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O-MINIMALISM

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Abstract. An ordered structure is called o-minimalistic if it has all the first-order features of an o-minimal structure. To any o-minimalistic structure, we can associate its Grothendieck ring, which in the non-o-minimal case is a non-trivial invariant. To study this invariant, we identify an o-minimalistic property, the Discrete Pigeonhole Principle, which in turn allows us to define discretely valued Euler characteristics. As an application, we study certain analytic subsets, called Taylor sets.

§1. Introduction.

Let L be a language containing a binary predicate $<$, to be interpreted as a dense linear ordering. We call an L-structure $\mathcal{M}$ *o-minimalistic*, if it is a model of $T^{\text{omin}} := T^{\text{omin}}(L)$, the collection of L-sentences that hold true in every o-minimal L-structure. In [7], we laid the groundwork to study o-minimalistic structures, by studying a well-known fragment of T^{omin}, the theory DCTC of definably complete/type complete structures: every one-variable definable subset has a (possibly infinite) supremum, and every one-sided type at every (possibly infinite) point is complete. We then showed that many results from o-minimality carry over to this more general situation, upon replacing 'finite' by 'discrete', such as the Monotonicity Theorem, dimension theory, Hardy structures. Cell decompositions cause more trouble, and we identified the subclass of *tame* structures, as those for which every definable subset can be partitioned in discretely many cells (here a cell is defined exactly as in the o-minimal context).

Although we do not yet know whether every o-minimalistic structure is tame,[1] we can show that every o-minimalistic structure has a tame, o-minimalistic reduct (Theorem 2.5). Neither do I do know whether DCTC is equal to T^{omin}, but in §3, I will formulate an o-minimalistic (first-order) property, the Discrete Pigeonhole Principle (DPP=any definable, injective map from a discrete set to itself is bijective), which does not obviously follow from DCTC. In fact, it is not clear if we can axiomatize o-minimalism by first-order conditions on one-variable formulae only (note that DPP is a priori not of this form).

So this paper investigates properties of o-minimalistic structures that are modified versions of the corresponding properties of o-minimal structures. Therefore, whereas most papers on generalizing o-minimality are searching for weakenings that would include certain tamely behaving structures, our hands are tied and we have to obey by the

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[1] Dolich has some ideas to construct a counterexample.

laws of o-minimalism. Thus, to the chagrin of some of my esteemed colleagues, we have to discard the structure $(\mathbb{Q}, <, +, \mathbb{Z})$ as it is not o-minimalistic, although it is definably complete and locally o-minimal. However, it fails to have the type completeness property at infinity, which forces every discrete set to be bounded. In §3, we study the Grothendieck ring of an o-minimalistic structure. It will follow from the DDP that this Grothendieck ring is equal to the ring of integers if and only if the structure is o-minimal (in which case it corresponds to the Euler characteristic). Using Grothendieck rings, we can also formulate a condition for an *ultra-o-minimal* structure $\mathcal{M}_\natural$, that is to say, an ultraproduct of o-minimal structures $\mathcal{M}_i$, to be o-minimal (no such criterion seemed to have existed before): this is the case if for each L-formula φ, there is a bound N_φ on the absolute value of the Euler characteristic of $\varphi(\mathcal{M}_i)$, independent of i (Theorem 3.20).

In §4, we study expansions of an o-minimalistic structure by a predicate that are again o-minimalistic. For discrete subsets, we get the notion of an *o-finitistic* set, that is to say, a set enjoying all first-order properties of an arbitrary finite set in an o-minimal structure. This notion is particularly interesting when it comes to classifying definable subsets up to 'virtual' isomorphism, that is to say, definable in some o-minimalistic expansion; the corresponding Grothendieck ring is called the *virtual Grothendieck ring* and studied in §5. However, a priori, the treatment depends on a choice of 'context', that is to say, of an ultra-o-minimal elementary extension. Using this technology, we associate in §6 to each definable, discrete subset of M a (discretely valued) Euler characteristic defined on its virtual Grothendieck ring. This allows us to calculate explicitly this virtual Grothendieck ring in the special case of a tame, o-minimalistic expansion of an ordered field admitting a power dominant discrete subset (Corollary 6.6).

The last section is an application to the study of analytic sets. In the o-minimal context, (sub)analytic sets are normally understood to be given by analytic functions supported on the unit box (often simply called *restricted analytic functions*), as the corresponding structure $\mathbb{R}_{\text{an}}$ is o-minimal, and admits quantifier elimination in an appropriate language by the seminal work of [2]. There is a good reason to restrict to compact support, as the global sine function defines $\mathbb{Z}$, and hence can never be part of an o-minimal expansion. Our approach here is to look at subsets of $\mathbb{R}^k$ that can be uniformly approximated on compact sets by $\mathbb{R}_{\text{an}}$-definable subsets. More precisely, we call a subset $X \subseteq \mathbb{R}^k$ a *Taylor* set, if the ultraproduct over all n of the truncations $X_{\rfloor n} := \{\mathbf{x} \in X \,|\, |\mathbf{x}| \leq n\}$ is definable in $\mathcal{R}^{\text{an}}_\natural$, where the latter structure is obtained as the ultraproduct of the scalings of $\mathbb{R}_{\text{an}}$ by a factor n (that is to say, for each n, the expansion of $\mathbb{R}$ by power series converging on $|\mathbf{x}| \leq n$). It follows from [2] that $\mathcal{R}^{\text{an}}_\natural$ is o-minimalistic. Any subset definable by a quantifier free formula using convergent power series, whence in particular, any globally analytic variety, is Taylor. A discrete subset is Taylor if and only if it is closed, and any such set satisfies the Discrete Pigeonhole Principle with respect to Taylor maps. However, we can now also define sets by analytic parameterization, like the spiral with polar coordinates $R = \exp\theta$, for $\theta > 0$ (in contrast, the spiral obtained by allowing θ to be negative as well is not Taylor!). We use our o-minimalistic results to give a geometric treatment of the class of Taylor sets: to a Taylor set X, we associate an $\mathcal{R}^{\text{an}}_\natural$-definable subset $X_\flat$, called its *protopower*, given as the ultraproduct of its truncations. We obtain a good dimension theory, a Monotonicity Theorem, a (partly conjectural, locally finite) cell-decomposition, and a corresponding Grothendieck ring, all indicative of the tameness of the class of Taylor sets, albeit not first-order.

Notations and conventions. Definable means definable with parameters. Throughout this paper, L denotes some language containing a distinguished binary relation symbol $<$ and any L-structure $\mathcal{M}$ will be (at least) a dense linear order without endpoints. When needed, $\mathtt{U}$ denotes some predicate (often unary), and we will write $(\mathcal{M}, X)$ for the $L(\mathtt{U})$-structure in which X is the interpretation of $\mathtt{U}$.

We will use the following ISO convention for intervals: *open* $]a,b[$ (which we always assume to be non-empty, that is to say, $a < b$), *closed* $[a,b]$ (including the singleton $\{a\} = [a,a]$), *half-open* $]a,b]$ or $[a,b[$, and their infinite variants like $]-\infty,a[$, $]-\infty,a]$, $]a,\infty[$, and $[a,\infty[$, with $a,b \in M$.

When taking ultraproducts, we rarely ever mention the underlying index set or (non-principal) ultrafilter. We use the notation introduced in [6], denoting ultraproducts with a subscript $\natural$. Thus, we write $\mathbb{N}_\natural$, $\mathbb{Z}_\natural$, and $\mathbb{R}_\natural$ for the ultrapower of the set of natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, and reals $\mathbb{R}$ respectively. On occasion we need the (countable) ultraproduct of the diagonal sequence $(n)_n$ in $\mathbb{N}_\natural$, which we denote suggestively by $\omega_\natural$.

§2. O-minimalism.

Let us briefly recall some notions from [7]. Let $\mathcal{O}$ be an (ordered) L-structure. We call $\mathcal{O}$ *o-minimalistic*, if it is a model of the theory T^{omin} of *o-minimalism*, given as the intersection of the theories $\operatorname{Th}(\mathcal{M})$ of all o-minimal L-structures $\mathcal{M}$.

2.1. LEMMA. *A reduct of an o-minimalistic structure is again o-minimalistic.*

PROOF. Let $L \subseteq L'$ be languages, let $\mathcal{M}'$ be an o-minimalistic L'-structure, and let $\mathcal{M} := \mathcal{M}'|_L$ be its L-reduct. To show that $\mathcal{M}$ is o-minimalistic, take a sentence in T^{omin}_L and let $\mathcal{N}'$ be any o-minimal L'-structure. Since its reduct $\mathcal{N}'|_L$ is also o-minimal, σ holds in the latter, whence also in $\mathcal{N}'$ itself. As this holds for all o-minimal L'-structures, σ also holds in $\mathcal{M}'$. Since σ only mentions L-symbols, it must therefore already hold in the reduct $\mathcal{M}$, as we needed to show. ⊣

We will call an ultraproduct of o-minimal L-structures an *ultra-o-minimal* structure.[2] Using a well-known elementarity criterion via ultraproducts, we have:

2.2. COROLLARY. *An L-structure is o-minimalistic if and only if it is elementary equivalent with (equivalently, an elementary substructure of) an ultra-o-minimal structure.* ⊣

This produces many examples of non-o-minimal o-minimalistic structures.

2.3. EXAMPLE. Let L be the language of ordered fields together with a unary predicate $\mathtt{U}$. For each n, let $\mathcal{R}_n := (\mathbb{R}, \{0,1,\dots,n\}$ be the expansion of the field $\mathbb{R}$. Since $\{0,1,2,\dots,n\}$ is finite, each $\mathcal{R}_n$ is o-minimal, and therefore their ultraproduct $\mathcal{R}_\natural$ is o-minimalistic by Corollary 2.2. The set $D := \mathtt{U}(\mathcal{R}_\natural)$ is discrete but not finite, so $\mathcal{R}_\natural$ cannot be o-minimal. Note that D contains $\mathbb{N}$ and that $\omega_\natural$ is its maximum. In fact, $D = (\mathbb{N}_\natural)_{\leq \omega_\natural}$.

An o-minimalistic field (with no additional structure), being definably complete, is o-minimal by [5, Corollary 1.5]. Any o-minimalistic structure whose underlying order is that of the reals, or more generally, admits the Heine-Borel property, must be o-minimal by [7, Corollary 2.4 and Remark 2.5]. We defined the *dimension* $\dim(X)$ of a definable

[2] O-minimality is preserved under elementary equivalence, whence under ultrapowers, but not necessarily under ultraproducts.

subset X as the largest d for which it contains a d-cell. Here a *cell* is defined in the same way as in the o-minimal case ([8, Chapt. 3, Def. 2.3]): it is either the graph of a continuous, definable map on a cell of lower arity, or the region between two such graphs. A definable subset $X \subseteq M^k$ has dimension zero if and only if it is discrete, whereas it has dimension k if and only if it has non-empty interior.

2.4. PROPOSITION. *In an ultra-o-minimal structure $\mathcal{M}$, a definable set has dimension e if and only if it is an ultraproduct of e-dimensional definable sets.*

PROOF. Suppose $\mathcal{M}$ is the ultraproduct of o-minimal structures $\mathcal{M}_i$, and let $X = \varphi(\mathcal{M})$ be a definable subset. By Łoś' Theorem, X is the ultraproduct of the definable sets $X_i := \varphi(\mathcal{M}_i)$. The result now follows from the definability of dimension: we leave the general case to the reader, but for the *planar* case (=definable subsets of M^2), observe that both being discrete or having non-empty interior are first-order definable properties, and hence pass through the ultraproduct by Łoś' Theorem. ⊣

Tameness. We call a definable subset $X \subseteq O^k$ *tame*, if there exists a definable map $c\colon X \to D$ with D a closed, bounded, and discrete subset, such that each fiber $c^{-1}(a)$ is a cell. We call $\mathcal{O}$ *tame*, if all $\mathcal{O}$-definable sets are tame. Any tame structure is a model of DCTC ([7, Lemma 9.2]). The class of tame subsets is closed under Boolean operations and projections, and so we can define the *tame reduct* $\mathcal{O}^{\text{tame}}$ of $\mathcal{O}$ in which the definable subsets are precisely the tame subsets (whence $\mathcal{O}^{\text{tame}}$ is in particular tame itself; see [7, Corollary 9.10]).

2.5. THEOREM. *If $\mathcal{M}$ is o-minimalistic, then so is $\mathcal{M}^{tame}$.*

PROOF. Let $\bar{L}$ be the language with a predicate for each tame subset of $\mathcal{M}$, so that $\mathcal{M}^{\text{tame}}$ is an $\bar{L}$-structure. Viewing $\mathcal{M}$ as a structure in the language having a predicate for each definable subset yields again a tame structure, since we added no new definable subsets (see Lemma 4.1 below). Therefore, upon replacing L by the latter language, we assume from the start that $\bar{L} \subseteq L$, and the result now follows from Lemma 2.1. ⊣

If $\mathcal{M}$ is a model of DCTC, then we defined in [7, §6.6] its Hardy structure $\mathbf{H}(\mathcal{M})$ as the set of all germs of one-variable, continuous, definable maps at infinity. We showed that it is again an L-structure, and, in fact, an elementary extension of $\mathcal{M}$. Hence, if $\mathcal{M}$ is o-minimalistic, then so is $\mathbf{H}(\mathcal{M})$. As we argued there, this gives rise to plenty of Vaughtian pairs, showing that o-minimalism has Vaughtian pairs.

2.6. *Remark.* We have the following puzzling fact that at least one among the following three statements holds:

2.6.i. there is a tame structure which is not o-minimalistic;
2.6.ii. there is an o-minimalistic structure which is not tame;
2.6.iii. any reduct of a tame structure is again tame.

Indeed, suppose both (2.6.i) and (2.6.iii) fail. So, by the latter, there is a tame structure $\mathcal{M}$ with a non-tame reduct $\bar{\mathcal{M}}$, and by the former, $\mathcal{M}$ is o-minimalistic, whence so is $\bar{\mathcal{M}}$ by Lemma 2.1. Hence $\bar{\mathcal{M}}$ is o-minimalistic but not tame. Note that (2.6.i) implies that DCTC is not equal to T^{omin}. I do not know whether any ultraproduct of tame structures is tame. If so, then (2.6.ii) fails, that is to say, any o-minimalistic structure is tame, since it is elementary equivalent by Corollary 2.2 with an ultra-o-minimal structure, and the latter would then be tame, whence so would the former be by [7, Proposition 9.3].

§**3. The Grothendieck ring of an o-minimalistic structure.** Given any first-order structure $\mathcal{N}$, we define its *Grothendieck ring* $\mathbf{Gr}(\mathcal{N})$ as follows. Given two formulae $\varphi(x)$ and $\psi(y)$ with parameters, with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$, we say that φ and ψ are *($\mathcal{N}$-)definably isomorphic*, if there exists a definable bijection $f \colon \varphi(\mathcal{N}) \to \psi(\mathcal{N})$. Let $\mathbf{Gr}(\mathcal{N})$ be the quotient of the free Abelian group generated by $\mathcal{N}$-definable isomorphism classes $\langle\varphi\rangle$ of formulae φ modulo the *scissor relations*

$$\text{(sciss)} \qquad \langle\varphi\rangle + \langle\psi\rangle - \langle\varphi \wedge \psi\rangle - \langle\varphi \vee \psi\rangle$$

where φ, ψ range over all pairs of formulae in the same free variables. See for instance [3, 4] for more details.

We will write $[\varphi]$ or $[Y]$ for the image of the formula φ, or the set Y defined by it, in $\mathbf{Gr}(\mathcal{N})$. Since we can always replace a definable subset with a definable copy that is disjoint from it, the scissor relations can be simplified, by only requiring them for disjoint unions: $[X \sqcup Y] = [X] + [Y]$. In particular, combining all terms with a positive sign as well as all terms with a negative sign by taking disjoint unions, we see that every element in the Grothendieck ring is of the form $[X] - [Y]$, for some definable subsets X and Y. To make $\mathbf{Gr}(\mathcal{N})$ into a ring, we define the product of two classes $[\varphi]$ and $[\psi]$ as the class of the *product* $\varphi(x) \wedge \psi(y)$ where x and y are disjoint sets of variables. One checks that this is well-defined and that the class of a point is the unit for multiplication, therefore denoted 1. Note that in terms of definable subsets, the product corresponds to the Cartesian product and the scissor relation to the usual inclusion/exclusion relation.

Variants are obtained by restricting the class of formulae/definable subsets. For our purposes, that is to say, working in an ordered L-structure $\mathcal{M}$, we will only do this for discrete subsets. Call a formula *discrete* if it defines a discrete subset. In a model $\mathcal{M} \models$ DCTC, discrete formulae are closed under Boolean combinations and products ([7, Corollary 2.9]), and if two discrete definable subsets are definably isomorphic, then the graph of this isomorphism is also given by a discrete formula. Therefore, the Grothendieck ring on discrete formulae is well-defined and will be denoted $\mathbf{Gr}_0(\mathcal{M})$. We have a canonical homomorphism $\mathbf{Gr}_0(\mathcal{M}) \to \mathbf{Gr}(\mathcal{M})$ with image the subring generated by classes of discrete formulae. The following is useful when dealing with Grothendieck rings:

3.1. LEMMA. *Two definable subsets X and Y in a first-order structure $\mathcal{N}$ have the same class in $\mathbf{Gr}(\mathcal{N})$ if and only if there exists a definable subset Z such that $X \sqcup Z$ and $Y \sqcup Z$ are definably isomorphic.*

PROOF. One direction is immediately, for if $X \sqcup Z$ and $Y \sqcup Z$ are definably isomorphic, then $[X] + [Z] = [X \sqcup Z] = [Y \sqcup Z] = [Y] + [Z]$ in $\mathbf{Gr}(\mathcal{N})$, from which it follows $[X] = [Y]$. Conversely, if $[X] = [Y]$, then by definition of scissor relations, there exist mutually disjoint, definable subsets $A_i, B_i, C_i, D_i \subseteq N^{p_i}$ such that

$$\langle X\rangle + \sum_i \langle A_i\rangle + \langle B_i\rangle - \langle A_i \sqcup B_i\rangle = \langle Y\rangle + \sum_i \langle C_i\rangle + \langle D_i\rangle - \langle C_i \sqcup D_i\rangle$$

in the free Abelian group on isomorphism classes. Bringing the terms with negative signs to the other side, we get an expression in which each term on the left hand side must also occur on the right hand side, that is to say, the collection of all isomorphism classes $\{\langle X\rangle, \langle A_i\rangle, \langle B_i\rangle, \langle C_i \sqcup D_i\rangle\}$ is the same as the collection of all isomorphism classes $\{\langle Y\rangle, \langle C_i\rangle, \langle D_i\rangle, \langle A_i \sqcup B_i\rangle\}$. By properties of disjoint union, we therefore get $\langle X \sqcup Z\rangle = \langle Y \sqcup Z\rangle$, where Z is the disjoint union of all definable subsets A_i, B_i, C_i, D_i. ⊣

If $\mathcal{M}$ is an expansion of an ordered, divisible Abelian group, then we have the following classes of open intervals. If $I = \,]a,b[\,$, then I is definably isomorphic to $]0, b-a[$ via the translation $x \mapsto x - a$. Moreover, $]0,a[$ is definably isomorphic to $]0,2a[$ via the map $x \mapsto 2x$. Hence the class $\mathbb{i}$ of $]2,a[$ is by (sciss) equal to the sum of the classes of $]0,a[\,$, $\{a\}$, and $]a,2a[\,$. In other words, $\mathbb{i} = 2\mathbb{i} + 1$, whence $\mathbb{i} = -1$ (the additive inverse of 1). Let $\mathbb{h}$ be the class of the unbounded interval $]0,\infty[\,$. By translation and/or the involution $x \mapsto -x$, any half unbounded interval is definably isomorphic with $]0,\infty[\,$. Finally, we put $\mathbb{L} := [M]$ (the so-called *Lefschetz class*). Since M is the disjoint union of $]-\infty,0[\,$, $\{0\}$, and $]0,\infty[\,$, we get

(lef) $$\mathbb{L} = 2\mathbb{h} + 1.$$

If M is moreover an ordered field, then taking the reciprocal makes $]0,1[$ and $]1,\infty[$ definably isomorphic, so that $\mathbb{h} = \mathbb{i} = -1$, and hence also $\mathbb{L} = -1$.

Under the assumption of an underlying ordered structure, whence a topology, we can also strengthen the definition by calling two definable subset *definably homeomorphic*, if there exists a definable (continuous) homeomorphism between them, and then build the Grothendieck ring, called the *strict Grothendieck ring* of $\mathcal{M}$ and denoted $\mathbf{Gr}^{\mathrm{s}}(\mathcal{M})$, on the free Abelian group generated by homeomorphism classes of definable subsets. Note that there is a canonical surjective homomorphism $\mathbf{Gr}^{\mathrm{s}}(\mathcal{M}) \to \mathbf{Gr}(\mathcal{M})$. In the o-minimal case, the monotonicity theorem implies that both variants are equal, but this might fail in the o-minimalistic case, since cell decompositions are no longer finite (but see Corollary 3.13 below). In fact, in the o-minimal case, the Grothendieck ring is extremely simple, as observed by Denef and Loeser ([8, Chap. 4, §2]):

3.2. PROPOSITION. *The Grothendieck ring of an o-minimal expansion of an ordered field is canonically isomorphic to the ring of integers $\mathbb{Z}$.*

PROOF. By the previous discussion, the class of any open interval is equal to -1. The graph of a function is definably isomorphic with its domain, and so the class of any 1-cell is equal to -1. Since a bounded planar 2-cell lies in between two 1-cells, it is definably isomorphic to an open box, and by definition of the multiplication in $\mathbf{Gr}(\mathcal{M})$, therefore its class is equal to $\mathbb{L}^2 = 1$. The unbounded case is analogous, and so is the case that the 2-cell lies in a higher Cartesian product. This argument easily extends to show that the class of a d-cell in $\mathbf{Gr}(\mathcal{M})$ is equal to $\mathbb{L}^d = (-1)^d$. By Cell Decomposition, every definable subset is a finite union of cells, and hence its class in $\mathbf{Gr}(\mathcal{M})$ is an integer (multiple of 1). ⊣

We denote the canonical homomorphism $\mathbf{Gr}(\mathcal{M}) \to \mathbb{Z}$ by $\chi_{\mathcal{M}}(\cdot)$ and call it the *Euler characteristic* of $\mathcal{M}$. Inspired by [1], we define the *Euler measure* of a definable subset X in an o-minimal structure $\mathcal{M}$ as the pair $\mu_{\mathcal{M}}(X) := (\dim(X), \chi_{\mathcal{M}}(X)) \in (\mathbb{N} \cup \{-\infty\}) \times \mathbb{Z}$, where we view the latter set in its lexicographical ordering.

In an arbitrary first-order structure, let us say, for definable subsets X and Y, that $X \preceq Y$ if and only if there exists a definable injection $X \to Y$. In general, this relation, even up to definable isomorphism, will fail to be symmetric (take for instance in the reals the sets $X = [0,1]$ and $Y = X \cup \{3/2\}$, where $x \mapsto x/2$ sends Y inside X), and therefore is in general only a partial pre-order. As we will discuss below in §3.14, it does induce a partial order on isomorphism classes of discrete, definable subsets in an o-minimalistic structure. In the o-minimal case, $\preceq$ is a total pre-order by the following (folklore) result. In some sense, the rest of the paper is an attempt to extend this result to the o-minimalistic case.

3.3. THEOREM. *In an o-minimal expansion of an ordered field, two definable sets X and Y are definably isomorphic if and only if $\mu_{\mathcal{M}}(X) = \mu_{\mathcal{M}}(Y)$. Moreover, $X \preceq Y$ if and only if $\dim(X) \leq \dim(Y)$ with the additional condition that $\chi_{\mathcal{M}}(X) \leq \chi_{\mathcal{M}}(Y)$ whenever both are finite.*

PROOF. The first statement is proven in [8, Chap. 8, 2.11]. So, suppose $X \preceq Y$. Since X is definably isomorphic with a subset of Y, its dimension is at most that of Y. If both are zero-dimensional, that is to say, finite, then the pigeonhole principle gives $\chi_{\mathcal{M}}(X) = |X| \leq |Y| = \chi_{\mathcal{M}}(Y)$.

Conversely, assume $\dim(X) \leq \dim(Y)$. If both are finite, the assertion is clear by the same argument, so assume they are both positive dimensional. Without loss of generality, by adding a (disjoint) cell of the correct dimension, we may then assume that they have both the same dimension $d \geq 1$. Let $e := \chi_{\mathcal{M}}(Y) - \chi_{\mathcal{M}}(X)$ and let F consist of e points disjoint from X if e is positive and of $-e$ open intervals disjoint from X if e is negative. Since $\chi(F) = e$, the Euler measure of $X \sqcup F$ and Y are the same, and hence they are definably isomorphic by the first assertion, from which it follows that $X \preceq Y$. ⊣

Let $\mathcal{M}$ be an ultra-o-minimal structure, say, realized as the ultraproduct of o-minimal structures $\mathcal{M}_i$. We define its *ultra-Euler characteristic* $\chi_{\mathcal{M}}(\cdot)$ as follows. Let $Y \subseteq M^n$ be a definable subset, say given by a formula $\varphi(x, \mathbf{b})$ with $\mathbf{b}$ a tuple of parameters realized as the ultraproduct of tuples $\mathbf{b}_i$ in each M_i. Let $Y_i := \varphi(\mathcal{M}_i, \mathbf{b}_i)$, so that Y is the ultraproduct of the Y_i, and let $\chi_{\mathcal{M}}(Y)$ now be the ultraproduct of the $\chi_{\mathcal{M}_i}(Y_i)$, viewed as an element of $\mathbb{Z}_\natural$. If X is definably isomorphic with Y, via a definable bijection with graph G, choose as above definable subsets X_i and G_i in $\mathcal{M}_i$ with ultraproduct equal to X and G respectively. By Łoś' Theorem, almost each G_i is the graph of a definable bijection between X_i and Y_i, and therefore $\chi_{\mathcal{M}_i}(X_i) = \chi_{\mathcal{M}_i}(Y_i)$ for almost all i, showing that $\chi_{\mathcal{M}}(X) = \chi_{\mathcal{M}}(Y)$. Similarly, we define the *ultra-Euler measure* $\mu_{\mathcal{M}}(X) := (\dim(X), \chi_{\mathcal{M}}(X))$. Since the ultra-Euler characteristic is easily seen to be also compatible with the scissor relations (sciss), we showed:

3.4. COROLLARY. *For an ultra-o-minimal structure $\mathcal{M}$, the ultra-Euler characteristic induces a homomorphism $\mathbf{Gr}(\mathcal{M}) \to \mathbb{Z}_\natural$.* ⊣

3.5. The Discrete Pigeonhole Principle. Before we proceed, we identify another o-minimalistic property, that is to say, a first-order property of o-minimal structures. For the remainder of this section, $\mathcal{M}$ will be an o-minimalistic structure.

3.6. PROPOSITION (Discrete Pigeonhole Principle). *If a definable map $f\colon Y \to Y$, for some $Y \subseteq M^n$, is injective and its image is co-discrete, meaning that $Y \setminus f(Y)$ is discrete, then it is a bijection. In particular, any definable map from a discrete subset D to itself is injective if and only if it is surjective.*

PROOF. For each formula $\varphi(x, y, \mathbf{z})$, we can express in a first-order way that if $\varphi(x, y, \mathbf{c})$, for some tuple $\mathbf{c}$ of parameters, defines the graph of an injective map $f\colon Y \to Y$ then

(DPP) $$Y \setminus f(Y) \text{ discrete implies } Y = f(Y).$$

Remains to show that (DPP) holds in any o-minimal structure $\mathcal{M}$. Indeed, if $D = Y \setminus f(Y)$, then $\chi_{\mathcal{M}}(Y) = \chi_{\mathcal{M}}(f(Y)) + \chi_{\mathcal{M}}(D)$. Since f is injective, Y and $f(Y)$ are definably isomorphic, whence have the same Euler characteristic, and so $\chi_{\mathcal{M}}(D) = 0$. But a discrete subset in an o-minimal structure is finite and its Euler characteristic is then just its cardinality, showing that $D = \emptyset$. One direction in the last assertion is immediate, and for the converse, assume $f\colon D \to D$ is surjective. For each $x \in D$, define $g(x)$ as the (lexicographical) minimum of $f^{-1}(x)$, so that $g\colon D \to D$ is an injective map, whence surjective by the above, and therefore necessarily the inverse of f. ⊣

At present, I do not know how to derive (DPP) from DCTC.

3.7. COROLLARY. *An o-minimalistic expansion of an ordered field is o-minimal if and only if its Grothendieck ring is isomorphic to* $\mathbb{Z}$.

PROOF. One direction is Proposition 3.2, so assume $\mathbf{Gr}(\mathcal{M}) = \mathbb{Z}$. Let D be a definable, discrete subset. By assumption, $[D] = n$ for some integer n. After removing n points, if n is positive, or adding $-n$ points, if negative, we may suppose $[D] = 0$. By Lemma 3.1, there exists a definable subset X such that X and $X \sqcup D$ are definably isomorphic. By (DPP), this forces $D = \emptyset$. ⊣

3.8. COROLLARY. *A monotone map* $f\colon D \to D$ *on an* $\mathcal{M}$*-definable, discrete subset* D *is either constant or an involution.*

PROOF. Suppose f is non-constant and hence f^2 is strictly increasing. So upon replacing f by its square, we may already assume that f is increasing, and we need to show that it is then the identity. Since f is injective, it is bijective by Proposition 3.6. Let h be the maximum of D, and suppose $f(d) = h$. If $d < h$, then $h = f(d) < f(h) \in D$, contradiction, showing that $f(h) = h$. If f is not the identity, then the set Q of all $d \in D$ for which $f(d) \neq d$ is non-empty, whence has a maximum, say, $u < h$. In particular, if $v := \sigma_D(u)$ is its immediate successor, then $f(u) < f(v) = v$, since $v \notin Q$, whence $f(u) < u$, since $u \in Q$. Since $u = f(a)$ for some $a \neq u$, then either $a < u$ or $v \leq a$, and hence $u = f(a) < f(u) < u$ or $v = f(v) \leq f(a) = u$, a contradiction either way. ⊣

3.9. *Remark.* Note that the map sending h to the minimum of D, and equal to σ_D otherwise is a definable permutation of D, but it obviously fails to be monotone. The map $x \mapsto \omega_\natural - x$ on $D = (\mathbb{N}_\natural)_{\leq \omega_\natural}$ as in Example 2.3 is a strictly decreasing involution. It is not hard to see that if an involution exists, it must be unique: indeed, if f and g are both decreasing, let a be the maximal element at which they disagree (it cannot be h since $f(h) = l = g(h)$), and assume $f(a) < g(a)$. Since $f(\sigma(a)) = g(\sigma(a)) < f(a) < g(a)$, it is now easy to see that $f(a)$ does not lie in the image of g, contradicting that g must be a bijection by (DPP).

3.10. PROPOSITION. *If* $\mathcal{M}$ *expands an ordered field, then there exists for every definable subset* $Y \subseteq M$*, two definable, discrete subsets* $D, E \subseteq Y$ *such that* $[Y] = [D] - [E]$ *in* $\mathbf{Gr}(\mathcal{M})$.

PROOF. Since the boundary ∂Y is discrete, we may remove it and assume Y is open, whence a disjoint union of open intervals by [7, Theorem 2.9]. Let us introduce some notation that will be useful later too, assuming Y is open. For $y \in Y$, let $l(y)$ and $h(y)$ be respectively the maximum of $(\partial Y)_{<y}$ and the minimum of $(\partial Y)_{>y}$ (allowing

$\pm\infty$). Hence $]l(y), h(y)[$ is the maximal interval in Y containing y, and we denote its barycenter by $m(y)$, where, in general, we define the *barycenter* of an interval $]a, b[$ as the midpoint $(a+b)/2$ if a and b are finite, or the point $a+1$ (respectively, $b-1$, or 0) if a (respectively, b, or both) is infinite. Let $L(Y)$, $M(Y)$ and $R(Y)$ consist respectively of all y less than, equal to, or greater than $m(y)$. Removing a maximal unbounded interval from Y if necessary (whose class is equal to -1 as already observed above), we may assume Y is bounded, so that $l(y)$ and $h(y)$ are always finite. Since the maps $f_Y : L(Y) \to Y : y \mapsto 2y - l(y)$ and $g_Y : L(Y) \to R(Y) : y \mapsto y + m(y)$ are bijections, $[Y] = [L(Y)] = [R(Y)]$. Since the scissor relations yield $[Y] = [L(Y)] + [M(Y)] + [R(Y)]$, we get $[Y] = -[M(Y)]$. By construction $M(Y)$ is discrete, and so we are done. $\dashv$

The proof gives the following more general result: given any discrete subset $D_0 \subseteq Y$, we can find disjoint discrete subsets $D, E \subseteq Y$ such that $D_0 \subseteq D$ and $[Y] = [D] - [E]$. Indeed, let $D := D_0 \cup (Y \cap \partial Y)$ and $E := M(Y \setminus D)$. If $\mathcal{M}$ merely expands an ordered group, then we have to also include the class $\mathbb{h}$, that is to say, in that case we can write $[Y] = e\mathbb{h} + [D] - [E]$, where $e \in \{0, 1, 2\}$ is the number of unbounded sides of Y. For higher arities, we need to make a tameness assumption:

3.11. COROLLARY. *Let X be a definable subset in an o-minimalistic expansion $\mathcal{M}$ of an ordered field. If ∂X is tame, then there exist definable, discrete subsets $D, E \subseteq X$ such that $[X] = [D] - [E]$ in $\mathbf{Gr}(\mathcal{M})$. In fact, the class of any tame subset in $\mathbf{Gr}(\mathcal{M})$ is of the form $[D] - [E]$, for some definable discrete subsets $D, E \subseteq M$.*

PROOF. We give again the proof only for X planar. There is nothing to show if X is discrete. Assume next that it has dimension one. Let $V := \mathrm{Vert}(X)$ be the vertical component of X. Since $\pi(V)$ is discrete, as we argued before, we can carry out the argument in the proof of Proposition 3.10 on each fiber separately to write $[V]$ as the difference of two discrete classes (we leave the details to the reader, but compare with the two-dimensional case below). Removing V from X, we may assume X has no vertical components. In particular, the set $N := \mathrm{Node}(X)$ of nodes of X is discrete by [7, Proposition 7.7]. Removing it, we may assume X has no nodes, so that every point lies on a unique optimal quasi-cell by [7, Corollary 7.16]. However, by assumption, X is tame, and hence there exists a cellular map $c \colon X \to D$. Given $x \in X$, let I_x be the domain $\pi(c^{-1}(c(x)))$ of the unique cell $c^{-1}(c(x))$ containing x. Let $L(X)$, $M(X)$, and $R(X)$ consist respectively of all $x \in X$ such that $\pi(x)$ lies in $L(I_x)$, $M(I_x)$, and $R(I_x)$ respectively (in the notation of the proof of Proposition 3.10). Define $f_X : L(X) \to X$ and $g_X : L(X) \to R(X)$ by sending x to the unique point on $c^{-1}(c(x))$ lying above respectively $f_{I_x}(\pi(x))$ and $g_{I_x}(\pi(x))$, showing that X, $L(X)$, and $R(X)$ are definably isomorphic. Since $M(X)$ is discrete and $[X] = [L(X)] + [R(X)] + [M(X)]$, we are done in this case.

If X has dimension two, its boundary has dimension at most one, and so we have already dealt with it by the previous case. Upon removing it, we may assume X is open. This time, we let $L(X)$, $M(X)$ and $R(X)$ be the union of respectively all $L(X[a])$, $M(X[a])$, and $R(X[a])$, for all $a \in \pi(X)$. The maps $(a, b) \mapsto f_{X[a]}(b)$ and $(a, b) \mapsto g_{X[a]}(b)$ put $L(X)$ in definable bijection with respectively X and $R(X)$ (with an obvious adjustment left to the reader if the fiber $X[a]$ is unbounded), and hence $[X] = -[M(X)]$. Since $M(X)$ has dimension at most one by [7, Proposition 5.1], we are done by induction. Without providing the details, we can extend this argument to higher dimensions, proving the last claim, where we also must use the fact proven below in Lemma 3.15 that definable discrete subsets are univalent in an ordered field. ⊣

3.12. *Remark.* We actually proved that if $c\colon X \to D$ is a cellular surjective map, then

$$[X] = \sum_{e=0}^{d} (-1)^e [D_e] \tag{1}$$

where $D_e = c(X^{(e)})$ consist of all $a \in D$ with e-dimensional fiber $c^{-1}(a)$, and where d is the dimension of X. We may reduce to the case that all fibers have the same dimension, and the assertion is then clear in the one-dimensional case, since the restriction of c to $M(X)$ is a bijection. Repeating the argument therefore to X, we get $[X] = -[M(X)] = [M(M(X))]$, and now $M(M(X))$ is definably isomorphic with D via c. Higher dimensions follow similarly by induction.

In particular, if $\mathcal{M}$ is a tame expansion of an ordered field, then its Grothendieck ring is generated by the definable discrete subsets of M, and the canonical homomorphism $\mathbf{Gr}_0(\mathcal{M}) \to \mathbf{Gr}(\mathcal{M})$ is surjective. Inspecting the above proof, we see that all isomorphisms involved are in fact homeomorphisms, and so the result also holds in the strict Grothendieck ring $\mathbf{Gr}^{\mathrm{s}}(\mathcal{M})$. Since any function with discrete domain is continuous, we showed:

3.13. COROLLARY. *For a tame, o-minimalistic expansion of an ordered field, its Grothendieck ring and its strict Grothendieck ring coincide.* ⊣

3.14. The partial order on $\mathfrak{D}(\mathcal{M})$. Let $\mathfrak{D}(\mathcal{M})$ denote the collection of isomorphism classes of definable, discrete subsets in an o-minimalistic structure $\mathcal{M}$. Recall that $X \preceq Y$ if there exists a definable injection $X \to Y$. We call a definable subset X *univalent*, if $X \preceq M$. By Theorem 3.3, every definable curve is univalent in an o-minimal structure. In this section, we study $\preceq$ on definable, discrete subsets.

3.15. LEMMA. *In an expansion of an ordered field, every definable, discrete subset is univalent.*

PROOF. By induction, it suffices to show that if $D \subseteq M^{n+1}$ is discrete and definable, then there is a definable, injective map $g\colon D \to M^n$. The set of lines connecting two points of D is again a discrete set (in the corresponding projective space) and hence we can find a hyperplane which is non-orthogonal to any of these lines. But then the restriction to D of the projection onto this hyperplane is injective. ⊣

Assume D and E are discrete, definable subsets with $D \preceq E$ and $E \preceq D$. Hence there are definable injections $D \to E$ and $E \to D$. By Proposition 3.6, both compositions are bijections, showing that D and E are definably isomorphic. Since transitivity is trivial, we showed that we get a partial order on $\mathfrak{D}(\mathcal{M})$. To obtain a partial order on the zero-dimensional Grothendieck ring $\mathbf{Gr}_0(\mathcal{M})$, we define $[D] \preceq [E]$, if there exists a definable, discrete subset A such that $D \sqcup A \preceq E \sqcup A$. To show that this well-defined, assume $[D] = [D']$ and $[E] = [E']$. By Lemma 3.1, there exist definable, discrete subsets F and G such that $D \sqcup F \cong D' \sqcup F$ and $E \sqcup G \cong E' \sqcup G$. Therefore,

$$D' \sqcup F \sqcup G \sqcup A \cong D \sqcup F \sqcup G \sqcup A \preceq E \sqcup F \sqcup G \sqcup A \cong E \sqcup F \sqcup G \sqcup A$$

since $D \sqcup A \preceq E \sqcup A$. We then extend this to a partial ordering on $\mathbf{Gr}_0(\mathcal{M})$ by linearity. In the o-minimal case, $\mathbf{Gr}_0(\mathcal{M})$ is just $\mathbb{Z}$ in its natural ordering.

In an expansion of an ordered group, let us call a definable, discrete set D *equidistant*, if the map $a \mapsto \sigma_D(a) - a$ is constant on all non-maximal elements of D, where σ_D is the successor function.

3.16. PROPOSITION. *In an o-minimalistic expansion $\mathcal{M}$ of an ordered field, any two definable equidistant subsets of M are comparable.*

PROOF. Let $D, E \subseteq M$ be definable equidistant subsets. Since they are bounded by [7, Corollary 4.2], we may assume after a translation that both have minimum equal to 0, and then after taking a scaling, that the distance between consecutive points in both is 1. Let m be the maximum of all $a \in D \cap E$ for which $D_{\leq a} = E_{\leq a}$. If m is non-maximal in either set, then $m+1$ lies both in D and in E by assumption, contradiction. Hence m is the maximum, say, of D, and therefore $D \subseteq E$, whence $D \preceq E$. ⊣

More generally, given a definable, discrete subset $D \subseteq M$ in an o-minimalistic expansion $\mathcal{M}$ of an ordered field, define the *derivative* D' of D as the set of all differences $\sigma_D(a) - a$, where a runs over all non-maximal elements of D. Hence an equidistant set is one whose derivative is a singleton. Since we have a surjective map $D \setminus \{\max D\} \to D' \colon a \mapsto \sigma_D(a) - a$, it follows from the next lemma that $D' \preceq D$.

3.17. LEMMA. *In an o-minimalistic structure $\mathcal{M}$, if $g \colon X \to M^k$ is a definable map, then $g(D) \preceq D$, for every discrete, definable subset $D \subseteq X$.*

PROOF. This follows by considering the injective map $g(D) \to D$ sending a to the minimum of $g^{-1}(a)$. ⊣

I do not expect $\preceq$ to be always total (although it can be made total by extending the class of isomorphisms as we shall see in Theorem 5.3 below). Since $D \preceq E$ implies $[D] \preceq [E]$, but not necessarily the converse, the former being total implies that the latter is too, but again, the converse is not clear. To construct potential counterexamples, let us introduce the following notation.

3.18. EXAMPLE (Discrete Overspill). Given a sequence $\mathbf{a} = (a_n)$ of real numbers, let $\mathbb{R}_\natural\langle\mathbf{a}\rangle$ be the ultraproduct of the $\mathcal{R}_n$, where each $\mathcal{R}_n$ is the expansion of the real field with a unary predicate $\mathtt{D}$ interpreting the first n elements $a_1, \dots, a_n$ in the sequence. Since each $\mathcal{R}_n$ is o-minimal, $\mathbb{R}_\natural\langle\mathbf{a}\rangle$ is o-minimalistic. Moreover, $\mathbf{a}$ is the "finite" part of the set $D_{\mathbf{a}} := \mathtt{D}(\mathbb{R}_\natural\langle\mathbf{a}\rangle)$ defined by $\mathtt{D}$, that is to say,

$$D_{\mathbf{a}} \cap \mathbb{R} = \{a_1, a_2, \dots, \}.$$

so that we refer to $\mathbb{R}_\natural\langle\mathbf{a}\rangle$ as the structure obtained from $\mathbf{a}$ by *discrete overspill* (for a related construction, see also §7 below).

In this notation, Example 2.3 is the discrete overspill $\mathbb{R}_\natural\langle\mathbb{N}\rangle$ of $\mathbb{N}$ listed in its natural order. I do not know whether $\preceq$ is total on it. Any countable subset can be enumerated, including $\mathbb{Q}$, although this enumeration might not be order preserving. Nonetheless, we get a structure $\mathbb{R}_\natural\langle\mathbf{q}\rangle$ with $D_{\mathbf{q}} \cap \mathbb{R} = \mathbb{Q}$ (the non-standard elements of $D_{\mathbf{q}}$ form a proper subset of $\mathbb{Q}_\natural$ and are harder to describe as they depend on the choice of enumeration). We can repeat this construction with more than one sequence, taking one unary predicate for each. Any structure obtained by discrete overspill is tame by [7, Remark 9.16].

3.19. EXAMPLE. Now, if we take two unary predicates, representing, say, the sequence of prime numbers $\mathbf{p}$ and the sequence of powers of two $\mathbf{t}$, then in $\mathbb{R}_\natural\langle\mathbf{p},\mathbf{t}\rangle$, it seems very unlikely that the discrete sets $D_\mathbf{p}$ and $D_\mathbf{t}$ are comparable. For if they were, they would have to be definably isomorphic by Lemma 5.2 below, as they have the same ultra-Euler characteristic (equal to $\omega_\natural$, the ultraproduct of the diagonal sequence $(n)_n$). It is easy to combine these two unary sets into a single one, by letting $a_{2n} := p_n$ and $a_{2n-1} := -t_n$, so that then $D_\mathbf{a} \cap (\mathbb{R}_\natural)_{\leq 0} = D_\mathbf{t}$ and $D_\mathbf{a} \cap (\mathbb{R}_\natural)_{\geq 0} = D_\mathbf{p}$, giving an example of a single discrete overspill $\mathbb{R}_\natural\langle\mathbf{a}\rangle$ in which $\preceq$ is most likely not total.

3.20. THEOREM (Euler O-minimality Criterion). *A necessary and sufficient condition for an ultra-o-minimal structure $\mathcal{M}_\natural$, given as the ultraproduct of o-minimal structures $\mathcal{M}_i$, to be o-minimal is that, for each formula φ without parameters, there exists an $N_\varphi \in \mathbb{N}$ such that $|\chi_{\mathcal{M}_i}(\varphi)| \leq N_\varphi$, for almost all i.*

PROOF. If $\mathcal{M}_\natural$ is o-minimal, then $\varphi(\mathcal{M}_\natural)$ is a disjoint union of N cells, whence by Łoś' Theorem, so are almost all $\varphi(\mathcal{M}_i)$. Since a cell has Euler characteristic ± 1, additivity yields $|\varphi(\mathcal{M}_i)| \leq N$, for almost all i. Conversely, let $Y_\natural \subseteq M_\natural$ be definable, say, given as the fiber of a $\emptyset$-definable subset $X_\natural \subseteq M_\natural^{1+n}$ over a tuple $\mathbf{b}_\natural$. Let $X_i \subseteq M_i^{1+n}$ be the corresponding $\emptyset$-definable subset, and choose $\mathbf{b}_i$ in M_i with ultraproduct $\mathbf{b}_\natural$, so that $Y_\natural$ is the ultraproduct of the $Y_i := X_i[\mathbf{b}_i]$. By the proof of [7, Theorem 8.3] (which in the o-minimal case does yield a finite cell decomposition), we can decompose each X_i as a disjoint union of $\emptyset$-definable subsets $X_i^{(e)}$ consisting of the union of all e-cells in a cell decomposition of X_i. In fact, this proof can be carried out in the theory DCTC, so that it holds uniformly in any DCTC $\models \mathcal{M}$. For instance, if $X = \varphi(\mathcal{M})$ is planar, then $X^{(2)}$ consists exactly of all interior points that do not lie on a vertical fiber containing some node of ∂X, whereas $X^{(0)}$ consists of all nodes of ∂X that belong to X, and $X^{(1)}$ of all remaining points. Let $\varphi^{(e)}$ define in each model $\mathcal{M} \models$ DCTC the set $X^{(e)}$, for $e \leq n+1$. Since each $X_i^{(e)}$ is a disjoint union of e-cells, its Euler characteristic is equal to $(-1)^e N_{i,e}$, where $N_{i,e}$ is the number of e-cells in the decomposition. By assumption (applied to the formula $\varphi^{(e)}$), this Euler characteristic is bounded in absolute value, whence so are the $N_{i,e}$, that is to say, there exist $N_e \in \mathbb{N}$ such that $N_{i,e} < N_e$ for all i. But then the fiber $X_i^{(e)}[\mathbf{b}_i]$ admits a decomposition in at most N_e cells. Since the union of the latter for all e is just Y_i, we showed that there is a uniform bound on the number of cells (whence intervals) in a decomposition of Y_i. Since this is now first-order expressible, $Y_\natural$ too is a finite union of intervals. $\dashv$

§**4. Expansions of o-minimalistic structures.** In this section, $\mathcal{M}$ will always denote an o-minimalistic structure. Since an expansion by definable sets does not alter the collection of definable sets, we immediately have:

4.1. LEMMA. *If X is definable, then $(\mathcal{M}, X)$ is again o-minimalistic.* $\dashv$

So we ask in more generality, what properties does a subset of an o-minimalistic structure need to have in order for the expansion to be again o-minimalistic? Let us call such a subset *o-minimalistic* (or, more correctly, *$\mathcal{M}$-o-minimalistic* as this depends on the surrounding structure), where we just proved that definable subsets are.

4.2. COROLLARY. *The image of an o-minimalistic subset under a definable map is again o-minimalistic, and so is its complement, its closure, its boundary, and its interior. More generally, any set definable from an o-minimalistic set is again o-minimalistic.*

PROOF. It suffices to prove the last assertion. Let X be o-minimalistic. Since $(\mathcal{M}, X)$ is o-minimalistic, any set definable in $(\mathcal{M}, X)$ is o-minimalistic (in the expansion, whence also in the reduct) by Lemma 4.1. $\dashv$

To define a weaker isomorphism relation, we introduce the following notation. Let X be a definable subset in a structure $\mathcal{N}$, say, defined by the formula (with parameters) φ, that is to say, $X = \varphi(\mathcal{N})$. If $\mathcal{N}'$ is an elementary extension of $\mathcal{N}$, then we set $X^{\mathcal{N}'} := \varphi(\mathcal{N}')$, and call it the *definitional extension* of X in $\mathcal{N}'$.

Let us call two $\mathcal{M}$-definable subsets *o-minimalistically isomorphic*, denoted $X \equiv Y$, if their definitional extensions have the same ultra-Euler measure in every ultra-o-minimal elementary extension $\mathcal{M} \preceq \mathcal{N}$, that is to say, if $\mu_{\mathcal{N}}(X^{\mathcal{N}}) = \mu_{\mathcal{N}}(Y^{\mathcal{N}})$. It is easy to see that this constitutes an equivalence relation on definable subsets.

4.3. PROPOSITION. *In an o-minimalistic expansion $\mathcal{M}$ of an ordered field, if two $\mathcal{M}$-definable subsets X and Y are o-minimalistically isomorphic, then there exists an o-minimalistic expansion of $\mathcal{M}$ in which they become definably isomorphic.*

PROOF. Suppose X and Y are o-minimalistically definable, and let $\mathcal{N}$ be some ultra-o-minimal elementary extension of $\mathcal{M}$, given as the ultraproduct of o-minimal structures $\mathcal{N}_i$. Let X_i and Y_i be $\mathcal{N}_i$-definable subsets with respective ultraproducts $X^{\mathcal{N}}$ and $Y^{\mathcal{N}}$. Since by Proposition 2.4 dimension is definable, $X^{\mathcal{N}}$ and $Y^{\mathcal{N}}$ have the same dimension, whence so do almost each X_i and Y_i by Łoś' Theorem. By assumption, they have also the same Euler characteristic for almost all i, so that they are definably isomorphic by Theorem 3.3. Hence, there exists for almost all i, a definable isomorphism $f_i\colon X_i \to Y_i$. Let $\Gamma_\natural$ be the ultraproduct of the graphs $\Gamma(f_i)$. Since almost all $(\mathcal{N}_i, \Gamma(f_i))$ are o-minimal, their ultraproduct $(\mathcal{N}, \Gamma_\natural)$ is o-minimalistic, whence so is $(\mathcal{M}, \Gamma)$, where Γ is the restriction of $\Gamma_\natural$ to $\mathcal{M}$. Moreover, by Łoś' Theorem, $\Gamma_\natural$ is the graph of a bijection $X^{\mathcal{N}} \to Y^{\mathcal{N}}$, and hence its restriction Γ is the graph of a bijection $X \to Y$, proving that X and Y are definably isomorphic in $(\mathcal{M}, \Gamma)$. $\dashv$

I do not know whether the converse is also true: if X and Y are definably isomorphic in some o-minimalistic expansion $\mathcal{M}'$, are they o-minimalistically isomorphic? They will have the same Euler characteristic in any (reduct of an) ultra-o-minimal elementary extension of $\mathcal{M}'$ by essentially the same argument, but what about ultra-o-minimal elementary extensions of $\mathcal{M}$ that are not such reducts? A related question is in case $\mathcal{M}$ itself is already ultra-o-minimal, if two sets have the same Euler characteristic, do their definitional extensions also have the same Euler characteristic in an ultra-o-minimal elementary extension? This would follow if Euler characteristic was definable, but at the moment, we can only prove a weaker version (see Theorem 6.7). Before we address these issues, we prove a result yielding non-trivial examples of o-minimalistically isomorphic sets that need not be definably isomorphic.

4.4. COROLLARY. *In an o-minimalistic expansion $\mathcal{M}$ of an ordered field, if two definable subsets X and Y have the same dimension and the same class in $\mathbf{Gr}(\mathcal{M})$, then they are o-minimalistically isomorphic.*

PROOF. By Lemma 3.1, there exists a definable subset Z such that $X \sqcup Z$ and $Y \sqcup Z$ are definably isomorphic. Let $\mathcal{M} \preceq \mathcal{N}$ be an ultra-o-minimal elementary extension.

Hence $X^{\mathcal{N}} \sqcup Z^{\mathcal{N}}$ and $Y^{\mathcal{N}} \sqcup Z^{\mathcal{N}}$ are definably isomorphic, and therefore

$$\chi_{\mathcal{N}}(X^{\mathcal{N}}) + \chi_{\mathcal{N}}(Z^{\mathcal{N}}) = \chi_{\mathcal{N}}(X^{\mathcal{N}} \sqcup Z^{\mathcal{N}}) = \chi_{\mathcal{N}}(Y^{\mathcal{N}} \sqcup Z^{\mathcal{N}}) = \chi_{\mathcal{N}}(Y^{\mathcal{N}}) + \chi_{\mathcal{N}}(Z^{\mathcal{N}})$$

showing that $X^{\mathcal{N}}$ and $Y^{\mathcal{N}}$ have the same ultra-Euler characteristic, as we needed to show. $\dashv$

4.5. Contexts and virtual isomorphisms. To overcome the difficulties alluded to above, we must make our definitions context-dependable in the following sense. Given an o-minimalistic structure $\mathcal{M}$, by a *context* for $\mathcal{M}$, we mean an ultra-o-minimal structure $\mathcal{N}$ that contains $\mathcal{M}$ as an elementary substructure (which always exists by Corollary 2.2). An expansion $\mathcal{M}'$ of $\mathcal{M}$ is then called *permissible* (with respect to the context $\mathcal{N}$), if $\mathcal{N}$ can be expanded to a context $\mathcal{N}'$, that is to say, $\mathcal{M}' \preceq \mathcal{N}'$ and $\mathcal{N}'$ is again ultra-o-minimal. If $\mathcal{M}$ itself is ultra-o-minimal, then we may take it as its own context, but even in this case, not every expansion will be permissible, as it may fail to be an ultraproduct.

From now on, we fix an o-minimalistic structure $\mathcal{M}$ and a context $\mathcal{N}$. We define a (context-dependable) *Euler characteristic* $\chi_{\mathcal{M}}(\cdot)$ (or, simply χ) by restricting the ultra-Euler characteristic of $\mathcal{N}$, that is to say, by setting $\chi(X) := \chi_{\mathcal{N}}(X^{\mathcal{N}})$, for any $\mathcal{M}$-definable subset X, and we define similarly its *Euler measure* $\mu(X) := (\dim(X), \chi(X))$. We say that two definable subsets are *virtually isomorphic*, if there exists a permissible expansion of $\mathcal{M}$ in which they become definably isomorphic. In particular, two definable subsets that are o-minimalistically isomorphic are also virtually isomorphic, but the converse is unclear. We can now prove an o-minimalistic analogue of Theorem 3.3.

4.6. THEOREM. *In an o-minimalistic expansion $\mathcal{M}$ of an ordered field, two definable subsets are virtually isomorphic if and only if they have the same Euler measure.*

PROOF. One direction is proven in the same way as Proposition 4.3, so assume X and Y are virtually isomorphic definable subsets. By assumption, $\mathcal{M} \preceq \mathcal{N}$ expands into o-minimalistic structures $\mathcal{M}' \preceq \mathcal{N}'$, with $\mathcal{N}'$ again ultra-o-minimal, such that X and Y are $\mathcal{M}'$-definably isomorphic. Let $\mathcal{N}'$ be the ultraproduct of o-minimal structures $\mathcal{N}'_i$. Since $X^{\mathcal{N}'}$ and $Y^{\mathcal{N}'}$ are definably isomorphic, so are almost all X_i and Y_i, where X_i and Y_i are $\mathcal{N}'_i$-definable subsets with respective ultraproducts $X^{\mathcal{N}'}$ and $Y^{\mathcal{N}'}$. In particular, X_i and Y_i have the same Euler measure for almost all i, by Theorem 3.3. Hence $X^{\mathcal{N}'}$ and $Y^{\mathcal{N}'}$ have the same ultra-Euler measure, by Proposition 2.4. Since both invariants remain the same in the reduct $\mathcal{N}$, elementarity then yields $\mu(X) = \mu(Y)$. $\dashv$

4.7. O-finitism. As we already mentioned in the introduction, in the o-minimalistic context, discrete sets play the role of finite sets, and so we briefly discuss the first-order aspects of this assertion. Given a (non-empty) collection of L-structures $\mathfrak{K}$, and a subset $X \subseteq N^p$ in some L-structure $\mathcal{N}$, we say that X is *$\mathfrak{K}$-finitistic*, if $(\mathcal{N}, X)$ satisfies every $L(\mathtt{U})$-sentence σ which holds in every expansion $(\mathcal{K}, F)$ of a structure $\mathcal{K} \in \mathfrak{K}$ by a finite set $F \subseteq K^p$. In case $\mathfrak{K}$ is the collection of o-minimal structures, we call X *o-finitistic*. Applying the definition just to L-sentences σ (not containing the predicate $\mathtt{U}$, so that $(\mathcal{K}, F) \models \sigma$ if and only if $\mathcal{K} \models \sigma$), we see that $\mathcal{N}$ is then necessarily o-minimalistic. Put differently, an o-finitistic set in an o-minimalistic structure is a model of *o-finitism*, that is to say, of the theory of a finite set in an o-minimal structure. By Proposition 3.6 and [7, Propositions 2.6 and 7.19], we have:

4.8. COROLLARY. *An o-finitistic set is discrete, closed, bounded, and locally definable, every non-empty intersection with an open interval has a maximum and a minimum, and every injective, definable self-map on it is an isomorphism.* ⊣

It seems unlikely that these properties characterize fully o-finitism. A complete axiomatization of o-finitism would be of interest in view of the following results.

4.9. THEOREM. *A subset $X \subseteq M^p$ is o-finitistic if and only if it is discrete and o-minimalistic. In particular, any definable, discrete subset in an o-minimalistic structure is o-finitistic.*

PROOF. Assume first that X is o-finitistic, whence discrete by Corollary 4.8. We have to show that given an $L(\mathtt{U})$-sentence σ holding true in every o-minimal $L(\mathtt{U})$-structure, then $(\mathcal{M}, X) \models \sigma$. Let $\mathcal{K}$ be an o-minimal structure and let $F \subseteq K^p$ be a finite subset. Hence $(\mathcal{K}, F)$ is also o-minimal and therefore satisfies σ. Since this holds for all such expansions, σ is true in $(\mathcal{M}, X)$ by o-finitism, as we needed to show.

Conversely, suppose $X \subseteq M^p$ is discrete and o-minimalistic, that is to say, $(\mathcal{M}, X)$ is o-minimalistic. To show that X is o-finitistic, let σ be a sentence true in every expansion $(\mathcal{K}, F)$ of an o-minimal structure $\mathcal{K}$ by a finite subset $F \subseteq K^p$. Consider the disjunction σ' of σ with the sentence expressing that the set defined by $\mathtt{U}$ is not discrete. Hence σ' is true in any o-minimal expansion $(\mathcal{K}, Y)$. Since X is o-minimalistic, this means that $(\mathcal{M}, X) \models \sigma'$, and since X is discrete, this in turn implies that σ is true in $(\mathcal{M}, X)$, as we needed to show. The last assertion then follows from Lemma 4.1. ⊣

Let us call a subset of an ultra-o-minimal structure *ultra-finite*, if it is the ultraproduct of finite subsets (such a set may fail to be definable, since the definition in each component may not be uniform).

4.10. THEOREM. *A subset $X \subseteq M^p$ is o-finitistic if and only if there exists an elementary extension $\mathcal{M} \preceq \mathcal{N}$ with $\mathcal{N}$ ultra-o-minimal and an ultra-finite subset $Y \subseteq N^p$, such that $X = Y \cap M^p$.*

PROOF. Suppose $\mathcal{N}$ and Y have the stated properties, and let $\mathcal{N}_i$ be o-minimal structures and $Y_i \subseteq N_i^p$ finite subsets, so that $\mathcal{N}$ and Y are their respective ultraproducts. Since $(\mathcal{N}_i, Y_i)$ is again o-minimal, their ultraproduct $(\mathcal{N}, Y)$ is o-minimalistic. Since $(\mathcal{M}, X)$ is then an elementary substructure, the latter is also o-minimalistic. Moreover, since Y is discrete, so must X be, and hence X is o-finitistic by Theorem 4.9. Conversely, by the same theorem, if X is o-finitistic, then $(\mathcal{M}, X)$ is o-minimalistic. Hence there exists an elementary extension $(\mathcal{N}, Y)$ which is ultra-o-minimal as an $L(\mathtt{U})$-structure by Corollary 2.2. Write $(\mathcal{N}, Y)$ as an ultraproduct of o-minimal structures $(\mathcal{N}_i, Y_i)$. Since X is discrete, so must Y be by elementarity, whence so are almost all Y_i by Łoś' Theorem. The latter means that almost all are in fact finite, showing that Y is ultra-finite, and the assertion follows since $X = Y \cap M^p$. ⊣

Next, we give a criterion for a subset $Y \subseteq M$ to be o-minimalistic. By [7, Theorem 2.9], its boundary ∂Y should be discrete, and $Y^\circ = Y \setminus \partial Y$ should be a disjoint union of open intervals. Given an arbitrary set $Y \subseteq M$, define its *enhanced boundary* ΔY as the set consisting of the pairs (y, ε) with $y \in \partial Y$ and ε equal to 0, 1, or -1, depending on whether respectively y, y^+, and/or y^- belongs to Y. Recall that a^+ (respectively, a^-) *belongs to* Y if there exists an open interval with left (respectively, right) endpoint a contained in Y (type completeness is then the assertion that a^+ belongs either to Y or to its complement). No fiber of an enhanced boundary can have more than two points and its projection is the ordinary boundary ∂Y. If Y is o-minimalistic, then ΔY must satisfy some extra conditions: it must be bounded, discrete and closed, and, by type completeness, if $(y, 1)$ belongs to it, then so must $(y', -1)$, where y' is the immediate successor of y in ∂Y.

4.11. THEOREM. *A subset $Y \subseteq M$ is o-minimalistic if and only if its enhanced boundary ΔY is o-finitistic and its interior is a disjoint union of open intervals.*

PROOF. Suppose Y is o-minimalistic, so that Y° is a disjoint union of open intervals. Since ΔY is definable from Y, it too is o-minimalistic by Corollary 4.2, whence o-finitistic by Theorem 4.9. To prove the converse, let $D := \partial Y = \pi(\Delta Y)$, a bounded, closed, discrete set, and let l be its minimum. Define $X \subseteq M$ as the set of all $x \in M$ such that one of the following three conditions holds

4.11.i. $(x, 0) \in \Delta Y$;
4.11.ii. $x > l$ and $(d, 1) \in \Delta Y$, where $d = \max D_{<x}$;
4.11.iii. $x < l$ and $(l, -1) \in \Delta Y$.

Since X is definable from ΔY, it is o-minimalistic by Corollary 4.2. Remains to show that $X = Y$. It follows from (4.11.i) that $X \cap D = Y \cap D$, so that it suffices to show that $X^\circ = Y^\circ$. Therefore, we may as well assume from the start that Y is open. Write $Y = \sqcup_n I_n$ as a disjoint union of open intervals, and let $]a, b[$ one of the I_n (we leave the unbounded case to the reader, for which one needs (4.11.iii)). In particular, $a \in D$ and a^+ belongs to Y, so that $(a, 1) \in \Delta Y$. By (4.11.ii), the entire interval $]a, b[$ lies in X, whence so does the whole of Y. Conversely, if $x \in X$, let $d := \max D_{<x}$, so that $(d, 1) \in \Delta Y$. Hence d^+ belongs to Y, and so d must be an endpoint of one of the I_n. The other endpoint must be bigger than d, and hence bigger than x, showing that $x \in I_n \subseteq Y$. ⊣

§**5. The virtual Grothendieck ring.** We fix again an o-minimalistic structure $\mathcal{M}$ and a context $\mathcal{N}$. We can use virtual isomorphisms instead of definable isomorphisms in the definition of the zero-dimensional or the full Grothendieck ring, that is to say, the quotient modulo the scissor relations of the free Abelian group on virtual isomorphism classes of respectively all discrete, definable subsets, and of all definable subsets yield the *virtual Grothendieck rings* $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$ and $\mathbf{Gr}^{\text{virt}}(\mathcal{M})$ respectively. We have surjective homomorphisms $\mathbf{Gr}_0(\mathcal{M}) \to \mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$ and $\mathbf{Gr}(\mathcal{M}) \to \mathbf{Gr}^{\text{virt}}(\mathcal{M})$.

5.1. COROLLARY. *Given an o-minimalistic expansion $\mathcal{M}$ of an ordered field, there exist embeddings $\mathbf{Gr}_0^{virt}(\mathcal{M}) \subseteq \mathbf{Gr}^{virt}(\mathcal{M}) \hookrightarrow \mathbb{Z}_\natural$, where $\mathbb{Z}_\natural$ is the ring of non-standard integers in the given context.*

PROOF. Since the Euler characteristic vanishes on any scissor relation, it induces by Theorem 4.6 a homomorphism $\chi\colon \mathbf{Gr}^{\text{virt}}(\mathcal{M}) \to \mathbb{Z}_\natural$. By the same result, its restriction to $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$ is injective. To see that χ is everywhere injective, assume $\chi(X) = \chi(Y)$ for some definable subsets X and Y. If they have the same dimension, then they are virtually isomorphic, again by Theorem 4.6. So assume X has dimension $d \geq 1$ and Y has lesser dimension. Let U be the difference of a d-dimensional box minus a $(d-1)$-dimensional sub-box, so that in particular $[U]$ vanishes, whence also $\chi(U)$. As X and $Y \sqcup U$ now have the same Euler measure, they are virtually isomorphic by Theorem 4.6, and hence $[X] = [Y] + [U] = [Y]$ in $\mathbf{Gr}^{\text{virt}}(\mathcal{M})$, as we needed to show. The injectivity of $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M}) \to \mathbf{Gr}^{\text{virt}}(\mathcal{M})$ is then also clear. ⊣

In particular, if $\mathcal{M}$ is moreover tame, then we have an equality of virtual Grothendieck rings $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M}) = \mathbf{Gr}^{\text{virt}}(\mathcal{M})$ by Corollary 3.11.

5.2. LEMMA. *If two discrete, $\mathcal{M}$-definable subsets with the same ultra-Euler characteristic are comparable, then they are definably isomorphic.*

PROOF. Suppose D and E are discrete, definable subsets with $D \preceq E$ and $\chi(D) = \chi(E)$. Upon replacing D by a definable copy, we may assume $D \subseteq E$. Taking ultra-Euler characteristics, we get $\chi(E \setminus D) = \chi(E) - \chi(D) = 0$. By Łoś' Theorem, the definitional expansion of $E \setminus D$ is empty, whence so is then $E \setminus D$ itself. ⊣

To obtain a 'virtual' generalization, we extend the partial order on $\mathfrak{D}(\mathcal{M})$ to a total order on $\mathfrak{D}^{\text{viso}}(\mathcal{M})$, the set of virtual isomorphism classes of definable, discrete subsets. First, given definable subsets X and Y, we say that $X \leq Y$, if $X \preceq_{\mathcal{M}'} Y$ in some permissible o-minimalistic expansion $\mathcal{M}'$ of $\mathcal{M}$. Clearly, if $X \preceq Y$, then $X \leq Y$. The following two results are the o-minimalistic analogues of Theorem 3.3.

5.3. THEOREM. *Given two $\mathcal{M}$-definable, discrete subsets F and G, we have $F \leq G$ if and only if $\chi(F) \leq \chi(G)$. In particular, $\leq$ is a total order on $\mathfrak{D}^{viso}(\mathcal{M})$.*

PROOF. Suppose first that $\chi(F) \leq \chi(G)$. Write $\mathcal{N}$ as the ultraproduct of o-minimal structures $\mathcal{N}_i$, and let F_i and G_i be finite sets with respective ultraproducts the definitional extensions $F^{\mathcal{N}}$ and $G^{\mathcal{N}}$ of F and G respectively. Since $\chi_{\mathcal{N}}(F^{\mathcal{N}}) \leq \chi_{\mathcal{N}}(G^{\mathcal{N}})$, the cardinality of F_i is at most that of G_i, for almost all i. In particular, there exists an injective map $F_i \to G_i$ for almost all i. Let $\Gamma_\natural$ be the ultraproduct of the graphs of these maps $F_i \to G_i$. Hence $\Gamma_\natural$ is ultra-finite and therefore its restriction Γ to $\mathcal{M}$ is o-finitistic by Theorem 4.10, whence o-minimalistic by Theorem 4.9. By Łoś' Theorem and elementarity, Γ is the graph of an injective map $F \to G$, showing that $F \preceq_{(\mathcal{M},\Gamma)} G$. Since $(\mathcal{M},\Gamma)$ is permissible, $F \leq G$. The converse goes along the same lines: suppose $F \preceq_{\mathcal{M}'} G$, for some permissible o-minimalistic expansion $\mathcal{M}'$ of $\mathcal{M}$. By definition, there is an ultra-o-minimal expansion $\mathcal{N}'$ of $\mathcal{N}$ with $\mathcal{M}' \preceq \mathcal{N}'$. Since $F^{\mathcal{N}} \preceq_{\mathcal{N}'} G^{\mathcal{N}}$, we have $\chi(F) = \chi_{\mathcal{N}'}(F^{\mathcal{N}}) \leq \chi_{\mathcal{N}'}(G^{\mathcal{N}}) = \chi(G)$. ⊣

5.4. PROPOSITION. *If $\mathcal{M}$ expands an ordered field, then $X \leq Y$ if and only if $\dim(X) \leq \dim(Y)$, for X and Y definable subsets with $\dim(Y) > 0$.*

PROOF. The direct implication is clear. For the converse, by definability of dimension, we may pass to the context of $\mathcal{M}$ and therefore already assume $\mathcal{M}$ is ultra-o-minimal, given as the ultraproduct of o-minimal structures $\mathcal{M}_i$. Let X_i and Y_i be definable subsets in $\mathcal{M}_i$ with respective ultraproducts X and Y. By Łoś' Theorem, $\dim(X_i) \leq \dim(Y_i)$, and hence $X_i \preceq Y_i$, by Theorem 3.3, for almost all i. Let $f_i \colon X_i \to Y_i$ be a definable injection and let $\Gamma_\natural$ be the ultraproduct of the graphs $\Gamma(f_i)$. Since each $(\mathcal{M}_i, \Gamma(f_i))$ is again o-minimal, $(\mathcal{M}, \Gamma_\natural)$ is ultra-o-minimal and hence in particular a permissible expansion. Since $\Gamma_\natural$ is the graph of an injective map by Łoś' Theorem, $X \preceq_{(\mathcal{M},\Gamma_\natural)} Y$, as we needed to show. ⊣

In particular, any definable, discrete subset D is virtually univalent, meaning that $D \leq M$.

5.5. COROLLARY. *[Virtual Pigeonhole Principle] Two $\mathcal{M}$-definable, discrete subsets D and E are virtually isomorphic if and only if, for some definable subset X, the sets $D \sqcup X$ and $E \sqcup X$ are virtually isomorphic, if and only if $[D] = [E]$ in $\mathbf{Gr}^{virt}(\mathcal{M})$.*

PROOF. One direction in the first equivalence is immediate, so assume $D \sqcup X$ and $E \sqcup X$ are virtually isomorphic. Passing to a permissible o-minimalistic expansion, we may assume that they are already definably isomorphic, say, by an isomorphism $f\colon D \sqcup X \to E \sqcup X$. By totality (Theorem 5.3), we may assume that $E \leq D$, and hence after taking another permissible o-minimalistic expansion, and replacing E with an isomorphic image, we may even assume that $E \subseteq D$. Therefore, the composition of f and the inclusion $E \sqcup X \subseteq D \sqcup X$ is a map with co-discrete image, and hence is surjective by (DPP). However, this can only be the case if $E = D$, as we needed to show. The last equivalence is now just Lemma 3.1. ⊣

5.6. COROLLARY. *The zero-dimensional, virtual Grothendieck ring $\mathbf{Gr}_0^{virt}(\mathcal{M})$ is an ordered ring with respect to $\leq$.*

PROOF. Every element in $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$ is of the form $[A] - [B]$, for some definable, discrete subsets A and B in the o-minimalistic structure $\mathcal{M}$. Therefore, for definable, discrete subsets A_i and B_i, with $i = 1, 2$, we set $[A_1] - [B_1] \leq [A_2] - [B_2]$ if and only if

$$A_1 \sqcup B_2 \leq A_2 \sqcup B_1. \tag{2}$$

To see that this is well-defined, suppose $[A_i] - [B_i] = [A'_i] - [B'_i]$, for $i = 1, 2$ and definable, discrete subsets A'_i and B'_i. Therefore, $[A_i \sqcup B'_i] = [A'_i \sqcup B_i]$, whence $A_i \sqcup B'_i$ and $A'_i \sqcup B_i$ are virtually isomorphic by Corollary 5.5. We have to show that assuming (2), the same inequality holds for the accented sets. Taking the disjoint union with $B'_1 \sqcup B'_2$ on both sides of (2), yields inequalities

$$\begin{aligned}(A_1 \sqcup B'_1) \sqcup B_2 \sqcup B'_2 &\leq (A_2 \sqcup B'_2) \sqcup B_1 \sqcup B'_1 \\ (A'_1 \sqcup B_1) \sqcup B_2 \sqcup B'_2 &\leq (A'_2 \sqcup B_2) \sqcup B_1 \sqcup B'_1 \\ (A'_1 \sqcup B'_2) \sqcup (B_1 \sqcup B_2) &\leq (A'_2 \sqcup B'_1) \sqcup (B_1 \sqcup B_2)\end{aligned}$$

which by another application of Corollary 5.5 then gives $A'_1 \sqcup B'_2 \leq A'_2 \sqcup B'_1$, as we needed to show. It is now easy to check that $\leq$ makes $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$ into a totally ordered ring. ⊣

5.7. COROLLARY. *Every o-finitistic subset defines a cut in $\mathfrak{D}^{viso}(\mathcal{M})$. In particular, we can put a total pre-order on the collection of o-finitistic subsets.*

PROOF. Let F be an o-finitistic subset of an o-minimalistic structure $\mathcal{M}$ and let $D \in \mathfrak{D}^{\text{viso}}(\mathcal{M})$ be arbitrary. Since $(\mathcal{M}, F)$ is o-minimalistic by Theorem 4.9, we can compare D and F in $\mathfrak{D}(\mathcal{M}, F)$ by Theorem 5.3. If G is another o-finitistic subset, then we set $F \leq G$ if and only if the lower cut in $\mathfrak{D}(\mathcal{M})$ determined by F is contained in the lower cut of G. ⊣

A note of caution: even if $F \leq G$ and $G \leq F$, for F and G o-finitistic subsets, they need not be virtually isomorphic. For instance, taking D as in Example 2.3, it is an o-finitistic subset of $\mathbb{R}_\natural$, and since $\mathfrak{D}(\mathbb{R}_\natural)$ is just $\mathbb{N}$ by o-minimality, its cut is ∞. However, $D \setminus \{\omega_\natural\}$ determines the same cut, whence $D \leq D \setminus \{\omega_\natural\} \leq D$, but we know that they cannot be definably isomorphic in any o-minimalistic expansion by

(DPP). In fact, it is not clear whether two given o-finitistic subsets live in a common o-minimalistic expansion, and therefore can be compared directly. This is also why we cannot (yet?) define a Grothendieck ring on o-finitistic subsets.

§**6. Discretely valued Euler characteristics.** In order to calculate the zero-dimensional virtual Grothendieck ring, we introduce a new type of Euler characteristic. Fix an o-minimalistic structure $\mathcal{M}$ and a context $\mathcal{N}$, and let D be a definable, discrete subset. In this section, we will always view D in its lexicographical order $\leq_{\text{lex}}$ (or, when there is no risk for confusion, simply denoted $\leq$).

6.1. COROLLARY. *Any definable subset of an $\mathcal{M}$-definable, discrete subset D is virtually isomorphic to an initial segment $D_{\leq a}$.*

PROOF. The set of initial segments is a maximal chain in $\mathfrak{D}^{\text{viso}}(\mathcal{M})$, since any two consecutive subsets in this chain differ by a single point. Hence, any definable subset $E \subseteq D$ must be a member of this chain up to virtual isomorphism. ⊣

Clearly, such an a must be unique, and so, given a non-empty definable subset $E \subseteq D$, we let $\chi_D(E)$ be the unique a such that E is virtually isomorphic with $D_{\leq a}$. We add a new symbol $\varnothing$ to D and set $\chi_D(\emptyset) := \varnothing$. For definable subsets $E_1, E_2 \subseteq D$, we have $E_1 \leq E_2$ if and only if $\chi_D(E_1) \leq \chi_D(E_2)$. Given a definable map g with domain D, we can define by Lemma 3.17 its *rank* as $\mathrm{rk}(g) := \chi_D(g(D))$. A map is constant if and only if its rank is minimal (that is to say, equal to the minimum of its domain). By (DPP), we immediately have:

6.2. COROLLARY. *An $\mathcal{M}$-definable map with discrete domain is injective if and only if its rank is maximal (that is to say, equal to the maximum of its domain).* ⊣

Assume now that $\mathcal{M}$ is an o-minimalistic expansion of an ordered field, so that in particular all definable discrete subsets are univalent (see Lemma 3.15). Let $D \subseteq M$ be definable and discrete, with minimal element l and maximal element h. For each n, we view the Cartesian power D^n as a definable subset of D^{n+1} via the map $\mathbf{a} \mapsto (l, \mathbf{a})$. We also need to take into consideration the empty set, and so we define $\varnothing$ to be lower than any element in any D^n, and we let D^∞ be the direct limit of the ordered sets $D^n \cup \{\varnothing\}$. Under this identification, the elements of $D^n \cup \{\varnothing\}$ form an initial segment in $D^{n+1} \cup \{\varnothing\}$ with respect to the lexicographical ordering. In particular, if $E \subseteq D^n$ is a non-empty definable subset, then $\chi_{D^n}(E) = \chi_{D^{n+1}}(E')$, where E' is the image of E in D^{n+1}. After identification therefore, we will view $\chi_{D^n}(E)$ simply as an element of D^∞, and we just denote it $\chi_D(E)$. More generally, given an arbitrary definable subset $X \subseteq M^n$, we define its *D-valued Euler characteristic* (or, simply *Euler characteristic*) $\chi_D(X) := \chi_{D^n}(X \cap D^n)$.

We define an addition and a multiplication on D^∞ as follows. First, let us define the *disjoint union* $A \sqcup B$ of two definable subsets $A, B \subseteq M^n$ as the definable subset in M^{n+1} consisting of all (a, l) and (b, h) with $a \in A$ and $b \in B$. For $a \in D^\infty$, we set $a \oplus \varnothing = \varnothing \oplus a = a$ and $a \otimes \varnothing = \varnothing \otimes a = \varnothing$. For the general case, assume $a, b \in D^n$, and let $a \oplus b$ be the Euler characteristic of the disjoint union $(D^n)_{\leq a} \sqcup (D^n)_{\leq b} \subseteq D^{n+1}$, and let $a \otimes b$ be the Euler characteristic of the Cartesian product $(D^n)_{\leq a} \times (D^n)_{\leq b} \subseteq D^{2n}$. One verifies that both operations are independent of the choice of n, making D^∞ into a commutative semi-ring, where the zero for $\oplus$ is $\varnothing$, and where the unit for $\otimes$ is l, the minimum of D. We even can define a subtraction: if $a \leq b$ in D^∞, then we define

$b \ominus a$ as the Euler characteristic of $D^n_{>a} \cap D^n_{\leq b}$, where n is sufficiently large so that $a, b \in D^n$. This allows us to define the (Grothendieck) group generated by $(D^\infty, \oplus)$, defined as all pairs (x, y) with $x, y \in D^\infty$ up to the equivalence $(x, y) \sim (x', y')$ if and only if $x \oplus y' = x' \oplus y$; the induced commutative ring will be denoted $\mathfrak{Z}(D)$, and called the ring of D-*integers*.

To turn this into a genuine Euler characteristic, recall the construction of the *induced structure* $\mathcal{D}_{\text{ind}}$ on a subset $D \subseteq M$ of a first-order structure: for each definable subset $X \subseteq M^n$, we have a predicate defining in $\mathcal{D}_{\text{ind}}$ the subset $M \cap D^n$. If $\mathcal{M}$ is an ordered structure, then so is $\mathcal{D}_{\text{ind}}$. If D is definable, then we have an induced homomorphism of Grothendieck rings $\mathbf{Gr}(\mathcal{D}_{\text{ind}}) \to \mathbf{Gr}_0(\mathcal{M})$. If instead of definable isomorphism, we take virtual isomorphism, we get the virtual variant $\mathbf{Gr}^{\text{virt}}(\mathcal{D}_{\text{ind}}) \to \mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$. By the Virtual Pigeonhole Principle (Corollary 5.5), this latter homomorphism is injective. To discuss when they are isomorphic, let us call D *power dominant*, if for every definable, discrete subset A, there is some n such that $A \leq D^n$.

6.3. PROPOSITION. *A definable, discrete subset $D \subseteq M$ is power dominant if and only if $\mathbf{Gr}^{virt}(\mathcal{D}_{ind}) \cong \mathbf{Gr}_0^{virt}(\mathcal{M})$.*

PROOF. Suppose first that D is power dominant and let A be an arbitrary definable, discrete subset. By assumption, there exists an n and a definable subset $B \subseteq D^n$, such that A is virtually isomorphic with B. Hence $[A] = [B]$ in $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$, proving that it lies in the image of $\mathbf{Gr}^{\text{virt}}(\mathcal{D}_{\text{ind}}) \to \mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$.

Conversely, assume that the latter map is surjective, and let A be an arbitrary definable, discrete subset. Hence, there exists an n and definable subsets $E, F \subseteq D^n$ such that $[A] = [E] - [F]$ in $\mathbf{Gr}_0^{\text{virt}}(\mathcal{M})$. By the Virtual Pigeonhole Principle (Corollary 5.5), this means that there is a virtual isomorphism $A \sqcup F \to E$. Hence the composition $A \subseteq A \sqcup F \to E \subseteq D^n$, shows that $A \leq D^n$. ⊣

To study the existence of power dominant sets, let us say, for D and E discrete, definable subsets, that $D \lll E$, if $D^n \leq E$ for all n. If neither $D \lll E$ nor $E \lll D$, then D and E are mutually power bounded, that is to say, there exist m and n such that $D \leq E^m$ and $E \leq D^m$, and we write $D \approx E$. Hence $\lll$ induces a total order relation on the set $\text{Arch}^{\text{pow}}(\mathcal{M})$ of $\approx$-classes of definable, discrete subsets of $\mathcal{M}$. The class of the empty set is the minimal element of $\text{Arch}^{\text{pow}}(\mathcal{M})$, the class of a singleton is the next smallest element, and the class of a two-element set is the next (and consists of all finite sets). For an o-minimal structure, these are the only three classes, whereas for a proper o-minimalistic structure, there must be at least one more class, of some infinite set. I do not know whether $\text{Arch}^{\text{pow}}(\mathcal{M})$ is always discretely ordered or even finite. In any case, it follows easily from the definitions that a class is maximal in $\text{Arch}^{\text{pow}}(\mathcal{M})$ if and only if it is the class of a power dominant set. Thus, the existence of a power dominant set corresponds to $\text{Arch}^{\text{pow}}(\mathcal{M})$ having a maximal element, which is especially interesting in view of Proposition 6.3 and its applications below. I conjecture that D as in Example 2.3 is power dominant (and a similar property for any set obtained by discrete overspill). This would follow from the following growth conjecture for an o-minimal L-expansion $\mathcal{R}$ of $\mathbb{R}$:

6.4. CONJECTURE. *There exists, for every formula φ in the language $L(\mathtt{U})$, some $n \in \mathbb{N}$, such that for any finite subset F, if $\varphi(\mathcal{R}, F)$ is finite, then it has cardinality at most $|F|^n$.*

Recall that $\varphi(\mathcal{R}, F)$ is the set defined by φ in the structure $(\mathcal{R}, F)$ in which we interpret the unary predicate $\mathtt{U}$ by F. Likewise, I conjecture that the following always produces a power dominant set: let $\mathcal{M}$ be o-minimal and let D be an infinite o-finitistic subset, then D is power dominant in the (o-minimalistic) expansion $(\mathcal{M}, D)$.

6.5. THEOREM. *Every definable, discrete subset $D \subseteq M$ induces a ring isomorphism $\mathbf{Gr}^{virt}(\mathcal{D}_{ind}) \cong \mathfrak{Z}(D)$ by sending the class of a definable subset to its D-valued Euler characteristic.*

PROOF. We already observed that the ring operations on $\mathfrak{Z}(D)$ are invariant under virtual isomorphism. It is now easy to see that they also respect the scissor relations (sciss) in the Grothendieck ring of $\mathcal{D}_{\text{ind}}$. Surjectivity follows since every element in $\mathfrak{Z}(D)$ is of the form $a \ominus b$ for some n and some $a, b \in D^n$, and hence is the image of $[(D^n)_{\leq a}] - [(D^n)_{\leq b}]$. To calculate the kernel, we can write a general element as $[E] - [F]$, with E, F definable subsets in $\mathcal{D}_{\text{ind}}$. Such an element lies in the kernel if $\chi_D(E) = \chi_D(F)$, which means that E and F are virtually isomorphic, whence $[E] = [F]$ in $\mathbf{Gr}^{\text{virt}}(\mathcal{D}_{\text{ind}})$. ⊣

Summarizing, we have the following diagram of homomorphisms among the various Grothendieck rings, for $\mathcal{M}$ an o-minimalistic expansion of an ordered field:

(3)
$$\begin{array}{ccccc}
\mathbf{Gr}(\mathcal{D}_{\text{ind}}) & \twoheadrightarrow & \mathbf{Gr}^{\text{virt}}(\mathcal{D}_{\text{ind}}) & \xrightarrow{\sim} & \mathfrak{Z}(D) \\
\downarrow & & \downarrow i & & \\
\mathbf{Gr}_0(\mathcal{M}) & \twoheadrightarrow & \mathbf{Gr}_0^{\text{virt}}(\mathcal{M}) & & \\
\downarrow & & \downarrow j & & \\
\mathbf{Gr}(\mathcal{M}) & \twoheadrightarrow & \mathbf{Gr}^{\text{virt}}(\mathcal{M}) & &
\end{array}$$

with i an isomorphism if D is power dominant by Proposition 6.3, and with j an isomorphism if $\mathcal{M}$ is tame, by Corollary 3.11, that is to say, we proved:

6.6. COROLLARY. *If $\mathcal{M}$ is a tame, o-minimalistic expansion of an ordered field admitting a definable, power dominant subset D, then its o-minimalistic Grothendieck ring $\mathbf{Gr}^{virt}(\mathcal{M})$ is isomorphic to the ring of D-integers $\mathfrak{Z}(D)$.* ⊣

If we would allow classes of o-finitistic subsets in $\text{Arch}^{\text{pow}}(\mathcal{M})$, then there never is a maximal element: let D be any definable, discrete subset (or even any o-finitistic subset). Take an ultra-o-minimal elementary extension $\mathcal{N}$, and choose $D_i \subseteq N_i$ such that their ultraproduct is $D^{\mathcal{N}}$. By the observation following Proposition 5.4, we can choose $A_i \subseteq N_i$ to be virtually isomorphic with D_i^i and let $A_\natural \subseteq N$ be their ultraproduct. By Theorem 4.10, the restriction $A_\natural \cap M$ is o-finitistic and satisfies by Łoś' Theorem $D^n \leq A$ for all n, that is to say, $D \lll A$.

6.7. THEOREM (O-minimalism of Euler characteristics). *Let $D \subseteq M$ be a definable, discrete subset of an o-minimalistic structure $\mathcal{M}$, and let $X \subseteq M^{n+k}$ be any definable subset. For each $e \in D^n$, the set of parameters $\mathbf{a} \in M^n$ such that $\chi_D(X[\mathbf{a}]) = e$ is o-minimalistic.*

PROOF. If $\mathbf{a}$ does not belong to D^n, then the fiber $X[\mathbf{a}]$ is empty, whence has Euler characteristic $\varnothing$. As these $\mathbf{a}$ form a definable subset, we may therefore replace X by

$X \cap D^{n+k}$ and assume already that X is a definable subset of D^n. Let $\mathcal{N}$ be the context and write it as the ultraproduct of o-minimal structures $\mathcal{N}_i$. Choose $D_i \subseteq N_i$, $e_i \in D_i^n$ and $X_i \subseteq D_i^{n+k}$ with respective ultraproducts $D^{\mathcal{N}}$, e, and $X^{\mathcal{N}}$. For each i, let $F_i \subseteq N_i^n$ be the (finite) set of parameters for which the fiber has the same cardinality as $(D_i^n)_{\leq e_i}$. Hence, for each $\mathbf{a} \in F_i$, there exists a bijection $f_{\mathbf{a}} \colon X_i[\mathbf{a}] \to (D_i^n)_{\leq e_i}$. Let $H_i \subseteq N_i^{3n}$ be the union of all $\{\mathbf{a}\} \times \Gamma(f_{\mathbf{a}})$, where $\mathbf{a}$ runs over all tuples in F_i. Let $F_\natural \subseteq N^n$ and $H_\natural \subseteq N^{3n}$ be their ultraproduct, so that both sets are ultra-finite. By Łoś' Theorem, for each $\mathbf{a} \in F_\natural$, the fiber $H_\natural[\mathbf{a}]$ is the graph of a bijection $X^{\mathcal{N}}[\mathbf{a}] \to \left((D^{\mathcal{N}})^n\right)_{\leq e}$. Therefore, $F := F_\natural \cap M^n$ consists precisely of those $\mathbf{a} \in M^n$ for which the fiber $X[\mathbf{a}]$ has D-valued Euler characteristic e in the expansion $(\mathcal{M}, H_\natural \cap M^{3n})$ whence in $\mathcal{M}$, as the former is o-minimalistic by Theorem 4.10. For the same reason, F is o-finitistic, whence o-minimalistic by Theorem 4.9, so that we are done. ⊣

6.8. *Remark.* In everything in this section on Euler characteristics, we may, by passing to a suitable permissible expansion, even assume that D is only o-finitistic.

6.9. Archimedean reducts. As before, let D be definable and discrete with respective minimum l and maximum h. By [7, Theorem 4.1.iii], we have a successor function $\sigma := \sigma_D$, defined on $D \setminus \{h\}$, with inverse σ^{-1} defined on $D \setminus \{l\}$. Let us write $e \ll d$, if $\sigma^n(e) < d$, for all $n \in \mathbb{N}$. If neither $d \ll e$ nor $e \ll d$, then $\sigma^n(d) = e$ for some $n \in \mathbb{Z}$, and we write $d \sim_D e$. The set of $\sim_D$-equivalence classes is totally ordered by $\ll$, and is called the *Archimedean reduct* $\operatorname{Arch}(D)$ of D.

6.10. THEOREM. *The Archimedean reduct* $\operatorname{Arch}(D)$ *of a definable, discrete subset* D *in an o-minimalistic structure* $\mathcal{M}$ *is dense.*

PROOF. This is clear if D is finite, since then there is only one Archimedean class, so assume it is infinite. If $\operatorname{Arch}(D)$ is not dense, there would exist $l \ll h$ in D so that for no $d \in D$ we have $l \ll d \ll h$. Therefore, upon replacing D with $D \cap [l, h]$, we may assume that $\operatorname{Arch}(D)$ consists of exactly two classes, those of l and h . By Corollary 2.2 (or, Theorem 4.10), we can embed $\mathcal{M}$ elementary in an ultra-o-minimal structure $\mathcal{N}$ so that D is the restriction of a (definable) ultra-finite set F in $\mathcal{N}$. Let $\mathcal{N}_i$ and F_i be respectively o-minimal structures and finite subsets in these with ultraproduct equal to $\mathcal{N}$ and F respectively. For each i, let $f_i \colon F_i \to F_i$ be the map reversing the (lexicographical) order and let $\Gamma_\natural$ be the ultraproduct of the graphs of the f_i. Since this is an ultra-finite set, its restriction Γ to $\mathcal{M}$ is an o-finitistic set by Theorem 4.10. By Łoś' Theorem, Γ is the graph of the order reversing permutation $f \colon D \to D$. In particular, f is definable in the o-minimalistic expansion $(\mathcal{M}, \Gamma)$ and maps any element in the class of l to an element in the class of h and vice versa. By definability, there is a maximal $a \in D$ such that $f(a) \geq a$. In particular, $f(a') < a'$, where a' is the successor of a in D. A moment's reflection then shows that then either $f(a) = a$ or $f(a) = a'$, which contradicts that no element is $\sim_D$-equivalent with its image. ⊣

6.11. *Remark.* Similarly, given $D, E \in \mathfrak{D}^{\text{viso}}(\mathcal{M})$, we can define $D \ll E$ if for every finite subset F, we have $D \cup F \leq E$. If neither $D \ll E$ nor $E \ll D$, then we say that D and E have the same *virtual Archimedean class*, and write $D \sim E$. This is equivalent with the existence of finite subsets F and G such that $D \cup F$ and $E \cup G$ are virtually isomorphic. The induced order $\ll$ on virtual Archimedean classes is dense: indeed, suppose $D \ll E$ and let $d := \chi_E(D)$ and $h := \chi_E(E)$ (i.e., the maximum of

E). By Theorem 6.10, since $d \ll h$, there is some $a \in D$ with $d \ll a \ll h$. It follows that $D \ll E_{\leq a} \ll E$.

§**7. Taylor sets.** In this section, we work in an expansion of $\mathbb{R}$ and its ultrapower $\mathbb{R}_\natural$, and we introduce some notation and terminology tailored to this situation. Recall that an element in $\mathbb{R}_\natural$ is called *infinitesimal* if its norm is smaller than $1/n$, for all positive n. The *standard part* of $\alpha \in \mathbb{R}_\natural$, denoted $\alpha_\sharp$, is the supremum of all $r \in \mathbb{R}$ with $r \leq \alpha$; if $\alpha_\sharp$ is not infinite (that is to say, if α is *bounded*), then $\alpha_\sharp - \alpha$ is infinitesimal and $\alpha_\sharp$ is the unique real number with this property. If α is a tuple $(\alpha_1, \dots, \alpha_k)$, then we define $\alpha_\sharp$ coordinate-wise as $(\alpha_{1\sharp}, \dots, \alpha_{k\sharp})$. For a subset $X \subseteq \mathbb{R}^k$, we write $X_\sharp$ for the set of all $\alpha_\sharp$ where α runs over all bounded elements of the ultrapower $X_\natural$ (so that $\pm\infty$ never belongs to $X_\sharp$), and, following the ideology from [6, §8], we call $X_\sharp$ the *catapower* of X. We note the following simple result from non-standard analysis:

7.1. LEMMA. *The catapower of a subset $X \subseteq \mathbb{R}^k$ is equal to its closure $\bar{X}$.*

PROOF. Suppose $\alpha \in X_\natural$ is bounded, given as the ultraproduct of elements $\mathbf{a}_n \in X$. Hence the ultraproduct of the sequence $\alpha_\sharp - \mathbf{a}_n$ is an infinitesimal, showing that the $\mathbf{a}_n$ are arbitrary close to $\alpha_\sharp$ for almost all n. Put differently, there exists a subsequence of $(\mathbf{a}_n)_n$ which converges to $\alpha_\sharp$, proving that $\alpha_\sharp$ lies in the closure of X. Conversely, if $\mathbf{b}$ lies in closure of X, then we can find a sequence $\mathbf{b}_n \in X$ converging to it, and by the same argument, $(\mathbf{b}_\natural)_\sharp = \mathbf{b}$, where $\mathbf{b}_\natural$ is the ultraproduct of the sequence $\mathbf{b}_n$. ⊣

In [6, Chapter 9], we also introduce the notion of a *protopower*. Since it was catered to deal with an additional ring structure, which is not needed here, we will use only the following simplified version: for $X \subseteq \mathbb{R}^k$, we define, for each n, its *truncation* $X_{\rfloor_n}$ as the set of points in X whose coordinates have norm at most n, where the norm of a point is defined as the maximum of the absolute values of its coordinates. We define the *protopower* $\mathbb{R}_\flat$ of $\mathbb{R}$ as the ultraproduct of the $\mathbb{R}_{\rfloor_n}$. We extend this to any subset $X \subseteq \mathbb{R}^k$, by calling the ultraproduct of the truncations $X_{\rfloor_n}$ the *protopower* of X, and denote it $X_\flat$. In other words, $X_\flat = X_\natural \cap \mathbb{R}^k_\flat$, where $X_\natural$ is the ultrapower of X. In particular, any protopower is bounded (in norm) by $\omega_\natural$. (To make this conform with the definitions in [6, §9], one actually has to take the protoproduct of the structures $(\mathbb{R}, \frac{1}{n}\,|\cdot|)$, and the Archimedean hull of our $\mathbb{R}_\flat$ is then equal to this latter protoproduct.)

By the *trace* of a subset $\Xi \subseteq \mathbb{R}^k_\natural$, denoted $\operatorname{tr}(\Xi)$, we mean the set of its real points, that is to say, $\operatorname{tr}(\Xi) = \Xi \cap \mathbb{R}^k$. If Ξ is definable by a formula φ in some expansion of $\mathbb{R}_\natural$, we may use the slightly ambiguous notation $\varphi(\mathbb{R})$ for its trace as well. The trace of a protopower $X_\flat$ is equal to X, that is to say, $X = \operatorname{tr}(X_\flat)$: indeed, $\mathbf{a} \in \mathbb{R}^k$ satisfies $\mathbf{a} \in X_\flat$, if and only if $\mathbf{a} \in X_{\rfloor_n}$ for almost all n (by Łoś' Theorem), if and only if $\mathbf{a} \in X$. For given $n \in \mathbb{N}$ and a k-ary function f, let us write $f_{\rfloor_n}$ for the *truncated* function defined by sending a point $\mathbf{a}$ to $f(\mathbf{a})$ if $|\mathbf{a}| \leq n$ and to zero otherwise (note that this is not the same as taking the truncation of the graph of f, since we allow values of arbitrary high norm).

Let L^{an} be the language of ordered fields together with a function symbol for each everywhere convergent power series (also referred to as a *globally analytic* function). Clearly, we may view $\mathbb{R}$ as an L^{an}-structure, but this is not very useful, since $\mathbb{Z}$ is definable in it (as the zero set of $\sin(\pi x)$), and therefore neither tame nor o-minimalistic. Instead, we approximate this L^{an}-structure on $\mathbb{R}$ as follows. Let $\mathcal{R}^{\mathrm{an}}_n$ be the L^{an}-structure on $\mathbb{R}$ where each function symbol corresponding to a convergent power series f is interpreted as its truncation $f_{\rfloor n}$. By [2], each $\mathcal{R}_n^{\text{an}}$ is o-minimal (where one usually denotes $\mathcal{R}_1^{\text{an}}$ by $\mathbb{R}_{\text{an}}$), and hence their ultraproduct $\mathcal{R}_\natural^{\text{an}}$ is o-minimalistic. Moreover, $\mathcal{R}_\natural^{\text{an}}$ is tame by [7, Corollary 9.15]. While not part of the signature, power series with a smaller radius of convergence can also be encoded, at least in one variable: using a combination of linear transformations $x \mapsto ax + b$, and the (inverse) trig functions $\tan x$ and $\arctan x$, any two open intervals (bounded or unbounded) are isomorphic via a globally analytic map. For instance, if f is defined on the open interval $]-1, 1[$, then $g(x) := f(\frac{2}{\pi}\arctan x)$ is globally analytic, and hence f is definable in L^{an}.

7.2. DEFINITION (Taylor sets). We call $X \subseteq \mathbb{R}^k$ a *Taylor* set, if there exists an L^{an}-formula $\varphi(\mathbf{x}, \mathbf{y})$ (without parameters), such that for each sufficiently large n, there exists a tuple of parameters $\mathbf{b}_n$ so that $X_{\rfloor n} = \varphi(\mathcal{R}_n^{\text{an}}, \mathbf{b}_n)$.

Modifying φ if necessary, we may even assume that this holds for all n, and that $|\mathbf{x}| \leq n$ is a conjunct in φ. If $\mathbf{b}_\natural$ is the ultraproduct of the $\mathbf{b}_n$, then the protopower $X_\flat$ is equal to $\varphi(\mathcal{R}_\natural^{\text{an}}, \mathbf{b}_\natural)$ by Łoś' Theorem, and hence $X = \text{tr}(X_\flat)$. Any set realized as a *protopower* of a Taylor set will be called an *analytic protopower*, giving a one-one correspondence between Taylor sets and analytic protopowers. We refer to the defining formula $\varphi(\mathbf{x}, \mathbf{b}_\natural)$ of $X_\flat$ as the *analytic* formula for X, and we express this by writing $X = \varphi(\mathbb{R})$ (this does not mean that X is definable, since the parameters might be non-standard; in the terminology of [7, §7.18], a Taylor set is in general only locally L^{an}-definable). Not every definable subset is an analytic protopower (equivalently, not every $L^{\text{an}}(\mathbb{R}_\natural)$-formula is analytic): let Θ be defined by $(\exists y)\, xy = 1 \wedge \sin(\pi y) = 0$. Its trace $\text{tr}(\Theta)$ is equal to the set of reciprocals of positive natural numbers and cannot be a Taylor set by Lemma 7.3 below. Any quantifier free $L^{\text{an}}(\mathbb{R}_\natural)$-formula is analytic, so that in particular, any globally real analytic variety is Taylor. Taylor sets are closed under (finite) Boolean combinations, but not under definable (analytic) images, nor under projections. In particular, the Taylor sets do not form a first-order structure.

7.3. LEMMA. *A real discrete subset is Taylor if and only if it is closed. Moreover, a discrete Taylor set intersects any bounded set in finitely many points.*

PROOF. If X is discrete, then $X_{\rfloor n}$ must be finite by o-minimality, and hence X cannot have an accumulation point whence is closed. Conversely, if X is discrete and closed, then it is the zero set of some analytic function f (taking sums of squares allows us to reduce to a single equation), and hence $X_{\rfloor n}$ is defined in $\mathcal{R}_n^{\text{an}}$ by $f_{\rfloor n}(\mathbf{x}) = 0$, and $|\mathbf{x}| \leq n$. ⊣

7.4. LEMMA. *A subset $X \subseteq \mathbb{R}^k$ is Taylor if and only if its protopower $X_\flat$ is $\mathcal{R}_\natural^{an}$-definable.*

PROOF. Recall that $X_\flat$ is the ultraproduct of the truncations $X_{\rfloor n}$. One direction has already been observed. Assume $X_\flat$ is $\mathcal{R}_\natural^{\text{an}}$-definable, say $X_\flat = \varphi(\mathcal{R}_\natural^{\text{an}}, \mathbf{b})$, for some L^{an}-formula φ and some tuple of parameters $\mathbf{b}$. Writing $\mathbf{b}$ as the ultraproduct of tuples $\mathbf{b}_n$, it follows from Łoś' Theorem that $X_{\rfloor n} = \varphi(\mathcal{R}_n^{\text{an}}, \mathbf{b}_n)$ for almost all n. Enlarging the tuple of parameters if necessary, we may assume that n is one of the entries of $\mathbf{b}_n$. Choosing for each n some $m > n$ such that $X_{\rfloor m} = \varphi(\mathcal{R}_m^{\text{an}}, \mathbf{b}_m)$, we get $X_{\rfloor n} = \varphi(\mathcal{R}_n^{\text{an}}, \mathbf{b}_m) \wedge |\mathbf{x}| \leq n$, showing that X is Taylor. ⊣

We can rephrase this as a criterion for analytic protopowers:

7.5. COROLLARY. *A protopower is analytic if and only if it is $\mathcal{R}_\natural^{an}$-definable if and only if its trace is Taylor.* $\dashv$

In terms of formulae, we might paraphrase this as: an $L^{\mathrm{an}}(\mathbb{R}_\natural)$-formula φ is analytic if and only if $\varphi(\mathbb{R}_\natural)$ is the ultraproduct of the $\varphi(\mathbb{R}_{\downharpoonleft n})$. Thus, an open interval in $\mathbb{R}_\natural$ is an (analytic) protopower if and only if its endpoints are either real or equal to $\pm\omega_\natural$: indeed, suppose $]\alpha,\beta[$ is a protopower, and let $\alpha_\sharp$ and $\beta_\sharp$ be the respective standard parts of α and β. Hence $I :=]\alpha,\beta[\cap\mathbb{R}$ is a (not necessarily open) interval with endpoints $\alpha_\sharp$ and $\beta_\sharp$. If $\alpha_\sharp$ is finite, then $I_{\downharpoonleft n}$ is an interval with left endpoint $\alpha_\sharp$ for n sufficiently large, and hence the same is true for the ultraproduct of these truncations. By Corollary 7.5, this forces $\alpha_\sharp = \alpha$. In the other case, the left endpoint of $I_{\downharpoonleft n}$ is $-n$, and hence their ultraproduct has left endpoint $-\omega_\natural$, showing that $\alpha = -\omega_\natural$. The same argument applies to β, proving the claim.

7.6. EXAMPLE. By Lemma 7.3, every closed, discrete subset, whence in particular any subset of $\mathbb{Z}$, is Taylor. To give a non-discrete example, consider the spiral $C \subseteq \mathbb{R}^2$ with parametric equations $x = \exp\tau\sin\tau$ and $y = \exp\tau\cos\tau$, for $\tau\in\mathbb{R}$. If $(x,y)\in C_{\downharpoonleft n}$, then $\exp\tau = \sqrt{x^2+y^2} \leq n\sqrt{2}$ and hence $\tau \leq \log(n\sqrt{2}) \leq n$. In particular, the negative values of τ can be larger in absolute value than n. Hence C is not Taylor. However, if C^+ is the 'positive' part, given by the same equations but only for $\tau \geq 0$, then $C^+_{\downharpoonleft n}$ is defined in $\mathcal{R}_n^{\mathrm{an}}$ by $x = \exp_{\downharpoonleft n}(\tau)\sin_{\downharpoonleft n}(\tau)$, $y = \exp_{\downharpoonleft n}(\tau)\cos_{\downharpoonleft n}(\tau)$, and $\tau \leq \log_{\downharpoonleft n}(n\sqrt{2})$, showing that C^+ is Taylor (see Corollary 7.10 below).

7.7. PROPOSITION. *The closure, interior, frontier, and boundary of a Taylor set is again Taylor.*

PROOF. Since all concepts are obtained by either taking closures or Boolean combinations, it suffices to show that the closure $\bar{X}$ of a Taylor set X is again Taylor. Let $\varphi(x,z)$ be an analytic formula for X, so that $X_{\downharpoonleft n} = \varphi(\mathcal{R}_n^{\mathrm{an}},\mathbf{b}_n)$, for some parameters $\mathbf{b}_n$ and all n. If $\psi(x,z)$ is the formula $(\forall a>0)(\exists y)\,|x-y|<a\wedge\varphi(y,z)$, then $\psi(\mathcal{R}_n^{\mathrm{an}},\mathbf{b}_n)$ defines the closure of $X_{\downharpoonleft n}$. It is now easy to check that the latter is equal to $\bar{X}_{\downharpoonleft n}$, showing that $\bar{X}$ is Taylor. $\dashv$

7.8. *Remark.* From the proof it is also clear that if $X_\flat$ is the protopower of X, then the closure $\overline{X_\flat}$ of $X_\flat$ is the protopower of $\bar{X}$, and the analogous properties for the other topological operations. Inspecting the above proofs and examples, we can single out the following geometric feature of Taylor sets.[3]

7.9. PROPOSITION. *Let $X\subseteq\mathbb{R}^{k+1}$ be a Taylor set and let $Y\subseteq\mathbb{R}^k$ be its projection onto the first k coordinates. If there exists $l\in\mathbb{N}$ such that $Y_{\downharpoonleft n}$ is contained in the projection of $X_{\downharpoonleft ln}$, for all sufficiently large n, then Y is again Taylor.*

PROOF. Let $\varphi(\mathbf{x},y,\mathbf{c}_\natural)$ be the analytic formula defining X, and choose tuples $\mathbf{c}_n$ with ultraproduct equal to $\mathbf{c}_\natural$. By definition, $X_{\downharpoonleft n}$ is defined in $\mathcal{R}_n^{\mathrm{an}}$ by $\varphi(\mathbf{x},y,\mathbf{c}_n)$. Let $\tilde{\varphi}(\mathbf{x},y,\mathbf{c}_\natural)$ be the formula obtained from φ by replacing every power series $f(\mathbf{x},y)$ occurring in it by the power series $f(\mathbf{x},ly)$, and put $\psi(\mathbf{x},\mathbf{c}_\natural) := (\exists y)\tilde{\varphi}(\mathbf{x},y,\mathbf{c}_\natural)$. I claim that ψ is an analytic formula with $\psi(\mathbb{R}) = Y$. To this end, we have to show that $Y_{\downharpoonleft n} = \psi(\mathcal{R}_n^{\mathrm{an}},\mathbf{c}_n)$, for almost all n. One inclusion is clear, so assume $\mathbf{a}\in Y_{\downharpoonleft n}$, for some n. Hence $|\mathbf{a}|\leq n$ and there exists $b\in\mathbb{R}$ such that $(\mathbf{a},b)\in X$. By assumption,

[3] The corresponding syntactic characterization of analytic formulae is not yet clear to me.

we can choose $|b| \leq ln$. Let $b' := b/l$, so that $|b'| \leq n$. Since then $\mathcal{R}_n^{\mathrm{an}} \models \tilde{\varphi}(\mathbf{a}, b')$, as the point $(\mathbf{a}, b')$ has norm at most n, whence agrees on any power series with its n-th truncation, we get $\mathcal{R}_n^{\mathrm{an}} \models \psi(\mathbf{a})$, as required. ⊣

Given a $\mathcal{C}_1$-function $f\colon \mathbb{R} \to \mathbb{R}$ on an open interval $]a, b[$, we say that f is *increasing* at b if $f'(b^-) > 0$, where $f'(b^-)$ denotes the left limit at b of the derivative f', with a similar definition for decreasing or at the left endpoint.

7.10. COROLLARY. *Let f be a power series converging on a half-open interval $[a, b[$. If f is increasing at b, then the curve $C \subseteq \mathbb{R}^2$ with polar equation $R = f(\theta)$, for $a \leq \theta < b$, is Taylor.*

PROOF. As discussed above, we may make an order-preserving, analytic change of variables so that f becomes convergent on $\mathbb{R}_{\geq 0}$. In particular, f is increasing at ∞, which by L'Hôpital's rule means that the limit of $f(x)/x$ for $x \to \infty$ exists and is positive. Hence, we may choose $l \in \mathbb{N}$ large enough so that $1/l < f(x)/x$ for all $x \geq l$. Let $X \subseteq \mathbb{R}^3$ be the semi-analytic set given by $x = f(z)\sin(z)$, $y = f(z)\cos(z)$, and $a \leq z < b$, so that C is just the projection of X onto the first two coordinates. By Proposition 7.9, it suffices to show that $C_{\downharpoonright n}$ is contained in the projection of $X_{\downharpoonright (l\sqrt{2})n}$, for all n. To this end, let $(a, b) \in C_{\downharpoonright n}$, so that $a = f(\theta)\sin\theta$ and $b = f(\theta)\cos\theta$, for some $\theta \geq 0$. In particular, $f(\theta) = \sqrt{a^2 + b^2} \leq n\sqrt{2}$. There is nothing to prove if $\theta \leq l$, so let $\theta > l$ and hence $1/l < f(\theta)/\theta$. The result now follows since $\theta < lf(\theta) \leq (l\sqrt{2})n$. ⊣

Of course, a similar criterion exists if the domain is open at the left endpoint, where the function now has to be decreasing. We already observed that a Taylor set is of the form $\varphi(\mathbb{R})$ for some $L^{\mathrm{an}}(\mathbb{R}_\natural)$-formula φ, that is to say, is a trace of an $\mathcal{R}_\natural^{\mathrm{an}}$-definable subset. For each such trace $X := \varphi(\mathbb{R})$, we can define its *dimension* $\dim(X)$ to be the dimension of $\varphi(\mathcal{R}_\natural^{\mathrm{an}})$. In general, this notion is not well behaved: the trace of the discrete, zero-dimensional set given by the formula $(\exists y > 0)\sin(\pi y) = 0 \wedge \sin(\pi xy) = 0$ is equal to $\mathbb{Q}$, a non-discrete set. Fortunately, Taylor sets behave tamely, as witnessed, for instance, by the following planar trichotomy (compare with [7, Theorem 7.4]):

7.11. THEOREM. *A non-empty Taylor subset $X \subseteq \mathbb{R}^2$ is either*

7.11.i. *zero-dimensional, discrete, and closed;*

7.11.ii. *one-dimensional, nowhere dense, but at least one projection has non-empty interior;*

7.11.iii. *two-dimensional with non-empty interior.*

PROOF. Let $X_\flat$ be the protopower of X and d its dimension. By Proposition 2.4, almost all truncations $X_{\downharpoonright n}$ have dimension d. Hence, if $d = 0$, then almost all (whence all) $X_{\downharpoonright n}$ are finite and X is closed and discrete. If $d = 2$, then almost all (whence all) $X_{\downharpoonright n}$ have non-empty interior, whence so does X. Finally, if $d = 1$, (almost) all $X_{\downharpoonright n}$ are nowhere dense, and some projection has interior. Therefore, X itself has the same properties. ⊣

In view of Remark 7.8, the dimension of the frontier $\mathrm{fr}(X)$ of a Taylor set X is strictly less than its dimension $\dim(X)$. Hence, by the same argument as [7, Corollary 7.11], we immediately get:

7.12. COROLLARY. *Any Taylor set is constructible.* ⊣

Next, we study maps in this context. For $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^l$, let us call a map $f\colon X \to Y$ *Taylor*, if its graph is a Taylor set.

7.13. COROLLARY. *The domain and image of a Taylor map are Taylor, and so is any fiber. Likewise, if the graph of an $\mathcal{R}^{an}_{\natural}$-definable map $\gamma\colon \Xi \to \Theta$ is a protopower, then so are Ξ and $\gamma(\Xi)$, as well as every fiber $\gamma^{-1}(\mathbf{b})$ with $\mathbf{b} \in Y$. Moreover, the trace of γ induces a Taylor map $g\colon \operatorname{tr}(\Xi) \to \operatorname{tr}(\gamma(\Xi))$, and any Taylor map is obtained in this way.*

PROOF. The first assertion follows from the last assertion by Corollary 7.5. So assume the graph $\Gamma(\gamma)$ is a protopower. Without loss of generality, we may assume $\Theta = \gamma(\Xi)$, that is to say, that γ is surjective. Let $G := \operatorname{tr}(\Gamma(\gamma))$ be the trace, so that the ultraproduct of the $G_{\rceil n}$ is equal to $\Gamma(\gamma)$, and let $X := \operatorname{tr}(\Xi)$ and $Y := \operatorname{tr}(\Theta)$ be the respective traces of domain and image. It follows that $X_{\rceil n}$ is defined by the formula $(\exists \mathbf{y})(\mathbf{x}, \mathbf{y}) \in G_{\rceil n}$, and hence X is Taylor (alternatively, use Proposition 7.9). Moreover, γ restricted to $X_{\rceil n}$ takes values inside $Y_{\rceil n}$, that is to say, induces a map $g_n\colon X_{\rceil n} \to Y_{\rceil n}$. It follows that the ultraproduct of the g_n is equal to γ. By Łoś' Theorem, almost all g_n are surjective. Therefore, the respective ultraproducts of $X_{\rceil n}$ and $Y_{\rceil n}$ are Ξ and Θ, proving that both sets are analytic protopowers by Corollary 7.5. Moreover, the union g of the g_n is the restriction of γ to X. Fix $\mathbf{b} \in Y$, let $\Phi := \gamma^{-1}(\mathbf{b})$ its fiber and $F := \operatorname{tr}(\Phi)$ the latter's trace. One checks that $F_{\rceil n} = g_n^{-1}(\mathbf{b})$, and hence the ultraproduct of the $F_{\rceil n}$ is equal to Φ, proving that Φ is an analytic protopower. ⊣

7.14. *Remark* (Taylor cell decomposition). In particular, a horizontal Taylor 1-cell in $\mathbb{R}^2$ must be the graph of a continuous, Taylor map, and similarly, a Taylor 2-cell in $\mathbb{R}^2$ is the region between two Taylor graphs. Let X be a Taylor set with protopower $X_\flat$. Since $\mathcal{R}^{\mathrm{an}}_{\natural}$ is tame, we can find a surjective, cellular map $\delta\colon X_\flat \to \Delta$ with Δ a discrete, closed set. I conjecture that we may take δ to be a protopower too. Assuming this, taking traces yields a Taylor map $d\colon X \to \operatorname{tr}(\Delta)$, whose fibers are all Taylor cells, and hence defined by means of continuous Taylor maps. Hypothetically, this yields a Taylor cell decomposition of X which is finite on each compact subset by Lemma 7.3.

7.15. COROLLARY. *Any discrete Taylor set D satisfies DPP in the sense that a Taylor map $D \to D$ is injective if and only if it is surjective.*

PROOF. Let $g\colon D \to D$ be Taylor and let $g_\flat\colon D_\flat \to D_\flat$ be its protopower, that is to say, given as the ultraproduct of the restrictions $g_n := g|_{D_{\rceil n}}$. By Łoś' Theorem, $g_\flat$ is injective (surjective) if and only if almost all g_n are, whence if and only if g is, and the result now follows easily from the o-minimalistic DPP (Proposition 3.6). ⊣

7.16. COROLLARY (Monotonicity for Taylor maps). *A Taylor map $g\colon X \to Y$ is continuous outside a set of dimension strictly less than the dimension of X. In particular, one-variable Taylor maps are monotone outside a discrete, closed (Taylor) subset.*

PROOF. We may assume, for the purposes of this proof that g is surjective, so that, in particular, both X and Y are Taylor, by Corollary 7.13. By the same result, taking protopowers yields a definable map $g_\flat\colon X_\flat \to Y_\flat$ whose restriction to X is equal to g. By the Monotonicity Theorem ([7, Theorem 3.2]), the set of discontinuities Δ of $g_\flat$ has dimension strictly less than $\dim(X_\flat) = \dim(X)$. Replacing Δ by its closure,[4] which does not change the dimension, we may assume Δ is closed. I claim that g is continuous outside the trace $D := \operatorname{tr}(\Delta)$. Indeed, if $a \in X \setminus D$, then by the non-standard criterion

[4] In fact, this is not needed since one can show that Δ is already closed.

for continuity, we have to show that for every α infinitesimally close to a, their images under $g_\natural$ remain infinitesimally close, where $g_\natural$ is the ultrapower of g. However, since $g_\flat$ is the ultrapower of the restrictions $g|_{X_{\downharpoonleft n}}$, both maps agree on bounded elements, and so we have to show that $g_\flat(a)$ and $g_\flat(\alpha)$ are infinitesimally close. This does hold indeed for α sufficiently close to a since $a \notin \Delta$ and Δ is closed.

In the one-variable case, we may choose Δ so that $g_\flat$ is monotone on any interval with endpoints in Δ, and clearly, g is then monotone on D. It follows from Lemma 7.3 and Theorem 7.11 that D is Taylor. ⊣

7.17. *Remark.* Using the discussion in [7, Remark 3.5], we can choose Δ in the above statement also to be Taylor in higher dimensions.

If $f\colon X \to Y$ is Taylor and bijective, then its inverse is also Taylor, and we will say that X and Y are *analytically isomorphic*. In the definition of a Grothendieck ring, it was not necessary that the collection of subsets formed a first-order structure, only that they were preserved under Boolean combinations and products. Since this is true also of Taylor sets, we can define the *analytic* Grothendieck ring $\mathbf{Gr}^{\text{an}}$ as the free Abelian group of analytic isomorphism classes of Taylor sets modulo the scissor relations.

7.18. PROPOSITION. *There is a natural homomorphism $\mathbf{Gr}^{an} \to \mathbf{Gr}(\mathcal{R}_\natural^{an})$ of Grothendieck rings sending the class of a Taylor set X to the class of its protopower $X_\flat$.*

PROOF. To show that the map $[X] \to [X_\flat]$ is well-defined, suppose $f\colon X \to Y$ is an analytic isomorphism. The ultraproduct of the truncations $f_n\colon X_{\downharpoonleft n} \to Y_{\downharpoonleft n}$ induces then a definable map $f_\flat\colon X_\flat \to Y_\flat$, and by Łoś' Theorem, this is again a bijection. ⊣

By Corollary 3.4, composition with the ultra-Euler-characteristic yields the *analytic Euler characteristic* $\chi^{\text{an}}(X)$ of a Taylor set X; by definition, it is the ultraproduct of the $\chi(X_{\downharpoonleft n})$. In particular, for D discrete and closed, $\chi^{\text{an}}(D)$ is the ultraproduct of the cardinalities of its truncations. For which D does there exist a *density* d such that $\chi^{\text{an}}(D)/\omega_\natural^d$ is a bounded element, and if so, what is its standard part?

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