exercise 12.5.12):

Theorem 12.2.10. *A proto-grading on a ring A is cyclically pure if and only if for each pair (n,s), there exists a bound $m := m(n,s)$ such that if $f_0,\dots,f_s$ are elements in A of proto-grade at most n, with f_0 in the ideal generated by the remaining f_i, then $f_0 = f_1g_1 + \cdots + f_sg_s$ for some g_i of proto-grade at most m.*

Moreover, the bound m can be chosen independent from s if the proto-grading is cyclically pure and Noetherian.

Flat proto-gradings. The following theorem generalizes the results on page 114.

Theorem 12.2.11. *For a proto-graded ring A, consider the following conditions:*

1. *for each n there exists a bound n' such that if $I,J \subseteq A$ are ideals of proto-grade at most n, then their colon ideal $(I:J)$ is generated*
   a. *by elements of proto-grade at most n';*
   b. *by n' elements of proto-grade at most n', that is to say, $(I:J)$ has proto-grade at most n';*
2. *for each triple (n,s,m), there exists a bound $n'' := n''(n,s,m)$ such that if $(\mathscr{L})$ is a homogeneous linear system of s equations in m variables with coefficients of proto-grade at most n, then the A-module of solutions of $(\mathscr{L})$ is generated*
   a. *by solutions with entries of proto-grade at most n'';*
   b. *by n'' solutions with entries of proto-grade at most n'';*
3. *for each n, there exists a bound n''' with the property that if I is an ideal of proto-grade at most n, then its module of syzygies is generated*
   a. *by syzygies with entries of proto-grade at most n''';*

b. by n''' syzygies with entries of proto-grade at most n'''.

If the proto-grading is flat, then (1a), (2a) and (3a) hold. If the proto-grading is moreover Noetherian, or more generally, coherent, then the proto-grading is flat if and only if (2b) holds if and only if (3b) holds. If on the other hand, the proto-grading is cyclically pure, then (1b) holds if and only if the proto-grading is faithfully flat and coherent.

Proof. Note that in view of ($\ddagger_{IB}$), conditions (3a) and (3b) are a special instance of (2a) and (2b) respectively, with $s = 1$ and $m = n$. We start by proving that flatness implies (1a). By way of contradiction, assume that for some n, no bound as asserted exists. Hence for each w, we can construct a counterexample consisting of two ideals I_w and J_w of proto-grade at most n, such that $(I_w : J_w)$ cannot be generated by elements of proto-grade at most w. In particular, there exists $x_w \in (I_w : J_w)$ not belonging to the ideal generated by $(I_w : J_w) \cap \Gamma_w(A)$. Let $A_\flat$ and $A_\natural$ be the respective protopower and ultrapower of A. By 12.2.2, we can find finitely generated ideals $I, J \subseteq A_\flat$ (in fact, of proto-grade at most n), such that $IA_\natural$ and $JA_\natural$ are the respective ultraproducts of the I_w and J_w. By Łos' Theorem, the ultraproduct $f \in A_\natural$ of the f_w belongs to $(IA_\natural : JA_\natural)$. By assumption, $A_\flat \to A_\natural$ is flat, so that $f \in (I : J)A_\natural$ by Theorem 5.6.16. Let $g_1, \dots, g_s \in (I : J)$ be such that f is a linear combination in $A_\natural$ of the g_i, and choose $N \geq n$ large enough so that all g_i have proto-grade at most N. Let g_{iw} be an approximation of g_i. Hence by Łos' Theorem, almost each g_{iw} has proto-grade at most N and belongs to $(I_w : J_w)$. Moreover, almost each f_w is a linear combination of the g_{iw}, contradicting our assumption whenever $w > N$. Note that if $A_\flat$ is coherent, then we may choose the g_i so that they generate $(I : J)$ (see for instance [22, Theorem 2.3.2] or Exercise 5.7.26). In that case, any element in $(I_w : J_w)$ is a linear combination of the approximations g_{iw}, that is to say, has proto-grade at most N for almost all w, from which we can now derive (1b) by a similar ad absurdum argument. That flatness implies (2a) and, under the additional coherency assumption, (2b) are proven in the same way, using instead Theorem 5.6.1.

To prove that (3b) yields flatness, we will verify the equational criterion for flatness as stated in Theorem 5.6.1. To this end, let $\mathbf{x}$ be a solution in $A_\natural$ of a linear equation $a_1t_1 + \cdots + a_st_s = 0$ with $a_i \in A_\flat$. Choose $n \geq s$ sufficiently large such that each a_i has proto-grade at most n, and choose approximations a_{iw} of each a_i of proto-grade at most n, and an approximation $\mathbf{x}_w$ of $\mathbf{x}$. By Łos' Theorem, $\mathbf{x}_w$ is a solution of $a_{1w}t_1 + \cdots + a_{sw}t_s = 0$, and hence in view of ($\ddagger_{IB}$) and (3b), there exists some n''' such that $\mathbf{x}_w$ is a linear combination of n''' solutions all of whose entries have proto-grade at most n'''. Hence the ultraproduct of these n''' solutions are solutions of $a_1t_1 + \cdots + a_st_s = 0$ in $A_\flat$, and $\mathbf{x}$ is an $A_\natural$-linear combination of these solutions, as we wanted to show.

So remains to show is that if (1b) holds and the proto-grading is cyclically pure, then it is also flat and coherent. To show that $A_\flat \to A_\natural$ is flat, we verify the colon criterion (Theorem 5.6.16). Let $I := (h_1, \dots, h_s)A_\flat$ be a finitely generated ideal, and let $a \in A_\flat$. We need to show that $(IA_\natural : a) = (I : a)A_\natural$. Choose $n \geq s$ so that both I and a have proto-grade at most n. Let $I_w := (h_{1w}, \dots, h_{sw})A$ and a_w be approximations of proto-grade at most n of I and a respectively. By (1b), almost all $(I_w : a_w)$ have proto-grade at most n', say, generated by the n' elements f_{iw} of proto-grade at most n'. By Theorem 12.2.10, there exists a bound n'' only depending on n' whence on n, and elements g_{ijw} of proto-grade at most n'' such that

$$a_w f_{iw} = g_{i1w}h_{1w} + \cdots + g_{isw}h_{sw}$$

for all $i = 1,\dots,n'$ and all w. Taking the respective ultraproducts of the f_{iw} and g_{ijw} yield elements f_i and g_{ij} in $A_\flat$. Moreover, by Łos' Theorem, we have, for all i, an identity $af_i = g_{i1}h_1 + \cdots + g_{is}h_s$ in $A_\natural$, whence in the subring $A_\flat$. This shows that $f_i \in (I : a)$. On the other hand, an easy argument on Łos' Theorem shows that $(IA_\natural : a) = (f_1,\dots,f_{n'})A_\natural$, from which it follows that $(IA_\natural : a) = (I : a)A_\natural$, as we wanted to show. This also shows that $(I : a)$ is finitely generated, from which it follows that $A_\flat$ is coherent by Exercise 5.7.26. □

Theorem 12.2.12. *Let $(R,\mathfrak{m})$ be a local ring with a faithfully flat, Noetherian proto-grading. For each n, there exists a bound $e := e(n)$ with the property that for any $f_0,\dots,f_s$ of proto-grade at most n, if f_0 lies in $(f_1,\dots,f_s)R + \mathfrak{m}^e$ then f lies already in $(f_1,\dots,f_s)R$.*

Proof. Suppose that for some n, no such bound exists, so that for each w, we can find a counterexample consisting of an ideal I_w generated by elements of proto-grade at most n and an element f_w of proto-grade at most n, so that f_w lies in $I_w + \mathfrak{m}^w$ but not in I_w. By Proposition 12.2.3, the I_w are generated by at most n' elements of proto-grade at most n', for some n' only depending on n. By 12.2.2, there exists $I \subseteq R_\flat$ such that $IR_\natural$ is the ultraproduct of the I_w. By Łos' Theorem, the ultraproduct f of the f_w does not belong to $IR_\natural$, whence a fortiori $f \notin I$. On the other hand, by Łos' Theorem, $f \in IR_\natural + \mathfrak{m}^N R_\natural$ for every N. By assumption $R_\flat \to R_\natural$ is faithfully flat, so that f belongs to $I + \mathfrak{m}^N R_\flat$, for all N. Since $R_\flat$ is Noetherian, the Krull Intersection Theorem (Theorem 1.4.11) yields $f \in I$, contradiction. □

12.3 Proto-gradings over the integers

In this section, we will discuss briefly the existence of some bounds over the integers originally due to Seidenberg (for instance, the bound proven in Corollary 12.3.1 below is shown to actually be doubly exponential in [66]), with improved bounds given by Aschenbrenner (the same bound is proven to be polynomial in [4]). More precisely, let $A := \mathbb{Z}[\xi]$, viewed in its Kronecker proto-grading. Since $A = A_\flat \subseteq A_\natural$ is faithfully flat by Corollary 5.6.3, the Kronecker proto-grading is faithfully flat, and therefore combining the previous results (see Exercise 12.5.14) yield:

Corollary 12.3.1. *There exists for each pair (m,n) a bound $n' := n'(m,n)$, such that if $I = (f_1,\dots,f_s)A$, with $A = \mathbb{Z}[\xi_1,\dots,\xi_m]$, is an ideal of Kronecker proto-grade at most n (that is to say, generated by n polynomials of degree at most n with coefficients of absolute value at most n), then the module of syzygies of I is generated by n' syzygies with entries of Kronecker proto-grade at most n'. Moreover, if f has*

Kronecker proto-grade at most n and belongs to I, then there exist g_i of Kronecker proto-grade at most n' such that $f = g_1 f_1 + \cdots + f_s g_s$. □

We already argued that the above is false if we take the degree proto-grading on $\mathbb{Z}[\xi]$, since this proto-grading fails to be faithful. However, Aschenbrenner observed that 12.2.11(2b) holds in this case, proving that the degree proto-grading is flat. I will give here an independent, direct proof of flatness, and hence via (2b), recover Aschenbrenner's result.

Theorem 12.3.2. *The affine proto-grading on $\mathbb{Z}[\xi]$ is flat. In particular, the module of syzygies of a tuple of polynomials of degree at most n is generated by n' tuples whose entries have degree at most n' for some n' only depending on n and the number of indeterminates.*

Proof. The last assertion follows from the first and Theorem 12.2.11. By Proposition 12.1.7, the protopower of $A := \mathbb{Z}[\xi]$ is $A_\flat = \mathbb{Z}_\natural[\xi]$, and we have to show that $\mathbb{Z}_\natural[\xi] \to A_\natural$ is flat. We will do this by means of the Tor criterion (Theorem 5.2.6), that is to say, by showing that

$$T := \operatorname{Tor}_1^{A_\flat}(A_\natural, A_\flat/I)$$

vanishes for every finitely generated ideal $I \subseteq A_\flat$. Towards a contradiction, suppose that τ is a non-zero element in T. Let $\mathbb{Q}_\natural$ be the ultrapower of $\mathbb{Q}$, so that by Łos' Theorem, it is the field of fractions of $\mathbb{Z}_\natural$. Viewing $B := \mathbb{Q}[\xi]$ in its affine proto-grading, we have $B_\flat = \mathbb{Q}_\natural[\xi] = A_\flat \otimes_{\mathbb{Z}_\natural} \mathbb{Q}_\natural$. Moreover, since any polynomial in B is of the form af with $a \in \mathbb{Q}$ and $f \in A$, we have $B_\natural = A_\natural \otimes_{\mathbb{Z}_\natural} \mathbb{Q}_\natural$. Since $B_\flat \to B_\natural$ is faithfully flat by Theorem 7.2.2, and since this is just the base change of $A_\flat \to A_\natural$ with respect to $\mathbb{Q}_\natural$ by our previous calculations, we get $T \otimes_{\mathbb{Z}_\natural} \mathbb{Q}_\natural = 0$ by Exercise 5.7.5. Therefore, there exists some non-zero $a \in \mathbb{Z}_\natural$ such that $a\tau = 0$ in T. Since $\mathbb{Z}_\natural$ is a Prüfer domain by Exercise 12.5.20, the polynomial ring $A_\flat = \mathbb{Z}_\natural[\xi]$ is coherent by [50, Proposition 3] or [22, Corollary 7.3.4]. In particular, I has a finitely generated module of syzygies, and hence there exists an exact sequence

$$A_\flat^m \xrightarrow{d_2} A_\flat^n \xrightarrow{d_1} A_\flat \to A_\flat/I \to 0.$$

By definition of a Tor module (see page 67), we can calculate T as the homology of the tensored complex

$$A_\natural^m \xrightarrow{d_2} A_\natural^n \xrightarrow{d_1} A_\natural.$$

In particular, τ is the image of a tuple $\mathbf{x} \in A_\natural^n$ such that $d_1(\mathbf{x}) = 0$. Moreover, $\mathbf{x}$ does not belong to $\operatorname{Im}(d_2)$, but $a\mathbf{x}$ does. Let $a_w \in \mathbb{Z}$, $\mathbf{x}_w \in A^n$ and d_{iw} be approximations of a, $\mathbf{x}$ and d_i respectively, yielding for almost all w a complex

$$A^m \xrightarrow{d_{1w}} A^n \xrightarrow{d_{2w}} A$$

such that $\mathbf{x}_w$ lies in the kernel of d_{1w} and not in the image of d_{2w}, whereas $a_w\mathbf{x}_w$ does lie in the image. Hence we can find tuples $\mathbf{y}_w$ and prime numbers $p_w \in \mathbb{Z}$, such that almost each $\mathbf{y}_w$ lies in the kernel of d_{1w} but not in the image of d_{2w}, yet $p_w\mathbf{y}_w$ does. Let $\pi \in \mathbb{Z}_\natural$ and $\mathbf{y} \in A_\natural^n$ be the respective ultraproducts of the p_w and the $\mathbf{y}_w$. Since $\mathbf{y}$ lies in the kernel of d_1 but outside the image of d_2 by Łos' Theorem, its image in T is a non-zero element, annihilated by π.

On the other hand, since $\mathbb{Z}_\natural/\pi\mathbb{Z}_\natural$ is the ultraproduct of the fields $\mathbb{Z}/p_w\mathbb{Z}$, the base change of $A_\flat/\pi A_\flat \to A_\natural/\pi A_\natural$ is faithfully flat by Theorem 7.2.2. Since π is an $A_\flat$-regular element, we have a short exact sequence

$$0 \to A_\natural \xrightarrow{\pi} A_\natural \to A_\natural/\pi A_\natural \to 0 \tag{12.4}$$

and by Exercise 12.5.21, also an exact sequence (derived from a degenerated spectral sequence)

$$\mathrm{Tor}_{i-1}^{A_\flat/\pi A_\flat}(A_\natural/\pi A_\natural, A_\flat/(I:\pi)) \to \mathrm{Tor}_i^{A_\flat}(A_\natural/\pi A_\natural, A_\flat/I) \to \mathrm{Tor}_i^{A_\flat/\pi A_\flat}(A_\natural/\pi A_\natural, A_\flat/(I+\pi A_\flat))$$

For $i = 2$, the two outer modules are zero by the flatness of $A_\flat/\pi A_\flat \to A_\natural/\pi A_\natural$, whence so is the inner module. Therefore, the relevant part of the long exact Tor sequence (5.2.5) associated to (12.4) becomes

$$0 = \mathrm{Tor}_2^{A_\flat}(A_\natural/\pi A_\natural, A_\flat/I) \to T \xrightarrow{\pi} T$$

showing that π is T-regular, contradicting the fact that $\pi\mathbf{y} = 0$ in T. □

Inspecting the above proof, we see that we may replace $\mathbb{Z}$ by an arbitrary one-dimensional normal unique factorization domain Z (note that Z is then in particular a Dedekind domain), and so we proved:

Corollary 12.3.3. *Let Z be a one-dimensional normal unique factorization domain and ξ a finite tuple of indeterminates. For each n, there exists a bound n' such that if I is an ideal in $Z[\xi]$ generated by polynomials of degree at most n, then the module of syzygies of I is generated by n' many tuples all of whose entries have degree at most n'.* □

The present proof seems to require that Z is a unique factorization domain, but perhaps this can be circumvented by using [64, Theorem 2] instead of Theorem 7.2.2, so that we may derive Corollary 12.3.3 for any Dedekind domain, or even for any Prüfer domain, thus recovering the result in [5, Theorem A].

12.4 Prime bounded proto-gradings

Let A be a proto-graded ring, with protopower $A_\flat$ and ultrapower $A_\natural$. Since the proto-grading may fail to be cyclically pure, not every ideal of $A_\flat$ is the contraction of an ideal of $A_\natural$. Among the contracted ideals in $A_\flat$, the following class is particularly nice:

Definition 12.4.1. An ideal $\mathfrak{a} \subseteq A_\flat$ is called *finitary* if it is of the form $IA_\natural \cap A_\flat$ for some finitely generated ideal $I \subseteq A_\flat$.

Note that a finitary ideal need not be finitely generated. If the proto-grading is cyclically pure and Noetherian, then any ideal in the protopower is finitary.

Any finitely generated ideal $I \subseteq A_\flat$ admits an *approximation* $I_w \subseteq A$, that is to say, ideals whose ultraproduct is equal to $IA_\natural$. We can extend this construction to finitary ideals of the form $\mathfrak{a} := IA_\natural \cap A_\flat$ with $I \subseteq A_\flat$ finitely generated, by defining an approximation I_w of $\mathfrak{a}$ to be any approximation of I. This makes sense since $\mathfrak{a}A_\natural = IA_\natural$. Note that for some n, almost all I_w have proto-grade at most n. Conversely, if I_w is a collection of ideals of proto-grade at most n, for some n, then $I_\natural \cap A_\flat$ is finitary by 12.2.2, with approximation I_w, where $I_\natural$ is the ultraproduct of the I_w. On occasion, we may refer to $I_\natural \cap A_\flat$ as the *finitary protoproduct* of the I_w. Note that, unlike the ideal given by 12.2.2, the finitary protoproduct is uniquely determined by the I_w.

The next definition generalizes the uniform primality results obtained previously in the case of a polynomial ring over a field (see page 115).

Definition 12.4.2. We call the proto-grading on A *prime bounded* if the extension of any finitary prime ideal of $A_\flat$ remains prime in $A_\natural$.

An easy example of a prime bounded proto-grading is the Kronecker proto-grading on $\mathbb{Z}[\xi]$: since the proto-power is then just $\mathbb{Z}[\xi]$ (by Proposition 12.1.7), the result follows from the general fact that any prime ideal in a ring remains prime in its ultrapower. In view of Theorem 7.3.4, the degree proto-grading on a polynomial ring over a field is prime bounded. The property of being prime bounded is again characterized by a certain uniformity result:

Theorem 12.4.3 (Uniform Primality). *For a proto-graded ring A, the following are equivalent:*

1. *The proto-grading is prime bounded;*
2. *for each n, there exists a bound n' with the following property: given an ideal I of proto-grade at most n, the ideal is prime if and only if for any two elements f and g of proto-grade at most n', if both do not belong to I, then neither does their product.*

Proof. Suppose the proto-grading is prime bounded, but no bound as in (2) exists. Hence for some n, we can find non-prime ideals $I_w \subseteq A$ of proto-grade at most n, having the property that if a product of two elements of proto-grade at most w belongs to I_w, then already one of them belongs to I_w. Let $\mathfrak{a}$ be the finitary protoproduct of the I_w, that is to say, let $I \subseteq A_\flat$ be an ideal given by 12.2.2, and put $\mathfrak{a} := IA_\natural \cap A_\flat$. I claim that $\mathfrak{a}$ is prime. Indeed, suppose we have elements $f, g \in A_\flat$ such that $fg \in \mathfrak{a}$. Choose n' large enough so that f and g have both proto-grade at most n'. Let f_w, g_w be respective approximations of f and g of proto-grade at most n'. Since $fg \in IA_\natural$, almost each $f_w g_w$ lies in I_w. For those w which are also bigger than n', we then have by assumption that one of the two, say f_w, belongs to I_w. It follows that f lies in $IA_\natural$, whence in $\mathfrak{a}$, proving the claim. By definition of prime boundedness, $\mathfrak{a}A_\natural = IA_\natural$ is then also a prime ideal. However, since the latter ideal is the ultraproduct of the I_w, almost all of these ideals must be prime ideals by Łos' Theorem, contradiction.

Conversely, suppose a bound to test primality as asserted in (2) exists and let $\mathfrak{p}$ be a finitary prime ideal in $A_\flat$. We want to show that $\mathfrak{p}A_\natural$ is also prime. Let $\mathfrak{p}_w \subseteq A$ be an approximation of $\mathfrak{p}$. If almost no $\mathfrak{p}_w$ is prime, then since almost all have uniformly bounded proto-grade, for some n, there exist elements $f_w, g_w \in A$ of proto-grade at most n not belonging to $\mathfrak{p}_w$ but whose product does. If f,g are their respective ultraproducts, then f and g already lie in $A_\flat$. Moreover, by Łos' Theorem, f and g do not belong to $\mathfrak{p}A_\natural$ but their product does. Since $\mathfrak{p}$ is finitary, it is equal to $\mathfrak{p}A_\natural \cap A_\flat$, and hence $fg \in \mathfrak{p}$. Therefore, at least one among f or g belongs to $\mathfrak{p}$, contradiction. Hence almost all $\mathfrak{p}_w$ must be prime ideals, whence so is their ultraproduct $\mathfrak{p}A_\natural$, as we wanted to show. □

Theorem 12.4.4. *Let A be a ring with a faithfully flat, prime bounded, Noetherian proto-grading, then there exists for each n a bound $e := e(n)$, such that for any choice of elements $a_1,\dots,a_s$ of proto-grade at most n, the ideal $I := (a_1,\dots,a_s)A$ has at most e minimal prime ideals, each of proto-grade at most e, its radical* $\operatorname{rad} I$ *has proto-grade at most e, and* $(\operatorname{rad} I)^e$ *lies inside I.*

Proof. We will prove both properties simultaneously. Assume no bound exists for some n, so that we can construct, after an application of Proposition 12.2.3, for each w, a counterexample I_w of proto-grade at most n with radical J_w, so that, respectively, J_w cannot be realized as the intersection of w prime ideals of proto-grade at most w, has proto-grade at least w, or $(J_w)^w$ is not contained in I_w. Let $\mathfrak{a}$ be the finitary protoproduct of the I_w, and let $\mathfrak{b}$ be its radical. Since the protopower $A_\flat$ is by assumption Noetherian, we can find some e such that $\mathfrak{b}^e \subset \mathfrak{a}$. From the inclusions $\mathfrak{b}^e A_\natural \subset \mathfrak{a}A_\natural \subset \mathfrak{b}A_\natural$, we conclude that both $\mathfrak{a}A_\natural$ and $\mathfrak{b}A_\natural$ have the same radical. Let $\mathfrak{p}_i$ be the (finitely many) minimal prime ideals of $\mathfrak{a}$. Since $\mathfrak{b} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s$ and $A_\flat \to A_\natural$ is flat by assumption, Exercise 5.7.4 yields

$$\mathfrak{b}A_\natural = \mathfrak{p}_1 A_\natural \cap \dots \cap \mathfrak{p}_s A_\natural. \tag{12.5}$$

Furthermore, each $\mathfrak{p}_i$ is finitary (since by faithful flatness it is equal to $\mathfrak{p}_i A_\natural \cap A_\flat$), and hence by prime boundedness, $\mathfrak{p}_i A_\natural$ is again a prime ideal. In particular, (12.5) shows that the extended ideal $\mathfrak{b}A_\natural$ is also radical.

Let $\mathfrak{b}_w$ and $\mathfrak{p}_{iw}$ be approximations of $\mathfrak{b}$ and $\mathfrak{p}_i$ respectively. By Łos' Theorem, $\mathfrak{b}_w = \mathfrak{p}_{1w} \cap \dots \cap \mathfrak{p}_{sw}$, almost all $\mathfrak{p}_{iw}$ are prime, and almost all $\mathfrak{b}_w$ are radical. Moreover, by Łos' Theorem, $\mathfrak{b}_w^e \subset I_w \subset \mathfrak{b}_w$, for almost all w. This shows that almost each $\mathfrak{b}_w$ is equal to the radical J_w of I_w and the $\mathfrak{p}_{iw}$ are minimal prime ideals of I_w, contradicting either of our assumptions. □

The next result provides a larger class of prime bounded proto-gradings, to which we therefore may apply the previous results.

Theorem 12.4.5. *Let Z be a one-dimensional domain and ξ a finite tuple of indeterminates. Then the affine proto-grading on $Z[\xi]$ is prime bounded.*

Proof. Set $A := Z[\xi]$, so that by Proposition 12.1.7 its protopower $A_\flat$ is equal to $Z_\natural[\xi]$, where $Z_\natural$ is the ultrapower of Z. Let $\mathfrak{p}$ be a finitary prime ideal of $Z_\natural[\xi]$ and let $\mathfrak{q} := \mathfrak{p} \cap Z$. Choose an approximation $\mathfrak{p}_w \subseteq A$ of $\mathfrak{p}$. Let $\mathfrak{q}_w := \mathfrak{p}_w \cap Z$. By Łos' Theorem, the ultraproduct of the $\mathfrak{q}_w$ is equal to $\mathfrak{p}A_\natural \cap Z_\natural = (\mathfrak{p}A_\natural \cap A_\flat) \cap Z_\natural = \mathfrak{p} \cap Z_\natural = \mathfrak{q}$.

We give a different argument, depending on whether $\mathfrak{q}$ is zero or not. In the first case, let $Q_\natural$ be the field of fractions of $Z_\natural$. In other words, $Q_\natural$ is the ultrapower of the field of fractions Q of Z. Since $\mathfrak{p} \cap Z_\natural = (0)$, the extended ideal $\mathfrak{p}Q_\natural[\xi]$ is also prime. Let $B_\natural$ be the ultrapower of the polynomial ring $Q[\xi]$. By Theorem 7.3.4, the extension $\mathfrak{p}B_\natural$ remains prime. Hence we are done in this case if we can show that

$$\mathfrak{p}A_\natural = \mathfrak{p}B_\natural \cap A_\natural.$$

To this end, let f be in the right hand side and let $f_w \in A$ be an approximation of f. It follows that almost each f_w lies in $\mathfrak{p}_w Q[\xi]$. Hence, for some non-zero $s_w \in Z$, $s_w f_w \in \mathfrak{p}_w$ for almost all w. Since s_w cannot belong to $\mathfrak{p}_w$, as almost all $\mathfrak{q}_w$ are zero, we must have $f_w \in \mathfrak{p}_w$, and therefore $f \in \mathfrak{p}A_\natural$, as we wanted to show.

In the remaining case, almost all $Z/\mathfrak{q}_w$ are fields, since Z is one-dimensional. Since $A_\natural/\mathfrak{q}A_\natural$ is then the ultraproduct of polynomial rings over fields, $\mathfrak{p}(A_\natural/\mathfrak{q}A_\natural)$ is prime, again by Theorem 7.3.4, whence so is $\mathfrak{p}A_\natural$ as we wanted to show. □

To extend this result to higher dimensions, using induction on the dimension, we need a version for protoproducts rather than just protopowers. The above proof, however, can easily be adjusted to accomodate for this more general setting; see Exercise 12.5.17. It is worthwhile to formulate the application of Theorem 12.4.3 to Theorem 12.4.5, or to its generalization to higher dimensions given by Exercise 12.5.17, as a separate theorem:

Theorem 12.4.6. *For every finite-dimensional domain A, for every finite tuple of indeterminates ξ, and for every postive integer n, there exists a bound n' with the following property: given an ideal $\mathfrak{p} \subseteq A[\xi]$ generated by n polynomials of degree at most n, if for any two polynomials of degree at most n' outside $\mathfrak{p}$, neither does their product belong to $\mathfrak{p}$, then $\mathfrak{p}$ is prime.* □

We can now also give an example of a prime-bounded proto-grading which is not faithfull: the affine proto-grading on $\mathbb{Z}[\xi]$ with ξ a single variable. Namely, in $\mathbb{Z}_\natural[\xi]$, let $\mathfrak{p}$ be the ideal generated by $1-2\xi$ and the intersection of all powers $2^n\mathbb{Z}_\natural[\xi]$. One checks that $\mathfrak{p}$ is a prime ideal, but its extension to the ultrapower $A_\natural$ of $\mathbb{Z}[\xi]$ is the unit ideal. In particular, this implies that $\mathfrak{p}$ cannot be a finitary ideal.

We finish this section with a uniform elimination result. To this end, we must first prove some form of transfer result:

Proposition 12.4.7. *Let A have a Noetherian, faithfully flat, prime bounded proto-grading and let $I \subset A_\flat$ be an ideal in its protopower, with approximation $I_w \subseteq A$.*

1. *I is prime (radical) if and only if almost all I_w are prime (radical).*
2. *I has height h if and only if almost all I_w have height h.*

Proof. Note that since the proto-grading is Noetherian and faithfully flat, I is finitary. If I is prime, then so is $IA_\natural$ by prime boundedness, and hence by Łos' Theorem, so are almost all I_w. Conversely, if almost all I_w are prime, then so is their ultraproduct $IA_\natural$, whence so is $I = IA_\natural \cap A_\flat$.

If I is radical, then it can be written as $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ with all $\mathfrak{p}_i$ prime ideals in $A_\flat$. Choose an approximation $\mathfrak{p}_{iw}$ of the $\mathfrak{p}_i$. Since

$$IA_\natural = \mathfrak{p}_1 A_\natural \cap \cdots \cap \mathfrak{p}_s A_\natural$$

by Exercise 5.7.4, Łos' Theorem yields $I_w = \mathfrak{p}_{1w} \cap \cdots \cap \mathfrak{p}_{sw}$, for almost all w. By the previous case, almost all $\mathfrak{p}_{iw}$ are prime, showing that almost all I_w are radical. Conversely, suppose almost all I_w are radical and $a^n \in I$. Choose $a_w \in R$ whose ultraproduct is a. By Łos' Theorem, I_w contains a_w^n, whence a_w, for almost all w. Hence a lies in $IA_\natural$, and hence by faithful flatness, in I, showing that I is radical.

To prove (2), assume first that I is prime, and hence, by the previous argument, so are then almost all I_w. We induct on the height h of I. If $h = 0$, then I is a minimal prime of $A_\flat$ and hence the extension of a minimal prime of A by Exercise 12.5.24. It follows that almost all I_w are equal to this minimal prime, and the assertion is clear in this case. So assume $h > 0$ and choose a height $h-1$ prime ideal $\mathfrak{p}$ inside I. If $\mathfrak{p}_w$ is an approximation of $\mathfrak{p}$, then by induction almost all are height $h-1$ prime ideals. By Łos' Theorem, $\mathfrak{p}_w \subsetneq I_w$, so that almost all I_w have height at least h. Choose some a in I but not in $\mathfrak{p}$ and let $a_w \in A$ be an approximation. In particular, I is a minimal prime of $\mathfrak{p} + aA_\flat$. Let $\mathfrak{g}_w$ be a minimal prime ideal of $\mathfrak{p}_w + a_w A$ contained in I_w (note that by Łos' Theorem, a_w lies in I_w, for almost all w). By the Krull Principal Ideal Theorem, almost all $\mathfrak{g}_w$ have height h. Choose n sufficiently large so that I and a both have proto-grade at most n, whence so do almost all I_w and a_w. By Theorem 12.4.4, therefore, almost all $\mathfrak{g}_w$ have proto-grade at most n', for some n' depending only on n. Let $\mathfrak{g}$ be their finitary protoproduct, that is to say, the ideal $\mathfrak{g}_\natural \cap A_\flat$, where $\mathfrak{g}_\natural$ is the ultraproduct of the $\mathfrak{g}_w$. By (1), the ideal $\mathfrak{g}$ is prime. By Łos' Theorem and faithful flatness, $\mathfrak{p} + aA_\flat \subseteq \mathfrak{g} \subset I$, so that $\mathfrak{g}$ and I, both being minimal prime ideals of $\mathfrak{p} + aA_\natural$, must be equal. Hence also almost all $\mathfrak{g}_w = I_w$ are equal, whence have height h. This proves (2) for I a prime ideal

Assume finally that I is arbitrary, of height h, and let $\mathfrak{p}$ be a minimal prime of I, with approximation $\mathfrak{p}_w$. By Łos' Theorem and what we already established, $\mathfrak{p}_w$ is a height h prime ideal containing I_w, for almost all w. It follows that almost each I_w has height at most h. If almost all I_w would have height less than h, then we can choose for each w, a minimal prime $\mathfrak{g}_w$ of I_w of that height and of proto-grade at most n, for some n independent from w, by Theorem 12.4.4, so that the same argument as before, the finitary protoproduct of these $\mathfrak{g}_w$ would be a prime ideal of height less than h containing I, contradiction. □

Theorem 12.4.8 (Uniform Elimination). *Let A have a Noetherian, faithfully flat, prime bounded proto-grading. Let ξ be a finite tuple of indeterminates and view $A[\xi]$ in its extended degree proto-grading. For each n, there exists a bound n' so that any prime ideal $\mathfrak{P} \subseteq A[\xi]$ of proto-grade at most n contracts to a prime ideal $\mathfrak{P} \cap R$ of proto-grade at most n'.*

Proof. By induction on the number of indeterminates, we only need to treat the case that ξ is a single variable. Suppose that for some n, no bound as claimed exists. Hence we can find prime ideals $\mathfrak{P}_w \subseteq A[\xi]$ of proto-grade at most n such that $\mathfrak{g}_w := \mathfrak{P}_w \cap A$ has proto-grade at least w. Let $B_\natural$ denote the ultrapower of $A[\xi]$. Recall that the protopower of $A[\xi]$ is equal to $A_\flat[\xi]$ by Proposition 12.1.7. Let $\mathfrak{P}_\natural$ and $\mathfrak{P} := \mathfrak{P}_\natural \cap A_\flat[\xi]$ be the respective ultraproduct and finitary protoproduct of the $\mathfrak{P}_w$ and put $\mathfrak{p} := \mathfrak{P} \cap A_\flat$. Whence both $\mathfrak{P}$ and $\mathfrak{p}$ are prime ideals. Choose an approximation $\mathfrak{p}_w \subseteq A$ of $\mathfrak{p}$. It follows from Proposition 12.4.7 and Łos' Theorem that almost each $\mathfrak{p}_w$ is a prime ideal contained in $\mathfrak{g}_w$, of proto-grade at most n, for some n independent of w.

If $\mathfrak{g}_\natural$ is the ultraproduct of the $\mathfrak{g}_w$, then by Łos' Theorem,

$$\mathfrak{g}_\natural = \mathfrak{P}B_\natural \cap A_\natural. \tag{12.6}$$

Let h be the height of $\mathfrak{P}$. By [18, Exercise 10.2], the contraction $\mathfrak{p}$ has height h if and only if $\mathfrak{P} = \mathfrak{p}A_\flat[\xi]$. Assuming that this is the case, we have by (12.6) that $\mathfrak{g}_\natural = \mathfrak{p}B_\natural \cap A_\natural$. However, by Łos' Theorem, this means that $\mathfrak{g}_w = \mathfrak{p}_w A[\xi] \cap A = \mathfrak{p}_w$, contradicting that the $\mathfrak{g}_w$ have unbounded proto-grade.

The remaining possibility is for $\mathfrak{p}$ to have height $h-1$. By Proposition 12.4.7, then so have almost all $\mathfrak{p}_w$. If $\mathfrak{p}_w \neq \mathfrak{g}_w$, then $\mathfrak{g}_w$ has height at least h. On the other hand, almost all $\mathfrak{P}_w$ have height h by Proposition 12.4.7, so that by another application of [18, Exercise 10.2], we have $\mathfrak{P}_w = \mathfrak{g}_w A[\xi]$. By Exercise 12.5.18, almost all $\mathfrak{g}_w$ then have proto-grade at most n, contradiction. □

Corollary 12.4.9. *Let A have a Noetherian, faithfully flat, prime bounded proto-grading. For each n, there exists n' such that if I and J are ideals of proto-grade at most n, then $I \cap J$ has proto-grade at most n'.*

Proof. Let $B := A[\xi]$ with ξ a single indeterminate and set $\mathfrak{a} := \xi IB + (1-\xi)JB$. By construction, $\mathfrak{a}$ has proto-grade at most $n+1$ in the extended degree proto-grading on B. Hence by Theorem 12.4.8, the proto-grade of $\mathfrak{a} \cap A$ is at most n', for some n' independent from I and J. However, it is easy to see that $I \cap J = \mathfrak{a} \cap A$. Indeed, if $a \in I \cap J$, then $a = \xi a + (1-\xi)a \in \mathfrak{a}$. Conversely, if $a \in \mathfrak{a} \cap A$, then $a = \xi f + (1-\xi)g$ with $f \in IB$ and $g \in JB$. Putting ξ equal to 0 and 1 respectively gives $a = g(0) \in J$ and $a = f(1) \in I$. □

12.5 Exercises

Ex 12.5.1

Prove 12.1.1.

Ex 12.5.2

Prove 12.1.2. More explicitly, if A is a proto-F-graded reduced ring, construct a unary function G (only depending on F) such that $B := A[\xi]$ is proto-G-graded. Show that we may weaken the assumption that A is reduced to A having finite nilpotency degree *N, that is to say, if $x \in A$ is nilpotent, then $x^N = 0$ (this always holds if A is Noetherian). Construct a counterexample of infinite nilpotency degree such that extended degree does not yield a proto-grading.*

Ex 12.5.3

Prove the assertions about the category of proto-graded rings from page 183. In particular, prove 12.1.3.
Show that taking protopowers is a functor on the category of proto-graded rings.

Ex 12.5.4

Let A be a proto-graded ring, and let B be an A-affine algebra. Let $B \cong A[\xi]/I$ and $B \cong A[\zeta]/J$ be two presentations of B as an A-affine algebra. Show that the affine proto-gradings on B induced by either presentation are equivalent.

Ex 12.5.5

Let (A,Γ) and (A,Γ') be two equivalent proto-gradings on A. Show that their respective protopowers are the same. Disprove by means of a counterexample the analogue for protoproducts, that is to say, a protoproduct which is no longer the same when we replace the proto-grading on each component by an equaivalent one.

Ex 12.5.6

Formulate and prove an analogue of 12.1.5 for protoproducts.

Ex 12.5.7

Show that a local ring with a Noetherian proto-grading has the same dimension as its protopower. In fact, the geometric dimension of an arbitrary proto-graded local ring is the same as the geometric dimension of its protopower.

Ex 12.5.8

Show that the embedding of a ring inside its ultrapower is always cyclically pure. Conclude that the same is therefore true for the embedding of a proto-graded ring in its protopower.

Ex 12.5.9

Use the criterion from Corollary 5.6.17 to show that for a cyclically pure proto-grading on a Noetherian ring A, the embedding $A \to A_\flat$ is faithfully flat.

Ex 12.5.10

Give details for all unproven implications in Theorem 12.2.11.

Ex 12.5.11

Prove the following stronger version of Corollary 12.4.9 directly using Exercise 5.7.4: for any ring A with a Noetherian, flat proto-grading, there exists, for each n, a bound n' such that if $I, J \subseteq A$ are ideals of proto-grade at most n, then their intersection $I \cap J$ has proto-grade at most n'.

Ex 12.5.12

Prove Theorem 12.2.10.

Ex 12.5.13

Show the following partial converse to Proposition 12.2.3: if, for each n, there exists a bound n' such that any ideal generated by elements of proto-grade at most n has proto-grade at most n', then for any m, any ideal in $A_\flat$ generated by elements of proto-grade at most m is finitely generated.

Ex 12.5.14

Let A be a ring with a Noetherian, faithfully flat proto-grading. Prove by combining the results of this chapter that there exists, for each n, a bound n', such that if $I = (f_1, \dots, f_s)A$ is an ideal of proto-grade at most n, then the module of syzygies of I (see the discussion following Corollary 5.2.8) is generated by n' syzygies with entries of proto-grade at most n. Moreover, if $f \in I$ and has proto-grade at most n, then there exist g_i of proto-garde at most n' such that $f = g_1 f_1 + \cdots + f_s g_s$.

***Ex 12.5.15**

Let R be an Artinian local ring. Show using [64, Theorem 1.2], that the R-affine proto-grading on an R-affine algebra B is faithfully flat and Noetherian.

Ex 12.5.16

Prove the following more general version of Theorem 12.2.12: for a ring A with a faithfully flat, Noetherian proto-grading, there exists, for each n, some $e := e(n)$ with the property that if $I \subseteq J$ are ideals of proto-grade at most n, and $f \in I + J^e$ has proto-grade at most n, then there is some $s \in J$ of proto-frade at most e such that $(1+s)f \in I$.

***Ex 12.5.17**

Show the following more general version of Theorem 12.4.5 (including the higher dimensional case): let ξ be a finite tuple of indeterminates and let Z_w be domains of dimension at most d. Since all $Z_w[\xi]$ are uniformily proto-graded in their affine proto-grading, we can take their protoproduct $A_\flat$ as defined on page 184, which is therefore equal to $Z_\natural[\xi]$, where $Z_\natural$ is the ultraproduct of the Z_w, by Proposition 12.1.7. Show, using induction on d in combination with the argument in the proof of the theorem, that any finitary prime ideal of $A_\flat$ extends to a prime ideal in the ultraproduct $A_\natural$ (with the obvious generalization of the notion finitary *to this more general setup). Now deduce Theorem 12.4.6.*

Ex 12.5.18

Show that given a proto-graded ring A and a finite tuple ξ of indeterminates, any ideal $I \subseteq A$ has proto-grade at most the proto-grade of its extension $IA[\xi]$, where we view $A[\xi]$in its extended degree proto-grading.

Additional exercises.

Ex 12.5.19

Show that if φ is a universal formula in the language of rings, and A_w are proto-F-graded rings satisfying φ, then their protoproduct $A_\flat$ also satisfies φ.

Ex 12.5.20

Show that an ultraproduct of Dedekind domains is a Prüfer *domain, meaning that any localization at a maximal ideal is a valuation ring (or, equivalently by [22, §1.4], that every finitely generated ideal is projective).*

Ex 12.5.21

(To solve this problem, some knowledge of spectral sequences is required). Let $A \to B$ be a ring homomorphism, M an A-module, and x an element in the kernel of $A \to B$. By [18, Exercise A3.45], there exists a spectral sequence

$$\mathrm{Tor}_i^{A/xA}(B, \mathrm{Tor}_j^A(A/xA, M)) \Rightarrow_j \mathrm{Tor}_{i+j}^A(B, M).$$

Show that if x is A-regular, then this spectral sequence is degenerated, and hence yields a long exact sequence

$$\cdots \to \mathrm{Tor}_{i+1}^{A/xA}(B, M/xM) \to \mathrm{Tor}_{i-1}^{A/xA}(B, (0:_M x)) \to \mathrm{Tor}_i^A(B, M)$$
$$\to \mathrm{Tor}_i^{A/xA}(B, M/xM) \to \mathrm{Tor}_{i-2}^{A/xA}(B, (0:_M x)) \to \mathrm{Tor}_{i-1}^A(B, M) \to \dots$$

for all $i \geq 2$, where $(0:_M x)$ denotes the submodule of all $m \in M$ such that $xm = 0$. Use this to prove that if x is an A-regular element in the kernel of $A \to B$ such that the induced map $A/xA \to B$ is flat, then $\mathrm{Tor}_i^A(B, \cdot)$ is identical zero for all $i \geq 2$.

Ex 12.5.22

Show, using Exercises 12.5.7 and 12.5.9, that if R is a regular local ring with a faithfully flat, Noetherian proto-grading, then $R_\flat$ is a regular local ring.

Ex 12.5.23

Given a proto-graded local ring R of finite embedding dimension, apart from the chromatic powers, we can also take certain 'diatonic' powers. Define the cataprotopower *of R to be the separated quotient of the protopower $R_\flat$. Show that it is a Noetherian local ring, and if the proto-grading is faithfully flat, then its completion is equal to $R_\sharp$. In that case, R is regular if and only if its cataprotopower is.*

Ex 12.5.24

Let A be a faithfully flat, Noetherian proto-graded ring with protopower $A_\flat$. Show that any associated prime ideal of $A_\flat$ is the extension of an associated prime ideal of A, whence its extension to the ultrapower $A_\natural$ remains prime. Conclude that any faithfully flat, Noetherian proto-grade on a one-dimensional Noetherian local ring is prime bounded.

12.6 Project: etale proto-gradings

This project is in essence the proto-graded version of Project 6.6: we will construct a proto-grading on the Henselization of a proto-graded Noetherian local ring $(R, \mathfrak{m})$, and give conditions under which this proto-grading is Noetherian and faithfully flat.

Define a proto-grading on $R^\sim$ by the condition that a Hensel element $y \in R^\sim$ lies in $\Gamma_n(R^\sim)$ if it admits a Hensel system $(\mathscr{H}, \mathbf{u})$ of length $N \leq n$, in which all polynomials have degree at most n, and all coefficients as well as all entries of $\mathbf{u}$ have proto-grade at most n.

12.6.1 *Show that this yields a proto-grading on $R^\sim$, called the* etale proto-grading *on $R^\sim$ extending the proto-grading on R. Show that $R \to R^\sim$ is a morphism of proto-graded rings.*

The following result enables us to calculate protopowers:

12.6.2 *Show that if R is a proto-graded Noetherian local ring and $R^\sim$ is viewed with its etale proto-grading extending the proto-grading on R, then we have an isomorphism*

$$(R^\sim)_\flat \cong (R_\flat)^\sim.$$

In particular, this already yields the first assertion in the next result:

12.6.3 *If R is a proto-graded local ring with a Noetherian proto-grading, then the etale proto-grading on $R^\sim$ is also Noetherian. If R is moreover regular and the proto-grading on R is faithfully flat, then the etale proto-grading is faithfully flat too.*

To prove the second assertion, you also need Exercise 12.5.22 and the Cohen-Macaulay criterion for flatness (Theorem 5.6.16). Conclude with the following result on uniform bounds in the ring of algebraic power series. More precisely, assuming 6.6.4, we now have the etale proto-grading on an algebraic power series ring $k[[\xi]]^{\mathrm{alg}}$ extending the affine proto-grading on the localization of $k[\xi]$ with respect to the maximal ideal generated by the indeterminates ξ, and we have:

12.6.4 *For each pair (n,m) there exists a bound $n' := n'(n,m)$ with the property that if k is an arbitrary field, $R := k[[\xi]]^{alg}$ with ξ an m-tuple of indeterminates, and $I := (f_1, \dots, f_s)R$ an ideal generated by elements f_i of etale proto-grade at most n, then the module of syzygies of I is generated by n' syzygies with entries of proto-grade at most n'. Moreover, if $f \in I$ has etale proto-grade at most n, then there exist algebraic power series g_i of etale proto-grade at most n' such that $f = g_1 f_1 + \cdots + f_s g_s$.*

Chapter 13
Asymptotic homological conjectures in mixed characteristic

In this final chapter, we discuss some of the homological conjectures.[1] We already encountered one of these conjectures—and proved it in equal characteristic; see Theorems 9.4.9 and 10.4.5—, when discussing big Cohen-Macaulay algebras: the Monomial Conjecture. In fact, Hochster has established most of the homological conjectures in equal characteristic by means of the existence of big Cohen-Macaulay modules. Hence probably the 'mother' of all homological conjectures in mixed characteristic is the very existence of a (balanced) big Cohen-Macaulay module (or, preferably, algebra); the best result to date is the existence of these up to dimension three (see [28], based upon the positive solution of the Direct Summand Conjecture in mixed characteristic in dimension three due to Heitmann [25]).

The ultraproduct method is a priori insufficiently powerful to derive the full versions of these conjectures from their equal characteristic counterparts. The idea is to transfer the proven theorems in equal characteristic to the mixed characteristic case via ultraproducts, but since properties only hold almost everywhere on the components, we will only be able to deduce 'asymptotic' versions. This roughly means that the conjecture holds for a particular ring of mixed characteristic if its residue characteristic is sufficiently large with respect to some other invariants associated to the particular problem. The first successful application of the ultraproduct method, however, goes back to the work of Ax-Kochen ([9]), in which they solve a conjecture of Artin over the p-adics (see Theorem 13.1.3). Inspired by this method, I derived asymptotic versions of various homological conjecturtes in mixed characteristic in [56, 61] where the lower bounds on the residue characteristic are in terms of the degrees of the polynomials defining the data. In the terminology of these notes, this is in essence a protoproduct method and will be discussed in §13.2. However, in [63], using instead cataproducts, lower bounds in terms of much more natural invariants (dimension, multiplicity, etc.) can be derived, and this will be discussed in §13.3. Since most of the arguments are outside the scope of these notes, we will most of

[1] Although now theorems in equal characteristic, many remain conjectures in mixed characteristic. Although there may be no consensus as to which conjectures count as 'homological', an extensive list of these conjectures and their interconnection can be found in Hochster's autorative treatise [26].

the time only discuss the method, and leave the details of the proofs to the cited sources.

13.1 The Ax-Kochen-Eršhov principle

One normally states this model-theoretic principle in terms of valued fields, but for our purposes, it is more natural to phrase it as a certain Lefschetz principle for discrete valuation rings, formulated as an isomorphism of certain ultra-discrete valuation rings (recall that the latter are simply ultraproducts of discrete valuation rings). In this formalism, the principle states:

Theorem 13.1.1 ([9, 19, 20]). *If V and V' are two Henselian ultra-discrete valuation rings of the same uncountable cardinality with isomorphic residue fields of characteristic zero, then $V \cong V'$.*

We will use this principle in the following form. For each p, let V_p be a complete discrete valuation ring of mixed characteristic, with residue field k_p of characteristic p. Let ξ be a single indeterminate, and put

$$V_p^{\mathrm{eq}} := k_p[[\xi]] \tag{13.1}$$

We have:

Corollary 13.1.2. *The ultraproduct of the V_p is isomorphic to the ultraproduct of the V_p^{eq}.*

Proof. As stated, one might need to assume the continuum hypothesis, but this can be avoided by taking an ultraproduct with respect to a larger underlying set than just the prime numbers. All we need is that the ultraproduct $V_\natural$ of the V_p has the same cardinality as the ultraproduct $W_\natural$ of the V_p^{eq}, and so we will for sake of simplicity just assume this. Since the residue field of both $V_\natural$ and $W_\natural$ is the field of characteristic zero $k_\natural$, given as the ultraproduct of the k_p, the desired isomorphism now follows immediately from Theorem 13.1.1. □

Artin's problem. A field K is called C_2 if for every homogeneous polynomial $f(\xi) \in K[\xi]$ of degree d in more than d^2 variables ξ, there exists a non-trivial solution in K. Lang proved in [38] that the field of fractions of $\mathbb{F}_p[[\xi]]$ is C_2, and Artin conjectured that the field of p-adics $\mathbb{Q}_p$ too is C_2. However, some counterexamples to the latter conjecture were found, and the optimal result is now:

Theorem 13.1.3. *For each d, there is a bound d' so that if p is a prime number bigger than d', then any homogeneous equation of degree d in more than d^2 variables has a non-trivial solution in $\mathbb{Q}_p$.*

Proof. The existence of a non-trivial solution of $f = 0$ in $\mathbb{Q}_p$ yields after clearing denominators a non-trivial solution in ring of p-adic integers, $\mathbb{Z}_p$. By Corollary 13.1.2, the ultraproduct of all $\mathbb{Z}_p$ is equal to the ultraproduct of the $\mathbb{F}_p[[\xi]]$. Since the assertion can be formulated by a first-order sentence (depending on d), which holds for all $\mathbb{F}_p[[\xi]]$, it holds in their ultraproduct, whence in almost all $\mathbb{Z}_p$, by a double application of Łos' Theorem. This shows that the exceptional set of prime numbers for this fixed d must lie outside any ultrafilter, whence must be finite by Exercise 1.5.3. □

13.2 Asymptotic homological conjectures via protoproducts

The extent to which Artin's question has been answered is indicative of what follows: the truth of a certain property can only be established for sufficiently large p, depending on the complexity of the data. This is best described using the formalism of proto-gradings from Chapter 12.

Affine proto-grade. We will work inside the class $\mathfrak{C}_{\text{DVR}}$ of local affine algebras over a complete discrete valuation ring. More precisely, a local ring $(R, \mathfrak{m})$ belongs to $\mathfrak{C}_{\text{DVR}}$, if its a local V-affine algebra with V a complete discrete valuation ring (recall that this means that R is a localization, with respect to a prime ideal containing the maximal ideal of V, of a finitely generated V-algebra). We will view R with its V-affine proto-grading. For instance, if R is the localization of $V[\xi]$ at the maximal ideal generated by the uniformizing parameter of V and the indeterminates ξ, then $\Gamma_n(R)$ consists of all fractions f/g with $f, g \in V[\xi]$ of degree at most n and $g(0)$ a unit in V.

Although any sequence of rings in $\mathfrak{C}_{\text{DVR}}$ is uniformly proto-graded (namely, proto-F-graded for the function $F(m,n) := m+n$), and hence their protoproduct is well-defined, we cannot expect in general for it to capture much of the information stored in the sequence. For instance, let S be the localization of $V[\xi, \zeta]$ at the maximal ideal generated by the indeterminates and the uniformizing parameter of the discrete valuation ring V, and put $R_n := S/(\xi^n - \zeta^{n-1})S$. By Exercise 13.4.1, the protoproduct $R_\flat$ of the R_n is isomorphic to the protopower $S_\flat$. Of course, what goes wrong in this example is that the defining ideals $(\xi^n - \zeta^{n-1})S$ have unbounded proto-grade.

As we will shall see shortly, we can avoid this phenomenon by introducing the following terminology. For an arbitrary member R of $\mathfrak{C}_{\text{DVR}}$, say of the form $(V[\xi]/I)_\mathfrak{p}$, we say that R itself has *affine proto-grade* at most n, if the number of indeterminates ξ is at most n, and both I and $\mathfrak{p}$ have proto-grade at most n (recall that the latter means that they are generated by at most n elements of degree at most n).[2] There is some ambiguity here in our definition of the affine proto-grade of a ring, because different affine presentations might yield different values. However, since we are only interested in uniform behavior, this will not matter. Depending on

[2] In the articles [56, 61] the affine proto-grade of R was called its *(degree) complexity*.

the situation, we will also make explicit what it means for some additional data to have bounded affine proto-grade.

Approximations and transfer. The method to derive asymptotic properties is a mixture of the methods from Chapter 7, using ultra-hulls, and Chapter 12, using protoproducts. Crucial to either method in deriving bounds was a certain flatness result, which in the present context becomes:

Theorem 13.2.1. *If R_w are members of $\mathfrak{C}_{DVR}$ of affine proto-grade at most n, for some n, then the protoproduct $R_\flat$ is a local $V_\natural$-affine algebra, with $V_\natural$ an ultra-discrete valuation ring, and the canonical map $R_\flat \to R_\natural$ to the ultraproduct is faithfully flat.*

Proof. Since $R_\flat \to R_\natural$ is by construction local, we only need to show its flatness. Since this is a local question, I claim that we may reduce to the following case: for each w, let V_w be a complete discrete valuation ring, and let $A_w := V_w[\xi]$, viewed with its affine proto-grading, then the natural homomorphism $A_\flat \to A_\natural$ is flat. Indeed, assuming this flatness result, let $I_w \subseteq \mathfrak{p}_w$ have affine proto-grade at most n, so that $R_w = (A_w/I_w)_{\mathfrak{p}_w}$. By 12.2.2, there exist ideals $I \subseteq \mathfrak{p} \subseteq A_\flat$ whose extension to $A_\natural$ are the ultraproducts of respectively the I_w and $\mathfrak{p}_w$. Since each $\mathfrak{p}_w$ contains the uniformizing parameter π_w of V_w, an application of Theorem 7.3.4 over the fields $V_w/\pi_w V_w$ yields that $\mathfrak{p}$ is prime. By base change, $(A_\flat)_\mathfrak{p} \to (A_\natural)_{\mathfrak{p}A_\natural}$ is then faithfully flat. By an argument similar to the one proving 12.1.5 (see Exercise 13.4.1), the protoproduct of the A_w/I_w is equal to $A_\flat/(IA_\natural \cap A_\flat)$. Since $I(A_\natural)_{\mathfrak{p}A_\natural} \cap (A_\flat)_\mathfrak{p} = I(A_\flat)_\mathfrak{p}$ by faithful flatness, we showed that $R_\flat = (A_\flat/I)_\mathfrak{p}$. Since $R_\natural = (A_\natural/IA_\natural)_{\mathfrak{p}A_\natural}$, flatness follows by base change. Note that $A_\flat = V_\natural[\xi]$ by Proposition 12.1.7, where $V_\natural$ is the ultraproduct of the V_w, proving that $R_\flat$ is a local $V_\natural$-affine algebra.

So remains to show that $A_\flat \to A_\natural$ is flat, and this can be done by a straightforward modification of the proof of Theorem 12.3.2 (see Exercise 13.4.2). □

As we have observed before, the converse process of an ultraproduct is an approximation. This also applies here. Let as above V_w be complete discrete valuation rings with ultraproduct $V_\natural$. Given an $V_\natural$-affine (local) algebra $R := (V_\natural[\xi]/I)_\mathfrak{p}$, with ξ a finite tuple of indeterminates, I a finitely generated ideal, and $\mathfrak{p}$ a prime ideal lying above the maximal ideal of $V_\natural$ and containing I. We define the approximations of R as follows. Put $A_w := V_w[\xi]$, and let $I_w \subseteq \mathfrak{p}_w \subseteq A_w$ be respective approximations of I and $\mathfrak{p}$. By the argument in the above proof, almost each $\mathfrak{p}_w$ is prime (see Exercise 13.4.4). Moreover, if R has affine proto-grade at most n, then almost all I_w and $\mathfrak{p}_w$ have proto-grade at most n. Hence almost all $R_w := (V_w[\xi]/I_w)_{\mathfrak{p}_w}$ are well-defined members of $\mathfrak{C}_{\mathrm{DVR}}$ and have proto-grade at most n. It is easy to check that their protoproduct $R_\flat$ is equal to R (see Exercise 13.4.5). We will therefore refer to the R_w as *approximations* of R. These approximations, however, depend on the choice of components V_w of $V_\natural$, a fact that has to be borne in mind.

We can also look at this construction from an ultra-hull perspective as follows. In this point of view, the ultraproduct $R_\natural$ of the R_w functions as an ultra-hull of R, called the *ultra-$V_\natural$-hull* of R (or more correctly, the relative $V_\natural$-hull $\mathsf{L}_{V_\natural}(R)$ in the terminology of Exercise 10.5.17). By Theorem 13.2.1, this ultra-hull is faithfully flat . Note that $V_\natural$ is no longer Noetherian, which will account for some of the difficulties below in developing the theory, but it is still a valuation domain (by Exercise 1.5.12) of embedding dimension one since its maximal ideal is generated by the ultraproduct π of the uniformizing parameters π_w. In particular, $V_\natural/\pi V_\natural$ is a field whence Noetherian, and hence any ideal in the above protoproduct $R_\flat$ containing π is finitely generated. This applies in particular to the maximal ideal of the protoproduct, showing that it has finite embedding dimension.

Let us specialize to the case we will encounter shortly. For each prime number p, let V_p be a complete discrete valuation ring of mixed characteristic with residue characteristic p, and let $V_\natural$ be their ultraproduct. Let R_p be a local V_p-affine algebra of affine proto-grade at most n, and let $R_\flat$ be their protoproduct. As we just proved, $R_\flat$ is a local $V_\natural$-affine algebra with approximations R_p and ultra-$V_\natural$-hull $R_\natural$. By Corollary 13.1.2, we may realize $V_\natural$ also as the ultraproduct of the complete discrete valuation rings V_p^{eq} of equal characteristic p (see (13.1)). Hence from this point of view, $R_\flat$ has approximations defined over the various V_p^{eq}, which we therefore denote by R_p^{eq}, and call *equal characteristic approximations* of the R_p (note that they have also bounded affine proto-grade). The ultraproduct of the R_p^{eq} will be denoted $R_\natural^{\mathrm{eq}}$. To distinguish it from $R_\natural$, we will call $R_\natural^{\mathrm{eq}}$ the *equal characteristic ultra-hull* of $R_\flat$, and $R_\natural$ its *mixed characteristic ultra-hull*. Similarly, if $x \in R_\flat$, then an *equal characteristic approximation* of x means an approximation of x viewed as an element in $R_\natural^{\mathrm{eq}}$, that is to say, elements $x_p \in R_p^{\mathrm{eq}}$ with ultraproduct equal to x. Since by construction $R_\flat$ is also the protoproduct of the R_p^{eq}, the canonical embedding $R_\flat \to R_\natural^{\mathrm{eq}}$ is again faithfully flat by Theorem 13.2.1. Of course, we may also reverse the process, going from equal to mixed characteristic instead, as explained in more detail in Exercise 13.4.3.

The fact that both ultra-hulls are faithfully flat over the (common) protoproduct will guarantee a fair amount of transfer between the R_p and their equal characteristic approximations. The following result is but an example of this.

Theorem 13.2.2. *For some n and for each prime number p, let R_p be a local V_p-affine algebra of affine proto-grade at most n over a mixed characteristic complete discrete valuation ring V_p of residue characteristic p, and let R_p^{eq} be an equal characteristic approximation of the R_p. Almost all R_p are regular if and only if almost all R_p^{eq} are regular.*

Proof. Because of symmetry, it suffices to show only one direction, and so we may assume that almost all R_p are regular. Since by assumption R_p has embedding dimension less than or equal to its affine proto-grade n, there are only finitely many possibilities for its dimension, and hence almost all R_p will have the same dimension, say, d. Let $\mathbf{x}_p$ be a regular system of parameters of R_p (see Definition 4.1.5). Note that in particular each $\mathbf{x}_p$ is an R_p-regular sequence (of length d) minimally generating the maximal ideal, by Proposition 4.2.3 and Theorem 4.2.6. By construction, we may choose its entries to be of proto-grade (=degree) at most n, for all p. Hence their ultraproduct $\mathbf{x} := (x_1,\dots,x_d)$ has entries in the protoproduct $R_\flat$. By faithfull flatness, moreover, $\mathbf{x}R_\flat = \mathbf{x}R_\natural \cap R_\flat$ is the maximal ideal of $R_\flat$. Let R_p^{eq}

be equal characteristic approximations of the R_p, let $R_\natural$ and $R_\natural^{\mathrm{eq}}$ be the respective ultraproducts of R_p and R_p^{eq}, and let $\mathbf{x}_p^{\mathrm{eq}} := (x_{1p}^{\mathrm{eq}}, \dots, x_{dp}^{\mathrm{eq}})$ be equal characteristic approximations of $\mathbf{x}$, that is to say, tuples in R_p^{eq} whose ultraproduct is equal to $\mathbf{x}$, where we view the latter as a tuple over $R_\natural^{\mathrm{eq}}$ via the canonical embedding $R_\flat \to R_\natural^{\mathrm{eq}}$. By Łos' Theorem, almost each $\mathbf{x}_p^{\mathrm{eq}}$ generates the maximal ideal of R_p^{eq}. So remains to show that almost each $\mathbf{x}_p^{\mathrm{eq}}$ is a regular sequence, whence a regular system of parameters. Fix some $i \leq d$, let $I_p^{\mathrm{eq}} := (x_{1p}^{\mathrm{eq}}, \dots, x_{i-1,p}^{\mathrm{eq}})R_p^{\mathrm{eq}}$, and, towards a contradiction, assume $z_p^{\mathrm{eq}} x_{ip}^{\mathrm{eq}} \in I_p^{\mathrm{eq}}$, for some z_p^{eq} not in I_p^{eq}. Let $I := (x_1, \dots, x_{i-1})R_\flat$. Since $\mathbf{x}$ is an $R_\natural$-regular sequence by Łos' Theorem, $(IR_\natural : x_i) = IR_\natural$. Since $R_\flat \to R_\natural$ is faithfully flat by Theorem 13.2.1, we get $(IR_\natural : x_i) = (I : x_i)R_\natural$ by Theorem 5.6.16. Faithful flatness then yields $(I : x_i) = I$. On the other hand, by Łos' Theorem, the ultraproduct $z \in R_\natural^{\mathrm{eq}}$ of the z_p^{eq} belongs to $(IR_\natural^{\mathrm{eq}} : x_i)$ but not to $IR_\natural^{\mathrm{eq}}$. Since also $R_\flat \to R_\natural^{\mathrm{eq}}$ is faithfully flat, another application of Theorem 5.6.16 yields $(IR_\natural^{\mathrm{eq}} : x_i) = (I : x_i)R_\natural^{\mathrm{eq}}$, which is then equal to $IR_\natural^{\mathrm{eq}}$ by what we just proved. Hence, z lies in $IR_\natural^{\mathrm{eq}}$, contradiction. □

To formulate analoguous transfer results for arbitrary rings, we have to also face the complications encountered in Chapter 11, where the ultraproduct (and hence the protoproduct and the equal characteristic approximations) may have larger geometric dimension than the components. To control this bad behavior, one has to also bound the parameter degree. In the present setup, this is in fact easier, as the below domain case shows; we refer to [61, §6] for a discussion of the general case.

Proposition 13.2.3. *For some n and for each prime number p, let R_p be a local V_p-affine algebra of affine proto-grade at most n over a mixed characteristic complete discrete valuation ring V_p of residue characteristic p, and let R_p^{eq} be an equal characteristic approximation of the R_p. Then almost all R_p are domains if and only if almost all R_p^{eq} are. Moreover, if this is the case, then almost all R_p and R_p^{eq} have the same dimension.*

Proof. It suffices to show that the protoproduct $R_\flat$ is a domain if and only if its approximations (of either type) are, and that almost all have the same dimension, equal to the geometric dimension of $R_\flat$. Since $R_\flat \subseteq R_\natural$, one direction in the equivalence is immediate. So assume $R_\flat$ is a domain of geometric dimension d, and we need to show that then so is $R_\natural$. Let V be the ultraproduct of the V_p, let $\pi \in V$ be a generator of its maximal ideal with approximations $\pi_p \in V_p$, and let Q be the field of fractions of V. If $\pi = 0$ in $R_\flat$, then $R_\flat$ is in fact a local affine algebra over the residue field $V/\pi V$, and hence $R_\natural$ is a domain by Theorem 7.3.4. Moreover, the R_p are then the approximations of $R_\flat$, whence almost all have dimension d by Corollary 7.3.3. So we may assume $\pi \neq 0$ in $R_\flat$. Since $R_\flat$ is a domain, π is then $R_\flat$-regular, whence $R_\flat$ is torsion-free over V by Exercise 13.4.7. In particular, $R_\flat \otimes_V Q$ is again a domain. By Theorem 7.3.4 once more, $R_\natural \otimes_V Q$ being the ultra-hull of $R_\flat \otimes_V Q$, is a domain too. On the other hand, since $R_\flat \to R_\natural$ is faithfully flat, $R_\natural$ is torsion-free over V by Theorem 5.6.16. Hence the natural map $R_\natural \to R_\natural \otimes_V Q$ is injective, showing that $R_\natural$ is a domain. Since π is $R_\flat$-regular, it is part of a system of parameters in $R_\flat$ by

Exercise 13.4.7. In particular, $R_\flat/\pi R_\flat$ has (geometric) dimension $d-1$. By Corollary 7.3.3 once more, almost all $R_p/\pi_p R_p$ have dimension $d-1$. Since almost all R_p are domains, almost all of them must therefore have dimension d. □

Asymptotic Direct Summand Conjecture. Let $\mathscr{P}$ be a property of Noetherian local rings and some additional finite amount of data (to be made precise in each case). In the terminology of page 207, we can now define what it means for a property to hold asymptotically.

Definition 13.2.4. We will say that property $\mathscr{P}$ holds *asymptotically in mixed characteristic*, if for each n, there exists n' only depending on n, such that a Noetherian local ring of mixed characteristic R in $\mathfrak{C}_{\mathrm{DVR}}$ satisfies $\mathscr{P}$ provided its residue characteristic p is at least n', where R and the additional data have affine proto-grade at most n.

We will illustrate this terminology by means of the Direct Summand Conjecture, which states that given a finite extension of local rings $R \to S$, if R is regular, then R is a direct summand of the R-module S (the reader should convince him/herself that this is weaker than saying that $R \to S$ is split). The Direct Summand Conjecture is related to another of the homological conjectures, to wit the Monomial Conjecture, which we already encountered before (see Theorems 9.4.9 and 10.4.5, and also 13.2.8 below):

Theorem 13.2.5 (Direct Summand Conjecture). *If S is a Noetherian local ring for which the Monomial Conjecture holds, then for any finite extension $R \subseteq S$ with R regular, R is a direct summand of S.*

In particular, the Direct Summand Conjecture holds for any Noetherian local ring of equal characteristic.

Proof. The second assertion follows from the first, in view of Theorem 10.4.5. To prove the first, one shows that R is a direct summand of S if and only if some regular system of parameters of R is monomial when viewed as a tuple in S; see [13, Lemma 9.2.2]. □

In mixed characteristic, the Direct Summand Conjecture is still wide open, but we can now show:

Theorem 13.2.6. *The Direct Summand Conjecture holds asymptotically in mixed characteristic.*

Proof. Let us be more precise as to the exact statement. There exists for each n a bound n' with the following property. Let V be a complete discrete valuation ring of mixed characteristic, let R and S be local V-affine algebras of affine proto-grade at most n, such that $R \subseteq S$ is a finite extension and R is regular. Now, if the residue characteristic of R is at least n', then R is a direct summand of S. Here we must view the affine proto-grade of S via its presentation as a finite R-module. In order to prove this, we suppose by way of contradiction that no such bound exists for n. Hence we can find for each prime number p a counterexample consisting of a complete discrete valuation ring V_p of residue characteristic p, and local V_p-affine algebras $R_p \subseteq S_p$ of affine proto-grade at most n with R_p regular and S_p finitely generated as an R_p-module, such that R_p is not a direct summand of S_p. Let $R_\flat \subseteq S_\flat$ be the respective protoproducts. It is not hard to show that this is again a finite extension. Let $V_\natural$ be the ultraproduct of the V_p, so that by the above discussion $R_\flat$ and $S_\flat$ are local $V_\natural$-affine algebras. Let R_p^{eq} and S_p^{eq} be the equal characteristic approximations of the R_p and S_p respectively, and let $R_\natural^{\text{eq}}$ and $S_\natural^{\text{eq}}$ be their respective ultraproducts. By Theorem 13.2.2, almost all R_p^{eq} are regular. Moreover, it is not hard to show that $R_p^{\text{eq}} \to S_p^{\text{eq}}$ is a finite extension for almost all p (see Exercise 13.4.6). By Theorem 13.2.5, almost each R_p^{eq} is a direct summand of S_p^{eq}, and by Łos' Theorem, this in turn implies that $R_\natural^{\text{eq}}$ is a direct summand of $S_\natural^{\text{eq}}$. By faithful flatness (Theorem 13.2.1), this then yields that $R_\flat$ is a direct summand of $S_\flat$, whence $R_\natural$ is a direct summand of $S_\natural$, and by Łos' Theorem, we finally derive the desired contradiction that almost each R_p is a direct summand of S_p (see Exercise 13.4.6). □

The asymptotic weak Monomial Conjecture and big Cohen-Macaulay algebras. To prove an asymptotic version of the Monomial Conjecture, we introduce the following terminology. Let $(R,\mathfrak{m})$ be a local ring of finite embedding dimension. Recall that the *Monomial Conjecture* holds in R if every system of parameters is monomial (see the discussion preceeding Theorem 9.4.9). In case R is an ultra-ring, we say that the *Ultra-monomial Conjecture* holds in R, if for every system of parameters $(x_1,\dots,x_d)$ we have

$$(x_1 \cdots x_d)^{\alpha-1} \notin (x_1^\alpha,\dots,x_d^\alpha)R. \tag{13.2}$$

for all $\alpha \in \mathbb{N}_\natural$ (for the definition of ultra-exponentation, see page 12).

By a *strong system of parameters* $\mathbf{x}$ in R, we mean a system of parameters of R which is also part of a minimal system of generators of $\mathfrak{m}$. In other words, if R has geometric dimension d and embedding dimension e, then a d-tuple $\mathbf{x}$ is a strong system of parameters if and only if $R/\mathbf{x}R$ has geometric dimension zero and embedding dimension $e-d$. We say that the *weak Monomial Conjecture* holds in R if some strong system of parameters is monomial. Even this weak version is not known to hold in general for Noetherian local rings of mixed characteristic (the monomial systems of parameters given by Remark 9.4.10 never generate the maximal ideal when $t > 1$).

Before we prove an asymptotic version of this weaker conjecture, we must introduce big Cohen-Macaulay algebras in the present setup. We call an R-algebra B a *big Cohen-Macaulay algebra* if some system of parameters is B-regular, and a *balanced big Cohen-Macaulay algebra* if any system of parameters is B-regular. The Monomial Conjecture holds in any local ring admitting a balanced big Cohen-Macaulay algebra by the argument in proof of Theorem 9.4.9.

Proposition 13.2.7. *For some n and for each prime number p, let R_p be a local V_p-affine domain of affine proto-grade at most n over a mixed characteristic complete discrete valuation ring V_p of residue characteristic p. Then the protoproduct $R_\flat$ of* *the R_p admits a balanced big Cohen-Macaulay algebra $\mathcal{B}(R_\flat)$. In particular, the Ultra-monomial Conjecture holds in $R_\flat$.*

Proof. Let R_p^{eq} be equal characteristic approximations of the R_p. By Proposition 13.2.3, almost all R_p^{eq} are domains, and hence almost each $(R_p^{\mathrm{eq}})^+$ is a balanced big Cohen-Macaulay algebra by Theorem 9.4.1. Let $\mathcal{B}(R_\flat)$ be the ultraproduct of the $(R_p^{\mathrm{eq}})^+$. Let d be the geometric dimension of $R_\flat$, and let $\mathbf{x}$ be a system of parameters in $R_\flat$ with equal characteristic approximation $\mathbf{x}_p^{\mathrm{eq}}$ (so that each $\mathbf{x}_p^{\mathrm{eq}}$ is a d-tuple in R_p^{eq}). By Proposition 13.2.3, almost all R_p^{eq} have dimension d, and hence an easy application of Łos' Theorem yields that almost each $\mathbf{x}_p^{\mathrm{eq}}$ is a system of parameters in R_p^{eq}, whence $(R_p^{\mathrm{eq}})^+$-regular. By Łos' Theorem, $\mathbf{x}$ is therefore $\mathcal{B}(R_\flat)$-regular, as we wanted to show. The last assertion now easily follows by the usual argument (see Exercise 13.4.10). □

Theorem 13.2.8. *The weak Monomial Conjecture holds asymptotically in mixed characteristic for domains.*

Proof. Suppose not, so that for some n, we can find for each p, a mixed characteristic complete discrete valuation ring V_p of residue characteristic p, and a local V_p-affine domain R_p of affine proto-grade at most n, such that any strong system of parameters fails to be monomial. Let $R_\flat$ and $R_\natural$ be the respective protoproduct and ultraproduct of the R_p. By Exercise 13.4.11, the embedding dimension of $R_\flat$ is equal to that of almost all R_p. Since almost all R_p have also the same geometric dimension as $R_\flat$ by Proposition 13.2.3, it follows that a strong system of parameters $\mathbf{x}$ in $R_\flat$ has an approximation $\mathbf{x}_p$ almost each of which is a strong system of parameters in R_p. Hence, by assumption, almost each $\mathbf{x}_p$ fails to satisfy (9.8) for at least one exponent, say $k = \alpha_p$. Let α be the ultraproduct of the α_p, so that by Łos' Theorem, (13.2) fails in $R_\natural$ for the ultra-exponent α. By faithful flatness, this failure already is witnessed in $R_\flat$, contradicting that the Ultra-monomial Conjecture holds in that ring by Proposition 13.2.7. □

In the papers [56, 61] many more homological conjectures are proven to hold asymptotically, including the Hochster-Roberts Conjecture (see §13.5).

13.3 Asymptotic homological conjectures via cataproducts

We now discuss a second method for obtaining asymptotic properties in mixed characteristic, via cataproducts. Moreover, this method, when applicable, will give sharper results, where the residue characteristic has to be only large with respect to some more natural invariants than the affine proto-grade, and where in fact we no longer need to assume that the ring is affine. Moreover, there is a second version, where this time not the residue characteristic, but the ramification index (see page 214 below) has to be sufficiently large. Unfortunately, asymptotic versions of some of the homological conjectures, like the Direct Summand and the Monomial Conjecture that were treated by the previous method, elude at present treatment by the cataproduct method.

Ramification. Let us first discuss how to go from mixed to equal characteristic by means of cataproducts. One way, of course, is already quite familiar to us: the cataproduct of local rings of different residue characteristic has (residue) characteristic zero. However, there is a second way. Given a local ring $(R,\mathfrak{m})$ of residue characteristic p, we call the $\mathfrak{m}$-adic order of p its *ramification index*, that is to say, the ramification index of R is the largest n such that $p \in \mathfrak{m}^n$. If the ramification index is one, we call R *unramified*, and if the ramification index is infinite (that is to say, p is an infinitesimal, including the case that $p = 0$, the equal characteristic case), we say that it is *infinitely ramified*. Since a Noetherian local ring does not have non-zero infinitesimals, being infinitely ramified is the same as having equal characteristic, but not so for arbitrary local rings, and here lies the clue to obtain equal characteristic cataproducts:

13.3.1 *Let R_w be mixed characteristic local rings of bounded embedding dimension, with residue characteristic p. If the R_w have unbounded ramification index (that is to say, for all n, almost all R_w have ramification index at least n), then the cataproduct $R_\sharp$ has equal characteristic p.*

Indeed, the ultraproduct is infinitely ramified by Łos' Theorem, whence the cataproduct has equal characteristic p, as it is Noetherian by Theorem 11.1.4. Balanced big Cohen-Macaulay algebras are available in this setup too:

13.3.2 *If an ultra-Noetherian local ring R has either equal characteristic or is infinitely ramified, then it admits a balanced big Cohen-Macaulay algebra $B(R)$.*

Indeed, under either assumption, the separated quotient $R_\sharp$ is an equal characteristic Noetherian local ring by Theorem 11.1.4. If $\mathfrak{p}$ is a maximal dimensional prime ideal in $R_\sharp$, then any system of parameters in R remains one in $R_\sharp$ whence in $R_\sharp/\mathfrak{p}$, and therefore is $B(R_\sharp/\mathfrak{p})$-regular by Theorem 10.4.4. Hence $B(R) := B(R_\sharp/\mathfrak{p})$ yields the desired balanced big Cohen-Macaulay algebra. □

Asymptotic Improved New Intersection Conjecture. The last of the homological conjectures we will discuss is an 'intersection' conjecture. The original conjecture, called the *Intersection Conjecture* was proven by Peskine and Szpiro in [44], using properties of the Frobenius in positive characteristic, and lifting the result to characteristic zero by means of Artin Approximation (virtually the same lifting technique as for HH-tight closure discussed in §8.6). Hochster and others (see, for instance, [21, 26, 27, 33]) formulated and subsequently proved generalizations of this result in equal characteristic, called 'new' and 'improved' intersection theorems. In fact, the New Intersection Theorem (whence also the original one) was established in mixed characteristic as well by Roberts in [47]. However, the most general of them all, the so-called Improved New Intersection Conjecture is only known to hold in equal characteristic. It is concerned with the length of a finite free complete with finite homology. Its asymptotic version reads:

Theorem 13.3.3 (Asymptotic Improved New Intersection Theorem). *For each triple of non-negative integers (m,r,l), there exists a bound $e(m,r,l)$ with the following property. Let R be a Noetherian local ring of mixed characteristic and let $F_\bullet$*

be a finite complex of finitely generated free R-modules. Assume R has embedding dimension at most m and each module in $F_\bullet$ has rank at most r.

If each $H_i(F_\bullet)$, for $i > 0$, has length at most l and if $H_0(F_\bullet)$ has a non-zero minimal generator generating a submodule of length at most l, then the dimension of R is less than or equal to the length of the complex $F_\bullet$, provided R has either residue characteristic or ramification index at least $e(m,r,l)$.

Proof. We will give the proof modulo one result, Theorem 13.3.4 below. Since the dimension of R is at most m, there is nothing to show for complexes of length m or higher. Suppose the result is false for some triple (m,r,l), so that we can find for each w a counterexample consisting of a d_w-dimensional mixed characteristic Noetherian local ring R_w of embedding dimension at most m such that each R_w has either residue characteristic or ramification index at least w, and a complex $F_{w\bullet}$ of length $s_w \leq m$ consisting of finitely generated free R_w-modules of rank at most r such that all its higher homology has length at most l and such that its cokernel admits a non-zero minimal generator μ_w generating a submodule of length at most l, but such that $s_w < d_w$. Let $R_\natural$ and $R_\sharp$ be the respective ultraproduct and cataproduct of the R_w, and let μ, s and d be the ultraproduct of the μ_w, s_w and d_w respectively. In particular, $s < d \leq m$ and almost all s_w and d_w are equal to s and d respectively. By Exercise 11.3.1, the geometric dimension of $R_\natural$ is at least d. Let $F_\bullet$ be the ultraproduct of the complexes $F_{w\bullet}$. Since the ranks are at most r, each module in $F_\bullet$ will be a free $R_\natural$-module of rank at most r. Since ultraproducts commute with homology and preserve uniformly bounded length by Exercise 1.5.10, the higher homology $H_i(F_\bullet)$ has finite length (at most l) and so has the $R_\natural$-submodule of $H_0(F_\bullet)$ generated by μ. In particular, $F_\bullet$ is acyclic when localized at a non-maximal prime ideal, so that s is at least the geometric dimension of $R_\natural$ by Theorem 13.3.4 below, and hence $s \geq d$, contradiction. □

For the homological terminology used in the next result, see page 65.

Theorem 13.3.4. *Let $(R,\mathfrak{m})$ be an ultra-Noetherian local ring, and assume R has either equal characteristic or is infinitely ramified. Let $F_\bullet$ be a finite complex of finitely generated free R-modules, and let M be its cokernel. If $F_\bullet$ is acyclic when localized at any prime ideal of R different from $\mathfrak{m}$, and if there exists a non-zero minimal generator of M whose annihilator is $\mathfrak{m}$-primary, then the geometric dimension of R is less than or equal to the length of $F_\bullet$.*

Proof. The proof is really just a modification of the classical proof (see [63, Corollary 10.9] for details). As with most homological conjectures, they become easy to prove if the ring is moreover Cohen-Macaulay, and in this particular instance, this is because of the Buchsbaum-Eisenbud Acyclicity criterion ([13, Theorem 9.1.6]). It was Hochster's ingenious observation that instead of the ring being Cohen-Macaulay, it suffices for the proofs to to through that there exists a balanced big Cohen-Macaulay module. In the present situation, this is indeed the case due to 13.3.2. □

13.4 Exercises

Ex 13.4.1

Let V_n be discrete valuation rings with uniformizing parameter π_n, let ξ be a finite tuple of indeterminates, and let S_n be the localization of $V_n[\xi]$ at the maximal ideal generated by ξ and π_n. Let $I_n \subseteq S_n$ be ideals and put $R_n := S/I_n$. Show that the protoproduct $R_\flat$ of the R_n is isomorphic to the image of $S_\flat$ inside the ultraproduct $R_\natural$. Use this to prove the claim on page 207 about the protoproduct of the given sequence, by showing that in this example the natural map $V_\natural[\xi,\zeta] \to R_\natural$ is in fact injective.

Ex 13.4.2

Give the details of the proof of Theorem 13.2.1.

Ex 13.4.3

Show, using Cohen's structure theorem, that any ultraproduct of complete discrete valuation rings W_p of equal characteristic p is the ultraproduct of complete discrete valuation rings W_p^{mix} of mixed characteristic, and that to any collection of local W_p-affine algebras S_p of affine proto-grade at most d, for some d, we can associate local W_p^{mix}-affine algebras S_p^{mix}, called mixed characteristic approximations *of the S_p. Show that the S_p are the equal characteristic approximations of the S_p^{mix}.*

Ex 13.4.4

Use the argument in the proof of Theorem 13.2.1 to show that if $\mathfrak{p}$ is a prime ideal in a V-affine algebra lying above the maximal ideal of an ultra-discrete valuation ring V, then almost all of its approximations $\mathfrak{p}_w$ are prime. Show also that $\mathfrak{p}$ is finitary in the sense used in Exercise 12.5.17.

We may paraphrase these results by saying that the affine proto-grading on the category $\mathfrak{C}_{DVR}$ is prime bounded above the maximal ideal of the discrete valuation ring.

Ex 13.4.5

Show that given an V-affine local algebra R, with V an ultra-discrete valuation ring, is isomorphic to the protoproduct of its approximations as defined in our discussion following Theorem 13.2.1.

Ex 13.4.6

Show that if $R \to S$ is a local homomorphism of local V-affine algebras over an ultra-discrete valuation ring V, with respective approximations R_p and S_p, then $R \to S$ is a finite extension if and only if almost all $R_p \to S_p$ are; moreover, under these assumptions, R is a direct summand of S if and only if almost each R_p is a direct summand of S_p.

Ex 13.4.7

Show that if V is an ultra-discrete valuation rings with maximal ideal generated by π, and R is a local V-affine algebra, then R is torsion-free over V (meaning that any non-zero element of V is R-regular) if and only if π is R-regular.

Show that if this is the case, then π is part of a system of parameters of R, or more concretely, that $R/\pi R$ has dimension $d-1$, where d is the geometric dimension of R. To this end, show that the separated quotient $R/\mathfrak{I}_R$ is isomorphic to $R/\mathfrak{I}_V R$, whence in particular is Noetherian, and that π is also $R/\mathfrak{I}_R$-regular.

Ex 13.4.8

Show using Theorem 11.1.10, in conjunction with 11.1.7 and Proposition 13.2.3, that there exists for each n a bound n' such that if $R \in \mathfrak{C}_{DVR}$ is a domain of proto-grade at most n, then its parameter degree is at most n'.

***Ex 13.4.9**

For each p, let R_p be a local V_p-affine algebra of affine proto-grade at most n over a mixed characteristic complete discrete valuation ring V_p of residue characteristic p, and let R_p^{eq} be an equal characteristic approximation of the R_p. Show that R_p and R_p^{eq} have isomorphic cataproducts. In order to prove this, generalize Exercise 12.5.23, by showing that both cataproducts are equal to the completion of the cataprotoproduct, that is to say, the separated quotient of the protoproduct $R_\flat$.

Ex 13.4.10

Prove in detail the last assertion of Proposition 13.2.7.

Ex 13.4.11

Show that a local $V_\natural$-affine ring over an ultra-discrete valuation ring V has the same embedding dimension as almost all of its approximations.

13.5 Project: Asymptotic Hochster-Roberts Conjecture

The goal of this project is to prove the following asymptotic version of the Hochter-Roberts Theorem.

Theorem 13.5.1. *For each n, there exists a bound n' such that if $R \to S$ is a cyclically pure homomorphism of mixed characteristic local rings in $\mathfrak{C}_{DVR}$ of affine proto-grade at most n, and if S is regular, then R is Cohen-Macaulay provided its residue characteristic is at least n'.*

In what follows, V denotes an ultra-discrete valuation ring and π is a generator of its maximal ideal. Let R be a local V-affine ring with approximations R_w. Again, a flatness result underlies the entire method, which you can prove using the equational criterion (Theorem 5.6.1):

13.5.2 *If almost all R_w are regular, then the natural map $R \to B(R)$ is faithfully flat.*

In [56, 61, 63], we introduced the following terminology: a local ring R is called *pseudo-regular* if it has the same embedding dimension as geometric dimension, and *pseudo-Cohen-Macaulay* if it has the same depth as geometric dimension, that is to say, if R admits a system of parameters which is also an R-regular sequence (here we use the naive definition of depth as being the maximal length of a regular sequence—in a more general setup this is the wrong definition, but it is fine in the present situation). Show the following two transfer results:

13.5.3 *If almost all R_w are regular, then R is pseudo-regular.*

13.5.4 *If R is a domain, then almost all R_w are Cohen-Macaulay if and only if R is pseudo-Cohen-Macaulay.*

Use the existence of balanced big Cohen-Macaulay algebras (and show a weak functoriality property for them) to prove the following:

13.5.5 *Let $R \to S$ be a local homomorphism of local V-affine domains. If $R/\pi R \to S/\pi S$ is cyclically pure and S is pseudo-regular, then R is pseudo-Cohen-Macaulay.*

Combining these results, we can now prove Theorem 13.5.1 by the usual ultraproduct method ad absurdum. The key point is to show that $R_\flat/\pi R_\flat \to S_\flat/\pi S_\flat$ is cyclically pure for the protoproducts $R_\flat$ and $S_\flat$ of the respective counterexamples.

References

1. André, M.: Localisation de la lissité formelle. Manuscripta Math. **13**, 297–307 (1974) 176
2. Artin, M.: Algebraic approximation of structures over complete local rings. Inst. Hautes Études Sci. Publ. Math. **36**, 23–58 (1969) 93, 142, 155, 157, 158
3. Artin, M., Rotthaus, C.: A structure theorem for power series rings. In: Algebraic Geometry and Commutative Algebra: in honor of M. Nagata, vol. I, pp. 35–44. Kinokuniya, Tokyo (1988) 169
4. Aschenbrenner, M.: Ideal membership in polynomial rings over the integers. J. Amer. Math. Soc. **17**(2), 407–441 (2004) 192
5. Aschenbrenner, M.: Bounds and definability in polynomial rings. Quart. J. Math. **56**(3), 263–300 (2005) 194
6. Aschenbrenner, M., Schoutens, H.: Lefschetz extensions, tight closure and big Cohen-Macaulay algebras. Israel J. Math. (to appear) 80, 153, 158, 162, 163
7. Atiyah, M., Macdonald, G.: Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass. (1969) 11, 26, 44, 47, 48, 66
8. Ax, J.: The elementary theory of finite fields. Ann. of Math. (2) **88**, 239–271 (1968) 13
9. Ax, J., Kochen, S.: Diophantine problems over local fields I, II. Amer. J. Math. **87**, 605–630, 631–648 (1965) 205, 206
10. Becker, J., Denef, J., van den Dries, L., Lipshitz, L.: Ultraproducts and approximation in local rings I. Invent. Math. **51**, 189–203 (1979) 157, 158, 166
11. Boutot, J.F.: Singularités rationelles et quotients par les groupes réductifs. Invent. Math. **88**, 65–68 (1987) 130
12. Briançon, J., Skoda, H.: Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de C^n. C. R. Acad. Sci. Paris **278**, 949–951 (1974) 128
13. Bruns, W., Herzog, J.: Cohen-Macaulay Rings. Cambridge University Press, Cambridge (1993) 80, 211, 215
14. Chase, S.: Direct products of modules. Trans. Amer. Math. Soc. **97**, 457–473 (1960) 86
15. Denef, J., Lipshitz, L.: Ultraproducts and approximation in local rings II. Math. Ann. **253**, 1–28 (1980) 157
16. Denef, J., Schoutens, H.: On the decidability of the existential theory of $\mathbb{F}_p[[t]]$. In: Valuation theory and its applications, Vol. II (Saskatoon, 1999), *Fields Inst. Commun.*, vol. 33, pp. 43–60. Amer. Math. Soc. (2003) 157
17. Ein, L., Lazarsfeld, R., Smith, K.: Uniform bounds and symbolic powers on smooth varieties. Invent. Math. **144**, 241–252 (2001) 131
18. Eisenbud, D.: Commutative Algebra with a View toward Algebraic Geometry, *Graduate Texts in Mathematics*, vol. 150. Springer-Verlag, New York (1995) 8, 14, 47, 62, 67, 74, 75, 79, 97, 102, 133, 199, 202
19. Eršhov, Y.: On the elementary theory of maximal normed fields I. Algebra i Logica **4**, 31–69 (1965) 206

20. Eršhov, Y.: On the elementary theory of maximal normed fields II. Algebra i Logica **5**, 8–40 (1966) 206
21. Evans, E., Griffith, P.: The syzygy problem. Ann. of Math. **114**, 323–333 (1981) 214
22. Glaz, S.: Commutative coherent rings, *Lect. Notes in Math.*, vol. 1371. Springer-Verlag, Berlin (1989) 191, 193, 202
23. Hara, N.: A characterization of rational singularities in terms of injectivity of Frobenius maps. Amer. J. Math. **120**, 981–996 (1998) 130
24. Hartshorne, R.: Algebraic Geometry. Springer-Verlag, New York (1977) 8, 15, 33, 36
25. Heitmann, R.: The direct summand conjecture in dimension three. Ann. of Math. **156**, 695–712 (2002) 205
26. Hochster, M.: Topics in the Homological Theory of Modules over Commutative Rings, *CBMS Regional Conf. Ser. in Math*, vol. 24. Amer. Math. Soc., Providence, RI (1975) 132, 205, 214
27. Hochster, M.: Canonical elements in local cohomology modules and the direct summand conjecture. J. Algebra **84**, 503–553 (1983) 214
28. Hochster, M.: Big Cohen-Macaulay algebras in dimension three via Heitmann's theorem. J. Algebra **254**, 395–408 (2002) 205
29. Hochster, M., Huneke, C.: Infinite integral extensions and big Cohen-Macaulay algebras. Ann. of Math. **135**, 53–89 (1992) 145
30. Hochster, M., Huneke, C.: Applications of the existence of big Cohen-Macaulay algebras. Adv. in Math. **113**, 45–117 (1995) 147
31. Hochster, M., Huneke, C.: Tight closure in equal characteristic zero (2000). Preprint on http://www.math.lsa.umich.edu/˜hochster/tcz.ps.Z 132
32. Hochster, M., Huneke, C.: Comparison of symbolic and ordinary powers of ideals. Invent. Math. **147**, 349–369 (2002) 131
33. Hochster, M., Roberts, J.: Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. Adv. in Math. **13**, 115–175 (1974) 214
34. Hodges, W.: Model Theory. Cambridge University Press, Cambridge (1993) 5, 8, 9, 10
35. Hübl, R., Swanson, I.: Discrete valuations centered on local domains. J. Pure Appl. Algebra **161**(1-2), 145–166 (2001) 178
36. Huneke, C.: Tight Closure and its Applications, *CBMS Regional Conf. Ser. in Math*, vol. 88. Amer. Math. Soc. (1996) 119, 124, 130, 132, 146, 152
37. Kempf, G.: Algebraic varieties, *London Mathematical Society Lecture Note Series*, vol. 172. Cambridge University Press, Cambridge (1993) 33, 36
38. Lang, S.: On quasi algebraic closure. Ann. of Math. **55**, 373–390 (1952) 206
39. Marker, D.: Introduction to the model theory of fields. In: Model Theory of Fields, *Lect. Notes in Logic*, vol. 5, pp. 1–37. Springer-Verlag, Berlin (1996) 5, 8
40. Matsumura, H.: Commutative Algebra. W.A. Benjamin (1970) 120
41. Matsumura, H.: Commutative Ring Theory. Cambridge University Press, Cambridge (1986) 11, 14, 44, 48, 51, 57, 58, 67, 74, 75, 79, 97, 99, 146, 176
42. Milne, J.: Etale Cohomology. 33. Princeton Math. (1980) 102
43. Nagata, M.: Local Rings. Interscience Tracts in Pure and Applied Mathematics. Interscience Publishers (1962) 102
44. Peskine, C., Szpiro, L.: Dimension projective finie et cohomologie etale. Inst. Hautes Études Sci. Publ. Math. **42**, 47–119 (1972) 132, 214
45. Popescu, D.: General Néron desingularization and approximation. Nagoya Math. J. **104**, 85–115 (1986) 155
46. Raynaud, M.: Anneaux Locaux Henséliens. Lect. Notes in Math. Springer-Verlag (1970) 102
47. Roberts, P.: Le théorème d'intersections. C. R. Acad. Sci. Paris **304**, 177–180 (1987) 214
48. Rothmaler, P.: Introduction to model theory, *Algebra, Logic and Applications*, vol. 15. Gordon and Breach Science Publishers, Amsterdam (2000) 5
49. Rotthaus, C.: On the approximation property of excellent rings. Invent. Math. **88**, 39–63 (1987) 161
50. Sabbagh, G.: Coherence of polynomial rings and bounds in polynomial ideals. J. Algebra **31**, 499–507 (1974) 189, 193

51. Schmidt, K.: Bounds and definability over fields. J. Reine Angew. Math. **377**, 18–39 (1987) 114
52. Schmidt, K., van den Dries, L.: Bounds in the theory of polynomial rings over fields. A non-standard approach. Invent. Math. **76**, 77–91 (1984) 107, 114
53. Schoutens, H.: Bounds in cohomology. Israel J. Math. **116**, 125–169 (2000) 107, 114
54. Schoutens, H.: Uniform bounds in algebraic geometry and commutative algebra. In: Connections between model theory and algebraic and analytic geometry, *Quad. Mat.*, vol. 6, pp. 43–93. Dept. Math., Seconda Univ. Napoli, Caserta (2000) 107, 114, 158
55. Schoutens, H.: Lefschetz principle applied to symbolic powers. J. Algebra Appl. **2**, 177–187 (2003) 131
56. Schoutens, H.: Mixed characteristic homological theorems in low degrees. C. R. Acad. Sci. Paris **336**, 463–466 (2003) 171, 205, 207, 213, 218
57. Schoutens, H.: Non-standard tight closure for affine $\mathbb{C}$-algebras. Manuscripta Math. **111**, 379–412 (2003) 107, 142, 143, 144, 145
58. Schoutens, H.: Projective dimension and the singular locus. Comm. Algebra **31**, 217–239 (2003) 86
59. Schoutens, H.: Log-terminal singularities and vanishing theorems via non-standard tight closure. J. Alg. Geom. **14**, 357–390 (2005) 130, 144
60. Schoutens, H.: Absolute bounds on the number of generators of Cohen-Macaulay ideals of height at most two. Bull. Soc. Math. Belg. **13**, 719–732 (2006) 174
61. Schoutens, H.: Asymptotic homological conjectures in mixed characteristic. Pacific J. Math. **230**, 427–468 (2007) 171, 205, 207, 210, 213, 218
62. Schoutens, H.: Classifying singularities up to analytic extensions of scalars (in preparation). ArXiv: AC/0508236 100
63. Schoutens, H.: Dimension theory for local rings of finite embedding dimension (in preparation). (2004) preprint 171, 174, 205, 215, 218
64. Schoutens, H.: Bounds in polynomial rings over Artinian local rings. Monatsh. Math. (to appear) 114, 190, 194, 201
65. Schoutens, H.: Pure subrings of regular rings are pseudo-rational. Trans. Amer. Math. Soc. (to appear) 130
66. Seidenberg, A.: Constructions in algebra. Trans. Amer. Math. Soc. **197**, 273–313 (1974) 187, 192
67. Smith, K.: Tight closure of parameter ideals. Invent. Math. **115**, 41–60 (1994) 146
68. Smith, K.: F-rational rings have rational singularities. Amer. J. Math. **119**, 159–180 (1997) 130
69. Spivakovsky, M.: A new proof of D. Popescu's theorem on smoothing of ring homomorphisms. J. Amer. Math. Soc. **12**, 381–444 (1999) 155
70. Swan, R.: Néron-Popescu desingularization (Spring 1995). Expanded notes from a Univ. of Chicago series of lectures 155
71. Swanson, I.: Linear equivalence of ideal topologies. Math. Z. **234**(4), 755–775 (2000) 178