Chapter 3
Flatness

To effectively apply ultraproducts to commutative algebra, we will use, as our main tool, flatness. Since it is neither as intuitive nor as transparent as many other concepts from commutative algebra, we review quickly some basic facts, and then discuss some flatness criteria that will be used later on. Flatness is an extremely important and versatile property, which underlies many deeper results in commutative algebra and algebraic geometry. In fact, I dare say that many a theorem or conjecture in commutative algebra can be recast as a certain flatness result; an instance is Proposition 6.4.6. With David Mumford, the great geometer, we observe:

> "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers."
>
> [22, p. 214]

3.1 Flatness

Flatness is in essence a homological notion, so we start off with reviewing some homological algebra.

3.1.1 Complexes.

Recall that by a *complex*, over some ring A, we mean a (possibly infinite) sequence of A-module homomorphisms $M_i \xrightarrow{d_i} M_{i-1}$, for $i \in \mathbb{Z}$, such that the composition of any two consecutive maps is zero. We often simply will say that

$$\dots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} M_{i-2} \xrightarrow{d_{i-2}} \dots \qquad (M_\bullet)$$

is a complex. The d_i are called the the *boundary maps* of the complex, and often are omitted from the notation. Of special interest are those complexes in which all modules from a certain point on, either on the left or on the right, are zero (which forces the corresponding maps to be zero as well). Such a complex will be called *bounded* from the left or right respectively. In that case, one often renumbers so that the first non-zero module is labeled with $i = 0$. If $M_\bullet$ is bounded from the left, one also might reverse the numbering, indicate this notationally by writing $M^\bullet$, and refer to this situation as a *co-complex* (and more generally, add for emphasis the prefix 'co-' to any object associated to it).

3.1.2 Homology.

Since the composition $d_{i+1} \circ d_i$ is zero, we have in particular an inclusion $\mathrm{Im}(d_{i+1}) \subseteq \mathrm{Ker}(d_i)$. To measure in how far this fails to be an equality, we define the *homology* $\mathrm{H}_\bullet(M_\bullet)$ of $M_\bullet$ as the collection of modules

$$\mathrm{H}_i(M_\bullet) := \mathrm{Ker}(d_i)/\,\mathrm{Im}(d_{i+1}).$$

If all homology modules are zero, $M_\bullet$ is called *exact*. More generally, we say that $M_\bullet$ is *exact at i* (or at M_i) if $\mathrm{H}_i(M_\bullet) = 0$. Note that $M_1 \xrightarrow{d_1} M_0 \to 0$ is exact (at zero) if and only if d_1 is surjective, and $0 \to M_0 \xrightarrow{d_0} M_{-1}$ is exact if and only if d_0 is injective. An exact complex is often also called an *exact sequence*. In particular, this terminology is compatible with the nomenclature for short exact sequences. If $M_\bullet$ is bounded from the right (indexed so that the last non-zero module is M_0), then the *cokernel* of $M_\bullet$ is the cokernel of $d_1 \colon M_1 \to M_0$. Put differently, the cokernel is simply the zero-th homology module $\mathrm{H}_0(M_\bullet)$. We say that $M_\bullet$ is *acyclic*, if all $\mathrm{H}_i(M_\bullet) = 0$ for $i > 0$. In that case, the *augmented* complex obtained by adding the cokernel of $M_\bullet$ to the right is then an exact sequence.

We will use the following property of ultraproducts on occasion, and although its proof is straightforward, it is instructive for learning to work with ultraproducts:

Theorem 3.1.1. *[Ultraproduct commutes with homology] For each w, let $M_{\bullet w}$ be a complex over a ring A_w and let $M_{\bullet\natural}$ and $A_\natural$ be the respective ultraproducts. Then $M_{\bullet\natural}$ is a complex over $A_\natural$ and its i-th homology $\mathrm{H}_i(M_{\bullet\natural})$ is isomorphic to the ultraproduct of the i-th homologies $\mathrm{H}_i(M_{\bullet w})$.*

Proof. It suffices to prove this at a fixed spot i, and so we may assume that $M_{\bullet w}$ is the complex

$$F_w \xrightarrow{e_w} G_w \xrightarrow{d_w} H_w.$$

Taking ultraproducts, we get a diagram $M_{\bullet\natural}$ of homomorphism of $A_\natural$-modules (we leave it to the reader to verify that the ultraproduct construction extends to the category of modules):

$$F_\natural \xrightarrow{e_\natural} G_\natural \xrightarrow{d_\natural} H_\natural$$

and it is an easy exercise on Łoś' Theorem that $d_\natural \circ e_\natural = 0$ since all $d_w \circ e_w = 0$. In other words, $M_{\bullet\natural}$ is a complex. Let I_w and Z_w be respectively the image of e_w and the kernel of d_w, and let $I_\natural$ and $Z_\natural$ be their respective ultraproducts. The homology of $M_{\bullet w}$ is given by Z_w/I_w, and we have to show that the homology of $M_{\bullet\natural}$ is isomorphic to the ultraproduct of the Z_w/I_w. An element $x_\natural \in G_\natural$ with approximations $x_w \in G_w$ belongs to $Z_\natural$ (respectively, to $I_\natural$) if and only if almost all $d_w(x_w) = 0$ in H_w (respectively, there exist $y_w \in F_w$ such that $x_w = e_w(y_w)$ for almost all w) if and only if $d_\natural(x_\natural) = 0$, that is to say, $x_\natural$ lies in the kernel of $d_\natural$ (respectively, $x_\natural = e_\natural(y_\natural)$ where $y_\natural \in F_\natural$ is the ultraproduct of the y_w, that is to say, $x_\natural$ lies in the image of $e_\natural$). Since the ultraproduct of the Z_w/I_w is isomorphic to $Z_\natural/I_\natural$ by the module analogue of 2.1.6, the claim follows. □

3.1.3 Flatness.

Let A be a ring and M an A-module. Recall that $\cdot \otimes_A M$, that is to say, tensoring with respect to M, is a right exact functor, meaning that given an exact sequence

$$0 \to N_2 \to N_1 \to N_0 \to 0 \tag{3.1}$$

we get an exact sequence

$$N_2 \otimes_A M \to N_1 \otimes_A M \to N_0 \otimes_A M \to 0. \tag{3.2}$$

See [7, Proposition 2.18], where one also can find a good introduction to tensor products. We now call a module M *flat* if any short exact sequence (3.1) remains exact after tensoring, that is to say, we may add an additional zero on the left of (3.2). Put differently, M is flat if and only if $N' \otimes_A M \to N \otimes_A M$ is injective whenever $N' \to N$ is an injective homomorphism of A-modules. By breaking down a long exact sequence into short exact sequences, we immediately get:

3.1.2 *An exact complex remains exact after tensoring with a flat module.*

Well-known examples of flat modules are free modules, and more generally projective modules. In particular, $A[\xi]$, being free over A, is flat as an A-module. The same is true for the power series ring $A[[\xi]]$. Any localization of A is flat, and more generally, any localization of a flat module is again flat. In fact, flatness is preserved under base change in the following sense:

3.1.3 *If M is a flat A-module, then M/IM is a flat A/I-module for each ideal $I \subseteq A$. More generally, if $A \to B$ is any homomorphism, then $M \otimes_A B$ is a flat B-module.*

Immediate from the definition and fact that tensoring with $M \otimes_A B$ over B is the same as tensoring with M over A. □

3.1.4 Tor modules.

Let M be an A-module. A *projective resolution* of M is a complex $P_\bullet$, bounded from the right, in which all the modules P_i are projective, and such that the augmented complex

$$P_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$$

is exact. Put differently, a projective resolution of M is an acyclic complex $P_\bullet$ of projective modules whose cokernel is equal to M. Tensoring this augmented complex with a second A-module N, yields a (possibly non-exact) complex

$$P_i \otimes_A N \to P_{i-1} \otimes_A N \to \cdots \to P_0 \otimes_A N \to M \otimes_A N \to 0.$$

The homology of the non-augmented part $P_\bullet \otimes N$ (that is to say, without the final module $M \otimes N$), is denoted

$$\mathrm{Tor}_i^A(M,N) := \mathrm{H}_i(P_\bullet \otimes_A N).$$

As the notation indicates, this does not depend on the choice of projective resolution $P_\bullet$. Moreover, we have for each i an isomorphism $\mathrm{Tor}_i^A(M,N) \cong \mathrm{Tor}_i^A(N,M)$ ([27, Appendix 3] or [69, Appendix B]). Since tensoring is right exact, $\mathrm{Tor}_0^A(M,N) \cong M \otimes_A N$. The next result is a general fact of 'derived functors' (Tor is indeed the *derived functor* of the tensor product as discussed for instance in [69, Appendix B]).

3.1.4 *Given a short exact sequence of A-modules*

$$0 \to N' \to N \to N'' \to 0,$$

we get for every A-module M, a long exact sequence

$$\cdots \to \mathrm{Tor}_{i+1}^A(M,N'') \xrightarrow{\delta_{i+1}} \mathrm{Tor}_i^A(M,N') \to \mathrm{Tor}_i^A(M,N) \to \mathrm{Tor}_i^A(M,N'') \xrightarrow{\delta_i} \mathrm{Tor}_{i-1}^A(M,N') \to \ldots$$

where the δ_i are the so-called connecting homomorphisms, *and the remaining maps are induced by the original maps.*

3.1.5 Tor-criterion for flatness.

We can now formulate a homological criterion for flatness (see for instance [69, Theorem 7.8]; more flatness criteria will be discussed in §3.3 below).

Theorem 3.1.5. *For an A-module M, the following are equivalent*

3.1.5.i. *M is flat;*

3.1.5.ii. $\mathrm{Tor}_i^A(M,N) = 0$ *for all* $i > 0$ *and all* A*-modules* N*;*
3.1.5.iii. $\mathrm{Tor}_1^A(M,A/I) = 0$ *for all finitely generated ideals* $I \subseteq A$. □

For Noetherian rings we can even restrict the test in (3.1.5.iii) to prime ideals (but see also Theorem 3.3.18 below, which reduces the test to a single ideal):

Corollary 3.1.6. *Let* A *be a Noetherian ring and* M *an* A*-module. If* $\mathrm{Tor}_1^A(M,A/\mathfrak{p})$ *vanishes for all prime ideals* $\mathfrak{p} \subseteq A$*, then* M *is flat. More generally, if, for some* $i \geq 1$*, every* $\mathrm{Tor}_i^A(M,A/\mathfrak{p})$ *vanishes for* $\mathfrak{p}$ *running over the prime ideals in* A*, then* $\mathrm{Tor}_i^A(M,N)$ *vanishes for all (finitely generated)* A*-modules* N*.*

Proof. The first assertion follows from the last by (3.1.5.iii). The last assertion, for finitely generated modules, follows from the fact that every such module N admits a *prime filtration*, that is to say, a finite ascending chain of submodules

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_e = N \tag{3.3}$$

such that each successive quotient N_j/N_{j-1} is isomorphic to the (cyclic) A-module $A/\mathfrak{p}_j$ for some prime ideal $\mathfrak{p}_j \subseteq A$, for $j = 1,\dots,e$ (see [69, Theorem 6.4]). By induction on j, one then derives from the long exact sequence (3.1.4) that $\mathrm{Tor}_i^A(M,N_j) = 0$, whence in particular $\mathrm{Tor}_i^A(M,N) = 0$. To prove the result for N arbitrary, one reduces to the case $i = 1$ by taking syzygies of M, and then applies Theorem 3.1.5. □

3.2 Faithful flatness

We call an A-module M *non-degenerated*, if $\mathfrak{m}M \neq M$ for all (maximal) ideals $\mathfrak{m}$ of A. By Nakayama's Lemma, we immediately get:

3.2.1 *Any finitely generated module over a local ring is non-degenerated.* □

3.2.1 Faithfully flat homomorphisms.

Of particular interest are the non-degenerated modules which are moreover flat, called *faithfully flat* modules. One has the following homological characterization of faithful flatness (see [69, Theorem 7.2] for a proof):

3.2.2 *For an* A*-module* M *to be faithfully flat, it is necessary and sufficient that an arbitrary complex* $N_\bullet$ *is exact if and only if* $N_\bullet \otimes_A M$ *is exact.* □

It is not hard to see that any free or projective module is faithfully flat. On the other hand, no proper localization of A is faithfully flat. The analogue of 3.1.3 holds: the base change of a faithfully flat module is again faithfully flat.

3.2.3 *If M is a faithfully flat A-module, then $M \otimes_A N$ is non-zero, for every non-zero A-module N.*

Indeed, let $N \neq 0$ and choose a non-zero element $n \in N$. Since $I := \mathrm{Ann}_A(n)$ is then a proper ideal, it is contained in some maximal ideal $\mathfrak{m} \subseteq A$. Note that $An \cong A/I$. Tensoring the induced inclusion $A/I \hookrightarrow N$ with M gives by assumption an injection $M/IM \hookrightarrow M \otimes_A N$. The first of these modules is non-zero, since $IM \subseteq \mathfrak{m}M \neq M$, whence so is the second, as we wanted to show. □

In most of our applications, the A-module has the additional structure of an A-algebra. In particular, we call a ring homomorphism $A \to B$ *(faithfully) flat* if B is (faithfully) flat as an A-module. Since by definition a local homomorphism of local rings $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a ring homomorphism with the additional property that $\mathfrak{m} \subseteq \mathfrak{n}$, we get immediately:

3.2.4 *Any local homomorphism which is flat, is faithfully flat.* □

Proposition 3.2.5. *A faithfully flat map is cyclically pure, whence, in particular, injective.*

Proof. We need to show that if $A \to B$ is faithfully flat, and $I \subseteq A$ an ideal, then $I = IB \cap A$. For I equal to the zero ideal, this just says that $A \to B$ is injective. Suppose this last statement is false, and let $a \in A$ be a non-zero element in the kernel of $A \to B$, that is to say, $a = 0$ in B. However, by 3.2.3, the module $aA \otimes_A B$ is non-zero, say, containing the non-zero element x. Hence x is of the form $ra \otimes b$ for some $r \in A$ and $b \in B$, and therefore equal to $r \otimes ab = r \otimes 0 = 0$, contradiction.

To prove the general case, note that B/IB is a flat A/I-module by 3.1.3. It is clearly also non-degenerated, so that applying our first argument to the natural homomorphism $A/I \to B/IB$ yields that it must be injective, which precisely means that $I = IB \cap A$. □

We can paraphrase the previous result as *faithful flatness preserves the ideal structure of a ring.* In particular, from its definition as the ascending chain condition on ideals, we get immediately the following Noetherianity criterion:

Corollary 3.2.6. *Let $A \to B$ be a faithfully flat, or more generally, a cyclically pure homomorphism. If B is Noetherian, then so is A.* □

A similar argument shows:

3.2.7 *If $R \to S$ is a faithfully flat, or more generally, a cyclically pure homomorphism of local rings, and if $I \subseteq R$ is minimally generated by e elements, then so is IS.*

Clearly, IS is generated by at most e elements. By way of contradiction, suppose it is generated by strictly fewer elements. By Nakayama's Lemma, we may choose these generators already in I. So there exists an ideal $J \subseteq I$, generated by less than e elements, such that $JS = IS$. However, by cyclic purity, we have $J = JS \cap R = IS \cap R = I$, contradicting that I requires at least e generators. □

If $A \to B$ is a flat or faithfully flat homomorphism, then we also will call the corresponding morphism $Y := \mathrm{Spec}(B) \to X := \mathrm{Spec}(A)$ *flat* or *faithfully flat* respectively.

Theorem 3.2.8. *A morphism $Y \to X$ of affine schemes is faithfully flat if and only if it is flat and surjective.*

Proof. Let $A \to B$ be the corresponding homomorphism. Assume $A \to B$ is faithfully flat, and let $\mathfrak{p} \subseteq A$ be a prime ideal. Surjectivity of the morphism amounts to showing that there is at least one prime ideal of B lying over $\mathfrak{p}$. The base change $A_\mathfrak{p} \to B_\mathfrak{p}$ is again faithfully flat, and hence in particular $\mathfrak{p}B_\mathfrak{p} \neq B_\mathfrak{p}$. In other words, the fiber ring $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is non-empty, which is what we wanted to prove (indeed, take any maximal ideal $\mathfrak{n}$ of $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ and let $\mathfrak{q} := \mathfrak{n} \cap B$; then verify that $\mathfrak{q} \cap A = \mathfrak{p}$.)

Conversely, assume $Y \to X$ is flat and surjective, and let $\mathfrak{m}$ be a maximal ideal of A. Let $\mathfrak{q} \subseteq B$ be a prime ideal such that $\mathfrak{m} = \mathfrak{q} \cap A$. Hence $\mathfrak{m}B \subseteq \mathfrak{q} \neq B$, showing that B is non-degenerated over A. □

3.2.2 Flatness and regular sequences

A finite sequence $(x_1, \dots, x_n)$ in a ring A is called *pre-regular*, if each x_i is a non-zero divisor on $A/(x_1, \dots, x_{i-1})A$; if $(x_1, \dots, x_n)A$ is moreover a proper ideal, then we say that $(x_1, \dots, x_n)$ is a *regular sequence*. If (x_1) is a regular sequence, that is to say, a non-zero divisor and a non-unit, then we also express this by saying that x_1 is an *A-regular element*

Proposition 3.2.9. *If $A \to B$ is a flat homomorphism and $\mathbf{x}$ is an A-pre-regular sequence, then $\mathbf{x}$ is also B-pre-regular. If $A \to B$ is faithfully flat, and $\mathbf{x}$ is an A-regular sequence, then $\mathbf{x}$ is also B-regular.*

Proof. We induct on the length n of $\mathbf{x} := (x_1, \dots, x_n)$. Assume first $n = 1$. Multiplication by x_1, that is to say, the homomorphism $A \xrightarrow{x_1} A$, is injective, whence remains so after tensoring with B by 3.1.3. It is not hard to see that the resulting homomorphism is again multiplication $B \xrightarrow{x_1} B$, showing that x_1 is B-regular. For $n > 1$, the base change $A/x_1A \to B/x_1B$ is flat, so that by induction $(x_2, \dots, x_n)$ is B/x_1B-regular. Hence we are done, since x_1 is B-regular by the previous argument. The last statement now follows from this, since then B is non-degenerated, and hence, in particular, $\mathbf{x}B \neq B$. □

Tor modules behave well under deformation by a regular sequence in the following sense.

Proposition 3.2.10. *Let $\mathbf{x}$ be a regular sequence in a ring A, and let M and N be two A-modules. If $\mathbf{x}$ is M-regular and $\mathbf{x}N = 0$, then we have for each i an isomorphism*

$$\operatorname{Tor}_i^A(M,N) \cong \operatorname{Tor}_i^{A/\mathbf{x}A}(M/\mathbf{x}M, N).$$

Proof. By induction on the length of the regular sequence, we may assume that we have a single A-regular and M-regular element x. Put $B := A/xA$. From the short exact sequence

$$0 \to A \xrightarrow{x} A \to B \to 0$$

we get after tensoring with M, a long exact sequence of Tor-modules as in 3.1.4. Since $\mathrm{Tor}_i^A(A,M)$ vanishes for all i, so must each $\mathrm{Tor}_i^A(M,B)$ in this long exact sequence for $i > 1$. Furthermore, the initial part of this long exact sequence is

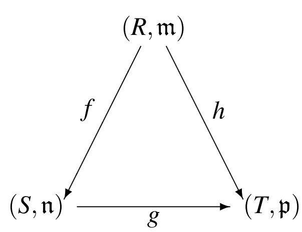
$$0 \to \mathrm{Tor}_1^A(M,B) \to M \xrightarrow{x} M \to M/xM \to 0$$

proving that $\mathrm{Tor}_1^A(M,B)$ too vanishes as x is M-regular. Now, let $P_\bullet$ be a projective resolution of M. The homology of $\bar{P}_\bullet := P_\bullet \otimes_A B$ is by definition $\mathrm{Tor}_i^A(M,B)$, and since we showed that this is zero, $\bar{P}_\bullet$ is exact, whence a projective resolution of M/xM. Hence we can calculate $\mathrm{Tor}_i^B(M/xM,N)$ as the homology of $\bar{P}_\bullet \otimes_B N$ (note that by assumption, N is a B-module). However, the latter complex is equal to $P_\bullet \otimes_A N$ (which we can use to calculate $\mathrm{Tor}_i^A(M,N)$), and hence both complexes have the same homology, as we wanted to show. □

3.2.3 *Scalar extensions*

Recall that a homomorphism $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ of local rings is called *unramified*, if $\mathfrak{m}S = \mathfrak{n}$, or equivalently, if the *closed fiber* $S/\mathfrak{m}S$ *is trivial.* A homomorphism which is at the same time is faithfully flat and unramified is sometimes called *formally etale*, although some authors in addition require that the residue field extension be separable. To not cause any confusion, we will call such a homomorphism a *scalar extension* (see below for the terminology). By [37, 0_{III} 10.3.1], for any Noetherian local ring R and any extension l of its residue field, there exists a scalar extension of R with residue field l; we will reprove this in Theorem 3.2.13 below.

Proposition 3.2.11. *Consider the following commutative triangle of local homomorphisms between Noetherian local rings*



(3.4)

If any two are scalar extensions, then so is the third.

Proof. It is clear that the composition of two scalar extensions is again scalar. Assume g and h are scalar extensions. Then f is faithfully flat by an easy argument using 3.2.2, and $\mathfrak{m}T = \mathfrak{p} = \mathfrak{n}T$. Since g is faithfully flat, we get $\mathfrak{m}S = \mathfrak{m}T \cap S = \mathfrak{n}T \cap S = \mathfrak{n}$ by Proposition 3.2.5, showing that f is also a scalar extension. Finally, assume f and h are scalar extensions. Let

$$\ldots R^{b_2} \to R^{b_1} \to R \to R/\mathfrak{m} \to 0 \tag{3.5}$$

be a free resolution of $R/\mathfrak{m}$. Since S is flat over R, tensoring yields a free resolution

$$\ldots S^{b_2} \to S^{b_1} \to S \to S/\mathfrak{m}S \to 0. \tag{3.6}$$

By assumption $S/\mathfrak{m}S$ is the residue field l of S. Therefore, $\mathrm{Tor}_\bullet^S(T,l)$ can be calculated as the homology of the complex

$$\ldots T^{b_2} \to T^{b_1} \to T \to T/\mathfrak{m}T \to 0 \tag{3.7}$$

obtained from (3.6) by the base change $S \to T$. However, (3.7) can also be obtained by tensoring (3.5) over R with T. Since T is flat over R, the sequence (3.7) is exact, whence, in particular, $\mathrm{Tor}_1^S(T,l) = 0$. By the Local Flatness Criterion (see Corollary 3.3.22 below), T is flat over S. Since $\mathfrak{n} = \mathfrak{m}S$ and $\mathfrak{p} = \mathfrak{m}T$, we get $\mathfrak{p} = \mathfrak{n}T$, showing that g, too, is a scalar extension. □

The following are examples of scalar extensions (for the notion of catapower, see Chapter 8; for the proof of (3.2.12.iii), see Corollary 3.3.3 and Theorem 8.1.15 below).

3.2.12 *Let R be a Noetherian local ring.*

3.2.12.i. *The natural map $R \to \widehat{R}$, given by completion, is a scalar extension.*
3.2.12.ii. *Any etale map is a scalar extension.*
3.2.12.iii. *The diagonal embeddings $R \to R_\natural$ and $R \to R_\sharp$ are scalar extensions, where $R_\natural$ and $R_\sharp$ are respectively an ultrapower and a catapower of R.* □

The next result, which extends Cohen's Structure Theorems, explains the terminology (for a version in mixed characteristic case, see [102]).

Theorem 3.2.13. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of equal characteristic with residue field k. Every extension $k \subseteq l$ of fields can be lifted to a faithfully flat extension $R \to \widehat{R_l}$ inducing the given extension on the residue fields, such that $\widehat{R_l}$ is a complete local ring with maximal ideal $\mathfrak{m}\widehat{R_l}$ and residue field l. In other words, $R \to \widehat{R_l}$ is a scalar extension.*

In fact, $\widehat{R_l}$ is a solution to the following universal property: any complete Noetherian local R-algebra T with residue field l has a unique structure of a local $\widehat{R_l}$-algebra. In particular, $\widehat{R_l}$ is uniquely determined by R and l up to isomorphism, and is called the complete scalar extension *of R along l.*

Proof. By Cohen's Structure Theorems, the completion $\widehat{R}$ of R is isomorphic to $k[[\xi]]/\mathfrak{a}$ for some ideal $\mathfrak{a}$ and some tuple of indeterminates ξ. Put

$$\widehat{R_l} := l[[\xi]]/\mathfrak{a}l[[\xi]],$$

that is to say, $R^{\frown}_{l}$ is the $\mathfrak{n}$-adic completion of $\widehat{R}\otimes_k l$, where $\mathfrak{n} := \mathfrak{m}(\widehat{R}\otimes_k l)$. It is now easy to check that this ring has the desired properties.

To prove the universal property, let T be any complete Noetherian local R-algebra, given by the local homomorphism $R \to T$. By the universal property of completions, we have a unique extension $k[[\xi]]/\mathfrak{a} \cong \widehat{R} \to T$, and by the universal properties of tensor product and completion, this uniquely extends to a homomorphism $R^{\frown}_{l} = l[[\xi]]/\mathfrak{a}l[[\xi]] \to T$. □

Note that complete scalar extension is actually a functor, that is to say, any local homomorphism $R \to S$ of Noetherian local rings whose residue fields are subfields of l extends to a local homomorphism $R^{\frown}_{l} \to S^{\frown}_{l}$. In particular, complete scalar extension commutes with homomorphic images:

$$(R/\mathfrak{a})^{\frown}_{l} \cong R^{\frown}_{l}/\mathfrak{a}R^{\frown}_{l}, \tag{3.8}$$

for all ideals $\mathfrak{a} \subseteq R$. Scalar extensions preserve many good properties:

3.2.14 *If $R \to S$ is a scalar extension, then R and S have the same dimension, and one is regular (respectively, Cohen-Macaulay) if and only if the other is.*

Indeed, the equality of dimension follows from (3.16) (see our discussion below). Since both have also the same embedding dimension by 3.2.7, the claim about regularity follows. In general, let $\mathbf{x}$ be a system of parameters in R. It follows that $\mathbf{x} = (x_1,\dots,x_d)$ is also a system of parameters in S. So if R is Cohen-Macaulay, $\mathbf{x}$ is an R-regular whence S-regular sequence, by Proposition 3.2.9, proving that S is Cohen-Macaulay. The converse follows from the fact that each $R/(x_1,\dots,x_i)R$ is a subring of $S/(x_1,\dots,x_i)S$ by Proposition 3.2.5. □

3.3 Flatness criteria

Because flatness will play such a crucial role in our later work, we want several ways of detecting it. In this section, we will see six such criteria.

3.3.1 Equational criterion for flatness

Our first criterion is very useful in applications (see for instance Theorem 4.4.3), and works without any hypothesis on the ring or module. To give a streamlined presentation, let us introduce the following terminology: given an A-module N, and tuples $\mathbf{b}_i$ in A^n, by an *N-linear combination* of the $\mathbf{b}_i$, we mean a tuple in N^n of the form $n_1\mathbf{b}_1 + \dots + n_s\mathbf{b}_s$ where $n_i \in N$. Of course, if N has the structure of an A-algebra, this is just the usual terminology. Given a (finite) homogeneous linear system of equations

$$L_1(t) = \dots = L_s(t) = 0 \tag{$\mathscr{L}$}$$

over A in the n variables t, we denote the A-submodule of N^n consisting of all solutions of $\mathscr{L}$ in N by $\mathrm{Sol}_N(\mathscr{L})$, and we let $f_{\mathscr{L}}\colon N^n \to N^s$ be the map given by substitution $\mathbf{x} \mapsto (L_1(\mathbf{x}), \dots, L_s(\mathbf{x}))$. In particular, we have an exact sequence

$$0 \to \mathrm{Sol}_N(\mathscr{L}) \to N^n \xrightarrow{f_{\mathscr{L}}} N^s. \qquad (\dagger_{\mathscr{L}/N})$$

Theorem 3.3.1. *A module M over a ring A is flat if and only if every solution in M of a homogeneous linear equation in finitely many variables over A is an M-linear combination of solutions in A. Moreover, instead of a single linear equation, we may take any finite, linear system of equations in the above criterion.*

Proof. We will only prove the first assertion, and leave the second for the reader. Let $L = 0$ be a homogeneous linear equation in n variables with coefficients in A. If M is flat, then the exact sequence $(\dagger_{L/A})$ remains exact after tensoring with M, that is to say,

$$0 \to \mathrm{Sol}_A(L) \otimes_A M \to M^n \xrightarrow{f_L} M,$$

and hence by comparison with $(\dagger_{L/M})$, we get

$$\mathrm{Sol}_M(L) = \mathrm{Sol}_A(L) \otimes_A M.$$

From this it follows easily that any tuple in $\mathrm{Sol}_M(L)$ is an M-linear combination of tuples in $\mathrm{Sol}_A(L)$, proving the direct implication.

Conversely, assume the condition on the solution sets of linear forms holds. To prove that M is flat, we will verify condition (3.1.5.iii) in Theorem 3.1.5. To this end, let $I := (a_1, \dots, a_k)A$ be a finitely generated ideal of A. Tensor the exact sequence $0 \to I \to A \to A/I \to 0$ with M to get by 3.1.4 an exact sequence

$$0 = \mathrm{Tor}_1^A(A, M) \to \mathrm{Tor}_1^A(A/I, M) \to I \otimes_A M \to M. \qquad (3.10)$$

Suppose y is an element in $I \otimes M$ that is mapped to zero in M. Writing $y = a_1 \otimes m_1 + \dots + a_k \otimes m_k$ for some $m_i \in M$, we get $a_1 m_1 + \dots + a_k m_k = 0$. Hence by assumption, there exist solutions $\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(s)} \in A^k$ of the linear equation $a_1 t_1 + \dots + a_k t_k = 0$, such that

$$(m_1, \dots, m_k) = n_1 \mathbf{b}^{(1)} + \dots + n_s \mathbf{b}^{(s)}$$

for some $n_i \in M$. Letting $b_i^{(j)}$ be the i-th entry of $\mathbf{b}^{(j)}$, we see that

$$y = \sum_{i=1}^{k} a_i \otimes m_i = \sum_{i=1}^{k} \sum_{j=1}^{s} a_i \otimes n_j b_i^{(j)} = \sum_{j=1}^{s} (\sum_{i=1}^{k} a_i b_i^{(j)}) \otimes n_j = \sum_{j=1}^{s} 0 \otimes n_j = 0.$$

Hence $I \otimes_A M \to M$ is injective, so that $\mathrm{Tor}_1^A(A/I, M)$ must be zero by (3.10). Since this holds for all finitely generated ideals $I \subseteq A$, we proved that M is flat by (3.1.5.iii). □

It is instructive to view the previous result from the following perspective. To a homogeneous linear equation $L = 0$, we associated an exact sequence $(\dagger_{L/N})$. The image of f_L is of the form IN where I is the ideal generated by the coefficients of the linear form defining L. In case $N = B$ is an A-algebra, this leads to the following extended exact sequence

$$0 \to \mathrm{Sol}_B(L) \to B^n \xrightarrow{f_L} B \to B/IB \to 0. \qquad (\ddagger_{IB})$$

This justifies calling $\mathrm{Sol}_B(L)$ the *module of syzygies* of IB (one checks that it only depends on the ideal I). Therefore, we may paraphrase the equational flatness criterion for algebras as follows:

3.3.2 *A ring homomorphism $A \to B$ is flat if and only if taking syzygies commutes with extension in the sense that the module of syzygies of IB is the extension to B of the module of syzygies of an arbitrary ideal $I \subseteq A$.*

Here is one application of the equational flatness criterion.

Corollary 3.3.3. *The diagonal embedding of a Noetherian ring inside its ultrapower is faithfully flat.*

Proof. Let A be a ring and $A_\natural$ an ultrapower of A. Recall that $A \to A_\natural$ is given by sending an element $a \in A$ to the ultraproduct $\mathrm{ulim}_{w\to\infty} a$ of the constant sequence. If $\mathfrak{m} \subseteq A$ is a maximal ideal, then $\mathfrak{m}A_\natural$ is its ultraproduct by 2.4.20, whence again maximal by Łoś' Theorem (Theorem 2.3.1), showing that $A_\natural$ is non-degenerated. To show it is also flat, we use the equational criterion. Let $L = 0$ be a homogeneous linear equation with coefficients in A. Let $\mathbf{a} \in A_\natural^n$ be a solution of $L = 0$ in $A_\natural$. Write $\mathbf{a}$ as an ultraproduct of tuples $\mathbf{a}_w \in A^n$. By Łoś' Theorem, almost each $\mathbf{a}_w \in \mathrm{Sol}_A(L)$. Hence $\mathbf{a}$ lies in the ultrapower of $\mathrm{Sol}_A(L)$. By Noetherianity, $\mathrm{Sol}_A(L)$ is finitely generated, and hence, its ultrapower is simply the $A_\natural$-module generated by $\mathrm{Sol}_A(L)$, so that we are done by Theorem 3.3.1. □

3.3.2 *Coherency criterion*

We can turn this into a criterion for coherency. Recall that a ring A is called *coherent*, if the solution set of any homogeneous linear equation over A is finitely generated. Clearly, Noetherian rings are coherent. We have:

Theorem 3.3.4. *A ring A is coherent if and only if the diagonal embedding into one of its ultrapowers is flat.*

Proof. The direct implication is proven by the same argument that proves Corollary 3.3.3, since we really only used that A is coherent in that argument. Conversely, suppose $A \to A_\natural$ is flat. Towards a contradiction, assume L is a linear form (in n indeterminates) over A whose solution set $\mathrm{Sol}_A(L)$ is infinitely generated. In particular, we can choose a sequence $\mathbf{a}_w$, for $w = 1, 2, \dots$, in $\mathrm{Sol}_A(L)$ which is contained in no finitely generated submodule of $\mathrm{Sol}_A(L)$. The ultraproduct $\mathbf{a}_\natural \in A_\natural^n$ of this sequence lies in $\mathrm{Sol}_{A_\natural}(L)$ by Łoś' Theorem. Hence, by Theorem 3.3.1, there exists a finitely generated submodule $H \subseteq \mathrm{Sol}_A(L)$ such that $\mathbf{a}_\natural \in H \cdot A_\natural$. Therefore, almost all $\mathbf{a}_j$ lie in H by Łoś' Theorem, contradiction. □

3.3.3 Quotient criterion for flatness.

The next criterion is derived from our Tor-criterion (Theorem 3.1.5):

Theorem 3.3.5. *Let $A \to B$ be a flat homomorphism, and let $I \subseteq B$ be an ideal. The induced homomorphism $A \to B/I$ is flat if and only if $\mathfrak{a}B \cap I = \mathfrak{a}I$ for all finitely generated ideals $\mathfrak{a} \subseteq A$.*

Moreover, if A is Noetherian, we only need to check the above criterion for $\mathfrak{a}$ a prime ideal of A.

Proof. From the exact sequence $0 \to I \to B \to B/I \to 0$ we get after tensoring with $A/\mathfrak{a}$ an exact sequence

$$0 = \mathrm{Tor}_1^A(B, A/\mathfrak{a}) \to \mathrm{Tor}_1^A(B/I, A/\mathfrak{a}) \to I/\mathfrak{a}I \to B/\mathfrak{a}B$$

where we used the flatness of B for the vanishing of the first module. The kernel of $I/\mathfrak{a}I \to B/\mathfrak{a}B$ is easily seen to be $(\mathfrak{a}B \cap I)/\mathfrak{a}I$. Hence $\mathrm{Tor}_1^A(B/I, A/\mathfrak{a})$ vanishes if and only if $\mathfrak{a}B \cap I = \mathfrak{a}I$. This proves by Theorem 3.1.5 the stated equivalence in the first assertion; the second assertion follows by the same argument, this time using Corollary 3.1.6. □

To put this criterion to use, we need another definition (for another application, see Theorem 8.2.1 below). The (A-) *content* of a polynomial $f \in A[\xi]$ (or a power series $f \in A[[\xi]]$) is by definition the ideal generated by its coefficients.

Corollary 3.3.6. *Let A be a Noetherian ring, let ξ be a finite tuple of indeterminates, and let B denote either $A[\xi]$ or $A[[\xi]]$. If $f \in B$ has content one, then B/fB is flat over A.*

Proof. The natural map $A \to B$ is flat. To verify the second criterion in Theorem 3.3.5, let $\mathfrak{p} \subseteq A$ be a prime ideal. The forward inclusion $\mathfrak{p}fB \subseteq \mathfrak{p}B \cap fB$ is immediate. To prove the other, take $g \in \mathfrak{p}B \cap fB$. In particular, $g = fh$ for some $h \in B$. Since $\mathfrak{p} \subseteq A$ is a prime ideal, so is $\mathfrak{p}B$ (this is a property of polynomial or power series rings, not of flatness!). Since f has content one, $f \notin \mathfrak{p}B$ whence $h \in \mathfrak{p}B$. This yields $g \in \mathfrak{p}fB$, as we needed to prove. □

3.3.4 Cohen-Macaulay criterion for flatness.

To formulate our next criterion, we need a definition.

Definition 3.3.7 (Big Cohen-Macaulay modules). Let R be a Noetherian local ring, and let M be an arbitrary R-module. We call M a *big Cohen-Macaulay module*, if there exists a system of parameters in R which is M-regular. If moreover every system of parameters is M-regular, then we call M a *balanced big Cohen-Macaulay* module.

It has become tradition to add the somehow redundant adjective 'big' to emphasize that the module is not necessarily finitely generated. It is one of the greatest open problems in homological algebra to show that every Noetherian local ring has at least one big Cohen-Macaulay module, and, as we shall see, this is known to be the case for any Noetherian local ring containing a field (see §6.4 and §7.4).[1] A Cohen-Macaulay local ring is clearly a balanced big Cohen-Macaulay module over itself, so the problem of the existence of these modules is only important for deriving results over Noetherian local rings with 'worse than Cohen-Macaulay' singularities.

Once one has a big Cohen-Macaulay module, one can always construct, using completion, a balanced big Cohen-Macaulay module from it (see for instance [17, Corollary 8.5.3]). Here is a criterion for a big Cohen-Macaulay module to be balanced taken from [6, Lemma 4.8] (recall that a regular sequence is called *permutable* if any permutation is again regular).

Proposition 3.3.8. *A big Cohen-Macaulay module M over a Noetherian local ring is balanced, if every M-regular sequence is permutable.* □

If R is a Cohen-Macaulay local ring, and M a flat R-module, then M is a balanced big Cohen-Macaulay module, since every system of parameters in R is R-regular, whence M-regular by Proposition 3.2.9. We have the following converse:

Theorem 3.3.9. *If M is a balanced big Cohen-Macaulay module over a regular local ring, then it is flat. More generally, over an arbitrary local Cohen-Macaulay ring, if M is a balanced big Cohen-Macaulay module of finite projective dimension, then it is flat.*

Proof. The first assertion is just a special case of the second since any module over a regular local ring has finite projective dimension. For simplicity, we will just prove the first, and leave the second as an exercise for the reader. So let M be a balanced big Cohen-Macaulay module over the d-dimensional regular local ring R. Since a finitely generated R-module N has finite projective dimension, all $\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$. Let e be maximal such that $\operatorname{Tor}_e^R(M,N) \neq 0$ for some

[1] A related question is even open in these cases: does there exist a 'small' Cohen-Macaulay module, that is to say, a finitely generated one, if the ring is moreover complete? There are counterexamples to the existence of a small Cohen-Macaulay module if the ring is not complete.

finitely generated R-module N. If $e=0$, then we are done by Theorem 3.1.5. So, by way of contradiction, assume $e \geq 1$. By Corollary 3.1.6, there exists a prime ideal $\mathfrak{p} \subseteq R$ such that $\operatorname{Tor}_e^R(M, R/\mathfrak{p}) \neq 0$. Let h be the height of $\mathfrak{p}$. Choose a system of parameters $(x_1, \dots, x_d)$ in R such that $\mathfrak{p}$ is a minimal prime of $I := (x_1, \dots, x_h)R$. Since (the image of) $\mathfrak{p}$ is then an associated prime of R/I, we get a short exact sequence

$$0 \to R/\mathfrak{p} \to R/I \to C \to 0$$

for some finitely generated R-module C. The relevant part of the long exact Tor sequence from 3.1.4, obtained by tensoring the above exact sequence with M, is

$$\operatorname{Tor}_{e+1}^R(M, C) \to \operatorname{Tor}_e^R(M, R/\mathfrak{p}) \to \operatorname{Tor}_e^R(M, R/I). \tag{3.12}$$

The first module in (3.12) is zero by the maximality of e. The last module is zero too since it is isomorphic to $\operatorname{Tor}_e^{R/I}(M/IM, R/I) = 0$ by Proposition 3.2.10 and the fact that $(x_1, \dots, x_d)$ is by assumption M-regular. Hence the middle module in (3.12) is also zero, contradiction. □

We derive the following criterion for Cohen-Macaulayness:

Corollary 3.3.10. *If X is an irreducible affine scheme of finite type over a field K, and $\phi\colon X \to \mathbb{A}_K^d$ is a Noether normalization, that is to say, a finite and surjective morphism, then X is Cohen-Macaulay if and only if ϕ is flat.*

Proof. Suppose $X = \operatorname{Spec}(B)$, so that ϕ corresponds to a finite and injective homomorphism $A \to B$, with $A := K[\xi_1, \dots, \xi_d]$ and B a d-dimensional affine domain. Let $\mathfrak{n}$ be a maximal ideal of B, and let $\mathfrak{m} := \mathfrak{n} \cap A$ be its contraction to A. Since flatness is a local property, it suffices to show that $A_\mathfrak{m} \to B_\mathfrak{n}$ is flat. Since $A/\mathfrak{m} \to B/\mathfrak{n}$ is finite and injective, and since the second ring is a field, so is the former by [69, §9 Lemma 1]. Hence $\mathfrak{m}$ is a maximal ideal of A, and $A_\mathfrak{m}$ is regular. Choose an ideal $I := (x_1, \dots, x_d)A$ whose image in $A_\mathfrak{m}$ is a parameter ideal. Since the natural homomorphism $A/I \to B/IB$ is finite, the latter ring is Artinian since the former is (note that $A/I = A_\mathfrak{m}/IA_\mathfrak{m}$). It follows that $IB_\mathfrak{n}$ is a parameter ideal in $B_\mathfrak{n}$.

Now, if B, whence also $B_\mathfrak{n}$ is Cohen-Macaulay, then $(x_1, \dots, x_d)$, being a system of parameters in $B_\mathfrak{n}$, is $B_\mathfrak{n}$-regular. This proves that $B_\mathfrak{n}$ is balanced big Cohen-Macaulay module over $A_\mathfrak{m}$, whence is flat by Theorem 3.3.9.

Conversely, assume $X \to \mathbb{A}_K^d$ is flat. Therefore, $A_\mathfrak{m} \to B_\mathfrak{n}$ is flat, and hence $(x_1, \dots, x_d)$ is a $B_\mathfrak{n}$-regular sequence by Proposition 3.2.9. Since we already showed that this sequence is a system of parameters, we see that $B_\mathfrak{n}$ is Cohen-Macaulay. Since this holds for all maximal prime ideals of B, we proved that B is Cohen-Macaulay. □

Remark 3.3.11. The above argument proves the following more general result in the local case: if $A \subseteq B$ is a finite and faithfully flat extension of local rings with A regular, then B is Cohen-Macaulay. For the converse, we can even formulate a stronger criterion; see Theorem 3.3.26 below.

We conclude with an application of the above Cohen-Macaulay criterion:

Corollary 3.3.12. *Any hypersurface in $\mathbb{A}^n_K$ is Cohen-Macaulay.*

Proof. Recall that a hypersurface Y is an affine closed subscheme of the form $\operatorname{Spec}(A/fA)$ with $A := K[\xi_1,\dots,\xi_n]$ and $f \in A$. Moreover, Y has dimension $n-1$, whence its Noether normalization is of the form $Y \to \mathbb{A}^{n-1}_K$. In fact, after a change of coordinates, we may assume that f is monic in ξ_n of degree d. It follows that A/fA is free over $A' := K[\xi_1,\dots,\xi_{n-1}]$ with basis $1,\xi_n,\dots,\xi_n^{d-1}$. Hence A/fA is flat over A', whence Cohen-Macaulay by Corollary 3.3.10. □

3.3.5 Colon criterion for flatness.

Recall that $(I:a)$ denotes the *colon ideal* of all $x \in A$ such that $ax \in I$. Colon ideals are related to cyclic modules in the following way:

3.3.13 *For any ideal $I \subseteq A$ and any element $a \in A$, we have an isomorphism $a(A/I) \cong A/(I:a)$.*

Indeed, the homomorphism $A \to A/I\colon x \mapsto ax$ has image $a(A/I)$ whereas its kernel is $(I:a)$. We already saw that faithfully flat homomorphisms preserve the ideal structure of a ring. Using colon ideals, we can even give the following criterion:

Theorem 3.3.14. *A homomorphism $A \to B$ is flat if and only if*

$$(IB:a) = (I:a)B$$

for all elements $a \in A$ and all (finitely generated) ideals $I \subseteq A$.

Proof. Suppose $A \to B$ is flat. In view of 3.3.13, we have an exact sequence

$$0 \to A/(I:a) \to A/I \to A/(I+aA) \to 0 \qquad (3.13)$$

which, when tensored with B gives the exact sequence

$$0 \to B/(I:a)B \to B/IB \xrightarrow{f} B/(IB+aB) \to 0.$$

However, the kernel of f is easily seen to be $a(B/IB)$, which is isomorphic to $B/(IB:a)$ by 3.3.13. Hence the inclusion $(I:a)B \subseteq (IB:a)$ must be an equality.

In view of Theorem 3.1.5, we need to show that $\operatorname{Tor}_1^A(B,A/J) = 0$ for every finitely generated ideal $J \subseteq A$ to prove the converse. We induct on the minimal number s of generators of J, where the case $s=0$ trivially holds. Write $J = I + aA$ with I an ideal generated by $s-1$ elements. Tensoring (3.13) with B, we get from 3.1.4 an exact sequence

$$0 = \operatorname{Tor}_1^A(B,A/I) \to \operatorname{Tor}_1^A(B,A/J) \xrightarrow{\delta} B/(I:a)B \to B/IB \xrightarrow{g} B/JB \to 0,$$

where the first module vanishes by induction. As above, the kernel of g is easily seen to be $B/(IB:a)$, so that our assumption on the colon ideals implies that δ is the zero map, whence $\mathrm{Tor}_1^A(B,A/J)=0$ as we wanted to show. □

Here is a nice 'descent type' application of this criterion:

Corollary 3.3.15. *Let $A\to B\to C$ be homomorphisms whose composition is flat. If $B\to C$ is cyclically pure, then $A\to B$ is flat. In fact, it suffices that $B\to C$ is* cyclically pure with respect to ideals extended from A*, that is to say, that $JB=JC\cap B$ for all ideals $J\subseteq A$.*

Proof. Given an ideal $I\subseteq A$ and an element $a\in A$, we need to show in view of Theorem 3.3.14 that $(IB:a)=(I:a)B$. One inclusion is immediate, so take y in $(IB:a)$. By the same theorem, we have $(IC:a)=(I:a)C$, so that y lies in $(I:a)C\cap B$ whence in $(I:a)B$ by cyclical purity. □

The next criterion will be useful when dealing with non-Noetherian algebras in the next chapter. We call an ideal J in a ring B *finitely related*, if it is of the form $J=(I:b)$ with $I\subseteq B$ a finitely generated ideal and $b\in B$.

Theorem 3.3.16. *Let A be a Noetherian ring and B an arbitrary A-algebra. Suppose $\mathscr{P}$ is a collection of prime ideals in B such that every proper, finitely related ideal of B is contained in some prime ideal belonging to $\mathscr{P}$. If $A\to B_{\mathfrak{p}}$ is flat for every $\mathfrak{p}\in\mathscr{P}$, then $A\to B$ is flat.*

Proof. By Theorem 3.3.14, we need to show that $(IB:a)=(I:a)B$ for all $I\subseteq A$ and $a\in A$. Put $J:=(I:a)$. Towards a contradiction, let x be an element in $(IB:a)$ but not in JB. Hence $(JB:x)$ is a proper, finitely related ideal, and hence contained in some $\mathfrak{p}\in\mathscr{P}$. However, $(IB_{\mathfrak{p}}:a)=JB_{\mathfrak{p}}$ by flatness and another application of Theorem 3.3.14, so that $x\in JB_{\mathfrak{p}}$, contradicting that $(JB:x)\subseteq\mathfrak{p}$. □

We can also derive a coherency criterion due to Chase ([21]):

Corollary 3.3.17. *A ring is coherent if and only if every finitely related ideal is finitely generated.*

Proof. The direct implication is a simple application of the coherency condition. For the converse, suppose every finitely related ideal is finitely generated. We will prove that $R\to R_\natural$ is flat, where $R_\natural$ is an ultrapower of R, from which it follows that R is coherent by Theorem 3.3.4. To prove flatness, we use the Colon Criterion, Theorem 3.3.14. To this end, let $I\subseteq R$ be finitely generated and let $a\in R$. We have to show that if b lies in $(IR_\natural:a)$ then it already lies in $(I:a)R_\natural$. Let b_w be an approximation of b. By Łoś' Theorem, almost each $b_w\in(I:a)$. By assumption, the colon ideal $(I:a)$ is finitely generated, say by $f_1,\dots,f_s$, and hence we can find c_{iw} such that $b_w=c_{1w}f_1+\dots+c_{sw}f_s$. Let $c_i\in R_\natural$ be the ultraproduct of the c_{iw}, for each $i=1,\dots,s$. By Łoś' Theorem, $b=c_1f_1+\dots+c_sf_s$, showing that it belongs to $(I:a)R_\natural$. □

3.3.6 Local criterion for flatness.

For finitely generated modules, we have the following criterion:

Theorem 3.3.18 (Local flatness theorem–finitely generated case). *Let R be a Noetherian local ring with residue field k. If M is a finitely generated R-module whose first Betti number vanishes, that is to say, if $\operatorname{Tor}_1^R(M,k) = 0$, then M is flat.*

Proof. Take a minimal free resolution

$$\cdots \to F_1 \to F_0 \to M \to 0$$

of M, that is to say, such that the kernel of each boundary map $d_i \colon F_i \to F_{i-1}$ lies inside $\mathfrak{m}F_i$. Therefore, since tensoring this complex with k yields the zero complex, the rank of F_i is equal to the *i-th Betti number* of M, that is to say, the vector space dimension of $\operatorname{Tor}_i^R(M,k)$. In particular, F_1 has rank zero, so that $M \cong F_0$ is free whence flat. □

There is a much stronger version of this result, where we may replace the condition that M is finitely generated over R by the condition that M is finitely generated over a Noetherian local R-algebra S (see for instance [69, Theorem 22.3] or [27, Theorem 6.8]). We will present here a new proof, for which we need to make some further definitions. The method is an extension of the work in [93], which primarily dealt with detecting finite projective dimension.

Let A be a (not necessarily Noetherian) ring, and let $\mathbf{mod}_A$ be the class of all finitely presented A-modules. We will call a subclass $\mathbf{N} \subseteq \mathbf{mod}_R$ a *deformation class* if it is closed under isomorphisms, direct summands, extensions, and deformations, that is to say, if it is closed under the following respective rules:[2]

- 3.3.18.i. if N belongs to $\mathbf{N}$ and $M \cong N$, then M belongs to $\mathbf{N}$;
- 3.3.18.ii. if $N \cong M \oplus M'$ belongs to $\mathbf{N}$, then so do M and M';
- 3.3.18.iii. if $0 \to K \to M \to N \to 0$ is an exact sequence in $\mathbf{mod}_R$ with $K, N \in \mathbf{N}$, then also $M \in \mathbf{N}$;
- 3.3.18.iv. if x is an M-regular element in the Jacobson radical of A and M/xM belongs to $\mathbf{N}$, then so does M.

Recall that the *Jacobson radical* of A is the intersection of all its maximal ideals; equivalently, it is the ideal of all x such that $1 + ax$ is unit for all a. Condition 3.3.18.iv holds vacuously, if the Jacobson radical is equal to the *nilradical*, the ideal of all nilpotent elements. Clearly, $\mathbf{mod}_A$ itself is a deformation class. We leave it as an easy exercise to show that:

3.3.19 *Any intersection of deformation classes is again a deformation class. In particular, any class $\mathbf{K} \subseteq \mathbf{mod}_A$ sits inside a smallest deformation class, called the* deformation class of $\mathbf{K}$. □

[2] A class satisfying the first three conditions is called a *net* in [93].

Let us call a subclass $\mathbf{K} \subseteq \mathbf{mod}_A$ *deformationally generating*, if its deformation class is equal to $\mathbf{mod}_A$, and *quasi-deformationally generating*, if its deformation class contains all cyclic modules of the form A/I with $I \subseteq A$ finitely generated. One easily shows, by induction on the number of generators, that if A is coherent, deformationally generating and quasi-deformationally generating are equivalent notions.

Proposition 3.3.20. *If R is a Noetherian local ring, then its residue field is deformationally generating.*

Proof. We need to show that any finitely generated module M belongs to the deformation class $\mathbf{N}$ generated by the residue field. Since any module generated by n elements is the extension of two modules generated by less than n elements, an induction on n using (3.3.18.iii) reduces to the case $n = 1$, that is to say, $M = R/\mathfrak{a}$. Suppose the assertion is false, and let $\mathfrak{a}$ be a maximal counterexample. If $\mathfrak{a}$ is not prime, then for $\mathfrak{p}$ a minimal prime ideal $\mathfrak{p}$ of $\mathfrak{a}$, we have an exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{a} \to R/\mathfrak{a}' \to 0$$

for some $\mathfrak{a}' \subseteq R$ strictly containing $\mathfrak{a}$. The two outer modules belong to $\mathbf{N}$ by maximality, whence so does the inner one by (3.3.18.iii), contradiction. Hence $\mathfrak{a}$ is a prime ideal, which therefore must be different from the maximal ideal of R. Let x be an element in the maximal ideal not in $\mathfrak{a}$. By maximality $R/(\mathfrak{a}+xR)$ belongs to $\mathbf{N}$, whence so does $R/\mathfrak{a}$ by (3.3.18.iv), since x is $R/\mathfrak{a}$-regular, contradiction again. □

The main flatness criterion of this section is:

Theorem 3.3.21. *Let $A \to B$ be a homomorphism sending the Jacobson radical of A inside that of B, and let $\mathbf{K} \subseteq \mathbf{mod}_A$ be quasi-deformationally generating. A coherent B-module Q is flat over A if and only if $\mathrm{Tor}_1^A(Q,M) = 0$ for all $M \in \mathbf{K}$.*

Proof. One direction is immediate, so we only need to show the direct implication. Define a functor $\mathscr{F}$ on $\mathbf{mod}_R$, by $\mathscr{F}(M) := \mathrm{Tor}_1^A(Q,M)$. By Theorem 3.1.5, it suffices to show that $\mathscr{F}$ vanishes on each A/I with $I \subseteq A$ finitely generated. This will follow as soon as we can show that $\mathscr{F}(M) = 0$ for all M in the deformation class $\mathbf{N}$ of $\mathbf{K}$. By induction on the rules (3.3.18.i)–(3.3.18.iv), it will suffice to show that each new module M in $\mathbf{N}$ obtained from an application of one of these rules vanishes again on $\mathscr{F}$. The case of rule (3.3.18.i) is trivial; for (3.3.18.ii), we use that $\mathscr{F}$ is additive; and for (3.3.18.iii), we are done by the long exact sequence of Tor (3.1.4). So remains to verify the claim for rule (3.3.18.iv), that is to say, assume x is an M-regular element in the Jacobson radical of A such that $\mathscr{F}(M/xM) = 0$. Applying 3.1.4 to the exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

we get part of a long exact sequence

$$\mathscr{F}(M) \xrightarrow{x} \mathscr{F}(M) \to \mathscr{F}(M/xM) = 0. \tag{3.14}$$

Since M is finitely presented, we have an exact sequence

$$F \to A^m \to A^n \to M \to 0$$

with F some (possibly infinitely generated) free A-module. Tensoring with Q yields a complex

$$F \otimes_A Q \to Q^m \to Q^n \to M \otimes_A Q \to 0 \tag{3.15}$$

whose first homology is by definition $\mathscr{F}(M)$. Since Q is a coherent module, so is any direct sum of Q by [35, Corollary 2.2.3], and hence the kernel of the morphism $Q^m \to Q^n$ in (3.15) is finitely generated by [35, Corollary 2.2.2]. Since $\mathscr{F}(M)$ is a quotient of this kernel, it, too, is finitely generated. By (3.14), we have an equality $\mathscr{F}(M) = x\mathscr{F}(M)$. By assumption, x belongs to the Jacobson radical of B, and hence, by Nakayama's Lemma, $\mathscr{F}(M) = 0$, as we needed to show. □

Combining Proposition 3.3.20 with Theorem 3.3.21 immediately gives the following well-known flatness criterion:

Corollary 3.3.22 (Local Flatness Criterion). *Let $R \to S$ be a local homomorphism of Noetherian local rings, and let k be the residue field of R. If M is a finitely generated S-module such that $\operatorname{Tor}_1^R(M,k) = 0$, then M is flat over R.* □

To extend this local flatness criterion to a larger class of rings, we make the following definition. Let us call a local ring R *ind-Noetherian*, if it is a direct limit of Noetherian local subrings R_i, indexed by a directed poset I, such that each $R_i \to R$ is a scalar extension (that is to say, faithfully flat and unramified; see §3.2.3). Clearly, any Noetherian local ring is ind-Noetherian (by taking $R_i = R$).

Lemma 3.3.23. *An ind-Noetherian local ring is coherent and has finite embedding dimension.*

Proof. Let $(R, \mathfrak{m})$ be ind-Noetherian. Since $\mathfrak{m}$ is in particular extended from a Noetherian local ring, it is finitely generated. We use Corollary 3.3.17 to prove coherency. To this end we must show that a finitely related ideal $(\mathfrak{a} : \mathfrak{b})$ is finitely generated. Since $\mathfrak{a}$ and $\mathfrak{b}$ are finitely generated, there exists a Noetherian local subring $S \subseteq R$ and ideals $I, J \subseteq S$ such that $S \to R$ is a scalar extension, and $\mathfrak{a} = IR$ and $\mathfrak{b} = JR$. Theorem 3.3.14 yields that $(I : J)R = (IR : JR) = (\mathfrak{a} : \mathfrak{b})$, whence in particular, is finitely generated. □

3.3.24 *If $R \to S$ is essentially of finite type and R is ind-Noetherian, then so is S.*

Indeed, S is isomorphic to the localization of $R[x]/(f_1, \dots, f_s)R[x]$ with respect to the ideal generated by the variables and by the maximal ideal of R. Hence, there is a directed subset $J \subseteq I$ such that $f_1, \dots, f_s$ are defined over each R_j with $j \in J$. It is now easy to see that the appropriate localization S_j of $R_j[x]/(f_1, \dots, f_s)R_j[x]$ forms a directed system with union equal to S, and each $S_j \to S$ is a scalar extension. □

Corollary 3.3.25. *Let $R \to S$ be a local homomorphism of ind-Noetherian rings. If Q is a finitely presented S-module such that $\operatorname{Tor}_1^R(Q,k) = 0$, where k is the residue field of R, then Q is flat over R. If Q is moreover Noetherian, then so is R.*

Proof. In view of Theorem 3.3.21, to prove the first assertion, we need to show that k is quasi-deformationally generating (note that S is coherent by Lemma 3.3.23, whence so is the finitely presented S-module Q). Let $\mathfrak{a} \subseteq R$ be a finitely generated ideal. Choose a Noetherian local subring T and an ideal $I \subseteq T$ such that $T \subseteq R$ is a scalar extension, and $IR = \mathfrak{a}$. By Proposition 3.3.20, the module T/I belongs to the deformation class of T-modules generated by the residue field l of T. Since each of the rules (3.3.18.i)–(3.3.18.iv) are preserved by faithfully flat extensions, $T/I \otimes_T R = R/\mathfrak{a}$ lies in the deformation class of $l \otimes_T R \cong k$, where the latter isomorphism follows from the unramifiedness of $T \to R$.

To prove that R is Noetherian, under the additional assumption that Q is Noetherian, let $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ be a chain of ideals in R. Choose i such that $\mathfrak{a}_i Q = \mathfrak{a}_j Q$ for all $j \geq i$. Hence $\mathfrak{a}_i/\mathfrak{a}_j \otimes Q = 0$, for $j \geq i$, and since Q is faithfully flat, as it is non-degenerated by 3.2.1, we get $\mathfrak{a}_i/\mathfrak{a}_j = 0$ by 3.2.3. □

3.3.7 Dimension criterion for flatness

If $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ is a local homomorphism of Noetherian local rings, then we have the following dimension inequality, with equality when $R \to S$ is flat (see [69, Theorem 15.1]):

$$\dim(S) \leq \dim(R) + \dim(S/\mathfrak{m}S). \tag{3.16}$$

Recall that we call $S/\mathfrak{m}S$ the *closed fiber* of $R \to S$: it defines the locus of all prime ideals in S which lie above $\mathfrak{m}$. Conversely, equality in (3.16) often implies flatness. We first discuss one well-known criterion, and then prove one new one.

Theorem 3.3.26. *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a homomorphism of Noetherian local rings, with R regular and S Cohen-Macaulay. Then $R \to S$ is flat if and only if we have equality in* (3.16).

Proof. One direction holds always, as we discussed above. So assume we have equality in (3.16), that is to say, $e = d + h$ where d, h, and e, are the respective dimension of R, the closed fiber $S/\mathfrak{m}S$, and S. Let $(x_1,\dots,x_d)$ be a system of parameters of R. Since $S/\mathfrak{m}S$ has dimension $h = e - d$, there exist $x_{d+1},\dots,x_e$ in S such that their image in $S/\mathfrak{m}S$ is a system of parameters. Hence $(x_1,\dots,x_e)$ is a system of parameters in S, whence is an S-regular sequence. In particular, $(x_1,\dots,x_d)$ is S-regular, showing that S is a balanced big Cohen-Macaulay R-module, and therefore is flat by Theorem 3.3.9. □

For our last criterion, which generalizes a flatness criterion due to Kollár [62, Theorem 8], we impose some regularity condition on the closed fiber, weakening instead the conditions on the rings themselves.

Theorem 3.3.27. *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume R is either an excellent normal local domain with perfect residue field, or an analytically irreducible domain with algebraically closed residue field. If the closed fiber is regular, of dimension* $\dim(S) - \dim(R)$, *then $R \to S$ is faithfully flat.*

Proof. Let d and e be the respective dimensions of R and S. We will induct on the dimension $h := e - d$ of the closed fiber. If $h = 0$, then $R \to S$ is in fact unramified. It suffices to prove this case under the additional assumption that both R and S are complete. Indeed, if $R \to S$ is arbitrary, then $\widehat{R} \to \widehat{S}$ satisfies again the hypotheses of the theorem and therefore would be faithfully flat. Hence $R \to S$ is faithfully flat by Proposition 3.2.11.

So assume R and S are complete and let l be the residue field of S. Either assumption on R implies that $R^{\widehat{}}_l$ is again a domain, of the same dimension as R (we leave this as an exercise to the reader; see [102, Corollary 3.10 and Proposition 3.11]). By the universal property of complete scalar extensions (Theorem 3.2.13—note that this result also holds in mixed characteristic, although we did not provide a proof in these notes; see [102, Corollary 3.3]), we get a local R-algebra homomorphism $R^{\widehat{}}_l \to S$. By [69, Theorem 8.4], this homomorphism is surjective. It is also injective, since $R^{\widehat{}}_l$ and S have the same dimension and $R^{\widehat{}}_l$ is a domain. Hence $R^{\widehat{}}_l \cong S$, so that $R \to S$ is a scalar extension, whence faithfully flat.

For the general case, $h > 0$, let $\tilde{R}$ be the localization of $R[\xi]$ at the ideal $\tilde{\mathfrak{m}}$ generated by $\mathfrak{m}$ and the variables $\xi := (\xi_1, \dots, \xi_h)$. By assumption, $\tilde{R}$ has the same dimension as S. Let $\mathbf{y}$ be an h-tuple whose image in the closed fiber is a regular system of parameters, that is to say, which generates $\mathfrak{n}(S/\mathfrak{m}S)$. Let $\tilde{R} \to S$ be the R-algebra homomorphism given by sending ξ to $\mathbf{y}$. Hence $\mathfrak{n} = \mathfrak{m}S + \mathbf{y}S = \tilde{\mathfrak{m}}S$, so that by the case $h = 0$, the homomorphism $\tilde{R} \to S$ is flat, whence so is $R \to S$. □

The requirement on R that we really need is that any complete scalar extension is again a domain, and for this, it suffices that the complete scalar extension over the algebraic closure of the residue field of R is a domain (see [102, Proposition 3.11]).