Appendix A
Henselizations

In this appendix, I have gathered some facts about Henselizations that can be found scattered in the literature (some sources dealing more extensively with Henselizations are [70, 71, 77, 106]). Hensel observed that solving an equation over the p-adics can be reduced to finding a root in the residue field, provided this root is simple. This property, now known as Hensel's Lemma—and a ring satisfying it, is called *Henselian*—, extends easily to any complete local ring; see Theorem A.1.1. Although any Noetherian local ring admits a uniquely defined, smallest complete overring, its completion—which inherits many of the good properties of the original ring, and in particular is Henselian—, the process introduces transcendental elements. The Henselization of a local ring is much closer to it than its completion, since it is a direct limit of finite etale extensions. As Eisenbud remarks

> "... [i]t can thus be used to give the same microscopic view of a variety as the completion, but without passing out of the category of algebraic varieties."
>
> [27, p. 186]

The main objective of this appendix is to give a direct construction of the Henselization which, to my knowledge, never appeared in print.[1]

A.1 Hensel's Lemma

A very important algebraic tool in studying local properties of a variety, or equivalently, properties of Noetherian local rings, is the completion $\widehat{R}$ of a Noetherian local ring R. It is again a Noetherian local ring, which inherits many of the properties of the original ring, and in fact, there is natural homomorphism $R \to \widehat{R}$, which is flat and unramified (recall that the latter means that the maximal ideal of

[1] Jan Denef, who was my promotor at the time, suggested the construction to me in 1981, which I then subsequently worked out and wrote up as part of my license thesis [87].

R extends to the maximal ideal of its completion $\widehat{R}$). Whereas there is no known classification of arbitrary Noetherian local rings, we do have many structure theorems, due mostly to Cohen, for complete Noetherian local rings. In particular, the equal characteristic complete regular local rings are completely classified by their residue field k and their dimension d: any such ring is isomorphic to the power series ring $k[[\xi_1, \dots, \xi_d]]$. Also extremely useful is the fact that we have an analogue of Noether normalization for complete Noetherian local domains: any such ring admits a regular subring over which it is finite. Another nice property of complete local rings is the following formal version of Newton's method for finding approximate roots.

Theorem A.1.1 (Hensel's Lemma). *Let $(R, \mathfrak{m})$ be a complete local ring with residue field k. Let $f \in R[t]$ be a monic polynomial in the single variable t, and let $\bar{f} \in k[t]$ denote its reduction modulo $\mathfrak{m}R[t]$. For every simple root $u \in k$ of $\bar{f} = 0$, we can find $a \in R$ such that $f(a) = 0$ and u is the image of a in k.*

Proof. Let $a_1 \in R$ be any lifting of u. Since $\bar{f}(u) = 0$, we get $f(a_1) \equiv 0 \mod \mathfrak{m}$. We will define elements $a_n \in R$ recursively such that $f(a_n) \equiv 0 \mod \mathfrak{m}^n$ and $a_n \equiv a_{n-1} \mod \mathfrak{m}^{n-1}$ for all $n > 1$. Suppose we already defined $a_1, \dots, a_n$ satisfying the above conditions. Consider the Taylor expansion

$$f(a_n + t) = f(a_n) + f'(a_n)t + g_n(t)t^2 \tag{A.1}$$

where $g_n \in R[t]$ is some polynomial. Since the image of a_n in k is equal to u, and since $\bar{f}'(u) \neq 0$ by assumption, $f'(a_n)$ does not lie in $\mathfrak{m}$ whence is a unit, say, with inverse u_n. Define $a_{n+1} := a_n - u_n f(a_n)$. Substituting $t = -u_n f(a_n)$ in (A.1), we get

$$f(a_{n+1}) \in (u_n f(a_n))^2 R \subseteq \mathfrak{m}^{2n},$$

as required.

To finish the proof, note that the sequence a_n is by construction Cauchy, and hence by assumption admits a limit $a \in R$ (whose residue is necessarily again equal to u). By continuity, $f(a)$ is equal to the limit of the $f(a_n)$ whence is zero. □

There are sharper versions of this result, where the root in the residue field need not be simple (see [27, Theorem 7.3]), or even involving systems of equations (see [13, §4.6]; but see also the next section).

A local ring satisfying the hypothesis of the above theorem is normally called a *Henselian* ring, although we will deviate from that practice in the next section. For some equivalent definitions, we refer once more to the literature [70, 71, 77, 106]. From a model-theoretic point of view, it is more convenient to work with Henselian local rings than with complete ones, since they form a first-order definable class (as is clear from the defining condition).

As with completion, there exists a 'smallest' Henselian overring. More precisely, for each Noetherian local ring R, there exists a Noetherian local R-algebra $R^{\sim}$, its *Henselization*, satisfying the following universal property: any local homomorphism $R \to H$ with H a Henselian local ring, factors uniquely through an

R-algebra homomorphism $R^{\sim} \to H$. Below, we will show the existence of such a Henselization by giving a concrete construction of $R^{\sim}$. Note that Theorem A.1.1 and the universal property imply that $R^{\sim}$ is a subring of $\widehat{R}$, and in particular, the latter is the completion of the former.

A.2 Construction of the Henselization

Let $(R, \mathfrak{m})$ be a Noetherian local ring. By a *Hensel system* over R of size N, we mean a pair $(\mathscr{H}, \mathbf{a})$ consisting of a system $\mathscr{H}(t)$ of N polynomial equations $h_1, \ldots, h_N \in R[t]$ in the N unknowns $t := (t_1, \ldots, t_N)$, and an approximate solution $\mathbf{a}$ modulo $\mathfrak{m}$ in R (meaning that $h_i(\mathbf{a}) \equiv 0 \mod \mathfrak{m}$ for all i), such that associated Jacobian matrix

$$\operatorname{Jac}(\mathscr{H}) := \begin{pmatrix} \partial h_1/\partial t_1 & \partial h_1/\partial t_2 & \ldots & \partial h_1/\partial t_N \\ \partial h_2/\partial t_1 & \partial h_2/\partial t_2 & \ldots & \partial h_2/\partial t_N \\ \vdots & \vdots & \ddots & \vdots \\ \partial h_N/\partial t_1 & \partial h_N/\partial t_2 & \ldots & \partial h_N/\partial t_N \end{pmatrix} \tag{A.2}$$

evaluated at $\mathbf{a}$ is invertible over R, that is to say, the *Jacobian determinant* $\det(\operatorname{Jac}(\mathscr{H}))$ evaluated at $\mathbf{a}$ is a unit in R. We express the latter condition also by saying that $\mathbf{a}$ is a *non-singular* approximate solution. An N-tuple $\mathbf{s}$ in some local R-algebra S is called a *solution* of the Hensel system $(\mathscr{H}, \mathbf{a})$, if it is a solution of the system $\mathscr{H}$ and $\mathbf{s} \equiv \mathbf{a} \mod \mathfrak{m}S$. Note that $(\mathscr{H}, \mathbf{s})$ is then a Hensel system over S, and therefore, we sometimes call $\mathscr{H}$ a Hensel system, without mentioning the (approximate) non-singular solution. A Hensel system of size $N = 1$ is just a Hensel equation together with a solution in the residue field, as in the statement of Hensel's lemma. In fact, R satisfies Hensel's lemma if and only if any Hensel system over R has a solution in R. The proof of this equivalence is not that easy (one can give for instance a proof using standard etale extensions as in [70]).

Instead, we alter out definition by calling a local ring R *Henselian*, if any Hensel system (of any size) over R has a solution in R. In conclusion, being Henselian in the new sense implies that in the old sense, and the converse also holds, but is harder to prove. An easy modification of the proof of Theorem A.1.1, left to the reader, shows that complete local rings are Henselian in this new sense. In fact, using multivariate Taylor expansion, we obtain the following stronger version.

A.2.1 *Any Hensel system $(\mathscr{H}, \mathbf{a})$ over R admits a unique solution in the completion $\widehat{R}$.* □

We call an element $r \in \widehat{R}$ a *Hensel element* (over R) if there exists a Hensel system $(\mathscr{H}, \mathbf{a})$ over R such that r is the first entry of the unique solution of this system in $\widehat{R}$. We will express this by saying that *$\mathscr{H}$ is a Hensel system for r*. Note that if $\mathbf{r} = (r_1, \ldots, r_N)$ is a solution of a Hensel system $\mathscr{H}$ over R, then any r_i is a Hensel element. This is true by definition for r_1. For $i > 1$, let $\mathscr{H}'$ be obtained by interchanging the unknowns t_1 and t_i, as well as, h_1 with h_i. It follows that $\mathscr{H}'$

is a Hensel system for $(r_i, r_2, \dots, r_{i-1}, r_1, r_{i+1}, \dots, r_N)$, showing that r_i is a Hensel element.

Let $R^\sim$ be the subset of $\widehat{R}$ of all Hensel elements. For given Hensel elements r and r', we construct from their associated Hensel systems $(\mathscr{H}(t), \mathbf{a})$ and $(\mathscr{H}'(t'), \mathbf{a}')$ of size N and N' respectively, a new Hensel system for $r+r'$ as follows: let $N'' := N + N' + 1$, let t'' be the N''-tuple of unknowns (u, t, t'), with u a single variable, and consider the system $\mathscr{H}''$ of N'' equations in t'' given by the equation $u = t_1 + t'_1$, and the systems $\mathscr{H}(t)$ and $\mathscr{H}'(t')$. One checks that $(\mathscr{H}'', a_1 + a'_1, \mathbf{a}, \mathbf{a}')$ is a Hensel system—since its Jacobian determinant is the product of the Jacobian determinants of $\mathscr{H}$ and $\mathscr{H}'$—whose unique solution in $\widehat{R}$ has first entry equal to $r + r'$, showing that the latter is again a Hensel element. The same argument can be used to prove that the product of Hensel elements is again a Hensel element. With little effort one actually shows:

A.2.2 *The collection of all Hensel elements is a local ring $R^\sim$ with maximal ideal $\mathfrak{m}R^\sim$. Moreover, $R^\sim$ is Henselian, with completion equal to $\widehat{R}$.*

Indeed, let $\mathfrak{m}^\sim := \mathfrak{m}\widehat{R} \cap R^\sim$. To show that $R^\sim$ is local with maximal ideal $\mathfrak{m}^\sim$, it suffices to show that any element $r \in R^\sim$ not in $\mathfrak{m}^\sim$ is a unit in $R^\sim$. Since r does not belong to $\mathfrak{m}\widehat{R}$, it has an inverse in $\widehat{R}$. Using an auxiliary variable u and the equation $t_1 u = 1$, it is now not hard to show that $1/r$ is an Hensel element. In particular, R, $R^\sim$ and $\widehat{R}$ all have the same residue field k. To prove that $R^\sim$ is Henselian, we must verify the multivariate Hensel lemma, that is to say, let $(\tilde{\mathscr{H}}(t), \mathbf{a})$ be a Hensel system over $R^\sim$. Since $R^\sim$ and R have the same residue field, we may choose $\mathbf{a}$ in R. By A.2.1, there exists a unique solution $\mathbf{r}$ over $\widehat{R}$ of this Hensel system. Remains to show that $\mathbf{r}$ has its entries already in $R^\sim$, and to this end, it suffices by the above discussion to construct a Hensel system over R of which $\mathbf{r}$ is part of a solution.

Let $\mathbf{s} = (s_1, \dots, s_d)$ be the tuple of coefficients in $R^\sim$ of the equations $\tilde{\mathscr{H}}$ (listed in a fixed order), and let $\mathscr{H}(t, u)$ be obtained from $\tilde{\mathscr{H}}$ by replacing each of these coefficients by a new variable u_i, so that $\tilde{\mathscr{H}}(t) = \mathscr{H}(t, \mathbf{s})$. For each s_i, choose $b_i \in R$ such that $s_i \equiv b_i \mod \mathfrak{m}\widehat{R}$. Let $(\mathscr{H}_i(u_i, t_i), (b_i, \mathbf{c}_i))$ be a Hensel system for each Hensel element s_i, with t_i a finite tuple of auxiliary unknowns and $\mathbf{c}_i$ a tuple of the corresponding length in R, for $i = 1, \dots, d$. One easily checks that the system $\mathscr{G}$ in the unknowns $t, u_1, t_1, \dots, u_d, t_d$ at the tuple $\mathbf{c} := (\mathbf{a}, b_1, \mathbf{c}_1, \dots, b_d, \mathbf{c}_d)$ given by $\mathscr{H}$ and all $\mathscr{H}_i$ is a Hensel system, since the Jacobian determinant of $(\mathscr{G}, \mathbf{c})$ is the product of the Jacobian determinants of $(\mathscr{H}, \mathbf{a})$ and the $(\mathscr{H}_i, (b_i, \mathbf{c}_i))$. By A.2.1, the unique solution of this Hensel system in $\widehat{R}$ must be of the form $(\mathbf{r}, s_1, \mathbf{r}_1, \dots, s_d, \mathbf{r}_d)$, for some $\mathbf{r}_i$ in $\widehat{R}$, showing that $\mathbf{r} \in R^\sim$. □

It is unfortunately less easy to prove that $R^\sim$ is also Noetherian, and we postpone the discussion until after we proved our main result:

Theorem A.2.3. *The ring $R^\sim$ satisfies the universal property of Henselization: any Henselian local R-algebra S admits a unique structure of $R^\sim$-algebra.*

Proof. We need to show that there exists a (unique) R-algebra homomorphism $R^\sim \to S$. Given $r \in R^\sim$, let $(\mathscr{H}, \mathbf{a})$ be a Hensel system admitting a solution with first entry r. Since $\mathbf{a}$ is an approximate solution of $\mathscr{H}$ in R, it remains so in S. By (the revised) definition of Henselian, the approximate solution $\mathbf{a}$ lifts uniquely to a solution $\mathbf{s}$ in S. We define the image of r in S now as the first entry of this solution $\mathbf{s}$. Uniqueness guarantees firstly that this is an R-algebra homomorphism, an secondly that it is unique. □

Returning to the issue of Noetherianity, we will use the local flatness criterion discussed §3.3.6. We start with the flatness of the Henselization:

Proposition A.2.4. *For any ideal $I \subseteq R$, the Henselization of R/I is isomorphic to $R^\sim/IR^\sim$. Moreover, $R \to R^\sim$ is faithfully flat, whence a scalar extension, and $R^\sim$ is ind-Noetherian.*

Proof. Let $S := R/I$. It is not hard to show that any homomorphic image of a Henselian local ring is again Henselian. Hence $R^\sim/IR^\sim$ is Henselian, and the universal property of Henselizations then yields a unique homomorphism $S^\sim \to R^\sim/IR^\sim$. The composition of this homomorphism with $R^\sim/IR^\sim \to \widehat{R}/I\widehat{R}$ is injective, since the latter is the completion of S. Hence $S^\sim \to R^\sim/IR^\sim$ must also be injective. To prove surjectivity, let $r \in R^\sim$ and let $\mathscr{H}$ be a Hensel system for r. The reduction modulo I of this Hensel system therefore has a unique solution in $S^\sim$, and by uniqueness, the first entry of this solution must map to the image of r in $R^\sim/IR^\sim$. This proves the first assertion, and in particular that $I\widehat{R} \cap R^\sim = IR^\sim$, for any ideal $I \subseteq R$. The second assertion then follows from the flatness of $R \to \widehat{R}$ and Corollary 3.3.15. Since $R \to R^\sim$ is unramified by A.2.2, it is therefore a scalar extension (see §3.2.3).

So remains to show that $R^\sim$ is ind-Noetherian (defined after Corollary 3.3.22). Let $\mathbf{x}$ be a finite tuple in $R^\sim$. As already remarked before, we can find a Hensel system $\mathscr{H}(t)$ over R such that $\mathbf{x}$ is part of its unique solution. Hence, if $S_\mathbf{x}$ is the localization of $R[t]/(\mathscr{H})$ with respect to the ideal generated by $\mathfrak{m}$, then $\mathbf{x}$ is already a tuple in $S_\mathbf{x}$. It follows from the construction of $R^\sim$, that $S_\mathbf{x}^\sim = R^\sim$. In particular, $S_\mathbf{x} \to R^\sim$ is a scalar extension by what we just proved, and $R^\sim$ is the direct limit of the $S_\mathbf{x}$. □

Theorem A.2.5. *The Henselization of a Noetherian local ring is again Noetherian.*

Proof. It suffices to show that $R^\sim \to \widehat{R}$ is faithfully flat, since $\widehat{R}$ is Noetherian. To obtain flatness, it suffices in view of Corollary 3.3.25 and Proposition A.2.4 to show that $\mathrm{Tor}_1^{R^\sim}(\widehat{R}, k) = 0$, where k is the residue field of R. To this end, let

$$R^m \to R^n \to R \to k \to 0 \tag{A.3}$$

be an exact sequence. By Proposition A.2.4, tensoring with $R^\sim$ yields an exact sequence

$$(R^\sim)^m \to (R^\sim)^n \to R^\sim \to k \to 0.$$

By definition, $\mathrm{Tor}_1^{R^\sim}(\widehat{R},k)$ is the homology of the complex obtained from tensoring this exact sequence with $\widehat{R}$, that is to say, of the complex

$$(\widehat{R})^m \to (\widehat{R})^n \to \widehat{R} \to k \to 0.$$

However, this latter complex is actually exact since it is obtained from tensoring (A.3) with the flat extension $\widehat{R}$, showing that $\mathrm{Tor}_1^{R^\sim}(\widehat{R},k) = 0$. □

A.3 Etale proto-grade

We conclude with a proto-graded version of the previous construction by constructing a proto-grading on the Henselization of a proto-graded Noetherian local ring $(R,\mathfrak{m})$, and giving conditions under which this proto-grading is Noetherian and faithfully flat. Define a proto-grading on $R^\sim$ by the condition that a Hensel element $y \in R^\sim$ has proto-grade at most n if it admits a Hensel system $(\mathscr{H},\mathbf{u})$ of length $N \leq n$, in which all polynomials have degree at most n, and all coefficients as well as all entries of $\mathbf{u}$ have proto-grade at most n.

A.3.1 *This yields a proto-grading on $R^\sim$, called the* etale proto-grading *on $R^\sim$, extending the proto-grading on R. Moreover, $R \to R^\sim$ is a morphism of proto-graded rings.*

The fact that this is a proto-grading follows from the proof that $R^\sim$ is a ring, since we explicitly constructed Hensel systems for sums, products, and inverses (of units). Since $t-a$ is a Hensel system of $a \in R$, its etale proto-grade is equal to its proto-grade in R, and the last assertion is now immediate. □

The following result enables us to calculate protopowers:

Proposition A.3.2. *If R is a proto-graded Noetherian local ring and $R^\sim$ is viewed with its etale proto-grading extending the proto-grading on R, then we have an isomorphism*

$$(R^\sim)_\flat \cong (R_\flat)^\sim.$$

Proof. A special instance of the above isomorphism is the fact that if R is Henselian, then so is $R_\flat$. We prove this first, and so, let $(\mathscr{H},\mathbf{u})$ be a Hensel system over $R_\flat$ of proto-grade at most n, say. Choose approximations $\mathscr{H}_w$ and $\mathbf{u}_w$ over R of proto-grade at most n, with respective ultraproduct $\mathscr{H}$ and $\mathbf{u}$. By Łoś' Theorem, almost all $(\mathscr{H}_w,\mathbf{u}_w)$ are Hensel systems. Since R is Henselian by assumption, these Hensel systems have a (unique) solution $\mathbf{x}_w$. By definition, the $\mathbf{x}_w$ have etale proto-grade at most n, and hence their ultraproduct $\mathbf{x}$ lies in $R_\flat$. By Łoś' Theorem, $\mathbf{x}$ is then a solution of the Hensel system $(\mathscr{H},\mathbf{u})$.

Let R now be an arbitrary proto-graded Noetherian local ring. The embedding $R \to R^\sim$ induces an embedding $R_\flat \to (R^\sim)_\flat$. By our previous argument, $(R^\sim)_\flat$ is Henselian, whence by the universal property of a Henselization, we have a unique $R_\flat$-algebra embedding $(R_\flat)^\sim \to (R^\sim)_\flat$. To see that this is surjective, let x be an element in $(R^\sim)_\flat$, say of etale proto-grade at most n. Choose an approximation $x_w \in R^\sim$ of proto-grade at most n. Hence, almost each x_w is the first entry of the unique solution $\mathbf{x}_w$ of a Hensel system $(\mathscr{H}_w, \mathbf{u}_w)$ over R of proto-grade at most n. Since the ultraproduct $(\mathscr{H}, \mathbf{u})$ of the $(\mathscr{H}_w, \mathbf{u}_w)$ is a Hensel system of proto-grade at most n, whence defined over $R_\flat$, the ultraproduct $\mathbf{x}$ of the $\mathbf{x}_w$ is a solution of etale proto-grade at most n, belonging therefore to $(R_\flat)^\sim$. Since x is its first entry, $x \in (R_\flat)^\sim$, as we wanted to show. □

Theorem A.3.3. *If R is a local ring with a Noetherian proto-grading, then the etale proto-grading on $R^\sim$ is also Noetherian. If R is moreover regular and the proto-grading on R is faithfully flat, then the etale proto-grading on $R^\sim$ is also faithfully flat.*

Proof. The first assertion follows from Proposition A.3.2 and Theorem A.2.5. To prove the second assertion, assuming that $(R, \mathfrak{m})$ is moreover regular, we first show that $R_\flat$ is also regular, by induction on the dimension d of R. Since the proto-grade is faithfully flat, $(R/I)_\flat = R_\flat/IR_\flat$ for all ideals $I \subseteq R$ by 9.1.7. Applied to $I = x_1R$, where $\mathfrak{m} = (x_1, \dots, x_d)R$, we have by induction that $(R/x_1R)_\flat = R_\flat/x_1R_\flat$ is regular, whence so is $R_\flat$, since x_1 is $R_\flat$-regular (as $R \to R_\flat$ is flat). Since $R_\flat \to (R_\flat)^\sim$ is a scalar extension by Proposition A.2.4, also $(R_\flat)^\sim$ is regular by 3.2.14. Hence, in view of Proposition A.3.2, we proved that $(R^\sim)_\flat$ is regular. Since $(x_1, \dots, x_d)$ is an $(R^\sim)_\natural$-regular sequence by the flatness of $R^\sim \to (R^\sim)_\natural$ (using Theorem A.2.5 and Corollary 3.3.3), the Cohen-Macaulay criterion for flatness (Theorem 3.3.9) together with Proposition 3.3.8, yields the desired flatness of $(R^\sim)_\flat \to (R^\sim)_\natural$. □

Let k be a field and ξ a finite tuple of indeterminates. For simplicity, we denote the Henselization of the localization of $k[\xi]$ with respect to the variables also by $k[\xi]^\sim$. A power series $f \in k[[\xi]]$ is called *algebraic* if it is a root of a non-zero polynomial in one variable with coefficients in $k[\xi]$. We denote the subring of algebraic power series by $k[[\xi]]^{\mathrm{alg}}$. The following result is well-known (see, for instance, [3, 77]).

A.3.4 *For any field k, the ring $k[[\xi]]^{alg}$ is equal to the Henselization $k[\xi]^\sim$ of $k[\xi]_\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal generated by the indeterminates.*

In particular, viewing $k[\xi]$ with its affine proto-grade given by degree (see (9.1.1.i)), we get an etale proto-grade on $k[[\xi]]^{\mathrm{alg}}$: an algebraic power series f has proto-grade at most n, if there exists a Hensel system in $k[\xi, t]$ for f of size at most n, such that the total degree of each polynomial in the system is at most n. Theorem 9.2.11, in conjunction with Theorem A.3.3 and Corollary 9.2.4, applied to this etale proto-grade on the ring of algebraic power series, immediately yields:

Theorem A.3.5. *For each pair (n,m) there exists a uniform bound $n' := n'(n,m)$ with the property that if k is an arbitrary field, $R := k[[\xi]]^{alg}$ the ring of algebraic power series with ξ an m-tuple of indeterminates, and $I := (f_1, \dots, f_s)R$ an ideal generated by elements f_i of etale proto-grade at most n, then I is generated by at most n' of the f_i, and its module of syzygies is generated by n' syzygies with entries of proto-grade at most n'. Moreover, if $f \in I$ has etale proto-grade at most n, then there exist algebraic power series g_i of etale proto-grade at most n' such that $f = g_1 f_1 + \cdots + f_s g_s$.* □