THE JOURNAL OF SYMBOLIC LOGIC
Volume 00, Number 0, XXX 0000

PROLEGOMENA TO O-MINIMALISM: DEFINABLE COMPLETENESS, TYPE COMPLETENESS, AND TAMENESS

HANS SCHOUTENS

Abstract. An ordered structure is called o-minimalistic if it has all the first-order features of an o-minimal structure. In this paper, we concentrate on a well-known fragment DCTC (Definable Completeness/Type Completeness) and generalize o-minimal properties to this more general situation (dimension theory, monotonicity, Hardy structures, quasi-cell decomposition) upon replacing finiteness by discreteness in all of these. Failure of cell decomposition leads to the related notion of a tame structure.

§**1. Introduction.** O-minimality has been studied extensively (see [9] for some of the literature). It also has been generalized in many ways (weak o-minimality, quasi-o-minimality, d-minimality, local o-minimality, o-minimal open cores, etc.) These generalizations attempt to bring into the fold certain ordered structures that fail some of the good finiteness properties of o-minimality, but still behave tamely.[1] We propose a different perspective, noting that, contrary to ultrapowers, ultraproducts of o-minimal structures need no longer be o-minimal; let us call them *ultra-o-minimal*. This leads to two natural questions: (i) under which conditions on the o-minimal components is an ultra-o-minimal structure again o-minimal? And (ii), what properties do ultra-o-minimal structures have? We give an answer to (i) in [7, Theorem 3.20], but in this paper we will be only concerned with (ii).

Given a language L with an order relation, let $T^{\text{omin}} := T^{\text{omin}}(L)$ be the intersection of the theories of all o-minimal L-structures. Models of T^{omin} will be called *o-minimalistic*; they are precisely the elementary substructures of ultra-o-minimal ones. O-minimalism is in essence a non-standard feature, as any o-minimalistic expansion of the reals is already o-minimal (Corollary 2.4). The present paper is intended as a preliminary to the study of o-minimalism, as we restrict our study to two well-known, elementary consequences of o-minimality, *definable completeness* (=every definable subset has an infimum) and *type completeness*[2] (=every one-sided type of a point, including the ones at infinity, is complete). We denote these axiom schemes on one-variable definable sets by DCTC. I do not know whether DCTC is equal to T^{omin}, but in [7], which will

Received by the editors July 24, 2011.
Partially supported by PSC-CUNY grant #62247-00 40

[1]The concept of 'tameness' is quite vague and often depends on the particular author's taste, the present author not excluded; see §9.

[2]This is a slightly stronger version of what is called in the literature *local o-minimality*, but which agrees with it in the case of an expansion of an ordered field.

0022-4812/00/0000-0000/$00.00

investigate o-minimalism proper, I will formulate a third o-minimalistic (first-order) property, the Discrete Pigeonhole Principle (DPP=any definable, injective map from a discrete set to itself is bijective), which as of yet, I do not know how to derive from DCTC. In fact, it is not clear if we can axiomatize o-minimalism by first-order conditions on one-variable formulae only (note that DPP is a priori not of this form).

We show that the weaker theory DCTC proves already many properties that resemble those of o-minimal structures, such as the Monotonicity Theorem (Theorem 3.2), Fiber Dimension Theorem (Corollary 7.2), Quasi-Cell Decomposition (Theorem 8.3), Hardy structures on germs at infinity (Theorem 6.10), etc. Since some of these have already been treated by others, we often only give details for two variables, leaving higher arities to the reader, commenting on it occasionally. As we will argue in [7, §4.7] in more detail, *o-finitism*, that is to say, the first-order properties of finite sets in o-minimal structures, includes discreteness, boundedness, and closedness. Moreover, under the DCTC assumption, discrete always implies bounded and closed. So, we will have to substitute 'discrete' for 'finite' in any of the above properties of o-minimal structures. Nonetheless, this program does not always pan out. For instance, while decomposing into cells, we seem to run into infinite disjunctions, leading to the notion of a *quasi-cell*, which is only locally a cell. However, there is a large class of definable subsets, called *tame* subsets, that have a 'definable' cell decomposition, that is to say, loosely speaking, they admit a cell decomposition in 'discretely' many cells (see §9 for the precise definition). A *tame* structure is then one in which every definable subset is tame, and we show that it is always at least a model of DCTC. Any model of DCTC which is an expansion of a field by one-variable functions is tame.

There is some overlap of the present paper with the preprint [2], independently written by Fornasiero. Apart from the additional assumption of an underlying field (which, as pointed out by the author, is not always necessary), he also derives a Monotonicity Theorem, develops a good dimension theory, and achieves a decomposition into what he calls *multi-cells*. It would be of interest to combine this with our quasi-cell decomposition theorem, and see whether this eliminates the need for quasi-cells. I am also grateful for Fornasiero's comments on an earlier version of this paper.

Notations and conventions. Definable always will mean definable with parameters, unless stated explicitly otherwise. Throughout this paper, L denotes some language containing a distinguished binary relation symbol $<$ and any L-structure $\mathcal{O}$ will be a dense linear order without endpoints. We will always view an L-structure in its *order topology* in which the basic open subsets are the open (possibly unbounded) intervals. Note that this is always a Hausdorff topology by density. The Cartesian powers O^n are equipped with the product topology, for which the open boxes form a basis, where an *open box* is any product of open intervals. We introduce two new symbols $-\infty$ and ∞, and, given an L-structure $\mathcal{O}$, we let $O_\infty := O \cup \{\pm\infty\}$. When needed, $\mathtt{U}$ denotes some predicate (often unary), and we will write $(\mathcal{O}, X)$ for the $L(\mathtt{U})$-structure in which X is the interpretation of $\mathtt{U}$. Recall that the *interior* Y° of a subset Y in a topological space O consists of all points $x \in Y$ for which there exists an open $U \subseteq Y$ with $x \in U$; the *exterior* is the interior of the complement $O \setminus Y$; the *closure* $\bar{Y}$ is the complement of the exterior; the *frontier* $\operatorname{fr}(Y)$ is the difference $\bar{Y} \setminus Y$; and the *boundary* is the difference $\partial Y := \bar{Y} \setminus Y^\circ$. Clearly, if Y is definable, then so are its interior, exterior, frontier, and boundary. A definable subset $X \subseteq O^n$ is called *definably connected* if it can not be written as a disjoint union of two open definable subsets of X. The image of a definably connected subset under a definable and continuous function is definably connected. Any interval is definably connected. It is an easy exercise that in an o-minimal structure, a subset of O is definably connected if and only if it is an interval.

We will use the following ISO convention for intervals: *open* $]a,b[$ (which we always assume to be non-empty, that is to say, $a < b$), *closed* $[a,b]$ (including the singleton $\{a\} = [a,a]$), *half-open* $]a,b]$ or $[a,b[$, and their infinite variants like $]-\infty,a[$, $]-\infty,a]$, $]a,\infty[$, and $[a,\infty[$, with $a,b \in O$. Note that the usage of ∞ here is only informal since these are definable subsets in the language without the extra constants $\pm\infty$ by formulae of the form $x < a$, etc. Given a subset $Y \subseteq O$ and a point $b \in O$, we will sometimes use notations like $Y_{\leq b} := Y \cap]-\infty,b]$ or $Y_{<b} := Y \cap]-\infty,b[$.

When taking ultraproducts, we rarely ever mention the underlying index set or (non-principal) ultrafilter. We use the notation introduced in [6], denoting ultraproducts with a subscript $\natural$. Thus, we write $\mathbb{N}_\natural$, $\mathbb{Z}_\natural$, and $\mathbb{R}_\natural$ for the ultrapower of the set of natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, and reals $\mathbb{R}$ respectively. On occasion we need the (countable) ultraproduct of the diagonal sequence $(n)_n$ in $\mathbb{N}_\natural$, which we denote suggestively by $\omega_\natural$.

§**2. The theory DCTC.** Recall that an (ordered) L-structure $\mathcal{M}$ is called *o-minimal*, if every definable subset $Y \subseteq M$ is a finite union of open intervals and points. It is called *definably complete*, if every definable subset in M has an infimum (possibly $\pm\infty$). Since intervals have this property, every definable subset in an o-minimal structure is definably complete, that is to say, definable completeness is an o-minimalistic property. By [3, Corollary 1.5], it is equivalent with M being definably connected, and also with the validity of the *Intermediate Value Theorem* (IVT is the statement that if $f\colon [a,b] \to M$ is a definable and continuous function, then f assumes each value between $f(a)$ and $f(b)$). Recall that $\mathcal{M}$ is called *locally o-minimal*, if for every definable subset $Y \subseteq M$ and every point $x \in M$, there exists an open interval I containing x such that $I \cap Y$ is a finite union of open intervals and points, and by shrinking I even further, we may even take $I \cap Y$ to be an interval (for more on local o-minimality, see [8, 10, 2]). However, it appears to me a flaw of the definition, that one only requires x to be a point in M, that is to say, excluding the case $x = \pm\infty$. In case $\mathcal{M}$ has also the structure of a field—the most studied case—we can take reciprocals, bringing ∞ to 0, and so there is no need for the more general concept, but in general, one should include infinite points in this definition. We can reformulate local o-minimality as follows. Given $Y \subseteq M$ and $a \in M$, we say that a^- *belongs to* Y, if there exists an open interval $]b,a[\subseteq Y$ (similarly, a^+ *belongs to* Y, if $]a,b[\subseteq Y$ for some $b > a$). Thinking of a^- as a partial type (that is to say, consisting of all formulae $b < x < a$ in the variable x, where b runs over all elements less than a), if Y is defined by a formula φ, then a^- belongs to Y if and only if any realization of the type of a^- in any elementary extension of $\mathcal{M}$ satisfies φ. Therefore, local o-minimality says that a^- is a complete type, meaning that if Y is definable, then a^- belongs either to Y or to $M \setminus Y$, for any $a \in M_\infty$. By taking complements, a^+ is then also complete. As mentioned, it is important to include the infinite points, where the two types $(-\infty)^+$ and ∞^- are defined in the obvious way. For this reason, we will refer to this property as *type completeness* instead of local o-minimality.

2.1. DEFINITION. For a fixed language L with a binary order symbol $<$, we define the theory DCTC as the extension of the theory of dense linear orders without endpoints by the two axiom schemes (one axiom for every formula) given by definable completeness and type completeness. In other words, in a model $\mathcal{M}$ of DCTC, any definable subset $Y \subseteq M$ has an infimum and its characteristic function has a left limit at each point. We will call an ultraproduct of o-minimal L-structures an *ultra-o-minimal* structure. It follows that such a structure is a model of DCTC, and more generally, any o-minimalistic structure is a model of DCTC. The converse is not clear, although we will see in [7] an axiom which is o-minimalistic, but is currently not known to follow from DCTC. For the remainder of this paper, we will investigate the theory DCTC.

2.2. EXAMPLE. Let L be the language of ordered fields together with a unary predicate $\mathtt{U}$. For each n, let $\mathcal{R}_n := (\mathbb{R}, \{0, 1, \dots, n\}$ be the expansion of the field $\mathbb{R}$. Since $\{0, 1, 2, \dots, n\}$ is finite, each $\mathcal{R}_n$ is o-minimal, and therefore their ultraproduct $\mathcal{R}_\natural$ is o-minimalistic, whence a model of DCTC. The set $D := \mathtt{U}(\mathcal{R}_\natural)$ is discrete but not finite, so $\mathcal{R}_\natural$ cannot be o-minimal. Note that $D = (\mathbb{N}_\natural)_{\leq \omega_\natural}$.

2.3. PROPOSITION. *Given a model $\mathcal{M}$ of DCTC, let $K \subseteq M$ be compact and $Y \subseteq M$ definable. If either K is open or Y is contained in K, then $K \cap Y$ is a finite union of intervals.*

PROOF. Given a definable subset $Y \subseteq M$, by assumption, we can find in the open case, for each $x \in K$, an open interval $I_x \subseteq K$ such that $Y \cap I_x$ is an interval. Since K is compact and the I_x cover K, there exist finitely many points $x_1, \dots, x_n \in K$ such that $K = I_{x_1} \cup \dots \cup I_{x_n}$ and hence $K \cap Y$ is a finite union of intervals. If K is arbitrary, then we cannot arrange for all I_x to be contained in K, and so we only get $K \subseteq I_{x_1} \cup \dots \cup I_{x_n}$. But since $Y \subseteq K$, the same conclusion can be drawn. $\dashv$

The next corollary improves [1, 2.13(3)] as it does not assume any underlying field structure.

2.4. COROLLARY. *If $\mathcal{M} \models$ DCTC with underlying order that of the reals, then it is o-minimal.*

PROOF. Identify M with $\mathbb{R}$, and let $Y \subseteq \mathbb{R}$ be definable. Depending whether $(-\infty)^+$ or ∞^- belong to Y or not, we may assume after possibly removing one or two unbounded intervals that Y is bounded, whence contained in some closed, bounded interval $K := [a, b]$. Hence $Y = Y \cap K$ is a finite union of intervals by Proposition 2.3. $\dashv$

2.5. *Remark.* From the proof it is clear that we have the following more general result: if a model of DCTC has the Heine-Borel property, meaning that any closed bounded set is compact, then it is o-minimal.

2.6. PROPOSITION. *For a definable subset $Y \subseteq M$ in a model $\mathcal{M} \models$ DCTC, we have:*

2.6.i. *The infimum of Y is either infinite or a topological boundary point.*
2.6.ii. *If $a, b \in \partial Y$ and $]a, b[\cap \partial Y = \emptyset$, then $]a, b[$ is either disjoint from or entirely contained in Y.*
2.6.iii. *If Y is definably connected, then it is an interval.*
2.6.iv. *Y either has a non-empty interior or is discrete.*
2.6.v. *If Y is discrete, then it has a minimum and a maximum, and it is closed and bounded.*

2.6.vi. *The topological boundary ∂Y is discrete, closed, and bounded.*

PROOF. To prove (2.6.i), let $l \in M$ be the infimum of Y. By type completeness, l^- either belongs to Y or to $M \setminus Y$. The former case is excluded since l is the infimum of Y. In particular, l is not an interior point of Y. If l^+ does not belong to Y, then l is an isolated point of Y, and hence belongs to the (topological) boundary. In the remaining case, l lies in the closure of Y, since some open interval $]l, x[$ lies inside Y. To prove (2.6.ii), suppose there exists $x \in\ ]a, b[\ \cap Y$. By type completeness, either x^- belongs to Y or to $M \setminus Y$. In the latter case, there exists $z < x$ such that $]z, x[$ is disjoint from Y. However, x is then not an interior point of Y, whence must belong to its topological boundary, contradiction. So x^- belongs to Y, which means that the set of all $z \in\ ]a, x[$ such that $]z, x[\ \subseteq Y$ is non-empty. The infimum of this set must be a topological boundary point of Y by (2.6.i), and hence must be equal to a, showing that $]a, x]\ \subseteq Y$. Arguing the same with x^+, then shows that also $[x, b[\ \subseteq Y$, as we needed to prove.

To prove (2.6.iii), let $Y \subseteq M$ be definably connected. Let l and h be its respective infimum and supremum (including the case that these are infinite). The case $l = h$ is trivial, so assume $l < h$. If there were some $x \in\ ]l, h[$ not belonging to Y, then Y would be the union of the two definable, non-empty, disjoint open subsets $Y_{<x}$ and $Y_{>x}$, contradiction, Hence, Y is an interval with endpoints l and h. To prove (2.6.iv), assume Y is not discrete. Hence there exists $a \in Y$ which is not isolated, that is to say, such that any open interval containing a has some other point in common with Y. If both a^- and a^+ belong to $M \setminus Y$, then there are $x < a < y$ such that $]x, a[$, $]a, y[$ are disjoint from Y, contradicting that a is not isolated. Hence, say, a^- belongs to Y and Y has non-empty interior.

Assume next that Y is discrete and let l be its infimum (including possibly the case $l = -\infty$). If l^+ belongs to Y, then $]l, z[\ \subseteq Y$ for some $z > l$, contradicting discreteness. So l^+ does not belong to Y, which forces $l \in Y$. In particular, l is finite, proving the first part of (2.6.v), and in particular, that Y is bounded. To show that Y is closed, suppose it is not. Let $x \notin Y$ be a point in its closure. Since $Y \cup \{x\}$ is definable but not discrete, it must have interior by (2.6.iv), so that some open interval I is contained in $Y \cup \{x\}$. But then $I \cap Y = I \setminus \{x\}$ is not discrete, contradiction. To see (2.6.vi), it suffices by (2.6.v) to show that ∂Y is discrete. Let $b \in \partial Y$. We have to show that b is an isolated point of ∂Y, and this will clearly hold if b is an isolated point of either Y or $M \setminus Y$. In the remaining case, exactly one from among b^- and b^+ belongs to Y, say, b^-. Hence, there exist $x < b < y$ so that $]x, b[\ \subseteq Y$ and $]b, y[\ \cap Y = \emptyset$. Since any point in $]x, b[$ is interior to Y and any point in $]b, y[$ exterior to Y, we get $\partial Y \cap\ ]x, y[\ = \{b\}$, as we needed to prove. ⊣

2.7. *Remark.* For any definable, discrete subset $Y \subseteq M$ in a model of DCTC, we can therefore define a successor function σ_Y on Y by letting $\sigma_Y(b)$ be the minimum of (the definable subset) $Y_{>b}$, for any non-maximal b in Y.

2.8. COROLLARY. *The theory DCTC is equivalent with type completeness and discrete definable completeness, where in the latter we only require that definable, discrete sets have a minimum.*

PROOF. Let $Y \subseteq M$ be a definable set in a model $\mathcal{M}$ of the weaker system from the assertion. Inspecting the argument in the proof of (2.6.vi), type completeness already implies that ∂Y is discrete. Hence ∂Y has an infimum b, and it is now not hard to show that b is also the infimum of Y. ⊣

2.9. COROLLARY. *In a model of DCTC, any finite union of one-variable definable, discrete subsets is discrete.* ⊣

Using (2.6.vi), (2.6.ii) and Remark 2.7, we get immediately the following structure theorem for one-variable definable subsets (compare this with the notion of *d-minimality* from [4], where one needs finitely many discrete subsets):

2.10. THEOREM. *An L-structure $\mathcal{M}$ is a model of DCTC if and only if every definable subset $Y \subseteq M$ is a disjoint union of open intervals and a single closed, bounded, discrete set.*

PROOF. We only need to prove the converse. Let us first show that M is definably connected. Indeed, if U_1 and U_2 are disjoint definable open sets covering M, then this would yield a covering of M by disjoint open intervals. However, considering what the endpoints would be, this can only be the trivial covering, showing that one of the U_i must be empty. By [3, Corollary 1.5], definable connectedness implies definable completeness. To prove type completeness, let $Y \subseteq M$ be definable and $b \in \partial Y$ (boundary points are the only points in which it can fail). There is nothing to prove if b is an isolated point of Y or of $M \setminus Y$, so assume it is not. Decompose $Y = U \cup D$ into definable subsets with U a disjoint union of open intervals and D closed, bounded, and discrete. Let $]p, q[$ and $]u, v[$ be the open interval in U immediately to the left and to the right of b respectively. Since b is not isolated but in the boundary, it must be equal to exactly one of q or u, that is to say, either $q = b < u$ or $q < u = b$. Say the latter holds, so that b^+ belongs to Y. Since $D \cup \{b\}$ is discrete, we can find an open interval I containing b such that $I \cap (D \cup \{b\}) = \{b\}$. Shrinking I if necessary, we can make it disjoint from $]p, q[$, and hence $I_{<b} \cap Y = \emptyset$, showing that b^- belongs to $M \setminus Y$. We need to verify this also at $b = \pm\infty$, where the same argument works in view of the boundedness of D. ⊣

§**3. Definable maps.** For the remainder of this paper, we will always work in a model $\mathcal{M}$ of DCTC, unless noted explicitly otherwise. Next we study definable maps, where we call a map $f\colon Y \subseteq M^n \to M^k$ *definable* if its graph $\Gamma(f) \subseteq M^{n+k}$ is a definable subset. Note that since its domain $Y \subseteq M^n$ is the projection of its graph, it too is definable. Similarly, the set $\Gamma^*(f)$ consisting of all $(f(x), x) \in M^{k+n}$ is definable and is called the *reverse graph* of f. If $k = n = 1$, we speak of *one-variable* maps.

3.1. LEMMA. *Given a definable map $f\colon Y \to M$,*

3.1.i. *if Y is discrete, then so is its image $f(Y)$;*
3.1.ii. *if $f(Y)$ and each fiber of f is discrete, then so is Y.*

PROOF. Suppose (3.1.i) does not hold, so that $Y \subseteq M$ is discrete but not $f(Y)$. Let H be the (non-empty, definable, discrete) subset of all $x \in Y$ such that $f(Y_{\geq x})$ is non-discrete, and let h be its maximum. Since h cannot be the maximal element of $Y \subseteq M^n$ lest $f(Y_{\geq h})$ be a singleton, we can find its successor $\sigma(h) \in Y$ by (4.1.iii). But $f(Y_{\geq h}) = \{f(h)\} \sqcup f(Y_{\geq \sigma(h)})$, so that neither $f(Y_{\geq \sigma(h)})$ can be discrete by Corollary 2.9, contradicting maximality.

Assume next that (3.1.ii) is false, so that Y is non-discrete, but $Z := f(Y)$ and all $f^{-1}(u)$ are discrete. This time, let H be the subset of all $x \in Z$ such that $f^{-1}(Z_{\geq x})$ is non-discrete, and let h be its maximum. Again h must be non-maximal in Z, and so admits an immediate successor $\sigma(h) \in Z$. Since both subsets on the right hand side of

$$f^{-1}(Z_{\geq h}) = f^{-1}(h) \sqcup f^{-1}(Z_{\geq \sigma(h)})$$

are discrete, the first by assumption and the second by maximality, so must their union be by Corollary 2.9, contradiction. ⊣

We will say that a one-variable function $f\colon Y \to M$ has a *jump discontinuity* at a point c if the left and right limit of f at c exist, but are different.

3.2. THEOREM (Monotonicity). *The set of discontinuities of a one-variable definable map $f\colon Y \to M$ is discrete, closed, and bounded, and consists entirely of jump discontinuities. Moreover, there is a definable discrete, closed, bounded subset $D \subseteq Y$ so that in between any two consecutive points of $D \cup \{\pm\infty\}$, the map is* monotone*, that is to say, either strictly increasing, strictly decreasing, or constant.*

PROOF. We start with proving that all discontinuities are jump discontinuities, or equivalently, that f has a left limit in each point $a \in Y$. For each $y < a$, let $w(y)$ be the supremum of $f(\,[y, a[\,)$ and let b be the infimum of $w(Y_{<a})$. I claim that b is the left limit of f at a. To this end, choose $p < b < q$, and we need to show that there is some $x < a$ with $p < f(x) < q$. If b^+ does not belong $w(Y_{<a})$, and therefore belongs to its complement, then b is an isolated point of $w(Y_{<a})$, implying that f takes constant value b on some interval $]y, a[\,$, so that b is indeed the left limit at a. In the remaining case, we can find $u > b$ such that $]b, u] \subseteq w(Y_{<a})$. We may choose $u < q$. In particular, $u = w(y)$ for some $y < a$. Since b is strictly less than the supremum $u = w(y)$, we can find $x \in [y, a[$ such that $b < f(x) \leq u$, whence $p < f(x) < q$, as required.

Let $C \subseteq Y$ be the definable subset given as the union of the interior of all fibers, that is to say, $x \in C$ if and only if x is an interior point of $f^{-1}(f(x))$. Being an open set, C is a disjoint union of open intervals, and f is constant, whence continuous on each of these open intervals. Every fiber of the restriction of f to $Y \setminus C$ must have empty interior, whence is discrete by (2.6.iv). So upon replacing f by this restriction, we may reduce to the case that f has discrete fibers. There is nothing to show if Y is discrete, and so without loss of generality, we may assume Y is an open interval. For fixed $a \in Y$, let L_a (respectively, H_a) be the set of all $x \in Y$ such that $f(x) < f(a)$ (respectively, $f(a) < f(x)$). Since Y is the disjoint union of L_a, H_a, and $f^{-1}(f(a))$ with the latter being discrete, a^- must belong to one of the first two sets by Corollary 2.9, and depending on which is the case, we will denote this symbolically by writing respectively $f(a^-) < f(a)$ or $f(a^-) > f(a)$ (with a similar convention for a^+). Let L_- (respectively, H_-, L_+, and H_+) be the set of all $a \in Y$ such that $f(a^-) < f(a)$ (respectively, $f(a^-) > f(a)$, $f(a^+) < f(a)$, and $f(a^+) > f(a)$), so that Y is the disjoint union of these four definable subsets. Let D be the union of the topological boundaries of these four sets, a discrete set by Corollary 2.9. If $b < c$ are consecutive elements in D, then $]b, c[$ must belong to exactly one of these four sets by (2.6.ii), say, to L_-. It is now easy to see that in that case f is strictly increasing on $]b, c[\,$. This then settles the last assertion.

Let S be the (definable) subset of all discontinuities of f. To prove that S is discrete, we need to show by (2.6.iv) that it has empty interior, and this will follow if we can show that any open interval $I \subseteq Y$ contains a point at which f is continuous. By what we just proved, by shrinking I if necessary, we may assume f is monotone on I, say, strictly increasing. Note that f is then in particular injective. By Lemma 3.1, the image $f(I)$ contains an open interval J. Since f is strictly increasing, $f^{-1}(J)$ is also an open interval, and f restricts to a bijection between $f^{-1}(J)$ and J. We leave it to the reader to verify that any strictly increasing bijection between intervals is continuous. ⊣

3.3. *Remark.* Given a definable map f and a point a, we denote its left and right limit simply by $f^-(a)$ or $f^+(a)$ respectively, even if these values are infinite (to be distinguished from the symbol $f(a^-)$ which occurred above in formulae of the form $f(a^-) < f(a)$). Note that we even have this property at $\pm\infty$, so that we can define $f^+(-\infty)$ and $f^-(\infty)$, which we then simply abbreviate as $f(-\infty)$ and $f(\infty)$ respectively.

3.4. COROLLARY. *A definable map $f\colon I \to M$ with domain an open interval I is continuous if and only if its graph is definably connected.*

PROOF. Let C be the graph of f. If f is not continuous, then it has a jump discontinuity at some point $a \in I$ by Theorem 3.2. Without loss of generality, we may assume $f^-(a) < f^+(a)$. Let c be some element between these two limits and different from $f(a)$. By definition of one-sided limit, there exist $p < a < q$ such that $f(x) < c$ whenever $p < x < a$, and $f(x) > c$ whenever $a < x < q$. Consider the two open subsets

$$U_- := (I_{<a} \times M) \cup (I_{<q} \times M_{<c})$$
$$U_+ := (I_{>a} \times M) \cup (I_{>p} \times M_{>c}).$$

It is not hard to check that C is contained in their union but disjoint from their intersection, showing that it is not definably connected.

Conversely, assume f is continuous but C is not definably connected, so that there exist definable open subsets U and U' whose union contains C but whose intersection is disjoint from C. Since the projections $\pi(C\cap U)$ and $\pi(C\cap U')$ onto the first coordinate are definable subsets partitioning I, they must have a common boundary point $b \in I$ by Proposition 2.6. Since $(b, f(b))$ belongs to either U or U', say, to U, there exists a box $J \times J' \subseteq U$ containing $(b, f(b))$. By continuity, we may assume $f(J) \subseteq J'$. This implies that $(x, f(x)) \in U$, for all $x \in J$, and hence that $J \subseteq \pi(C \cap U)$, contradicting that b is a boundary point of the latter. ⊣

3.5. *Remark.* Without proof, we claim that the above results extend to arbitrary dimensions: given a definable map $f\colon X \subseteq M^n \to M^k$, the set of discontinuities of f is nowhere dense in X. For instance (with terminology to be defined below), if $X \subseteq M^2$ has dimension two, for each $a, b \in M$, let D_a and E_b be some discrete sets, as given by Theorem 3.2, such that between any two consecutive points the respective maps $y \mapsto f(a, y)$ and $x \mapsto f(x, b)$ are continuous and monotone. Let D and E be the respective union of all $\{a\} \times D_a$ and all $E_b \times \{b\}$. By Corollary 7.2 and Proposition 5.1 below, both D and E are one-dimensional, closed subsets, and hence $X' := X \setminus (D \cup E)$ is open and dense in X. It is now not hard to show that f is continuous on X' (see [9, Chapt. 3, Lemma 2.16]).

We can also strengthen this for expansions of fields by proving the same results with 'continuous' replaced by 'differentiable', or more generally, by $\mathcal{C}_n$ (see, for instance, [2, §7.4]).

§**4. Discrete sets.** As before, $\mathcal{M} \models$ DCTC. We start our analysis of multi-variable definable subsets, with a special emphasis on definable subsets of the plane M^2, called *planar* subsets, and only address the general case through some sporadic remarks. Since projections play an important role, we introduce some notation. Fix n and let $\sigma \subseteq \{1, \ldots, n\}$ of size $|\sigma| := e$. We let $\pi_\sigma \colon M^n \to M^e$ be the projection $(a_1, \ldots, a_n) \mapsto (a_{i_1}, \ldots, a_{i_e})$, where $\sigma = \{i_1 < i_2 < \cdots < i_e\}$. When σ is a singleton $\{i\}$, we just write π_i for the projection onto the i-th coordinate. Given a tuple $\mathbf{a} = (a_1, \ldots, a_e) \in M_e$, the ($\sigma$-)*fiber* of X above $\mathbf{a}$ is the set

$$X_\sigma[\mathbf{a}] := \pi_{\sigma^c}\left(\pi_\sigma^{-1}(\mathbf{a}) \cap X\right),$$

where σ^c is the complement of σ. In other words, $X_\sigma[\mathbf{a}]$ is the set of all $\mathbf{b} \in M^{n-e}$ such that $\tilde{\mathbf{b}} \in X$, where $\tilde{\mathbf{b}}$ is obtained from $\mathbf{b}$ by inserting a_{i_k} at the k-th spot. In case σ is of the form $\{1, \ldots, e\}$, for some e, we omit σ from the notation, since the length of the tuple $\mathbf{a}$ then determines the projection, and we refer to it as a *principal projection*, with a similarly nomenclature for fibers. Thus, for example, the *principal fiber* $X[a]$ is the set $X_1[a]$ of all $n-1$-tuples $\mathbf{b}$ such that $(a, \mathbf{b}) \in X$. Recall that by (2.6.v) any definable discrete subset of M is closed and bounded. The same is true in higher dimensions, for which we first prove:

4.1. THEOREM. *A definable subset $X \subseteq M^n$ is discrete if and only if all projections $\pi_1(X), \ldots, \pi_n(X)$ are discrete.*

PROOF. Suppose all projections are discrete and let $(a_1, \ldots, a_n) \in X$. Hence we can find open intervals I_k, for $k = 1, \ldots, n$, such that $I_k \cap \pi_k(X) = \{a_k\}$. The open box $I_1 \times \cdots \times I_n$ then intersects X only in the point $(a_1, \ldots, a_n)$, proving that X is discrete. To prove the converse, we will induct on n, proving simultaneously the following three properties for $X \subseteq M^n$ discrete:

4.1.i. $\pi_1(X), \ldots, \pi_n(X)$ are discrete;
4.1.ii. X with the induced lexicographical ordering has a minimal element;
4.1.iii. for this ordering, there exists a definable map σ_X of X, sending every non-maximal element in X to its immediate successor.

All three properties have been established by Proposition 2.6 when $n = 1$, so assume they hold for $n - 1$. Assume towards a contradiction that $\pi_1(X)$ is not discrete. For each $a \in \pi_1(X)$, the fiber $X[a]$ (that is to say, the set of all $\mathbf{b} \in M^{n-1}$ such that $(a, \mathbf{b}) \in X$), is discrete since $a \times X[a] \subseteq X$. By the induction hypothesis for (4.1.ii), in its lexicographical order, $X[a]$ has a minimum, denoted $f(a)$, yielding a definable map $f \colon \pi_1(X) \to M^{n-1}$ whose graph lies in X. By Theorem 3.2, each composition $\pi_i \circ f \colon \pi_1(X) \to M$, for $i = 1, \ldots, n-1$, is continuous outside a discrete set. The union of these discrete sets is again discrete by Corollary 2.9, and hence, since $\pi_1(X)$ is assumed non-discrete, there is a common point a at which all f_i are continuous, whence also f. By the discreteness of X, we can find an open interval I and an open box $U \subseteq M^{n-1}$ containing respectively a and $f(a)$ such that $(I \times U) \cap X = \{(a, f(a))\}$. By continuity, we can find an open interval $J \subseteq I$ containing a such that $f(J) \subseteq U$. However, this means that for any $u \in J$ different from a, we have $f(u) \in U$, whence $(u, f(u)) \in (I \times U) \cap X = \{(a, f(a))\}$, forcing $u = a$, contradiction.

To prove (4.1.ii), we now have established that $\pi_1(X)$ is discrete, whence has a minimum l. The minimum of X in the lexicographical ordering is then easily seen to be $(l, \min(X[l]))$. To define σ_X, let $\mathbf{a} = (a, \mathbf{b}) \in X$. For $\mathbf{a} \neq \max(X)$, either

$\mathbf{b}$ is not the maximum of $X[a]$ and we set $\sigma(\mathbf{a}) := (a, \mathbf{b}')$ where $\mathbf{b}' := \sigma_{X[a]}(\mathbf{b})$; or otherwise, a is not the maximum of $\pi_1(X)$ and we set $\sigma(\mathbf{a}) := (a', \min(X[a']))$, where $a' := \sigma_{\pi_1(X)}(a)$. Note that the existence of a' and $\mathbf{b}'$ follow from the induction hypothesis on (4.1.iii). We leave it to the reader to verify that σ_X has the required properties. ⊣

4.2. COROLLARY. *An $\mathcal{M}$-definable, discrete subset is closed and bounded.*

PROOF. Let $X \subseteq M^n$ be a definable, discrete subset. By Theorem 4.1, all $\pi_i(X)$ are discrete, whence bounded and closed by (2.6.v). It is now easy to deduce from this that so is then X. ⊣

4.3. COROLLARY. *The image under a definable map of an $\mathcal{M}$-definable discrete subset is again discrete.*

PROOF. If $f: X \subseteq M^n \to M$ is definable with X discrete, then the graph of f is also discrete (as a subset of M^{n+1}). Since $f(X)$ is the projection of this graph, it is discrete by Theorem 4.1. ⊣

4.4. COROLLARY. *A definable subset $X \subseteq M^n$ is discrete if and only if for some (equivalently, for all) $\sigma \subseteq \{1, \dots, n\}$, the projection $\pi_\sigma(X)$ as well as each fiber $X_\sigma[\mathbf{a}]$ is discrete.*

PROOF. The converse implication is easy, and for the direct, suppose X is discrete. We may assume, after renumbering, that $\sigma = \{1, \dots, e\}$. Since each fiber $X_\sigma[\mathbf{a}]$ is homeomorphic to the subset $\mathbf{a} \times X_\sigma[\mathbf{a}] \subseteq X$, for $\mathbf{a} \in M^e$, and since the latter is discrete, so is the former. On the other hand, each $\pi_i(X)$, for $i \leq e$, is discrete by Theorem 4.1, and since these are just the projections of $\pi_\sigma(X)$, the latter is also discrete, by the same theorem. ⊣

Suppose $\mathcal{M} \models$ DCTC is an expansion of an ordered group—which is therefore Abelian and divisible by [3, Proposition 2.2]. We define the absolute value $|a|$ as the maximum of a and $-a$, for any $a \in M$. We call a map $f: X \to X$, for $X \subseteq M$, *contractive*, if

$$|f(x) - f(y)| < |x - y|, \tag{1}$$

for all $x \neq y \in X$. We say that f is *weakly contractive*, if instead we have only a weak inequality in (1). Recall that a *fixed point* of f is a point $x \in X$ such that $f(x) = x$. If f is contractive, it can have at most one fixed point.

4.5. THEOREM (Fixed Point Theorem). *Suppose $\mathcal{M} \models$ DCTC expands an ordered group, and let $f: D \to D$ be a definable map on a discrete, definable subset $D \subseteq M$. If f is contractive, it has a unique fixed point. If f is weakly contractive, then f^2 has a fixed point.*

PROOF. We treat both cases simultaneously. Assume f does not have a fixed point. In particular, $f(l) > l$, where l is the minimum of D. Hence the set of $x \in D$ such that $x < f(x)$ is non-empty, whence has a maximal element u. Clearly, $u < h$, where h is the maximum of D, and hence u has an immediate successor $v := \sigma_D(u)$ by (4.1.iii). By maximality, we must have $f(v) < v$. Hence $v \leq f(u)$ and $u \leq f(v)$, and therefore $v - u \leq |f(u) - f(v)|$, leading to a contradiction in the contractive case with (1), showing that f must have a fixed point, necessarily unique. In the weak contractive case, we must have an equality in the latter inequality, whence also in the former two, that is to say, $f(u) = v$ and $f(v) = u$. Hence, u and v are fixed points of f^2. $\dashv$

§**5. Sets with non-empty interior.** We continue to work in a model $\mathcal{M}$ of DCTC. Shortly, we will introduce the notion of dimension, and whereas the discrete sets are those with minimal dimension (=zero), the sets with non-empty interior will be those of maximal dimension. Note that the non-empty definable subsets of M are exactly of one of these two types by (2.6.iv).

5.1. PROPOSITION. *A definable subset $X \subseteq M^n$ has non-empty interior if and only if the set of points $a \in M$ such that the fiber $X[a]$ has non-empty interior is non-discrete.*

PROOF. If X has non-empty interior, it contains an open box, and the assertion is clear. For the converse, note that, since we can pick definably the first open interval inside a definable non-discrete subset of M by the properties proven in Proposition 2.6, we may reduce to the case that $\pi(X)$ is an open interval and each fiber $X[a]$ for $a \in \pi(X)$ is an open box, where $\pi\colon M^n \to M$ is the projection onto the first coordinate. The proof for $n > 2$ is practically identical to that for $n = 2$, and so, for simplicity, we assume $n = 2$. Let $l(a)$ and $h(a)$ be respectively the infimum and supremum of X_a, so that $l, h\colon \pi(X) \to M_\infty$ are definable maps. The subset of $\pi(X)$ where either function takes an infinite value is definable, whence it or its complement contains an open interval, so that we can either assume that l is either finite everywhere or equal to $-\infty$ everywhere, and a similar dichotomy for h. The infinite cases can be treated by a similar argument, so we will only deal with the case that they are both finite (this is a practice we will follow often in our proofs). By Theorem 3.2, there is a point $a \in \pi(X)$ at which both l and h are continuous. Fix some $c < l(a) < p < q < h(a) < d$, so that by continuity, we can find $u < a < v$ so that $l(\,]u, v[\,) \subseteq \,]c, p[$ and $h(\,]u, v[\,) \subseteq \,]q, d[$. I claim that $]u, v[\,\times\,]p, q[$ is entirely contained in X. Indeed, if $u < x < v$ and $p < y < q$, then from $c < l(x) < p < y < q < h(x) < d$, we get $y \in X[x]$, that is to say, $(x, y) \in X$. $\dashv$

By a simple inductive argument, we get the following analogue of Corollary 4.4:

5.2. COROLLARY. *A definable subset $X \subseteq M^n$ has non-empty interior if and only if for some (equivalently, for all) $\sigma \subseteq \{1, \dots, n\}$, the set of points $\mathbf{a}$ for which $X_\sigma[\mathbf{a}]$ has non-empty interior, has non-empty interior.* $\dashv$

5.3. COROLLARY. *A finite union of definable subsets of M^n has non-empty interior if and only if one of the subsets has non-empty interior.*

PROOF. One direction is immediate, and to prove the other we may by induction reduce to the case of two definable subsets $X_1, X_2 \subseteq M^n$ whose union $X := X_1 \cup X_2$ has non-empty interior. We induct on n, where the case $n = 1$ follows from Corollary 2.9 and (2.6.iv). Let $W \subseteq M$ be the subset of all points $a \in M$ for which the fiber $X[a] \subseteq M^{n-1}$ has non-empty interior. By Proposition 5.1, the interior of W is non-empty. Since $X[a] = (X_1)[a] \cup (X_2)[a]$, our induction hypothesis implies that for $a \in W$, at least one of $(X_i)[a]$ has non-empty interior, in which case we put a in W_i. In particular, $W = W_1 \cup W_2$ so that at least one of the W_i has non-empty interior, say, W_1. By Proposition 5.1, this then implies that X_1 has non-empty interior. $\dashv$

§**6. Planar cells and arcs.** For the remainder of our analysis of multi-variable definable sets, apart from separate remarks, we restrict to planar subsets. Given an ordered structure $\mathcal{O}$, let us define a 2-*cell* in O^2 as a definable subset C of the following form: suppose I is an open interval, called the *domain* of the cell, and $f, g\colon I \to O$ are definable, continuous maps such that $f < g$ (meaning that $f(x) < g(x)$ for all $x \in I$). Let C be the subset of all $(x, y) \in O^2$ with $x \in I$ and $f(x) \diamond_1 y \diamond_2 g(x)$, where $\diamond_i$ is either no condition or a strict inequality (when we only have at most one inequality, we get an example of an *unbounded* cell; the remaining ones are call *bounded*, and in arguments we often only treat the latter case and leave the former with almost identical arguments to the reader). Any 2-cell is open. By a 1-*cell* $C \subseteq O^2$, we mean either the graph of a continuous definable map f with domain an open interval I, or a Cartesian product $x \times I$. We call the former *horizontal* and the latter *vertical*. Finally, by a 0-*cell*, we mean a point. We may combine all these definitions into a single definition: a cell C is determined by elements $a < b$ and definable, continuous maps $f < g\colon O \to O$, as the set of all pairs (x, y) such that $a \diamond_1 x \diamond_2 b$ and $f(x) \diamond_3 y \diamond_4 g(x)$, where each $\diamond_i$ is either no condition, equality or strict inequality. Moreover, if C is non-empty, then it is a d-cell, where d is equal to two minus the number of equality signs among the $\diamond_i$. We sometimes use some suggestive notation like $C(I; f < g)$ to denote, for instance, the cell given by $x \in I$ and $f(x) < y < g(x)$. If C is a cell with domain I and $J \subseteq I$ is an open interval, then we call $C \cap (J \times O)$ the *restriction* of C to J. Any restriction of a cell is again a cell, and so is any principal projection.

6.1. *Remark.* For higher arity, we likewise define cells inductively: we say that $C \subseteq O^n$ is a d-*cell* if either C is the graph of a definable, continuous function with domain some d-cell in O^{n-1}, or otherwise, is the region strictly between two such graphs with common domain some $(d-1)$-cell in O^{n-1}. As we shall see below in Remark 7.3, the d in d-cell refers to the dimension of the cell.

6.2. Arcs. As before, let $\mathcal{M} \models$ DCTC. Given a definable subset $X \subseteq M^n$, a point $P = (a, b) \in M^2$, and a definable map $h\colon Y \to M$ such that $a \in Y$ and $h^-(a) = b$, we will say that P_h^- *belongs to* X, if there exists an open interval $]u, a[\ \subseteq I$ so that the graph of the restriction of h lies inside X. By Theorem 3.2, we may shrink $]u, a[$ so that h is continuous on that interval, and so we could as well view this as a property of the horizontal 1-cell C defined by h. Note that P lies in the closure of C. Moreover, we only need a to lie in the closure of Y to make this work. So, given a 1-cell C such that P lies in its closure, we say that P_C^- *belongs to* X if P_h^- does, where h is the definable, continuous map determining C, in case C is a horizontal cell, or if b^- belongs to $X[a]$ in case C is a vertical cell. Of course, we can make a similar definition for P_h^+ or P_C^+. The following result essentially shows that viewed as a type, P_C^- is complete:

6.3. LEMMA. *Given a planar subset $X \subseteq M^2$, a 1-cell $C \subseteq M^2$, and a point P in the closure of C, either P_C^- belongs to X or it belongs to its complement.*

PROOF. Let $P = (a, b)$ be in the closure of C. If C is a vertical cell, then P_C^- belongs to X if and only if b^- belongs to $X[a]$, and so we are done in this case by type completeness. In the horizontal case, there exists a definable, continuous map $h\colon\ ]u, a[\ \to M$ whose graph is contained in C. By type completeness, either a^- belongs to $\pi(X \cap C)$ or to its complement. In the former case, after increasing u if necessary, we have $]u, a[\ \subseteq \pi(X \cap C)$, whence $(x, h(x)) \in X$ for every $x \in\ ]u, a[$.

In the latter case, $]u,a[$ is disjoint from $\pi(X\cap C)$, and hence $(x,h(x))\notin X$ for every $x\in\ ]u,a[$. ⊣

Using this, it is not hard to show that the following is an equivalence relation (and in particular symmetric): given a point $P\in M^2$ and 1-cells $V,W\subseteq M^2$ such that P lies in each of their closures, we say that $V\equiv_{P^-}W$, if P_V^- belongs to W. By a *left arc* at P, we mean an $(\equiv_{P^-})$-equivalence class of 1-cells whose closure contains P; and a similar definition for $V\equiv_{P^+}W$ and *right arc*. It is now easy to see that P_V^- belongs to some definable subset $X\subseteq M^2$ if and only if P_W^- belongs to it, for any $W\equiv_{P^-}V$, so that we may make sense of the expression P_α *belongs to* X, for any left (or right) arc α at P. There are two unique equivalence classes containing a vertical cell, called respectively the *lower* and *upper vertical arc*; the remaining ones are called *horizontal*. Given two left horizontal arcs α and β at P, we can find a common domain $I=\ ]u,a[$ and definable continuous functions f and g on I, such that α and β are the respective equivalence classes of the graphs of f and g. Let I_-, $I_=$ and I_+ be the subsets of all $x\in I$ such that $f(x)$ is less than, equal to, or bigger than $g(x)$ respectively. If $\alpha\neq\beta$, then a cannot be in the closure of $I_=$, so that upon shrinking even further, we may assume $I_=$ is empty. Hence a^- belongs either to I_- or I_+ and we express this by saying that $\alpha<_{P^-}\beta$ and $\alpha>_{P^-}\beta$ respectively. This yields a well-defined total order relation $<_{P^-}$ on the left horizontal arcs at a point P. To include the vertical arcs, we declare the lower one to be smaller than any horizontal left arc and the upper one to be bigger than any.

6.4. PROPOSITION. *Let $X\subseteq M^2$ be a definable subset, and $P\in M^2$ a point. The set of all left arcs α at P such that P_α belongs to X has an infimum β (with respect to the order $<_{P^-}$). If β is not vertical, then P_β belongs to ∂X.*

PROOF. Since a point is either interior, exterior or a boundary point, we may upon replacing X by its complement, reduce to the case that $P=(a,b)$ is either interior or a boundary point. In the former case, the lower vertical arc is clearly minimal, so assume $P\in\partial X$. In what follows, α always denotes a left arc at P. Consider the set $L_\emptyset$ of all $x<a$ such that $X[x]\cap J$ is empty for some open interval J containing b. If a^- belongs to $L_\emptyset$, then no P_α belongs to X so that the upper vertical arc is the minimum. So we may assume that the $X[x]\cap J$ are non-empty for x close to a from the left. If b^- belongs to $X[a]$, then the lower vertical arc is the infimum, so assume b^- belongs to $M\setminus X[a]$. Hence we may shrink J so that $J\cap(X[a])_{<b}$ is empty. For each $x<a$, let $f(x)$ be the infimum of $X[x]\cap J$. On a sufficiently small open interval $]u,a[$, the function f is continuous, whence defines a 1-cell V. Since $J\cap(X[a])_{<b}=\emptyset$, the left limit $f^-(a)$ must be equal to b, showing that (a,b) lies in the closure of V, and hence the equivalence class of V at P^- is a left arc β. It is now easy to show that β is the required infimum, and that it is contained in the boundary ∂X. ⊣

6.5. COROLLARY. *If $C\subseteq M^2$ is a definable subset without interior, then so is its closure, that is to say, C is nowhere dense.*

PROOF. Suppose $P=(a,b)$ is an interior point of the closure $\bar{C}$, so that there exists an open box $U\subseteq\bar{C}$ containing P. By Proposition 5.1, the fibers $C[x]$ for x close to a must be discrete. By Proposition 6.4, the infimum α of all left arcs at P belonging to $C\setminus C[x]$ exists, and by discreteness of the surrounding fibers, it must be a minimum, whence also belong to C. Similarly, the infimum β of all left arcs at P belonging to C

and strictly bigger than α is also a minimum. Choose an open interval $]u, a[$ such that α and β are represented by the respective continuous, definable maps $f, g\colon\ ]u, a[\ \to M$. Enlarging u if needed, we may assume $f < g$, so that the 2-cell $S := C(\,]u, a[\,; f < g)$ is disjoint from C. Since S is open and P lies in its closure, $S \cap U$ is non-empty. Since $(S \cap U) \cap C = \emptyset$, no point of $S \cap U$ can lie in the closure of C, contradiction. ⊣

6.6. Hardy structures. We now extend this to infinity in the obvious way: given two horizontal cells V and W with domain an interval unbounded to the right, we say that $V \equiv_\infty W$ if their restrictions to some interval $]u, \infty[$ are equal. Let $\mathbf{H}(\mathcal{M})$ be the set of all equivalence classes of cells defined on an open interval unbounded to the right. Note that any definable map $f\colon Y \to M$ whose domain is unbounded to the right yields an equivalence class in $\mathbf{H}(\mathcal{M})$, denoted $[f]$, since f is continuous by Theorem 3.2 on some open interval $]u, \infty[\ \subseteq Y$. Given a definable subset $X \subseteq M^2$, we can say, as before, that ∞_α *belongs to* X, if ∞^- belongs to the set of all $x \in Y$ such that $(x, f(x)) \in X$, for some f with arc α. However, in this case we can do more and make $\mathbf{H}(\mathcal{M})$ into an L-structure: if $\underline{c}$ is a constant symbol, then we interpret it in $\mathbf{H}(\mathcal{M})$ as the class of the constant function with value $c := \underline{c}^{\mathcal{M}}$; if $\underline{F}$ is an n-ary function symbol, and $\alpha_1, \dots, \alpha_n \in \mathbf{H}(\mathcal{M})$, then $\underline{F}(\alpha_1, \dots, \alpha_n)$ is the class given by the definable map $\underline{F}(g_1, \dots, g_n)$, where the g_i are definable functions with domain $I :=\ ]u, \infty[$ such that $[g_i] = \alpha_i$; if $\underline{R}$ is an n-ary predicate symbol, then $\underline{R}(\alpha_1, \dots, \alpha_n)$ holds in $\mathbf{H}(\mathcal{M})$ if and only if ∞^- belongs to the set of all $x \in I$ such that $\underline{R}(g_1(x), \dots, g_n(x))$ holds in $\mathcal{M}$.

6.7. DEFINITION. We call this L-structure on $\mathbf{H}(\mathcal{M})$ the *Hardy structure* of $\mathcal{M}$. In particular, by the same argument as above, $<$ interprets a total order on $\mathbf{H}(\mathcal{M})$, making it into a densely ordered structure without endpoints (note that the notion of vertical arc makes no sense in this context).

By induction on the complexity of formulae, we easily can show:

6.8. LEMMA. *Let $\varphi(x_1, \dots, x_n)$ be a formula with parameters from M and let $X \subseteq M^n$ be the set defined by it. For given arcs $\alpha_1, \dots, \alpha_n \in \mathbf{H}(\mathcal{M})$, we have $\mathbf{H}(\mathcal{M}) \models \varphi(\alpha_1, \dots, \alpha_n)$ if and only if there is a $u \in M$ such that $(g_1(x), \dots, g_n(x)) \in X$, for all $x > u$, where each g_i is some continuous function defined on $]u, \infty[$ representing the arc α_i.* ⊣

Since a continuous function with values in a discrete set must be constant, Lemma 6.8 yields:

6.9. COROLLARY. *If a discrete subset $D \subseteq \mathbf{H}(\mathcal{M})^n$ is definable with parameters in $\mathcal{M}$, then $D \subseteq M^n$.* ⊣

6.10. THEOREM. *There is a canonical elementary embedding $\mathcal{M} \to \mathbf{H}(\mathcal{M})$. In particular, $\mathbf{H}(\mathcal{M}) \models$ DCTC.*

PROOF. The map $M \to \mathbf{H}(\mathcal{M})$ sending an element $a \in M$ to the class of the corresponding constant function is easily seen to be an elementary embedding. ⊣

These two results together show that if $\mathcal{M}$ is a non-o-minimal model of DCTC, then $(\mathcal{M}, \mathbf{H}(\mathcal{M}))$ is a Vaughtian pair (see, for instance, [5, Proposition 9.3]). In particular, DCTC has Vaughtian pairs.

6.11. *Remark.* We can think of $\mathbf{H}(\mathcal{M})$ as a sort of *protoproduct*, in the meaning of a 'controlled' subring of an ultraproduct as studied in [6, Chapter 9]. Namely, endowing the set M with an ultrafilter containing all right unbounded open intervals, then $\mathbf{H}(\mathcal{M})$ consists of all elements in the ultrapower $\mathcal{M}_\natural$ given by definable maps (whereas an arbitrary element is given by any map).

We also can define a *standard part* operator, at least on the subset $\mathbf{H}^{\text{fin}}(\mathcal{M})$ of all *finite* elements, that is to say, the set of all arcs α at infinity represented by some definable, continuous map $f\colon\ ]u,\infty[\ \to M$ such that $f(\infty) \in M$ (see Remark 3.3 for the definition). Indeed, the value of $f(\infty)$ only depends on α, thus yielding a standard part map $\mathbf{H}^{\text{fin}}(\mathcal{M}) \to M$. Note, however, that as $\mathcal{M}$ is not definable in $\mathbf{H}(\mathcal{M})$, neither is $\mathbf{H}^{\text{fin}}(\mathcal{M})$.

§**7. Planar curves.** As always, $\mathcal{M}$ is a model of DCTC.

7.1. Dimension. Let us say that a non-empty definable subset $X \subseteq M^2$ has *dimension zero* if it is discrete, and *dimension two*, if it has non-empty interior. In the remaining case, we will put $\dim(X) = 1$ and call X a (generalized) *planar curve*. We will assign to the empty set dimension $-\infty$, in order to make the following formula work (with the usual conventions that $-\infty + n = -\infty$):

7.2. COROLLARY. *Given a definable subset $X \subseteq M^2$, let F_e be the set of all $a \in M$ for which the fiber $X[a]$ has dimension e, for $e = 0, 1$. Then each F_e is definable and the dimension of X is equal to the maximum of all $e + \dim(F_e)$.*

PROOF. Being discrete and having interior are definable properties, whence so is being a planar curve, showing that each F_e is definable. The formula then follows by inspecting the various cases by means of Corollary 4.4 and Proposition 5.1. ⊣

7.3. *Remark.* There are several ways of extending this definition to larger arity, and the usual one is to define the dimension of a definable subset $X \subseteq M^n$ as the largest d such that the image of X under some projection $\pi\colon M^n \to M^d$ has non-empty interior. It follows that a d-cell has dimension d.

We may rephrase the previous result as a trichotomy theorem for planar subsets:

7.4. THEOREM (Planar Trichotomy). *Any planar subset of $\mathcal{M}$ either*

7.4.i. *is discrete, closed, and bounded;*

7.4.ii. *is nowhere dense, but at least one projection onto a coordinate axis has non-empty interior;*

7.4.iii. *has non-empty interior.*

PROOF. We only need to show that (7.4.ii) is equivalent with having dimension one. The converse is clear from Corollaries 7.2 and 6.5, and for the direct implication, we must show that a definable subset satisfying (7.4.ii) cannot be discrete, and this follows from Theorem 4.1. ⊣

Immediate from the definitions and Corollary 5.3, we have:

7.5. COROLLARY. *The dimension of a union $X_1 \cup \cdots \cup X_n \subseteq M^2$ of definable subsets is the maximum of the dimensions of the X_i.* ⊣

7.6. Nodes. Let $S \subseteq M^2$ be an arbitrary subset. We call a point $P \in S$ a *node*, if for every open box B containing P, there is an open sub-box $I \times J \subseteq B$ containing P and some point $x \in I$ such that $S[x] \cap J$ is not a singleton. We denote the set of nodes of S by $\operatorname{Node}(S)$. We call a node an *edge*, if in the above condition $S[x] \cap J$ can be made empty. By an argument similar to the one proving Corollary 3.4, one shows that a function on an open interval h is continuous if and only if its graph has no edges (since it is a graph, it cannot have any other type of nodes). Note that the closure of a 1-cell C has at most two edges: indeed, if C is given as the graph of a definable, continuous function h on an interval $]a,b[$, then $\bar{C} \setminus C$ consists of those points among $(a, h^+(a))$ and $(b, h^-(b))$ that are finite (in the notation of Remark 3.3), and these are then the edges of $\bar{C}$.

Assume now that C is a planar curve. The isolated points of C are edges, and they form a discrete, closed, and bounded subset. Another special case of an edge is any point lying on an open interval inside a vertical fiber $C[a]$. Let $\operatorname{Vert}(C)$ be the set of all such edges, called the *vertical component* of C. Note that $\operatorname{Vert}(C)$ is equal to the union of the interiors of all fibers, that is to say, $\operatorname{Vert}(C) = \bigcup_a (C[a])^\circ$, and hence in particular is definable.

7.7. PROPOSITION. *The set of nodes of a planar curve in $\mathcal{M}$ is the union of its vertical component and a discrete set.*

PROOF. Let $C \subseteq M^2$ be a planar curve. Replacing C by $C \setminus \operatorname{Vert}(C)$, we may assume its vertical component is empty. Assume towards a contradiction that $N := \operatorname{Node}(C)$ is not discrete. Therefore, $\pi(N)$ cannot be discrete by Corollary 4.4, and hence contains an open interval I. For each $x \in I$, let $h(x)$ be the minimal $y \in C[x]$ such that $(x,y) \in N$. By Theorem 3.2, we may shrink I so that h becomes a continuous function on I. In particular, its graph V is a 1-cell contained in N. For each $x \in I$, let $l(x)$ and $u(x)$ be the respective predecessor and successor in $C[x]$ of $h(x)$ (if $h(x)$ is always an extremal element of $C[x]$ then we can adjust the argument accordingly, and so we just assume that $l(x) < h(x) < u(x)$ always exist). Since $(x, h(x))$ is a node and h is continuous, for $y < x$ sufficiently close to x, and J an open interval such that $J \cap C[x] = \{h(x)\}$, the intersection $J \cap C[y]$ contains at least one other element besides $h(y)$, necessarily either $l(y)$ or $u(y)$. By type completeness, either $l(y)$ belongs to all $J \cap C[y]$, for all y sufficiently close to the left of x, or otherwise $u(y)$ does. In particular, for a fixed $x \in I$, we have $h(x) = l^-(x)$ or $h(x) = u^-(x)$. Shrinking I if necessary, type completeness then reduces to the case that one of these alternatives happens for every $x \in I$, say, $h(x) = l^-(x)$ for all $x \in I$. Shrinking I even further, we may assume that l is continuous on I, and hence $l = h$ on I, contradiction. ⊣

7.8. LEMMA. *A point P on a planar curve $C \subseteq M^2$ is not a node if and only if there is some open box B containing P such that $C \cap B$ is a horizontal 1-cell. On the other hand, P is an edge if and only if it does not belong to any horizontal cell inside C.*

PROOF. If $P = (a,b) \notin \operatorname{Node}(C)$, there exist open intervals I and J containing respectively a and b such that $C[x] \cap J$ is a singleton $\{f(x)\}$, for every $x \in I$, and this property is preserved for any sub-box of $I \times J$ containing P. Hence $f \colon I \to J$ is a definable map with $f(a) = b$. Shrinking I if necessary, we may assume by Theorem 3.2 that f is continuous on I with a possible exception at a. As already observed, f is also continuous at a lest $(a, f(a))$ be a node. Hence the graph of f is a cell equal to $(I \times J) \cap C$. If P is not a node, then by definition, no intersection with an open box around P can be a cell. The second assertion is obvious. ⊣

7.9. *Remark.* This allows us to generalize the notion to higher arity: let us say that a point P on a definable subset $X \subseteq M^n$ is *strongly e-regular*, for some $e \leq n$, if there exists an open box B containing P such that $B \cap X$ is an e-cell. When $n = 2$, a point is strongly 2-regular if and only if it is interior, and strongly 0-regular if and only if it is isolated. The previous result then says that on a planar curve, a point is strongly 1-regular if and only if it is not a node. As with cells, this definition of regularity has a directional bias: nodes are really critical points with respect to projection onto the first coordinate. To break this bias, just taking permutations of the variables does not give enough transformations to turn some point on a curve in a non-nodal position, as for instance the origin on the curve given by (t, t) if $t \leq 0$ and $(-t, -2t)$ if $t \geq 0$. However, if we assume that there is an underlying ordered group, then we say that a point $x \in X \subseteq M^n$ is *e-regular*, if after a translation bringing x to the origin O, we can find a rotation ρ such that $\rho(O)$ is strongly e-regular in $\rho(X)$, where by a *rotation* of M^n, we mean a linear map $\rho\colon M^n \to M^n$ given by an invertible matrix of determinant one over $\mathbb{Q}$ (by [3, Proposition 2.2], any model of DCTC expanding a group is divisible, whence admits a natural structure of a $\mathbb{Q}$-vector space).

7.10. PROPOSITION. *A definable subset $X \subseteq M^2$ has the same dimension as that of its closure $\bar{X}$, whereas the dimension of its frontier $\mathrm{fr}(X)$ is strictly less.*

PROOF. If X is discrete, then it is closed by Corollary 4.2, and so $\mathrm{fr}(X) = \emptyset$, proving the assertion in this case. If X has dimension one, then so does $\bar{X}$ by Corollary 6.5. Let $V := \mathrm{Vert}(C)$ be the vertical component of C and let $\pi(V)$ be its projection. Since $\pi(V)$ is discrete by Proposition 5.1, the boundary ∂V is equal to the union of all $\partial(X[a])$, whence is discrete by Corollary 4.4. Hence, upon removing V from X, we may reduce to the case that X has no vertical components. Suppose towards a contradiction that $\mathrm{fr}(X)$ is a planar curve. By Proposition 7.7, the set of nodes on $\bar{X}$ and on $\mathrm{fr}(X)$ are both discrete sets, and so, there exists a $P \in \mathrm{fr}(X)$ which is not a node on $\mathrm{fr}(X)$ nor on $\bar{X}$. By Lemma 7.8, there exists an open box B containing P such that both $B \cap \mathrm{fr}(X)$ and $B \cap \bar{X}$ are cells, and therefore the inclusion $B \cap \mathrm{fr}(X) \subseteq B \cap \bar{X}$ must be an equality. In particular, $B \cap X$ is empty, contradicting that P lies in the closure of X.

Finally, if X has dimension two, then so must $\bar{X}$. Let $Y := X^\circ$ and $Z := X \setminus Y$. Since $\bar{X} = \bar{Y} \cup \bar{Z}$, we have $\mathrm{fr}(X) = (\bar{Y} \setminus X) \cup (\bar{Z} \setminus X)$, so that it suffices to show that neither of these two differences has interior. The first one, $\bar{Y} \setminus X$, is equal to ∂Y whence has no interior, being the boundary of an open set. By construction, Z has no interior, and hence by Corollary 6.5, neither does its closure. ⊣

Recall that a *constructible* subset is a finite Boolean combination of open subsets, and hence every one-variable definable subset is constructible. This is still true in higher dimensions: by an easy induction on the dimension, and using that the closure is obtained by adjoining the frontier, Proposition 7.10 yields:

7.11. COROLLARY. *Every $\mathcal{M}$-definable subset is constructible.* ⊣

7.12. COROLLARY. *The boundary of a two-dimensional, planar subset has dimension at most one.*

PROOF. Let $X \subseteq M^2$ have dimension two. Its boundary ∂X is the union of its frontier $\mathrm{fr}(X)$ and $X \setminus X^\circ$. The former has dimension at most one by Proposition 7.10 and the latter has no interior. The result now follows from Corollary 5.3. ⊣

Recall that a subset in a topological space is called *codense* if its complement is dense.

7.13. COROLLARY. *If Y is a codense definable subset of a non-empty definable subset $X \subseteq M^2$, then $\dim(Y) < \dim(X)$.*

PROOF. If X is discrete, then it is closed by Corollary 4.2, and hence its only codense subset is the empty set. If X and Y both have dimension two, then Y° is disjoint from the closure of $X \setminus Y$, contradicting that Y is codense in X. So remains the case that X is a curve. If Y is codense in X, then it must be contained in the frontier of $X \setminus Y$, and the latter has dimension strictly less than one by Proposition 7.10. ⊣

7.14. Quasi-cells. To obtain a cell decomposition as in the o-minimal case, we must generalize the notion of 1-cell by the following equivalent conditions:

7.15. LEMMA. *We call a subset $S \subseteq M^2$ a (horizontal)* 1-quasi-cell *if it satisfies one of the following equivalent conditions:*

7.15.i. *S is a union of mutually intersecting 1-cells in M^2 and has no nodes;*

7.15.ii. *S is the graph of a continuous map $h\colon \pi(S) \to M$ which is* locally definable*, meaning that its restriction to any open interval in its domain is definable.*

Moreover, $\pi(S)$ is then open and convex, and S is a 1-cell if and only if $\pi(S)$ is definable.

PROOF. The implication (7.15.ii) ⇒ (7.15.i) is easy, since the graph of a continuous function has no nodes. To show (7.15.i) ⇒ (7.15.ii), suppose S has no nodes, so that in particular, no vertical cell lies inside S. Fix $a_1, a_2 \in \pi(X)$ and choose 1-cells $C_1 \subseteq S$ and $C_2 \subseteq S$ containing a_1 and a_2 respectively. Let $I_k := \pi(C_k)$ and let h_k be the definable (continuous) function on I_k whose graph is C_k. Let $I := I_1 \cup I_2$. Since $C_1 \cap C_2$ is non-empty, so is $I_1 \cap I_2$, showing that I is an interval. Let H be the subset of $I_1 \cap I_2$ on which h_1 and h_2 agree, that is to say, $H = \pi(C_1 \cap C_2)$. For $a \in H$ with common value b, since (a, b) is not a node, there exist open intervals U and V containing respectively a and b such that $S[x] \cap V$ is a singleton, for all $x \in U$. Shrinking U if necessary, continuity allows us to assume that $h_k(U) \subseteq V$, so that $(x, h_k(x))$ both lie in $S[x] \cap V$ for $x \in U$, whence must be equal. This shows that $U \subseteq H$, and hence that H is open. Let $a \in \partial H$. Since H is open, a does not belong to H whereas either a^- or a^+ does. If a lies in $I_1 \cap I_2$, then $a \notin H$ implies $h_1(a) \neq h_2(a)$, and by continuity, this remains the case on some open around a, contradicting that either a^- or a^+ belongs to H. Hence $a \notin I_1 \cap I_2$. This means that the only boundary points of H are the endpoints of the interval $I_1 \cap I_2$, proving that h_1 and h_2 agree on this interval. Let $h(x)$ be equal to $h_1(x)$ if $x \in I_1$ and to $h_2(x)$ otherwise. Since the graph of h is then equal to $C_1 \cup C_2 \subseteq S$, whence contains no nodes, h is continuous. It is not hard to see that $\pi(S)$ is open and convex. The last assertion then follows since $\pi(S)$ is a disjoint union of open intervals by Theorem 2.10, whence, being also convex, a single open interval. ⊣

Although we should also entertain the notion of vertical quasi-cells (see Remark 7.17 below), they do not occur in the analysis of planar subsets. Given a curve C without nodes and a quasi-cell $S \subseteq C$, we say that S is *optimal* in C, if no quasi-cell inside C strictly contains S.

7.16. COROLLARY. *Any point on a planar curve $C \subseteq M^2$ without nodes lies on a (uniquely determined) optimal quasi-cell in C. In particular, C is a disjoint union of quasi-cells.*

PROOF. Fix $P \in C$. By Lemma 7.8, there exists a cell $V \subseteq C$ containing P. Let S be the union of all cells inside C containing P. Since $S \subseteq C$, has no nodes, it is a quasi-cell by Lemma 7.15. Suppose $S' \subseteq C$ is a quasi-cell containing S and let $P' \in S'$. By Lemma 7.15, there exists a cell $V' \subseteq S'$ containing both P and P'. By construction, we then have $V' \subseteq S$, whence $P' \in S$, showing that $S = S'$ is optimal. ⊣

If the curve has nodes, then to preserve uniqueness of optimal quasi-cells, we have to amend this definition as follows: for an arbitrary planar curve C, a horizontal 1-quasi-cell S is called *optimal* if $S \subseteq C$ contains no node of C and is maximal with this property.

Finally, we define the notion of a 2-*quasi-cell* $S \subseteq M^2$ given as the region between two continuous, locally definable maps defined on an open, convex subset of M (again called the *domain* of the quasi-cell), or an unbounded variant as in the case of 2-cells. More precisely, let $V \subseteq M$ be an open convex subset, $f, g\colon V \to M$ continuous and locally definable with $f < g$, then S consist of all (x, y) such that $x \in V$ and $f(x) \diamond y \diamond' g(x)$, with $\diamond, \diamond'$ strict inequality or no condition. By definition of local definability, the restriction of a quasi-cell S to an open interval $I \subseteq V$, that is to say, $S \cap (I \times M)$ is a cell, and hence every 2-quasi-cell is the union of 2-cells and therefore open. Moreover, we can arrange for all these cells in this union to contain a given fixed point of the quasi-cell.

7.17. *Remark.* The definition of an arbitrary d-quasi-cell is entirely similar: simply replace in the recursive definition from Remark 6.1 ‘cell’ by ‘quasi-cell’ and ‘definable, continuous map’ by ‘locally definable, continuous map’ everywhere.

7.18. Locally definable subsets. Quasi-cells are particular instances of locally definable subsets, which we now briefly study. In an arbitrary ordered structure $\mathcal{O}$, we call a subset $X \subseteq O^n$ *locally definable* if for each point $P \in O^n_\infty$, there exists an open box B containing P such that $B \cap X$ is definable. It is important to include in this definition also the *infinite points* of O^n_∞, that is to say, points with at least one coordinate equal to $\pm\infty$, where, just as an example, we mean by an *open box around an infinite point* like $(0, \infty, -\infty)$ one of the form $]u, v[\, \times\,]p, \infty[\, \times\,]-\infty, q[$, with $u < 0 < v$. It is also important to note that the definition applies to all points, not just to those belonging to X. In fact, the condition is void if P is either an interior or an exterior point, since then some open box is entirely contained in or entirely disjoint from X. So we only need to verify local definability at boundary points and at infinite points. Therefore, any clopen is locally definable. It is not hard to show that a finite Boolean combination of locally definable sets is again locally definable. Moreover, the interior, closure and exterior of a locally definable subset are again locally definable.

To see that a 1-quasi-cell $S \subseteq M^2$ is locally definable, write it as the graph of a locally definable, continuous map $f\colon V \to M$. We leave the infinite points again to the reader, so fix $P = (a, b)$. If a is exterior to V, then P lies in the exterior of S, and there is nothing to show. If a lies in the interior of V, then there is some open interval $I \subseteq V$ containing a, and $S \cap (I \times M)$ is the graph of f restricted at I by (7.15.ii), and the same is true if a is a boundary point of V, say on the left, by choosing $I = \,]a, q[\, \subseteq V$ for some $q > a$. Using this, it is not hard to see that 2-quasi-cells are also locally definable.

7.19. PROPOSITION. *A discrete set is locally definable in $\mathcal{M}$ if and only if it is closed and bounded.*

PROOF. Let $D \subseteq M^n$ be discrete. If D is not closed, then there is a $P \in \partial D$ not belonging to D. But then the intersection $D \cap B$ with any open box B containing P will have P in its closure, so that $D \cap B$ is not closed, whence cannot be definable by Corollary 4.2. Similarly, if D is not bounded, say, in the first coordinate on the right, then its intersection with any open box of the form $]p, \infty[\times B'$ will still be unbounded, whence not definable by Corollary 4.2. Suppose therefore D is closed and bounded. To check local definability at a boundary point P, as it belongs to D by closedness, there is an open box B such that $D \cap B = \{P\}$. To check at an infinite point, we can find an open box around P which is disjoint from D. ⊣

7.20. COROLLARY. *The topological boundary of a locally definable subset $Y \subseteq M$ is a discrete, closed, bounded set.*

PROOF. If the locally definable set ∂Y has non-empty interior, it would contain an open interval I and we may shrink this so that $F := I \cap Y$ is definable. Since $\partial F = I \cap \partial Y = I$, we get a contradiction with (2.6.vi). Hence ∂Y has no interior, and so, for $b \in \partial Y$ and an open interval I containing b such that $I \cap \partial Y$ is definable, the latter set, having no interior, must be discrete by (2.6.iv), and hence, shrinking I further if necessary, $I \cap \partial Y = \{b\}$. Hence ∂Y is discrete, whence bounded and closed by Proposition 7.19. ⊣

Given an arbitrary ordered structure $\mathcal{M}$, let $\mathcal{M}^{\text{loc}}$ be the structure generated by the locally definable subsets of $\mathcal{M}$ (formally, we have a language with an n-ary predicate $\mathtt{X}$ for any locally definable subset $X \subseteq M^n$, and we interpret $\mathtt{X}(\mathcal{M}^{\text{loc}})$ as the subset X). Since the class of locally closed subsets is closed under projection, fibers, and finite Boolean combinations, the definable subsets of $\mathcal{M}^{\text{loc}}$ are precisely the locally $\mathcal{M}$-definable subsets.

7.21. COROLLARY. *The reduct $\mathcal{M}^{loc}$ is type complete.*

PROOF. Given a one-variable definable subset of $\mathcal{M}^{\text{loc}}$, whence a locally definable subset $Y \subseteq M$, and a point $a \in M_\infty$, we may choose an open interval I around a such that $Y \cap I$ is definable. Since a^- belongs to $Y \cap I$ or to its complement, the same is true with respect to Y, proving type completeness. ⊣

Since bounded clopens are locally definable but have no infimum, definable completeness usually fails and $\mathcal{M}^{\text{loc}}$ is in general not a model of DCTC.

§**8. Planar cell decomposition.** In o-minimality, *cell decomposition* is the property that we can partition any given definable subset X into a disjoint union of cells. Every point is a 0-cell but writing X as a union of its points should not qualify as a cell decomposition. Slightly less worse, if X is planar, then each fiber $X[a]$ is a disjoint union of intervals and points, so that we can partition X into points and vertical cells. Of course, in the o-minimal context these pathologies are avoided by demanding the partition be finite. For arbitrary models of DCTC, however, we can no longer enforce finiteness, and so to exclude any unwanted partitions, we must impose some weaker restrictions. Moreover, at present, I do not see how to avoid having to use quasi-cells (but see §9 below).

We again work in a fixed model $\mathcal{M}$ of DCTC. Let us introduce the following terminology, which we give only for the planar case (but can easily be extended to larger arity, see Remark 8.2 below). First we extend the definition of dimension to arbitrary subsets of the plane (which is not necessarily well-behaved if the subset is not definable) by the same characterization: a non-empty subset $B \subseteq M^2$ has *dimension* 2, if it has non-empty interior; *dimension* 1, if it has empty interior but is non-discrete; and *dimension zero* if it is discrete. We can also define the *local dimension* $\dim_P(B)$ of B at a point $P \in M^2$ as the minimal dimension of $B \cap U$ where U runs over all open boxes containing P. Note that $\dim_P(B) \geq 0$ if $P \in \bar{B}$. It follows that the dimension of B is the maximum of its local dimensions at all points. It is not hard to see that the dimension of B is the largest e for which it contains an e-cell. In particular, a 2-quasi-cell has dimension 2, whereas a 1-quasi-cell has dimension one. More generally, by Corollaries 2.9 and 5.3 and the local nature of dimension, we showed that:

8.1. LEMMA. *The dimension of a finite union of e_i-quasi-cells is equal to the maximum of the e_i.* ⊣

Given a collection $\mathcal{B}$ of (not necessarily definable) subsets of M^2, we say that a definable subset $X \subseteq M^2$ has a *$\mathcal{B}$-decomposition*, if there exists a partition $X = \bigsqcup_{i \in I} B_i$ with all $B_i \in \mathcal{B}$, with the additional property that if $X^{(e)}$ denotes the union of all e-dimensional B_i in this partition, then $X^{(e)}$ is definable and has dimension at most e, for $e = 0, 1, 2$ (whence of dimension e if and only if it is non-empty). Put simply, in a decomposition there cannot be too many lower dimensional subsets. By a *cell* decomposition (respectively, a *quasi-cell* decomposition) we mean a $\mathcal{B}$-decomposition where $\mathcal{B}$ is the collection of all (quasi-)cells. By Lemma 8.1, any finite partition into quasi-cells is a cell decomposition (as there can be no quasi-cell in a finite decomposition since each subset in the partition is then definable).

8.2. *Remark.* For higher arities, we define the dimension of a subset $B \subseteq M^n$ to be the largest d such that it contains a d-cell (in case B is not definable, this might be different from the largest d such that the projection of B onto some M^d has non-empty interior, but both are equal in the definable case). The definition of $\mathcal{B}$-decomposition for a definable set $X \subseteq M^n$ now easily generalizes: it is a partition of X into sets from $\mathcal{B}$ such that the union of all sets in this partition of a fixed dimension e is a definable subset of dimension at most e.

8.3. THEOREM. *In $\mathcal{M}$, any planar subset has a quasi-cell decomposition.*

PROOF. Let $X \subseteq M^2$ be a planar subset. There is nothing to show if X has dimension zero. If X is a curve, then $\mathrm{Vert}(X)$ is a disjoint union of vertical cells (see the proof of Proposition 7.10). So after removing it from X, we may assume X has no vertical components. In that case, $\mathrm{Node}(X)$ is discrete by Proposition 7.7, and so after removing it, we may assume X has no nodes, and so we are done by Corollary 7.16.

So remains the case that X is 2-dimensional. Let $C := \partial X$ be its boundary. Since C has dimension at most one by Corollary 7.12, and so can be decomposed into disjoint quasi-cells by what we just argued, we may assume, after removing it, that X is moreover open. The projection $\pi(N)$ of the set $N := \mathrm{Node}(C)$ of all nodes is discrete by Proposition 7.7 and Theorem 4.1, and therefore $X \cap (\pi(N) \times M)$ can be partitioned into vertical cells. So remains to deal with points $(a, b) \in X$ such that $a \notin \pi(N)$. Since X is open, b is an interior point of $X[a]$. Let l and h be respectively the maximum of $(C[a])_{<b}$ and the minimum of $(C[a])_{>b}$, so that the open interval $]l, h[$ lies inside $X[a]$ and contains b (we leave the case that one of these endpoints is infinite to the reader). By choice, neither (a, l) nor (a, h) is a node of C, so that by Corollary 7.16, there are (unique) optimal 1-quasi-cells $L, H \subseteq C$ containing (a, l) and (a, h) respectively, say, given as the graphs of locally definable, continuous maps $f\colon V \to M$ and $g\colon W \to M$. Consider all open intervals $I \subseteq V \cap W$ containing a such that the 2-cell $C(I;\, f|_I < g|_I)$ lies entirely inside X, and let $Z \subseteq M$ be the union of all these intervals. Hence Z is open and convex, and $C(Z;\, f|_Z < g|_Z)$ is an (optimal) 2-quasi-cell inside X, by Lemma 7.15. To show that this construction produces a disjoint union of quasi-cells, we need to show that if (a', b') is any point in S, then the above procedure yields exactly the same quasi-cell containing (a', b'). Indeed, by convexity, we can find an open interval $I \subseteq V$ containing a and a'. Since the intersections of F and G with $I \times M$ are 1-cells, $C(I;\, f|_I < g|_I) = S \cap (I \times M)$ is a 2-cell contained in X, whence must lie inside S by construction.

To show that this is a decomposition, we induct on the dimension d of X, where the case $d = 0$ is trivial. In the above, at various stages, we had to remove some subsets of X of dimension strictly less than d, and partition each separately. Since each of these finitely many exceptional sets was definable, so is their union and by Lemma 8.1, has dimension strictly less than d. Hence the complement $X^{(d)}$, consisting of all d-quasi-cells in the partition, is also definable. After removing $X^{(d)}$, we are left with a definable subset of strictly less dimension, and so we are done by induction. $\dashv$

The proof gives in fact some stronger results, where for the sake of brevity, we will view any point as a 0-quasi-cell:

8.4. *Remark.* Keeping track of the various (quasi-)cells, we actually showed that we may partition X in quasi-cells S_i, such that each $\bar{S}_i \cap X$ is a disjoint union of S_i and some of the other S_j.

8.5. *Remark.* For a definable subset $Y \subseteq M$, define a *left outer point* (respectively, a *right outer point*) as a point $b \in Y$ such that $(\partial Y)_{\leq b}$ (respectively, $(\partial Y)_{\geq b}$) is finite. For a subset $X \subseteq M^2$, we call (a, b) an *outer point*, if b is an outer point in $X[a]$. We can now choose a partition $X = \bigsqcup_i S_i$ in quasi-cells S_i such that any quasi-cell S_i containing an outer point is a cell.

Indeed, for a fixed n, let $f_n\colon M \to X$ be defined by letting $f_n(x)$ be the n-th left outer point in $\partial(X[x])$ (note that this is a definable function by (4.1.iii)). The set where f_n takes finite values is definable, and after removing a discrete set, f_n is continuous on it, whence defines a cell above each open interval in its domain. So, given a (left) outer point (a, b), if n is the cardinality of $\partial(X[a])_{\leq b}$, then (a, b) lies either on the graph of f_n or between the graphs of f_n and f_{n+1}, and therefore, on a cell.

8.6. COROLLARY. *Any planar subset lying inside a compact subset admits a finite partition into cells.*

PROOF. We induct on the dimension d of the definable subset $X \subseteq M^2$ contained in the compact set K. By Corollary 4.2, if $d = 0$, then X is closed, whence compact, and discrete, whence finite. So assume X is a curve. Let D be the $a \in M$ for which $X[a]$ has non-empty interior, so that D is a discrete subset of the compact $\pi_1(K)$, whence finite. By Proposition 2.3, since each $X[a]$ lies in the compact set $\pi_2(K)$, it is a finite union of intervals, and hence $\mathrm{Vert}(X)$, being the union of all $X[a]$ for $a \in D$, is a finite union of cells. Removing it, we reduced to the case that X has no vertical components. In particular, each fiber is discrete whence finite, and so is $\mathrm{Node}(X)$ by Proposition 7.7. Upon removing the latter, we may assume X has no nodes. Apply the quasi-cell decomposition procedure from Theorem 8.3 to X, which we will call here the *canonical quasi-cell decomposition*. Since all its fibers are finite, every point is an outer point in the terminology of Remark 8.5, and hence belongs to a cell. In other words, the partition does not contain quasi-cells and so is a cell decomposition. Suppose there are infinitely many 1-cells in this partition. Since each cell inside K must have exactly two boundary points, and ∂X is finite, there must be at least two of these boundary points common to infinitely many cells. But then, by continuity, any vertical line in between these two points would intersect each of these cells in a different point, contradicting that the fibers are all finite.

So remains the case that $d = 2$. We follow again the procedure, and its notation, from Theorem 8.3 to produce the canonical quasi-cell decomposition of X. All the parts of X we will throw out in this procedure are lower dimensional and hence have finite cell decomposition by induction. So we may reduce to the case that X is open and that $(a, b) \in X$ has the property that no node of ∂X lies in the closure of $X[a]$. The definable maps $f\colon V \to M$ and $g\colon W \to M$ as in the proof of Theorem 8.3 are uniquely determined by (a, b) and X, and their graphs are part of a cell decomposition of ∂X. In particular, by induction, there are only finitely many choices for f and g. Since the 2-cell $C(Z;\, f|_Z < g|_Z)$ in the canonical cell decomposition containing (a, b) is uniquely determined by the property that Z is the largest open interval in $V \cap W$ such that the 2-cell lies inside X, there are only finitely many possibilities for it. $\dashv$

8.7. *Remark.* The analogue of Proposition 2.3 also holds: if $X \subseteq M^2$ is definable and $K \subseteq M^2$ is compact and open, then $X \cap K$ admits a finite cell decomposition. Indeed, choose for each $a \in K$ and open box U_a containing a and contained in K. Since each $X \cap U_a$ is definable and contained in K, it admits a finite cell decomposition by the previous result. But by compactness, K is the union of finitely many of the U_a, proving the claim.

Both the Corollary and the Remark extend to higher arities, the detail of which we leave to the reader.

§**9. Tameness.** The quasi-cell decomposition version given by Theorem 8.3 is not very useful in applications. Moreover, the non-definable nature of quasi-cells is a serious obstacle. Perhaps quasi-cells never occur, but in the absence of a proof of this, we make the following definitions, for $\mathcal{O}$ any (ordered) L-structure. Let us call a definable map $c\colon X \to Y$ *pre-cellular*, if every fiber $c^{-1}(y)$ is a cell. Note that the non-empty fibers of c then constitute a partition of X into cells. Injective maps are pre-cellular, but the resulting partition in cells is clearly not a decomposition if X has positive dimension. To guarantee that we get a cell decomposition, we require moreover that the image of c be discrete, bounded, and closed, and we call such a map then *cellular*. In particular, we may assume, if we wish to do so, that the cellular map $c\colon X \to D$ is surjective, where D is discrete, bounded, and closed.

Assume now that $\mathcal{O} \models$ DCTC, and let $c\colon X \to D$ be cellular. The collection $X^{(e)}$ of all fibers $c^{-1}(y)$ of dimension e is a definable subset, for each e, since we can express in a first-order way whether a fiber $c^{-1}(y)$ has dimension e (for instance, if X is planar, then having interior or being discrete are elementary properties). If $X^{(e)}$ is non-empty, then its dimension is equal to e by Corollary 7.2 and the fact that D is discrete, showing that we have indeed a cell decomposition. Note that in particular, the graph of c and

X have the same dimension. We have the following converse: given a definable subset $X \subseteq O^k$ and a cellular map $c\colon X \to Y$ such that $X = \bigsqcup_{y \in c(X)} c^{-1}(y)$ is a cell decomposition, then c is cellular, that is to say, $c(X)$ is discrete. It suffices to show that each $c(X^{(e)})$ is discrete, for $e \leq k$, and this follows from Corollary 7.2. It is not clear whether every definable subset admits a cell decomposition of this type, and so we make the following definition:

9.1. DEFINITION. A definable subset $X \subseteq O^n$ in an ordered structure $\mathcal{O}$ is called *tame* if it is the domain of a cellular map. If every definable subset in $\mathcal{O}$ is tame, then we call $\mathcal{O}$ *tame*.

9.2. LEMMA. *Any tame structure is a model of DCTC.*

PROOF. Let $\mathcal{O}$ be tame. By Theorem 2.10, it suffices to show that any definable subset $Y \subseteq O$ is a disjoint union of open intervals and a single discrete, bounded, and closed subset. Let $c\colon Y \to D$ be cellular, with D discrete, bounded, and closed. Hence, each fiber $c^{-1}(a)$ must be a one-variable cell, that is to say, either a point or an open interval. Let E be the subset of all $a \in D$ for which $c^{-1}(a)$ is a point. Hence $Y \setminus c^{-1}(E)$ is a disjoint union of open intervals, so that upon removing them, we may assume $E = D$, so that c is a bijection. Since D is discrete, c^{-1} is continuous. Therefore, $Y = \mathrm{Im}(c^{-1})$ is closed and bounded by [3, Prop. 1.10] (which we may invoke by definable completeness). Moreover, by the same result, any closed subset is again mapped to a closed subset, showing that c^{-1}, whence also c, is a homeomorphism. In particular, Y is discrete, as we needed to show. $\dashv$

Clearly any cell is tame. Since a (principal) fiber of a cell is again a cell, the same holds for tame subsets. Since a principal projection of a cell is a cell, the collection of tame subsets is closed under principal projections (we will generalize this in Corollary 9.13 below). Any finite cell decomposition is easily seen to be given by a cellular map, and hence in particular, any o-minimal structure is tame.

9.3. PROPOSITION. *Suppose $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are elementary equivalent L-structures. If $\mathcal{O}$ is tame, then so is $\tilde{\mathcal{O}}$.*

PROOF. Since both structures have isomorphic ultraproducts, we only need to show that tameness is preserved under elementary substructures and extensions. The former is easy, so assume $\mathcal{O}$ is a tame elementary substructure of $\tilde{\mathcal{O}}$ and let $\tilde{X}$ be a definable subset in $\tilde{O}^n$. Since tameness is preserved under fibers, we may assume that $\tilde{X}$ is definable without parameters, say $\tilde{X} = \varphi(\tilde{\mathcal{O}})$. By assumption, there exists a cellular map $c\colon \varphi(\mathcal{O}) \to D$, that is to say, formulae γ and δ, with $\gamma(\mathcal{O})$ the graph of a map all of whose fibers are cells of dimension at most n and whose image is the discrete, closed, bounded subset $\delta(\mathcal{O})$. Since all this is first-order, it must also hold in $\tilde{\mathcal{O}}$, so that $\gamma(\tilde{\mathcal{O}})$ is the graph of a cellular map $\tilde{c}\colon \tilde{X} \to \delta(\tilde{\mathcal{O}})$. $\dashv$

By Theorem 6.10, the associated Hardy structure of a tame structure is therefore again tame.

9.4. PROPOSITION. *In a reduct of a tame structure, every definable set admits a cell decomposition.*

PROOF. Let $\mathcal{O}$ be a tame structure, $\bar{\mathcal{O}}$ some reduct, and X an $\bar{\mathcal{O}}$-definable subset. By assumption, there exists an $\mathcal{O}$-definable cellular map $c\colon X \to D$, the fibers of which yield an $\mathcal{O}$-cell decomposition of X. We will need to show how we can turn this into an $\bar{\mathcal{O}}$-cell decomposition. As always, we only treat the planar case, $X \subseteq O^2$. There is nothing to show if X is discrete, so assume it is a curve. We already argued that its vertical component $\mathrm{Vert}(X)$ admits a cell decomposition, and so we may remove it. The remaining set of nodes is discrete, and hence may be removed as well, so that we are left with the case that X has no nodes. By Corollary 7.16, every point of X lies on a unique optimal quasi-cell. Hence if $C := c^{-1}(d)$ is one of the cells in the above decomposition, then it is contained in a unique $\bar{\mathcal{O}}$-quasi-cell S. As C is then the restriction of S to I, it is $\bar{\mathcal{O}}$-definable by Lemma 7.15.

If X has dimension two, we may assume it is open after removing its boundary, as we already dealt with curves. By Theorem 8.3, there exists an $\bar{\mathcal{O}}$-quasi-cell decomposition of X. Following that proof, we may assume, after removing all points lying on a vertical line containing a node of ∂X, that any quasi-cell S in this decomposition is open, and its boundary consists of quasi-cells of ∂X. By the one-dimensional case, the latter decompose into $\bar{\mathcal{O}}$-cells, whence so does S. ⊣

As before, we work again a model $\mathcal{M}$ of DCTC.

9.5. LEMMA. *Every one-variable $\mathcal{M}$-definable subset is tame.*

PROOF. Most proofs involving tameness will require some coding of disjoint unions, and as we will gloss over this issue below, let me do the proof in detail here. For ease of discussion, let us assume $Y \subseteq M$ is bounded (the unbounded case is only slightly more complicated and left to the reader). Assume 0 and 1 are distinct elements in M. Define $c\colon Y \to M^2$ by letting $c(y)$ be equal to $(y, 0)$, in case $y \in \partial Y$; and equal to $(x, 1)$ where x is the maximum of $(\partial Y)_{<y}$, in the remaining case. The fiber $c^{-1}(d, e)$ is either a point in ∂Y (when $e = 0$), or the interval $]d, \sigma_{\partial Y}(d)[\ \subseteq Y$. Since its image is contained in $\partial Y \times \{0, 1\}$, the map c is cellular by (2.6.v). ⊣

9.6. *Remark.* As it will be of use later, note that by the above argument, we can refine the cell decomposition given by c as follows: for any discrete subset D containing ∂Y, we can construct a cellular map $c_D\colon Y \to M^2$ whose cells have endpoints in D.

9.7. PROPOSITION. *Let $g\colon X \to M^n$ be a definable map with finite image. If every fiber $g^{-1}(a)$ is tame, then so is X.*

PROOF. Let $A := g(X)$, so that A is finite. By assumption, there exists for each $a \in A$, a cellular map $c_a\colon g^{-1}(a) \to D_a$ with $D_a \subseteq M^e$ discrete (for some large enough e). Let D be the union of all $\{a\} \times D_a \subseteq M^{n+e}$, for $a \in A$. It follows from Corollary 4.4 that D is discrete. Define $c\colon X \to D$ by the rule $c(x) = (g(x), c_{g(x)}(x))$. To see that this is cellular, note that the fiber over a point $(a, d) \in D$ is equal to $c_a^{-1}(d)$ for $d \in D_a$, whence is by assumption a cell. ⊣

9.8. THEOREM. *The collection of tame $\mathcal{M}$-definable subsets is closed under (finite) Boolean combinations.*

PROOF. For simplicity, we only prove this for planar subsets, and leave the general case to the reader (by an induction on the arity). Since the complement of a cell $V \subseteq M^2$ is a finite union of cells, it is tame. For instance, if $V = C(\,]a, b[\,; f < g)$, then its complement consists of the four 2-cells $]-\infty, a[\ \times M$, $]b, \infty[\ \times M$, $C(I; -\infty < f)$ and $C(I; g < \infty)$, and the four 1-cells, $a \times M$, $b \times M$ and the graphs of f and g. Since any union can be written as a disjoint union by taking complements, an application of Proposition 9.7 then reduces to showing that the intersection of two cells V_1 and V_2 in M^2 is tame. This is trivial if either one is discrete, whence a singleton. Suppose V_1 is a 1-cell, given by the definable, continuous map $f_1 : I_1 \to M$. Let Y be the subset of all $x \in I_1$ such that $(x, f_1(x))$ belongs to V_2. Choose a cellular map $c\colon Y \to D$ (by Lemma 9.5, or, for higher arities, by induction). Its composition with the (bijective) projection $V_1 \cap V_2 \to Y$ is then also cellular.

Suppose next that $V_i = C(I_i; f_i < g_i)$ are both 2-cells, assumed once more for simplicity to be bounded. Let $I := I_1 \cap I_2$ and for $x \in I$, let $f(x)$ be the maximum of $f_1(x)$ and $f_2(x)$, and let $g(x)$ be the minimum of $g_1(x)$ and $g_2(x)$. Note that f and g are continuous on I. Let Y consist of all $x \in I$ for which $f(x) < g(x)$, and let $c\colon Y \to D$ be cellular. The composition of c with the projection $V_1 \cap V_2 \to Y$ is again cellular, since its fibers are the cells $C(c^{-1}(a); f < g)$. ⊣

9.10. EXAMPLE. It is important in this result that the structure is already a model of DCTC. For instance, let D be the subset of the ultrapower $\mathbb{R}_\natural$ of the reals (viewed as an ordered field) consisting of all elements of the form n or $\omega_\natural - n$, for $n \in \mathbb{N}$. Note that D is closed, bounded, and discrete, and hence tame. However, $(\mathbb{R}_\natural, D)$ is not tame, since $\mathbb{N} = D_{<\omega_\natural/2}$ is definable but fails to have a supremum, and so, $(\mathbb{R}_\natural, D)$ is not even a model of DCTC.

It is not hard to show that the product of two cells is again a cell. Therefore, the product of two tame subsets is again tame. Similarly, the fiber of a cell is again a cell, and hence if $X \subseteq M^n$ is tame, then so is each fiber $X[\mathbf{a}]$. Together with Theorem 9.8 and the fact that a principal projection of a tame subset is again tame, we showed that the collection of tame subsets determines a first-order structure on M (in the sense of [9, Chapt. 1, 2.1], with a predicate for every tame subset of M^n). Calling this induced structure on M the *tame reduct* of $\mathcal{M}$ and denoting it $\mathcal{M}^{\text{tame}}$, is justified by:

9.10. COROLLARY. *If $\mathcal{M} \models$ DCTC, then $\mathcal{M}^{tame}$ is tame, whence in particular a model of DCTC.*

PROOF. The definable subsets of $\mathcal{M}^{\text{tame}}$ are precisely the tame definable subsets of $\mathcal{M}$, so that in particular, $\mathcal{M}$ and $\mathcal{M}^{\text{tame}}$ have the same cells. So remains to show that if $c\colon X \to D$ is cellular in $\mathcal{M}$, then it is also cellular in $\mathcal{M}^{\text{tame}}$. Discrete sets are tame by definition, and the graph $\Gamma(c)$ of c is tame via the projection $\Gamma(c) \to D$. In particular, c is $\mathcal{M}^{\text{tame}}$-definable, and since its fibers are cells, we are done. The last assertion then follows from Lemma 9.2. ⊣

We also have the following joint cell decomposition:

9.11. COROLLARY. *Given tame subsets $Y_1, \dots, Y_n$ of a tame set X in $\mathcal{M}$, there exists a cellular map $c\colon X \to D$, such that for each i, the restriction of c to Y_i is also cellular.*

PROOF. Since any Boolean combination of tame subsets is again tame by Theorem 9.8, we may reduce first to the case that all Y_i are disjoint, and then by induction, that we have a single tame subset $Y \subseteq X$. Since $X \setminus Y$ is tame too, we have cellular maps $d\colon Y \to D$ and $d'\colon X \setminus Y \to D'$, and their disjoint union is then the desired cellular map. ⊣

We call a definable map *tame*, if its graph is. Note that its domain then must also be tame. As already observed in the previous proof, cellular maps are tame. To characterize tame maps, we make the following observation/definition: given a cellular map $c\colon X \subseteq M^n \to D$, for $e \leq n$, let $X_c^{(e)} = X^{(e)}$ be the union of all e-dimensional fibers $c^{-1}(a)$. Since dimension is definable, so is each $X^{(e)}$, and hence the restriction of c to $X^{(e)}$ is also cellular, proving in particular that each $X^{(e)}$ is tame.

9.12. THEOREM. *A definable map $f\colon X \to M^k$ is tame if and only if X is tame, and the restriction of f to the set of its discontinuities is also tame. In particular, a definable, continuous map with tame domain is tame.*

PROOF. If f is tame, then f is $\mathcal{M}^{\text{tame}}$-definable, and hence so is its set of discontinuities X', proving that X' is tame. Since the graph of $f|_{X'}$ is $\Gamma(f) \cap (X' \times M^k)$, the restricted map is again tame by Theorem 9.8. For the converse, $U := X \setminus X'$ is tame by Theorem 9.8, so that we have a cellular map $c\colon U \to D$. Since f is continuous on U, the composition of c with the principal projection $\Gamma(f|_U) \to U$ is also cellular, showing that $\Gamma(f|_U)$ is tame. Since by assumption, the graph of the restriction to X' is tame, so is $\Gamma(f) = \Gamma(f|_U) \cup \Gamma(f|_{X'})$ by Theorem 9.8, showing that f is tame. ⊣

A tame map is $\mathcal{M}^{\text{tame}}$-definable, and hence so its its image, proving:

9.13. COROLLARY. *If the domain of a definable, continuous map in $\mathcal{M}$ is tame, then so is its image. More generally, the image of a tame subset under a tame map is again tame.* ⊣

Let us call a definable map $f\colon X \subseteq M^n \to M^k$ *almost continuous*, if its set of discontinuities is discrete. By Theorem 3.2, any one-variable definable map is almost continuous. Given a definable map $f\colon X \to M^k$, let us inductively define $D_i(f) \subseteq X$, by setting $D_0(f) := X$, and by setting $D_i(f)$, for $i > 0$, equal to the set of discontinuities of the restriction of f to $D_{i-1}(f)$. By Remark 3.5, each $D_i(f)$ has strictly lesser dimension than $D_{i-1}(f)$, and hence $D_n(f)$ is empty for n bigger than the dimension of X. Hence f is (almost) continuous if $D_1(f)$ is empty (respectively, discrete). Since the domain of a tame function is tame, an easy inductive argument using Theorem 9.12 immediately yields:

9.14. COROLLARY. *An almost continuous map in $\mathcal{M}$ (for instance, a one-variable map) with tame domain is tame. In particular, a definable map f is tame if and only if all $D_i(f)$ are tame.* ⊣

Let us say that an ordered structure is *almost continuous*, if apart from a binary predicate denoting the order, all other symbols represent almost continuous functions.

9.15. COROLLARY. *If $\mathcal{M}$ is almost continuous, then it is tame.*

PROOF. Since the collection of tame subsets is closed under Boolean operations, projections, and products by Theorem 9.8, we only have to verify that the ones defined by unnested atomic formulae are tame. Since by assumption the only predicate is the inequality sign, and the set it defines is a cell, we only have to look at formulae of the form $f(x) = g(x)$ or $f(x) < g(x)$, with f, g function symbols. Since f and g are total functions representing almost continuous maps, their graphs are tame by Corollary 9.14, whence so is their intersection by Theorem 9.8. The projection of the latter is the set defined by $f(x) = g(x)$, proving that is a tame subset. Let F and G be the subsets of

M^{n+2} of all $(a, f(a), c)$ and all $(a, b, g(a))$ respectively, with $a \in M^n$ and $b, c \in M$. Since these are just products of the respective graphs and M, both are tame, and so is the subset E of all $(a, b, c) \in M^{n+2}$ with $a < b$. Therefore, by another application of Theorem 9.8, the intersection $F \cap G \cap E$ is tame, and so is its projection, which is just the set defined by the relation $f(a) < g(a)$. ⊣

9.16. *Remark.* More generally, by the same argument, if $\mathcal{M} \models$ DCTC is an expansion of a tame structure by tame functions and by predicates defining tame subsets, then $\mathcal{M}$ itself is tame.

REFERENCES

[1] ALF DOLICH, CHRIS MILLER, and CHARLIE STEINHORN, *Structures having o-minimal open core*, ***Trans. Amer. Math. Soc.***, vol. 362 (2010), no. 3, pp. 1371–1411.

[2] ANTONGIULIO FORNASIERO, *Locally o-minimal structures and structures with locally o-minimal core*, preprint, 2011.

[3] CHRIS MILLER, *Expansions of dense linear orders with the intermediate value property*, ***J. Symbolic Logic***, vol. 66 (2001), no. 4, pp. 1783–1790.

[4] ———, *Tameness in expansions of the real field*, ***Logic Colloquium '01***, Lect. Notes Log., vol. 20, Assoc. Symbol. Logic, Urbana, IL, 2005, pp. 281–316.

[5] ANAND PILLAY, ***An introduction to stability theory***, Oxford Logic Guides, vol. 8, The Clarendon Press Oxford University Press, New York, 1983.

[6] HANS SCHOUTENS, ***The use of ultraproducts in commutative algebra***, Lecture Notes in Mathematics, vol. 1999, Springer-Verlag, 2010.

[7] ———, *O-minimalism*, submitted, 2011.

[8] CARLO TOFFALORI and KATHRYN VOZORIS, *Notes on local o-minimality*, ***MLQ Math. Log. Q.Math. Log. Q.***, vol. 55 (2009), no. 6, pp. 617–632.

[9] LOU VAN DEN DRIES, ***Tame topology and o-minimal structures***, LMS Lect. Note Series, Cambridge University Press, Berlin, 1998.

[10] KATHRYN VOZORIS, *The complex field with a predicate for the integers*, ***Ph.D. thesis***, University of Illinois at Chicago, 2007.