T-MINIMALITY

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ABSTRACT. A model theoretic minimality notion for structures with a definable topology, called t-minimality, is introduced. Cells are defined in analogy with the o-minimal or the p-adic case. It is shown that any definable set can be written as a finite union of cells, provided definable Skolem functions exist. This allows for the definition of the dimension of a definable set, and some basic properties of dimension are derived. In particular, dimension is preserved under definable bijections. Under some mild topological conditions on the definable topology, every definable function is continuous outside a set without interior. As a consequence, one can write the domain of the function as a union of finitely many cells, such that the restriction of the function to each such cell is continuous.

Examples of t-minimal structures are o-minimal structures and p-adic fields, so that we recover the Cell Decomposition theorems in each of these setups.

1. INTRODUCTION

It has by now become clear that the notion of o-minimality, first formally introduced by PILLAY-STEINHORN in [13, 14] but already implicitly present in VAN DEN DRIES's work [16], is a fundamental one, generalizing to a great extent the geometric methods used in real algebraic geometry. Among its many advantages was the ease with which a Cell Decomposition could be proved. This reduces in many instances the study of real definable sets to that of the more manageable cells. Around the same time, departing from MACINTYRE's Quantifier Elimination [9] (and even recovering it) DENEF proved in [2, 3, 4] a Cell Decomposition Theorem for p-adic definable sets. A notion analogous to o-minimality for the p-adics, called *p-minimality*, was subsequently introduced by HASKELL and MACPHERSON in [6]; a non-trivial example appeared in their joint paper [7] with VAN DEN DRIES.[1]

It has always appeared to me that these phenomena should be instances of a same principle. Namely, the main underlying idea in all cases is that definable subsets of the affine line have, up to a finite set, a *simple* structure. This fact then translates into a *tame* geometry in higher dimensions.[2] 'Simple' means here to be a finite union of *basic building blocks*. In the o-minimal case, the basic building blocks are open intervals; in the p-adic case, open annuli (an *annulus* is an open disk minus a closed disk) and cosets of the subgroup of n-th powers of the multiplicative group. The main observation to make is that the collection of all open intervals (respectively, of all open annuli and all non-zero n-th powers) is itself a definable

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[1]These authors also introduced a similar notion for algebraically closed valued fields (called *c-minimality*); it appears to also fit in the present framework, although I did not yet check the details.

[2]I will not attempt to make precise the notion of tameness–neither has done GROTHENDIECK–, but suffices it to say that it entails certain finiteness and uniformity conditions.

set, that is to say, open intervals (or annuli and n-th powers) occur as the fibers of a definable family. In the real case, one such definable family suffices; in the p-adic case, countably many are required, one for each n.

The obvious generalization is therefore to look at any first order structure M, in some language $\mathcal{L}$, which admits a distinguised definable set $\Delta \subset M^{s+1}$, such that its fibers over the projection $\pi\colon M^{s+1} \to M^s$ onto the first s coordinates, are to play the role of basic building blocks. In both the real and the valued field case, the building blocks can be chosen to be open and to form a basis for the canonical topology. It seems reasonable to require the same in the general case. This leads to the definition of a *definable topology* on M, as the collection of all fibers of some definable set Δ. In order to deal with instances in which several definable families of basic opens are required, as is the case for the p-adics, I introduce in the penultimate section *ind-definable* topologies. In this more general setup, the same statements can be proven with minor modifications.

A first order structure M with a (ind-)definable topology is then called *t-minimal*, if every definable subset of M is a finite union of singletons and Δ-fibers. Unlike the o-minimal case, it is not clear that the property of being t-minimal is preserved under elementary equivalence. Therefore, in analogy with the minimal case, we will say that M is *strongly t-minimal*, if every structure which is elementary equivalent to M is t-minimal (in the corresponding definable topology). Alternatively, we can call M strongly t-minimal, if the decomposition of definable sets in the affine line into Δ-fibers and points is uniform in the following sense. Let $A \subset M^{m+1}$ be a definable set. For $\mathbf{u} \in M^m$, write $A(\mathbf{u})$ for the set of all $x \in M$ such that $(\mathbf{u}, x) \in A$. Then M is strongly t-minimal if, and only if, the number of Δ-fibers and singletons needed in a decomposition of $A(\mathbf{u})$ is bounded, independently from $\mathbf{u} \in M^m$.

In the real case, a *cell* in $\mathbb{R}^m$ is defined recursively as all points between two definable continuous maps with domain a cell in $\mathbb{R}^{m-1}$ (allowing these maps to take values $\pm\infty$ as well), or as the graph of such a definable map. At the same time, the dimension of a cell is defined. However, the notion of 'betweenness' is lost in our general setting, and we replace it simply by a definable collection of Δ-fibers. Moreover, continuity does not behave as well either, so we temporary drop it from the definition of cells; see Definition 4.2 for details. I introduce the notion of a locally dense definable topology, in which continuity behaves properly, so that under this additional requirement, cells defined by means of *continuous* definable maps are open. This way we also recover Denef's notion of cell in the p-adic case. One cannot hope for Cell Decomposition (that is to say, the fact that every definable set is a finite union of cells) to hold in an arbitrary t-minimal structure. The main result of this paper is the following general Cell Decomposition Theorem (see Theorems 4.3 and 6.8).

Main Theorem. *Let M be a strongly t-minimal structure with definable Skolem functions. Then every definable set is a finite union of cells (in a uniform way).*

If M is moreover locally dense, then we can take all the cells in such a decomposition to be continuous. If f is any definable map, then there exists a (continuous) cell decomposition of its domain, such that f is continuous when restricted to each cell.

Admitting definable Skolem functions, albeit a non-trivial property, is neither too strong a property in this setup, as is made evident by Proposition 4.6. Namely, Cell Decomposition implies the existence of definable Skolem functions, provided the distinguished set Δ itself admits a definable section, that is to say, a definable map $s\colon M^s \to \Delta$ with $\pi \circ s$ the identity. In any case, both the reals and the p-adics admit definable Skolem functions, so that we obtain a unified proof for the Cell Decomposition Theorem in both cases.[3]

A unified treatment has been an important motivation for the author. Another motivation lies in the hope that t-minimality will prove to be as versatile a tool as o-minimality to study definable sets in (expansions of) valued fields, thus generalizing Denef's and Cluckers' results, or even in more general contexts. Moreover, Denef has shown how to obtain Quantifier Elimination via Cell Decomposition. Namely, assuming that the topology Δ is quantifier free definable, we then only need to resolve quantifiers for (graphs of) definable maps, for then cells, whence arbitrary definable sets, are quantifier free definable. Therefore, t-minimality in conjunction with a sufficient understanding of definable maps, will lead to some Quantifier Elimination results. Theorem 8.10 on elimination of imaginaries, is a first step towards such a general theory; unfortunately, it is not applicable in the p-adic case in view of the fact that the norm topology is totally disconnected.

Some further historic notes. Structures with definable topologies were introduced by Pillay in [12] and were called there *first order topological structures*. The present notion is slightly more restrictive, since we require some additional axioms to hold ((2.2.1)-(2.2.3)), to avoid trivial cases. The reader should be warned that in Pillay's paper, the term t-minimal is used in a different way then in the present paper,[4] albeit not without some similarity. However, Pillay's notion is essentially a stability theoretic one. In [10], Mathews further investigates this stability version of t-minimality and gives a general framework in which Cell Decomposition can be described (to make the connection with the present notion, use Lemma 3.6 and Remark 6.9 below). The present approach differs sufficiently from his, and is both more elementary and more general.

After writing this manuscript, I found out that Mourgues recently gave a proof in [11] of Cell Decomposition in p-minimal fields using definable Skolem functions. It should be noted that our present notion of t-minimality encompasses the p-minimal one, but potentially treats a larger class of examples of expansions of valued fields.

It goes without saying that many ideas in this paper are borrowed from the work of others on o-minimality. Especially van den Dries's book [17] has been a great source of inspiration for me and I strongly recommend it to anyone interested in o-minimality.

2. Definable Topologies

Let M be a set and W a subset of M^{m+n}. Let $\pi\colon M^{m+n} \to M^m$ be the projection onto the first m coordinates. For any $\mathbf{u} \in M^m$, we will write $W(\mathbf{u})$ to denote the set of all $\mathbf{x} \in M^n$, such that $(\mathbf{u}, \mathbf{x}) \in W$. We call $W(\mathbf{u})$ the *fiber* of W at $\mathbf{u}$, where in this terminology it is tacitly understood that we always take projection onto the

[3] It should be pointed out that Denef's Theorem in [2] or [4] states more than just the fact that definable sets are finite unions of cells. It also shows how to resolve definable maps, which is another key ingredient to prove Quantifier Elimination.

[4] I only realized this clash in terminology after the first version of this paper was written. However, since I did not want to further exhaust the alphabet for yet another kind of minimality, I decided to stick with my first choice.

first m coordinates. If $\pi\colon M^{m+n} \to M^m$ is an arbitrary projection, then we write $W_\pi(\mathbf{u})$ for the fiber with respect to π. More precisely, if $I \subset \{1,\dots,m+n\}$ has cardinality m, then we denote by $\pi_I^{(m+n)}$ the projection $M^{m+n} \to M^m$ onto the I-variables, that is to say, the map

$$(x_1,\dots,x_{m+n}) \mapsto (x_i \mid i \in I). \tag{1}$$

Set $\pi = \pi_I^{(m+n)}$. Enumerate I in ascending order as $\{i_1,\dots,i_m\}$ and its complement J as $\{j_1,\dots,j_n\}$. For $\mathbf{u} = (u_1,\dots,u_m) \in M^m$, we let $W_\pi(\mathbf{u})$ denote the set of all $\mathbf{v} = (v_1,\dots,v_n) \in M^n$, such that $(x_1,\dots,x_{m+n}) \in W$, where $x_{i_k} = u_k$ and $x_{j_l} = v_l$, for all k and l. In other words, the projection $\pi_J^{(m+n)}$ puts the sets $W \cap \pi^{-1}(\mathbf{u})$ and $W_\pi(\mathbf{u})$ in bijective correspondence.

2.1. Definition. Let M be a first order structure in some language $\mathcal{L}$. With a *$\mathcal{L}$-definable set*, or simply a *definable set*, we will always mean a subset A of some Cartesian power M^n which is first order definable by means of an $\mathcal{L}$-formula **with** parameters. If the defining formula has no parameters, then we say that A is *$\emptyset$-definable*.

2.2. Definition (Definable Topology)**.** We say that M has a *definable topology*, if there exists a $\emptyset$-definable set $\Delta \subset M^{s+1}$ with the following three properties.

2.2.1. For any $\mathbf{u} \in M^s$, the fiber $\Delta(\mathbf{u})$ consists of at least two elements.
2.2.2. For any two points $a, b \in M$ with $a \neq b$, we can find $\mathbf{u} \in M^s$, such that $a \in \Delta(\mathbf{u})$ but $b \notin \Delta(\mathbf{u})$.
2.2.3. For any two parameters $\mathbf{u}_1, \mathbf{u}_2 \in M^s$ and any point $a \in \Delta(\mathbf{u}_1) \cap \Delta(\mathbf{u}_2)$, we can find $\mathbf{v} \in M^s$, such that $\Delta(\mathbf{v})$ contains a and is contained in $\Delta(\mathbf{u}_1) \cap \Delta(\mathbf{u}_2)$.

By abuse of terminology, we will also call Δ a *definable topology*.

2.3. Lemma. *Let M be a structure with a definable topology Δ. Then every Δ-fiber $\Delta(\mathbf{u})$ is infinite.*

Proof. Suppose not, so that $\Delta(\mathbf{u}) = \{a_1,\dots,a_m\}$. By (2.2.2), we can find $\mathbf{u}_i \in M^s$, for $i = 2,\dots,m$, so that $a_1 \in \Delta(\mathbf{u}_i)$, but $a_i \notin \Delta(\mathbf{u}_i)$. By repeated use of (2.2.3), we can find $\mathbf{u}' \in M^s$ so that

$$\Delta(\mathbf{u}') \subset \Delta(\mathbf{u}) \cap \Delta(\mathbf{u}_2) \cap \dots \cap \Delta(\mathbf{u}_m). \tag{2}$$

where we use that the right hand side is non-empty, since it contains a_1. However, this means that $\Delta(\mathbf{u}') = \{a_1\}$, a contradiction with (2.2.1). □

2.4. Definition. Given a definable topology Δ, we define a topology on each Cartesian product M^n as follows. A non-empty set (not necessarily definable) $U \subset M^n$ is *Δ-open* (or just *open*, whenever the definable topology is clear from the context), if for each point $\mathbf{x} \in U$, we can find $\mathbf{u}_1,\dots,\mathbf{u}_n \in M^s$, such that

$$\mathbf{x} \in \Delta(\mathbf{u}_1) \times \dots \times \Delta(\mathbf{u}_n) \subset U. \tag{3}$$

Clearly arbitrary unions of open sets are open, and by (2.2.3), finite intersections of open sets are also open, so that we get indeed a topology. In particular, each fiber $\Delta(\mathbf{u})$ is open and the collection of all fibers of Δ forms a basis of open sets for M. The topology on any Cartesian power M^n is then just the product topology. It follows from (2.2.2) that this is a T_1 topology.

Let M^* be an $\mathcal{L}$-structure which is elementary equivalent with M, that is to say, a model of $\mathrm{Th}_\emptyset(M)$. Since Δ is $\emptyset$-definable, it is definable by means of a formula φ without parameters. Let Δ^* be the definable set in $(M^*)^{s+1}$ given by the same formula φ. Since the three conditions (2.2.1)–(2.2.3) are first order, it follows that Δ^* is a definable topology on M^*.

2.5. Definition (t-minimality)**.** We say that M, or more precisely, M with its definable topology Δ, is *t-minimal*, if any definable set $A \subset M$ is a finite union of points and Δ-fibers.

2.6. **O-minimal Structures.** Observe that $\mathbb{R}$ with its usually topology arises from a definable topology. Namely, let Δ be the subset of $\mathbb{R}^3$ consisting of all (x_1, x_2, y) satisfying one of the following three conditions

$$x_1 < y < x_2 \quad \wedge \quad x_1 < x_2 \tag{4}$$

$$y < x_2 \quad \wedge \quad x_1 = x_2 \tag{5}$$

$$x_1 < y \quad \wedge \quad x_1 > x_2. \tag{6}$$

If $a < b$, then $\Delta(a, b)$ is the open interval (a, b). If $a = b$, then $\Delta(a, b) = (-\infty, b)$. If $a > b$, then $\Delta(a, b) = (a, +\infty)$. There is of course some arbitrariness in our choice of Δ, but the essential point is that any open interval (of finite or infinite length) occurs as a Δ-fiber. It follows that the topology on each $\mathbb{R}^m$ is the Euclidean topology. Moreover, any expansion of the order on $\mathbb{R}$ is t-minimal if, and only if, it is o-minimal.

This construction can be made to work for an arbitrary o-minimal structure R, showing that t-minimality is a generalization of o-minimality. In fact, under some additional assumptions, we also have a converse; see Corollary 6.12.

It might perhaps come as a surprise that minimal structures are not t-minimal. This a consequence of the observation made below that definable topologies are unstable. We can see this in the case of an algebraically closed field K also more directly as follows. An non-empty definable open subset of K is cofinite, so a definable topology Δ should have fibers consisting of all points minus an arbitrary number of points. However, any definable family of fibers, whence in particular the family of Δ-fibers, has bounded Euler characteristic, contradicting this requirement. A similar obstruction caused by cofinite sets shows that Pressburger arithmetic $(\mathbb{Z}, +, \equiv_n)$ is not t-minimal.

2.7. Theorem. *If M is a first order structure admitting a definable topology, then M is unstable.*

Proof. Suppose $\Delta \subset M^{s+1}$ is a definable topology. Choose some $\mathbf{u}_1 \in M^s$. By (2.2.1), we can find at least two different points $a_1, a_1' \in \Delta(\mathbf{u}_1)$. By (2.2.2), we can find some $\mathbf{u}_2' \in M^s$, so that $a_1 \in \Delta(\mathbf{u}_2')$ and $a_1' \notin \Delta(\mathbf{u}_2')$. Finally, by (2.2.3), we can find some $\mathbf{u}_2 \in M^s$, so that $a_1 \in \Delta(\mathbf{u}_2)$ and $\Delta(\mathbf{u}_2) \subset \Delta(\mathbf{u}_1) \cap \Delta(\mathbf{u}_2')$. In particular, we have that $\Delta(\mathbf{u}_2) \subsetneq \Delta(\mathbf{u}_1)$, since the former does not contain a_1'. Choosing two different points in $\Delta(\mathbf{u}_2)$ and repeating the above argument, we can build an infinite strictly descending chain

$$\Delta(\mathbf{u}_1) \supsetneq \Delta(\mathbf{u}_2) \supsetneq \Delta(\mathbf{u}_3) \supsetneq \dots \tag{7}$$

It follows from [8, §6.7] that M is unstable. □

3. Strong t-minimality

Being t-minimal is preserved in elementary substructures.

3.1. Lemma. *Let $M \prec M^*$ be an elementary extension of ($\mathcal{L}$-)structures and let Δ be a definable topology (on both structures). If M^* is t-minimal, then so is M.*

Proof. Let $\delta(x, y)$ be the formula without parameters defining Δ. Let $\varphi(x, \mathbf{a})$ be a formula in the single free variable x, with parameters $\mathbf{a}$ from M. Let $A \subset M$ (respectively, $A^* \subset M^*$) be the set defined by $\varphi(x, \mathbf{a})$ in M (respectively, in M^*). By t-minimality, we can write

$$A^* = \{c_1^*, \dots, c_p^*\} \cup \Delta(\mathbf{u}_1^*) \cup \dots \cup \Delta(\mathbf{u}_q^*). \tag{8}$$

Let σ be the following sentence (with parameter $\mathbf{a}$)

$$(\exists z_i, \mathbf{y_j})(\forall x)[\varphi(x, \mathbf{a}) \leftrightarrow \bigvee_{i=1}^{p} x = z_i \vee \bigvee_{j=1}^{q} \delta(x, \mathbf{y_j})] \tag{9}$$

Equality (8) shows that σ holds in M^* (with $z_i = c_i^*$ and $\mathbf{y_j} = \mathbf{u}_\mathbf{j}^*$). Therefore, σ holds in M, say for $z_i = c_i$ and $\mathbf{y_j} = \mathbf{u_j}$. It follows that A is the union of the $\Delta(\mathbf{u}_j)$ and the points c_i. □

However, in general, t-minimality is not preserved under elementary equivalence. We therefore make the following definition.

3.2. Definition (Strong t-minimality)**.** We say that M with its definable topology Δ, is *strongly t-minimal*, if every $\mathcal{L}$-structure M^* elementary equivalent to M, is t-minimal.

3.3. Proposition. *Let M be a structure with a definable topology. In order for M to be strongly t-minimal, it suffices that each ultrapower of M is t-minimal.*

Proof. Assume that the condition in the above statement is satisfied. Let M^* be an arbitrary structure elementary equivalent with M. By the Keisler-Shelah Theorem ([8, Theorem 9.5.7]), some ultrapower N of M is isomorphic with some ultrapower of M^*. By assumption, N is t-minimal. Since $M^* \prec N$, it follows from Lemma 3.1 that M^* is t-minimal. □

Strong t-minimality is a uniform version of t-minimality, as the following theorem shows.

3.4. Theorem. *Let M be a structure with a definable topology Δ. A necessary and sufficient condition for M to be strongly t-minimal is that every definable family of subsets in M admits a uniformly bounded decomposition in singletons and Δ-fibers, that is to say, for every definable set $A \subset M^{m+1}$, there is a bound $N = N(A)$, such that each fiber $A(\mathbf{u}) \subset M$ with $\mathbf{u} \in M^m$ can be written as a union of at most N singletons and Δ-fibers.*

Proof. Suppose first that M is strongly t-minimal, but A provides a counterexample to the assertion in the theorem. Therefore, we can find for each $p \in \mathbb{N}$, a tuple $\mathbf{u}_p$ such that any decomposition of the fiber $A(\mathbf{u}_p)$ in singletons and Δ-fibers needs at least $p + 1$ sets. Let M_∞ be an ultrapower of M (with respect to some non-principal ultrafilter on $\mathbb{N}$) and let $\mathbf{u}_\infty$ be the tuple in M_∞ corresponding to the sequence $(\mathbf{u}_p \mid p \in \mathbb{N})$. By the t-minimality of M_∞, we can write $A(\mathbf{u}_\infty)$ as a union of a finite set $\{b_\infty^1, \dots, b_\infty^q\}$ and finitely many Δ-fibers $\Delta(\mathbf{a}_\infty^1), \dots, \Delta(\mathbf{a}_\infty^{q'})$. Choose sequences $b_p^1, \dots, b_p^q \in M$ and $\mathbf{a}_p^1, \dots, \mathbf{a}_p^{q'} \in M^s$, for all $p = 1, 2, \dots$, such that the image of $(b_p^i \mid p \in \mathbb{N})$ and $(\mathbf{a}_p^i \mid p \in \mathbb{N})$ in M_∞ is b_∞^i and $\mathbf{a}_\infty^i$ respectively. By Łos's Theorem, we can write almost all $A(\mathbf{u}_p)$ as a union of $\{b_p^1, \dots, b_p^q\}$ and the opens $\Delta(\mathbf{a}_p^1), \dots, \Delta(\mathbf{a}_p^{q'})$, contradicting our assumption on p.

Conversely, suppose every definable family of subsets of M admits a uniformly bounded decomposition in singletons and Δ-fibers. In particular, M is t-minimal. By Proposition 3.3, we have to show that M_∞ is t-minimal, where M_∞ is an arbitrary ultrapower of M (with respect to some non-principal ultrafilter on a set I). Let $\alpha(u, x)$ be a formula without parameters and let $\mathbf{u}_\infty$ be an m-tuple of parameters in M_∞. Let A_∞ be the set defined over M_∞ by the formula $\alpha(\mathbf{u}_\infty, x)$. Choose m-tuples $\mathbf{u}_p$ in M, for $p \in I$, so that the image of the sequence $(\mathbf{u}_p \mid p \in I)$ in M_∞ is precisely $\mathbf{u}_\infty$. Let B be the definable subset of M^{m+1} defined by α. By our assumption, there is a bound N on the number of singletons and Δ-fibers needed to decompose any fiber of B. In particular, we can find, for each $p \in I$, elements $b_p^1, \dots, b_p^q \in M$ and tuples $\mathbf{a}_p^1, \dots, \mathbf{a}_p^{q'} \in M^s$ such that

$$B(\mathbf{u}_p) = \{b_p^1, \dots, b_p^q\} \cup \Delta(\mathbf{a}_p^1) \cup \dots \cup \Delta(\mathbf{a}_p^{q'}). \tag{10}$$

Therefore, if we denote by b_∞^i and $\mathbf{a}_\infty^i$ the respective image of the sequences $(b_p^i \mid p \in I)$ and $(\mathbf{a}_p^i \mid p \in I)$ in M_∞, then

$$A_\infty = B(\mathbf{u}_\infty) = \{b_\infty^1, \dots, b_\infty^q\} \cup \Delta(\mathbf{a}_\infty^1) \cup \dots \cup \Delta(\mathbf{a}_\infty^{q'}). \tag{11}$$

□

3.5. Definition. Let A be a set in a topological space X. We will denote the *closure* of A by $\overline{A}$ and its *interior* by $\operatorname{int} A$. The *(topological) boundary* of A is the difference $\overline{A} - \operatorname{int} A$ and is denoted ∂A.

3.6. Lemma. *Let M be a t-minimal structure. Any definable set $A \subset M$ has a finite boundary ∂A. Moreover, if M is strongly t-minimal, then there is a uniform bound on the number of boundary points of any member of a definable family of subsets in M.*

Proof. The second statement will follow immediately from the proof and Theorem 3.4. Therefore, let Δ be a definable topology on M for which M is t-minimal and let $A \subset M$ be a definable set. Since

$$\overline{A} = \{\, x \in M \mid \forall \mathbf{u} \in M^s \colon x \in \Delta(\mathbf{u}) \to A \cap \Delta(\mathbf{u}) \neq \emptyset \,\} \tag{12}$$

we see that $\overline{A}$ is definable. Similarly, $\operatorname{int} A$ is definable and therefore so is ∂A. In particular, t-minimality implies that ∂A is a finite union of points and Δ-fibers. To finish the proof, we only need to show that no $\Delta(\mathbf{u})$ can lie entirely inside ∂A. So assume $\Delta(\mathbf{u}) \subset \partial A$. By t-minimality, we can write A as a union of a finite set F and finitely many $\Delta(\mathbf{v}_i)$. Suppose there is some i, such that $\Delta(\mathbf{u}) \cap \Delta(\mathbf{v}_i)$ is non-empty and choose some point x in this intersection. Since $\Delta(\mathbf{v}_i) \subset A$, we have that $x \in \operatorname{int} A$. On the other hand $x \in \Delta(\mathbf{u}) \subset \partial A$, so that $x \notin \operatorname{int} A$, contradiction. Therefore, $\Delta(\mathbf{u}) \cap A$ lies in F whence is finite. Since any Δ-fiber is infinite, we may choose some $x \in \Delta(\mathbf{u})$ outside F. By (2.2.2), we can find $\Delta(\mathbf{v}) \subset \Delta(\mathbf{u})$, such that $x \in \Delta(\mathbf{v})$ but $\Delta(\mathbf{v}) \cap F = \emptyset$. Therefore, $\Delta(\mathbf{v}) \cap A = \emptyset$, showing that x does not lie in $\overline{A}$, contradicting that $x \in \Delta(\mathbf{u}) \subset \partial A$. □

4. Cell Decomposition

In this section, I will show that every definable set can be written as a finite union of cells, under the extra assumption that we have definable Skolem functions. This is not a weak assumption, but by Proposition 4.6 it follows that we can not really hope for less. Cell Decomposition will allow for the definition of the dimension of an arbitrary definable set; this will be discussed in the next section.

4.1. Definition. Let M be a first order structure. We say that M *has definable sections* or *definable Skolem functions*, if for every definable set $A \subset M^{n+t}$ and every projection $\pi\colon M^{n+t} \to M^t$, we can find a definable map $f\colon \pi(A) \to A$, such that $\pi \circ f$ is the identity on $\pi(A)$.

Let $f\colon M^m \to M^n$ be a definable map and let $\Gamma \subset M^{m+n}$ denote its graph. Let $\pi\colon M^{m+n} \to M^n$ be the projection on the last n coordinates. If M has definable Skolem functions, then there exists a definable map $s_0\colon \pi(\Gamma) \to \Gamma$ such that $\pi \circ s_0$ is the identity. Note that $\pi(\Gamma) = \operatorname{Im} f$. Let s be the composition

$$\operatorname{Im} f \xrightarrow{s_0} \Gamma \subset M^{m+n} \twoheadrightarrow M^m \tag{13}$$

where the latter map is just the projection on the first m coordinates. It follows that $f \circ s$ is the identity. In other words, if M has definable Skolem functions, then any definable map has a definable section.

We recall our convention for denoting projections. Let $n \in \mathbb{N}$ and let I be a set of indices contained in $\{1, \dots, n\}$ of cardinality a. Let $\pi_I^{(n)}\colon M^n \to M^a$ denote the projection onto the variables with index in I, that is to say, $\pi_I^{(n)}(x_1, \dots, x_n) = (x_i \mid i \in I)$. If $I = \emptyset$, then $a = 0$, and we think of M^0 as a singleton (once and for all fixed).

4.2. Definition (Cell)**.** Let M have a definable topology $\Delta \subset M^{s+1}$. The definition of a d-dimensional *cell* $C \subset M^n$ will be given by induction on n. Moreover, we define the notion of the type of a cell, denoted $\operatorname{type}(C)$, to be either 1 or 0. If $n = 1$ then a 0-dimensional cell is just a singleton and a 1-dimensional cell is just a Δ-fiber. Moreover, a singleton has type zero and a Δ-fiber has type one. For $C \subset M^n$, we define each of the types separately. A d-dimensional cell of type one is of the form

$$C = \{\, (\mathbf{x}, y) \in M^n \mid \mathbf{x} \in D \text{ and } y \in \Delta(a(\mathbf{x})) \,\} \tag{14}$$

with D a $(d-1)$-dimensional cell (of either type) in M^{n-1} and $a\colon M^{n-1} \to M^s$ a definable map. Note that if $n = 1$, then this just means that a is a point in M^s. Therefore C is just a Δ-fiber, so that we recover the $n = 1$ case.

A d-dimensional cell of type zero is of the form

$$C = \{\, (\mathbf{x}, y) \in M^n \mid \mathbf{x} \in D \text{ and } a(\mathbf{x}) = y \,\} \tag{15}$$

with D a d-dimensional cell (of either type) in M^{n-1} and $a\colon M^{n-1} \to M$ a definable map.

We will denote the dimension of a cell C by $\dim C$. Observe that $\pi^{(n)}_{\{1,\dots,i\}}$ is the projection onto the first i coordinates. Unraveling the recursive definition of cell, we therefore see that

$$\dim C = \sum_{i=1}^{n} \operatorname{type}(\pi^{(n)}_{\{1,\dots,i\}}(C)). \tag{16}$$

This implies that the dimension of a cell is well-defined, that is to say, is independent of the particular description (14) or (15).

Note that Δ itself is an example of a cell, of type 1 and of dimension $s+1$ (for definable map just take the identity map on M^s).

4.3. Theorem (Cell Decomposition)**.** *Let M be a strongly t-minimal structure admitting definable Skolem functions. Then any definable set is a union of cells. If, moreover, every subset of M can be written as a disjoint union of Δ-fibers and points, then any definable set can be written as a disjoint union of cells.*

Proof. The second assertion will follow immediately from our proof of the first assertion. To prove the first assertion, let $A \subset M^{n+1}$ be a definable set. We will induct on n. If $n = 0$, then by definition of t-minimality, A is a union of cells (and in case of the second assertion, even a disjoint union). Hence assume $n > 0$. Let $\pi \colon M^{n+1} \to M^n$ be the projection on the n first coordinates. Since $\pi(A)$ is a definable subset of M^n, we can write it by induction as a union of cells $\pi(A) = D_1 \cup \cdots \cup D_p$. All $A \cap (D_i \times M)$ are definable and their union is A. Therefore, upon replacing A by one of the $A \cap (D_i \times M)$, we may assume that $\pi(A)$ is a cell D. By Theorem 3.4, there exist $p, q \in \mathbb{N}$, so that each fiber $A(\mathbf{u})$, with $\mathbf{u} \in D$, is the union of at most p distinct Δ-fibers and at most q singletons. Choose p as small as possible and then q as small as possible. We call the thus obtained pair (p, q) the *decomposition number* of A. We will induct (in the lexicographical ordering) on this decomposition number. Assume first that $p = 0$. Let s be a definable section of the map $\pi|_A$, that is to say, a definable map $s \colon \pi(A) \to A$ with $\pi \circ s$ the identity on $\pi(A)$. Let $C \subset M^{n+1}$ be the cell of type 0 given by $(\mathbf{x}, y) \in C$, if and only if, $\mathbf{x} \in D$ and $y = s(\mathbf{x})$. It follows that $C \subset A$ and that $A - C$ has decomposition number $(0, q-1)$. Therefore, by induction, we can write $A - C$ as a union of cells and we are done in this case.

Assume therefore that A has decomposition number (p, q) with $p > 1$. Let $W \subset M^{n+s}$ be the (definable) set of all tuples $(\mathbf{x}, \mathbf{u})$, such that $\Delta(\mathbf{u}) \subset A(\mathbf{x})$ and such that there exist $\mathbf{u}_1, \dots, \mathbf{u}_{p-1} \in M^s$ and $a_1, \dots, a_q \in M$ with

$$A(\mathbf{x}) - \Delta(\mathbf{u}) = \{a_1, \dots, a_q\} \cup \Delta(\mathbf{u}_1) \cup \cdots \cup \Delta(\mathbf{u}_{p-1}). \tag{17}$$

Let $\theta \colon M^{n+s} \to M^n$ be the projection on the first n coordinates. Let $d \colon \theta(W) \to W$ be a definable section of $\theta|_W$ and let $\tilde{d} \colon \theta(W) \to M^s$ be the composition of d followed by projection onto the last s coordinates. Since $\theta(W)$ is a definable set in M^n, we may write it by induction as the union of cells $D_1, \dots, D_m$. Let C_i be the cell of type 1 consisting of all $(\mathbf{x}, y)$ such that $\mathbf{x} \in D_i$ and $y \in \Delta(\tilde{d}(\mathbf{x}))$. Set

$$A' = A - (C_1 \cup \cdots \cup C_m). \tag{18}$$

I claim that the decomposition number of A' is at most $(p-1, q)$, so that by induction A' is a finite union of cells, proving the theorem. To prove the claim, let $(\mathbf{x}, y) \in A'$. If $\mathbf{x} \notin \theta(W)$, then this can only mean that $A(\mathbf{x})$ was finite and the claim follows. So we may assume that $\mathbf{x}$ lies in $\theta(W)$ whence in some D_i. Write $d(\mathbf{x})$ as $(\mathbf{x}, \mathbf{u})$, for some $\mathbf{u} \in M^s$. Note that $\Delta(\mathbf{u}) = C_i(\mathbf{x})$. Together with (17), we see that $A'(\mathbf{x}) = A(\mathbf{x}) - \Delta(\mathbf{u})$ is the union of at most $p-1$ open Δ-fibers and at most q points. Therefore, A' has decomposition number at most $(p-1, q)$ as claimed. $\square$

4.4. *Remark.* Strong t-minimality yields actually a uniform version of Cell Decomposition. One can easily show that if $C \subset M^{m+n}$ is a cell and $\mathbf{u} \in M^m$ lies in $\pi(C)$, where $\pi\colon M^{m+n} \to M^m$ is the projection onto the first m coordinates, then $C(\mathbf{u})$ is a cell in M^n. Therefore, if $A \subset M^{m+n}$ is a definable set admitting a cell decomposition $A = C_1 \cup \cdots \cup C_s$, then $A(\mathbf{u}) = C_1(\mathbf{u}) \cup \cdots \cup C_s(\mathbf{u})$ is a cell decomposition of each fiber $A(\mathbf{u})$. In particular, the number of cells needed is uniformly bounded by s in the family of fibers $A(\mathbf{u})$. We therefore proved (just put $N(A) = s$ below) that M is *algebraically* (or, *model-theoretically*) *bounded* in the following sense.

4.5. Corollary (Algebraic Boundedness)**.** *Let M be a strongly t-minimal structure admitting definable Skolem functions. Let $A \subset M^{m+n}$ be a definable set. There exists a number $N = N(A)$, such that each fiber $A(\mathbf{u})$ having more than N elements is infinite.*

There is a partial converse to Cell Decomposition.

4.6. Proposition. *Let M be a structure with a definable topology Δ. Suppose that $\Delta \subset M^{s+1} \twoheadrightarrow M^s$ has a definable section and that every definable set can be written as a finite union of cells. Then M has definable Skolem functions and is strongly t-minimal.*

Proof. To prove the existence of definable Skolem functions, it suffices to show that given $A \subset M^{n+m}$, we can find $s\colon \pi(A) \to A$ such that $\pi \circ s$ is the identity map, for $\pi\colon M^{n+m} \to M^n$ the projection onto the first n coordinates. An easy inductive argument then reduces to the case that $m = 1$.

I claim that if each $A_i \subset M^{n+1} \twoheadrightarrow M^n$ admits a definable section s_i, for $i = 1, \dots, m$, then so does $A = A_1 \cup \cdots \cup A_k$. Indeed, let $s\colon \pi(A) \to A$ be defined as follows. For every $\mathbf{x} \in \pi(A)$, there is a unique $i \in \{1, \dots, k\}$, such that $\mathbf{x} \in \pi(A_i)$ but $\mathbf{x}$ does not lie in any of the sets $\pi(A_1), \dots, \pi(A_{i-1})$. Set $s(\mathbf{x}) = s_i(\mathbf{x})$. One checks that s is a definable section for A.

Therefore, we may assume, in view of our hypothesis, that A is a cell. The case of a cell of type 0 is immediate. So assume that A has type 1. Hence we can find a cell $C \subset M^n$ and a definable function $a\colon M^n \to M^s$, such that A consists of all tuples $(\mathbf{x}, y)$, for which $\mathbf{x} \in C$ and $y \in \Delta(a(\mathbf{x}))$. Let $\delta\colon M^s \to \Delta$ be a definable section for Δ. Then $\delta \circ a$ is a definable section for A (note that $C = \pi(A)$).

Remains to prove that M is strongly t-minimal. Let $A \subset M^{m+1}$. By assumption, we can write A as a finite union of cells $A = C_1 \cup \cdots \cup C_s$. By Remark 4.4,

$$A(\mathbf{u}) = C_1(\mathbf{u}) \cup \cdots \cup C_s(\mathbf{u})$$

is a cell decomposition of $A(\mathbf{u})$, for each tuple $\mathbf{u}$. Since $C_i(\mathbf{u})$ is a cell in M, it is either a Δ-fiber or a point. In conclusion, we showed that there is a uniform bound on the number of points and Δ-fibers needed to write each fiber of A, so that by Theorem 3.4, M is strongly t-minimal. □

5. Dimension

We would like to define the dimension of an arbitrary definable set as the maximum of the dimensions of the cells in a cell decomposition of the set. In order to make this definition, we need to show that this is independent from the particular decomposition. To this end, we need to study more closely cells and their dimension.

5.1. Lemma. *Let M be a structure with a definable topology. Let C be a cell in M^n. Let I be a set of indices in $\{1,\dots,n\}$, such that for all $i \notin I$, we have that $\pi^{(n)}_{\{1,\dots,i\}}(C)$ has type 0. Then $\pi^{(n)}_I(C)$ is again a cell and $\pi^{(n)}_I$ restricted to C is injective. Moreover,*

$$\dim C = \dim \pi^{(n)}_I(C). \tag{19}$$

Proof. For each $i \in \{1,\dots,n\}$, let ϵ_i denote the type of $\pi^{(n)}_{\{1,\dots,i\}}(C)$. Unravelling the recursive definition of cells, we find for each i a definable map c_i on M^{i-1} with values in M when $\epsilon_i = 0$, and with values in M^s when $\epsilon_i = 1$, such that $\mathbf{x} = (x_1,\dots,x_n) \in C$ if, and only if, $\mathcal{P}_i(x_1,\dots,x_i)$ holds, for all $i = 1,\dots,n$, where $\mathcal{P}_i(x_1,\dots,x_i)$ is the expression $x_i = c_i(x_1,\dots,x_{i-1})$ when $\epsilon_i = 0$, and the expression $x_i \in \Delta(c_i(x_1,\dots,x_{i-1}))$ when $\epsilon_i = 1$. For $\mathbf{x} = (x_1,\dots,x_n)$, let $\mathbf{y} = \pi^{(n)}_I(\mathbf{x})$, so that each y_i is some $x_{j(i)}$. Let $\mathcal{Q}_i(\mathbf{y})$ for $i \in I$ be the expression obtained from $\mathcal{P}_i$ as follows. For each $l \notin I$ with $l < i$, replace each (formal) occurrence of x_l in $\mathcal{P}_i(x_1,\dots,x_i)$ by $c_l(x_1,\dots,x_{l-1})$. Note that by assumption $\epsilon_l = 0$ for $l \notin I$, so that if $\mathbf{x} \in C$ then x_l is indeed equal to $c_l(x_1,\dots,x_{l-1})$. After finitely many such substitutions all variables x_l with $l \notin I$ will have been eliminated, so that we get an expression $\mathcal{P}'_i$ in the $x_{j(i)}$, for $i \in I$. Let $\mathcal{Q}_j$ now be the formal expression obtained by replacing in $\mathcal{P}'_i$ these $x_{j(i)}$ by the corresponding y_i. It is now easy to check that $\pi^{(n)}_I(C)$ consists of all tuples $\mathbf{y}$ which satisfy all $\mathcal{Q}_i(\mathbf{y})$. Moreover, these expressions show that $\pi^{(n)}_I(C)$ is in fact a cell. Since we only projected out variables x_l for which $\epsilon_l = 0$, it follows that C and $\pi^{(n)}_I(C)$ have the same dimension by (16). Finally, since each projection corresponding to a variable x_l with $l \notin I$ is in fact a projection from the graph of the function c_l, it is clear that $\pi^{(n)}_I$ is injective on C. □

Let us denote for an arbitrary cell $C \subset M^n$ by $I(C)$ the set of all indices $i \in \{1,\dots,n\}$ for which $\operatorname{type}(\pi^{(n)}_{\{1,\dots,i\}}(C)) = 1$. Let d be its cardinality, so that $\dim C = d$ by (16).

5.2. Corollary. *Let M be a structure with a definable topology. Let $C \subset M^n$ be a cell. Let $i \in \{1,\dots,n\}$ and let I be a subset of $\{1,\dots,n\}$ containing $I(C) \cap \{1,\dots,i\}$. Then $\pi^{(n)}_I(C)$ is again a cell.*

Proof. Let $J = I \cup (I(C) \cap \{i+1,\dots,n\}$ and e the cardinality of J. By Lemma 5.1, $\pi^{(n)}_J(C)$ is a cell in M^e, since $I(C) \subset J$. But then $\pi^{(n)}_I(C)$ is obtained from $\pi^{(n)}_J(C)$ by projecting onto an initial fragment of the variables. However, such a projection always preserves cells by the very definition of cells. □

5.3. Lemma. *Let M be a structure with a definable topology. Let $C, D_1,\dots,D_m$ be cells in M^n. If $C = D_1 \cup \dots \cup D_m$, then $\dim C$ is equal to the maximum of the $\dim D_i$. In particular, if D is a cell with $D \subset C$, then $\dim D \le \dim C$.*

Proof. The last assertion follows immediately from the first by taking $D_1 = D$ and $D_2 = C$. To prove the first assertion, let $i \notin I(C)$, so that $\operatorname{type}(\pi^{(n)}_{\{1,\dots,i\}}(C)) = 0$. This means that all fibers of the projection $\pi^{(n)}_{\{1,\dots,i\}}(C) \twoheadrightarrow \pi^{(n)}_{\{1,\dots,i-1\}}(C)$ are finite. Therefore, the same must hold for each fiber of $\pi^{(n)}_{\{1,\dots,i\}}(D_j) \twoheadrightarrow \pi^{(n)}_{\{1,\dots,i-1\}}(D_j)$ and for all j. This in turn means that $\operatorname{type}(\pi^{(n)}_{\{1,\dots,i\}}(D_j)) = 0$, for all j, so that

$i \notin I(D_j)$. It follows from (16) that $\dim D_j$ is at most $\dim C$, for all j. Moreover, by Lemma 5.1, $\pi^{(n)}_{I(C)}(C)$ and each $\pi^{(n)}_{I(C)}(D_j)$ are cells, of the same dimension as C and D_j respectively. Therefore, without loss of generality we may assume that $I(C) = \{1, \dots, n\}$ and $\dim C = n$. We now only have to show that it is impossible that every D_j has dimension strictly less than n. Towards a contradiction, suppose that each $I(D_j)$ is a proper subset of $\{1, \dots, n\}$. Choose for each $j = 1, \dots, m$ some $i(j) \notin I(D_j)$. There exists therefore a definable function $a_j\colon M^{i(j)-1} \to M^s$, such that $\pi^{(n)}_{\{1,\dots,i(j)\}}(D_j)$ consists of all tuples $(\mathbf{x}, y)$ with $\mathbf{x} \in \pi^{(n)}_{\{1,\dots,i(j)-1\}}(D_j)$ and $y = a_j(\mathbf{x})$. On the other hand, since $I(C) = \{1, \dots, n\}$, there exist definable functions $b_i\colon M^{i-1} \to M^s$, for $i = 1, \dots, n$, such that $\pi^{(n)}_{\{1,\dots,i\}}(C)$ consists of all tuples $(\mathbf{x}, y)$ with $\mathbf{x} \in \pi^{(n)}_{\{1,\dots,i-1\}}(C)$ and $y \in \Delta(b_i(\mathbf{x}))$. For each i, let J_i be the collection of all $j \in \{1, \dots, m\}$ for which $i(j) = i$. We will successively choose $x_i \in M$. Choose x_1 different from all a_j with $j \in J_1$ but inside $\Delta(b_1)$ (note that b_1 and each a_j for $j \in J_1$ are constant functions). This is possible since Δ-fibers are always infinite. Next choose x_2 different from all $a_j(x_1)$, for $j \in J_2$, but inside $\Delta(b_2(x_1))$. Proceeding this way, we arrive at an n-tuple $\mathbf{x} = (x_1, \dots, x_n)$, which by construction belongs to C, but for each $j = 1, \dots, m$, we have that $a_j(x_1, \dots, x_{i(j)-1}) \neq x_{i(j)}$. Therefore, $\mathbf{x} \notin D_j$, for every j, contradicting that C is the union of all D_j. □

5.4. Corollary. *Let M be a strongly t-minimal structure with definable Skolem functions. Let $C_1, \dots, C_p$ and $D_1, \dots, D_q$ be cells in M^n. If*

$$\bigcup_{i=1}^{p} C_i = \bigcup_{j=1}^{q} D_j \tag{20}$$

then the maximum of the $\dim C_i$ *is equal to the maximum of the* $\dim D_j$.

Proof. By Cell Decomposition 4.3, we can write each $C_i \cap D_j$ as the union of finitely many cells $A^{(ij)}_k$. Therefore, from (20), it follows that

$$C_i = \bigcup_{j,k} A^{(ij)}_k. \tag{21}$$

By Lemma 5.3, the dimension of C_i, for a fixed i, equals the maximum of all $\dim A^{(ij)}_k$, where the maximum is taken over all j and k. Therefore, the maximum of all $\dim C_i$ equals the maximum of all $\dim A^{(ij)}_k$. By the same argument, this maximum is then also the maximum of all $\dim D_j$. □

5.5. Definition. Let M be a strongly t-minimal structure with definable Skolem functions and let $A \subset M^m$ be a definable subset. We define the *dimension* of A, denoted $\dim A$, as the maximum of all $\dim C_i$, where the C_i are finitely many cells whose union equals A. By Theorem 4.3 at least one such cell decomposition $A = C_1 \cup \dots \cup C_s$ exists and by Corollary 5.4, the value of $\dim A$ is independent of the cell decomposition. We take the convention that the empty set has dimension $-\infty$.

The following facts are easily verified using the results on the dimension of cells.

5.6. Corollary. *Let M be a strongly t-minimal structure with definable Skolem functions. Let A and B be definable subsets of M^m. Then the following holds.*

5.6.1. *If* $A \subset B$*, then* $\dim A \leq \dim B \leq m$*.*
5.6.2. *The dimension of* A *is zero if, and only if,* A *is finite.*
5.6.3. *The dimension of* $A \cup B$ *is equal to the maximum of* $\dim A$ *and* $\dim B$*.*

5.7. Proposition. *Let* M *be a strongly t-minimal structure with definable Skolem functions. Let* $f\colon M^n \to M^m$ *be a definable map and* $A \subset M^n$ *a definable subset. Then* $\dim f(A) \leq \dim A$*.*

Proof. Let us first prove the statement for f a projection map. By an inductive argument, it suffices to consider the case that f projects out a single variable, that is to say, that $f = \pi^{(n)}_{\{1,\dots,i-1,i+1,\dots,n\}}$, for some $i \in \{1,\dots,n\}$. We will induct on n (the case $n = 1$ is trivial). Set $d = \dim C$ and let $D = \pi^{(n)}_{\{1,\dots,i-1,i+1,\dots,n\}}(C)$. We want to show that $\dim D \leq d$. If $i \notin I(C)$, then the result follows from Lemma 5.1. So we may assume that $i \in I(C)$. Let $C' = \pi^{(n)}_{I(C)}(C)$, so that by Lemma 5.1, this is a d-dimensional cell in M^d. Consider $\pi^{(n-1)}_{I(D)}(D)$. This set can also be written as $\pi^{(d)}_{\{1,\dots,j-1,j+1,\dots,d\}}(C')$, for some $j \in \{1,\dots,d\}$. If $d < n$, then by our induction hypothesis, $\pi^{(n-1)}_{I(D)}(D)$ has dimension at most $\dim C' = d$. By another application of Lemma 5.1, $\pi^{(n-1)}_{I(D)}(D)$ has dimension equal to $\dim D$. This shows the desired inequality, at least in case $d < n$. However, if $d = n$, then the result is trivial, since $D \subset M^{d-1}$ can have dimension at most $d - 1$.

Consider now the general case of an arbitrary definable map f. Let $\Gamma \subset M^{n+m}$ be the graph of f. Let $D = \Gamma \cap (C \times M^m)$. One checks that D is a cell, and since the indices $m+1,\dots,m+n$ do not belong to $I(D)$, it follows from Lemma 5.1 that $\dim D = \dim \pi^{(n+m)}_{\{1,\dots,n\}}(D)$. However, $\pi^{(n+m)}_{\{1,\dots,n\}}(D)$ is just C, so that D and C have the same dimension. On the other hand, we see that $\pi^{(n+m)}_{\{m+1,\dots,m+n\}}(D) = f(C)$, so that the assertion follows from the special case of a projection, applied to the cell D. □

5.8. Corollary. *Let* M *be a strongly t-minimal structure with definable Skolem functions. If two definable sets* A *and* B *are* definably isomorphic*, that is to say, if there exists a definable bijection between them, then they have the same dimension.*

Proof. Let $f\colon A \to B$ be a definable bijection. Its inverse $g\colon B \to A$ is therefore also definable. By Proposition 5.7, we get that $\dim f(A) \leq \dim A$ and $\dim g(B) \leq \dim B$. Since $g(B) = A$ and $f(A) = B$, the statement follows. □

5.9. *Remark.* The converse will in general be false. For instance, over the ordered field of the reals, two sets are definably isomorphic if, and only if, they have the same dimension and the same Euler characteristic. However, Cluckers has shown in [1] that over the p-adics, two infinite definable sets are definably isomorphic if, and only if, they have the same dimension.

5.10. Theorem. *Let* M *be a strongly t-minimal structure with definable Skolem functions. Let* $f\colon A \subset M^m \to M^n$ *be a definable map. For each* $k \in \{1,\dots,m\}$*, let* $\mathrm{F}_k(f)$ *be the set of all* $\mathbf{y} \in M^n$ *for which* $\dim f^{-1}(\mathbf{y}) = k$*. Then* $\mathrm{F}_k(f)$ *is a definable set and* m *is equal to the maximum of the numbers* $k + \dim \mathrm{F}_k(f)$*, for* $k = 1,\dots,m$*.*

Proof. I claim that it suffices to prove the following version of the statement. Let A be a definable subset of M^m and let $\pi\colon M^m \to M^n$ be a projection map. Let

$\mathrm{F}_k(A;\pi)$ be the set of all $\mathbf{y} \in M^n$ for which $\dim A_\pi(\mathbf{y}) = k$. Recall from 2.2 that $A_\pi(\mathbf{y})$ is definably isomorphic with $\pi^{-1}(\mathbf{y}) \cap A$ (via projection). The assertion is that each $\mathrm{F}_k(A;\pi)$ is definable and, moreover, $\dim A$ is equal to the maximum of the numbers $k + \dim \mathrm{F}_k(A;\pi)$, for $k = 1, \dots, m$. Assuming the claim, let's prove the theorem. Let Γ be the graph of f. As in the proof of Proposition 5.7, Γ is a cell of dimension m. Let $\pi \colon M^{m+n} \to M^n$ be the projection onto the last n coordinates. Then $\Gamma_\pi(\mathbf{y})$, for $\mathbf{y} \in M^m$, is equal to $f^{-1}(\mathbf{y})$, so that $\mathrm{F}_k(\Gamma;\pi) = \mathrm{F}_k(f)$, for all $k = 1, \dots, m$, and we are done by the claim.

So remains to show the claim. Suppose $\pi = \pi_I^{(m)}$, with $I \subset \{1, \dots, m\}$ of cardinality n. Let σ be a permutation of $\{1, \dots, m\}$, so that $\sigma(I) = \{1, \dots, n\}$. Let f_σ be the map $(x_1, \dots, x_m) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(m)})$. Since this is a definable bijection, we have by Corollary 5.8 that A and $f_\sigma(A)$ have the same dimension. Moreover

$$\pi^{-1}(\mathbf{y}) = f_\sigma^{-1}((\pi^{(m)}_{\{1,\dots,n\}})^{-1}(\mathbf{y})), \tag{22}$$

for all $\mathbf{y} \in M^n$. Therefore, also $A_\pi(\mathbf{y})$ and $f_\sigma(A)(\mathbf{y})$ are definably isomorphic whence have the same dimension, for all $\mathbf{y} \in M^n$. So we may assume without loss of generality that $\pi = \pi^{(m)}_{\{1,\dots,n\}}$. Using Cell Decomposition 4.3 and Corollary 5.6, we may moreover assume that A is a cell. Just observe that if $A = A_1 \cup \dots A_l$, then $\mathrm{F}(A;\pi) = \mathrm{F}(A_1;\pi) \cup \dots \cup \mathrm{F}(A_l;\pi)$.

For A a cell, the equality

$$\dim A(\mathbf{y}) = \sum_{i=n+1}^{m} \mathrm{type}(\pi^{(m)}_{\{1,\dots,i\}}(A)), \tag{23}$$

for every $\mathbf{y} \in \pi(A)$, is almost immediate by an inductive argument on the definition of cell. In particular, the dimension of a non-empty fiber is independent of $\mathbf{y}$, say equal to d. On the other hand, $\pi(A)$ is again a cell, of dimension $\dim A - d$. Since $\mathrm{F}_d(A;\pi) = \pi(A)$, whereas all other $\mathrm{F}_k(A;\pi) = \emptyset$, the claim follows. □

6. Continuity

In o-minimality, an important fact is that every definable subset A of M^m of maximal dimension m has non-empty interior. In fact, if A is an m-dimensional cell, then it is open. This fails in general for t-minimal structures for the following two reasons. Firstly, cells in an o-minimal context are defined by means of continuous definable functions instead of just definable ones. For this reason, we need to introduce continuous cells. Nonetheless, in an o-minimal structure, even using the present weaker notion of cell, one still has that every cell has non-empty interior, since a definable function is continuous on a dense set of its domain. This might fail in arbitrary t-minimal structures, since the nesting of Δ-fibers might be too sparse. To be more precise, one needs that the family of Δ-fibers admits sufficiently many containments, in the following sense.

6.1. Definition (Local Density)**.** Let M be a structure with a definable topology $\Delta \subset M^{s+1}$. We call Δ *locally dense*, if the following two conditions are satisfied for each x and $\mathbf{b}$ with $x \in \Delta(\mathbf{b})$.

6.1.1. There exists a non-empty open $U \subset M^s$, such that for each $\mathbf{u} \in U$, we have $x \in \Delta(\mathbf{u}) \subset \Delta(\mathbf{b})$.

6.1.2. There exists a tuple $\mathbf{a}$ and an open $V \subset M^s$ with $\mathbf{b} \in V$, such that for each $\mathbf{v} \in V$, we have $x \in \Delta(\mathbf{a}) \subset \Delta(\mathbf{v})$.

6.2. Definition (Continuous Cell)**.** Let Δ be a definable topology. A *continuous cell* $C \subset M^m$ is defined recursively in the same way as an arbitrary cell with the following modification. If $m = 1$ then any cell is continuous. If $m > 1$, then a continuous cell C of type 1, consists of all tuples $(\mathbf{x}, y) \in M^m$, for which $\mathbf{x} \in D$ and $y \in \Delta(a(\mathbf{x}))$, where D is a continuous cell in M^{m-1} and $a\colon M^{m-1} \to M^s$ is a continuous definable map. A continuous cell C of type 0, consists of all tuples $(\mathbf{x}, y) \in M^m$, for which $\mathbf{x} \in D$ and $y = a(\mathbf{x})$, where D is a continuous cell in M^{m-1} and $a\colon M^{m-1} \to M$ is a continuous definable map. In other words, the same definition as for ordinary cells, except that only continuous definable maps can be used.

We will prove simultaneously several theorems on continuous cells, including a decomposition theorem in continuous cells, by induction on the dimension m of the embedding space. Although some of these theorems are weaker versions of others, it is necessary to establish the theorems in the order they are listed for the induction to work.

In what follows, M is a strongly t-minimal and locally dense $\mathcal{L}$-structure with definable Skolem functions.

6.3. Theorem. *The intersection of two continuous cells in* M^m *is a finite union of continuous cells in* M^m*.*

6.4. Theorem. *Any definable subset* $A \subset M^m$ *can be written as a finite union of continuous cells.*

6.5. Theorem. *Any continuous cell* $A \subset M^m$ *of dimension* m *is open.*

6.6. Theorem. *If* A *is a definable subset of* M^m *of dimension* m*, then its topological boundary* ∂A *has dimension strictly less than* m*.*

6.7. Theorem. *If* A *is a* d*-dimensional definable subset of* M^m *(with* $d \geq 0$*) and* $f\colon A \to M^n$ *a definable map, then the subset* D *of discontinuities of* f *is definable and has dimension strictly less than* d*.*

6.8. Theorem. *If* A *is a definable subset of* M^m *and* $f\colon A \to M^n$ *a definable map, then there exists a (finite) decomposition* $A = C_1 \cup \cdots \cup C_s$ *in continuous cells, such that* f *is continuous on each* C_i*.*

Each of these statements holds trivially if $m = 0$ (where M^0 is a fixed singleton, necessarily equipped with the discrete topology). Note that by convention, the empty set has dimension $-\infty$. Therefore, in proving these theorems, we may assume that $m > 0$ and that they all hold for lesser values of the embedding space. Also note that for $m = 1$, Theorems 6.3–6.5 hold by definition and Theorem 6.6 is Lemma 3.6. However in order to prove the two remaining theorems in case $m = 1$, we need the extra assumption that the topology is locally dense. Before we start the inductive proofs, one last remark on the two last statements. In order to prove these, it suffices to let $n = 1$, since there are definable maps $f_i\colon A \to M$, such that $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$, for all $\mathbf{x} \in A$ and f is continuous in some point $\mathbf{x}$ if, and only if, each f_i is.

Proof of Theorem 6.3. I will only consider the case that both cells are of type 1; the remaining cases admit similar proofs the details of which are left to the reader. Therefore, let C_1 and C_2 be continuous cells in M^m of type 1. Let $D_i \subset M^{m-1}$ be a continuous cell and $a_i\colon D_i \to M^s$ a continuous definable map, so that C_i consists of all tuples $(\mathbf{x}, y)$ with $\mathbf{x} \in D_i$ and $y \in \Delta(a_i(\mathbf{x}))$. Since by our induction hypothesis $D_1 \cap D_2$ is a finite union of continuous cells, we may assume that $D = D_1 = D_2$ is a continuous cell. For each $\mathbf{x} \in D$, we can write $\Delta(a_1(\mathbf{x})) \cap \Delta(a_2(\mathbf{x}))$ as a finite union of Δ-fibers and singletons by t-minimality. Moreover, by strong t-minimality, the number of singletons and Δ-fibers needed in this decomposition is uniformly bounded. Therefore, using definable Skolem functions, there exist definable maps $b_i \colon D \to M^s$ and $b_{p+i} \colon D \to M$, so that, for each $\mathbf{x} \in D$, we have

$$\Delta(a_1(\mathbf{x})) \cap \Delta(a_2(\mathbf{x})) = \Delta(b_1(\mathbf{x})) \cup \Delta(b_p(\mathbf{x})) \cup \{b_{p+1}(\mathbf{x}), \dots, b_{p+q}(\mathbf{x})\}. \tag{24}$$

By the induction hypothesis on Theorem 6.8, we can find for each $i = 1, \dots, p+q$ a decomposition of D in continuous cells, so that b_i is continuous on each cell. Again by induction, after taking the intersection of all these cells, we may assume that one such decomposition $D = D_1 \cup \dots \cup D_w$ in continuous cells has been obtained on which all b_i are simultaneously continuous. Therefore, $C_1 \cap C_2$ will be the union of all cells V_{ij}, for $i = 1, \dots, p+q$ and $j = 1, \dots, w$, where V_{ij} consists of all tuples $(\mathbf{x}, y)$ with $\mathbf{x} \in D_j$ and $y \in \Delta(b_i(\mathbf{x}))$ or $y = b_i(\mathbf{x})$, according to whether $i \leq p$ or $p < i \leq p+q$. □

Proof of Theorem 6.4. By ordinary Cell Decomposition (Theorem 4.3), we may assume that A is a cell. Suppose $D \subset M^{m-1}$ is a cell and $a \colon D \to M^t$ is a definable function, where $t = s$ if A has type 1 and $t = 1$ if A has type 0, so that A consists of all tuples $(\mathbf{x}, y) \in M^m$ with $\mathbf{x} \in D$ and $y \in \Delta(a(\mathbf{x}))$ (respectively, $y = a(\mathbf{x})$), if A has type 1 (respectively, type 0). By induction, we may decompose D in continuous cells and therefore, without loss of generality, we may already assume that D is a continuous cell. Moreover, by an argument similar to the one in the previous proof, for which we might have to split up D further, we may also assume that a is continuous on D. This shows that A is a continuous cell, as required. □

Proof of Theorem 6.5. Let $(\mathbf{x}, y)$ be an arbitrary point of the m-dimensional cell A, necessarily of type 1. Choose a continuous cell $D \subset M^{m-1}$ and a continuous definable map $a \colon D \to M^s$, such that $(\mathbf{x}, y)$ belongs to A if, and only if, $\mathbf{x} \in D$ and $y \in \Delta(a(\mathbf{x}))$. By induction, D is open. Let Z be the subset of M^{2s} consisting of all pairs $(\mathbf{u}, \mathbf{v})$ for which $y \in \Delta(\mathbf{u}) \subset \Delta(\mathbf{v})$. By Condition (6.1.2) of local density applied to the situation $y \in \Delta(a(\mathbf{x}))$, we can find $\mathbf{u}$ with $y \in \Delta(\mathbf{u})$, such that $a(\mathbf{x})$ lies in the interior of $Z(\mathbf{u})$. Let U be an open containing $a(\mathbf{x})$ and contained in $Z(\mathbf{u})$. Since a is continuous, $a^{-1}(U)$ is open in D whence in M^{m-1}. Moreover, $\mathbf{x} \in a^{-1}(U)$, so that $a^{-1}(U) \times \Delta(\mathbf{u})$ contains $(\mathbf{x}, y)$. Let $(\mathbf{s}, t)$ be an arbitrary point in $a^{-1}(U) \times \Delta(\mathbf{u})$. Hence $a(\mathbf{s}) \in U \subset Z(\mathbf{u})$ implies that $\Delta(\mathbf{u}) \subset \Delta(a(\mathbf{s}))$. Since $t \in \Delta(\mathbf{u})$, we conclude that $t \in \Delta(a(\mathbf{s}))$ and therefore $(\mathbf{s}, t) \in A$. In conclusion, $a^{-1}(U) \times \Delta(\mathbf{u}) \subset A$, showing that $(\mathbf{x}, y)$ is an interior point of A. □

Proof of Theorem 6.6. Since the closure and the interior of a definable set are again definable, so is its topological boundary. By Theorem 6.4 (which at this point has been proven for subsets of M^m), we can write A as a finite union of continuous cells. By the already proven Theorem 6.5, each cell in this decomposition of dimension m is open. In particular, this shows that we can write A as the union of a (definable) open set U and a definable set B of dimension less than m. Necessarily, U lies in the interior of A. Assume that ∂A also has dimension m. By the same argument, ∂A contains then a non-empty open V. In particular, $V - B$ must have dimension m as well by Corollary 5.6. By another application of Theorem 6.5, we can find a non-empty open W inside $V - B$. Since W lies in the closure of A, it must meet A. If $W \cap U$ is non-empty, then any point in this intersection would belong to the interior of A, since it lies in U. This, however, is impossible since W lies in the boundary of A. In summary, $W \cap U = \emptyset$, and, by construction $W \cap B = \emptyset$, contradicting that W meets A. □

Proof of Theorem 6.7. By what we said above, we may take $n = 1$, so that $f\colon A \to M$. Apart from the standing induction on m, we will perform a second induction on the dimension d of A. If $d = 0$, then $D = \emptyset$, since a function on a discrete set is everywhere continuous. So assume $d > 0$. If $d < m$, then by Lemma 5.1, there is some projection $\pi\colon M^m \to M^d$, such that its restriction to A is an isomorphism (and in fact a homeomorphism) $\gamma\colon A \to \pi(A)$. By our induction hypothesis on m, the discontinuities of the definable map $f \circ \gamma^{-1}\colon \pi(A) \to M$ lie in a set of dimension strictly smaller than d. Since γ induces a definable isomorphism between D and this latter set, the theorem is proven in this case. Therefore, we may assume that $d = m$. By the already established versions of Theorems 6.4 and 6.5, D contains a non-empty open definable subset U. For each $\mathbf{u} \in U$, we can find a Δ-fiber $\Delta_{\mathbf{x}}$ containing $f(\mathbf{x})$, with the property that no open containing $\mathbf{x}$ is mapped by f entirely inside $\Delta_{\mathbf{x}}$, exhibiting that f is discontinuous at $\mathbf{x}$. In other words, $\mathbf{x}$ lies on the boundary of $f^{-1}(\Delta_{\mathbf{x}})$. Since the existence of $\Delta_{\mathbf{x}}$ is first order expressible, we can find using definable Skolem functions, a definable map $g\colon U \to M^s$ so that we may take $\Delta_{\mathbf{x}} = \Delta(g(\mathbf{x}))$.

Let $W \subset M^{s+m}$ be the set of all tuples $(\mathbf{x}, \mathbf{v})$ with $\mathbf{x} \in U$, such that

$$f(\mathbf{x}) \in \Delta(\mathbf{v}) \subset \Delta(g(\mathbf{x})). \tag{25}$$

By Property (6.1.1) of local density applied to $f(\mathbf{x}) \in \Delta(g(\mathbf{x}))$, each fiber $W(\mathbf{x})$, for $\mathbf{x} \in U$, contains a non-empty open of M^s, whence has dimension s. Therefore, by Theorem 5.10, W has dimension $s + m$. Let $\pi\colon M^{s+m} \to M^s$ be the projection onto the last s coordinates. If each fiber $W_\pi(\mathbf{v})$ would have dimension strictly less than m, then by that same theorem, we would get that W has dimension less than $m + s$. Therefore, for some $\mathbf{v} \in M^s$, the fiber $W_\pi(\mathbf{v})$ is m-dimensional, whence contains a definable open V (by a similar argument as before). Let $\mathbf{x} \in V$. Since then $(\mathbf{x}, \mathbf{v}) \in W$, we get that $f(\mathbf{x}) \in \Delta(\mathbf{v}) \subset \Delta(g(\mathbf{x}))$. Since f is discontinuous at $\mathbf{x}$, so that $\mathbf{x}$ lies on the boundary of $f^{-1}(\Delta(g(\mathbf{x})))$, it follows that $\mathbf{x}$ also lies on the boundary of $f^{-1}(\Delta(\mathbf{v}))$. Since this holds for all $\mathbf{x} \in V$, we showed that V is contained in the boundary of $f^{-1}(\Delta(\mathbf{v}))$. However, this contradicts Theorem 6.6 in dimension m, proving the theorem. □

Proof of Theorem 6.8. By Theorem 6.7 just proved, the set D of discontinuities of f has dimension strictly less than m. Applying the induction hypothesis to the restriction $f|_B$, we find a decomposition of B in finitely many continuous cells, so that f is continuous when restricted to every such cell. Finally, by Theorem 6.4, we can write the complement $-D$ as a finite union of continuous cells as well, and the theorem follows. □

6.9. *Remark.* Theorem 6.5 together with Lemma 5.1 shows that any continuous cell in our sense is a cell in the sense of [10].

6.10. *Remark.* One shows that if $C \subset M^{m+n}$ is a continuous cell and $\mathbf{u} \in M^m$ lies in $\pi(C)$, where $\pi\colon M^{m+n} \to M^m$ is the projection onto the first m coordinates, then $C(\mathbf{u})$ is a continuous cell in M^n. Therefore, as in Remark 4.4, if $A \subset M^{m+n}$

is a definable set and f a definable map with domain A, and if $A = C_1 \cup \cdots \cup C_s$ is a decomposition in continuous cells on which f is continuous, then $A(\mathbf{u}) = C_1(\mathbf{u}) \cup \cdots \cup C_s(\mathbf{u})$ is a decomposition in continuous cells of each fiber $A(\mathbf{u})$ and $f_{\mathbf{u}}$ is continuous on each $C_i(\mathbf{u})$, where $f_{\mathbf{u}}$ is the map on $A(\mathbf{u})$ given by $\mathbf{x} \mapsto f(\mathbf{u}, \mathbf{x})$. In particular, the number of continuous cells needed is uniformly bounded by s in the family of fibers $A(\mathbf{u})$. In particular, each fiber $A(\mathbf{u})$ has at most s isolated points, so that we proved the following relative version of Corollary 4.5.

6.11. Corollary. *Let M be a strongly t-minimal and locally dense $\mathcal{L}$-structure with definable Skolem functions. Let $A \subset M^{m+n}$ be a definable set. There exists a number $N = N(A)$, such that each fiber $A(\mathbf{u})$ has at most N isolated points.*

As a final application, we obtain the following strong connection between o-minimality an t-minimality.

6.12. Corollary. *Let M be a strongly t-minimal and locally dense $\mathcal{L}$-structure with definable Skolem functions. If M admits a definable (total) order relation which induces the same topology as its definable topology, then M is o-minimal.*

Proof. Let $f\colon M \to M$ be a definable map. By Theorem 6.7, the set D of discontinuities of f is finite. This applies in particular to the characteristic function $f = \chi_A$ of a definable set $A \subset M$. More precisely, fix two different points $0, 1 \in M$ and set $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ in the remaining case. Let $x_1 < \cdots < x_n$ be the discontinuities of χ_A. Since these are also the discontinuities of χ_A in the order topology, we get that A is a union of some of the intervals $]x_i, x_{i+1}[$ (where $x_0 = -\infty$ and $x_{n+1} = +\infty$) and some of the points $x_1, \ldots, x_n$. This shows that M is o-minimal. □

Note that this argument actual shows that an (infinite) ordered structure M is o-minimal if, and only if, each definable map $f\colon M \to M$ has finitely many discontinuities.

7. Ind-Definable Topologies

The notion of a definable topology Δ is too weak if the language is infinite, since the formula defining Δ only quotes finitely many symbols. For instance, to obtain cell decomposition in the p-adics, one has to work with the language of valued fields together with a predicate for every n-th power subgroup of the multiplicative group. This is because the p-adics only have Quantifier Elimination in this language, as shown by Macintyre in [9]. The following definition repairs this short coming.

7.1. Definition (Ind-definable Topology)**.** Let M be a first order structure in a language $\mathcal{L}$. We say that M has an *ind-definable topology*, if there exists a collection $\mathcal{D}$ of $\emptyset$-definable subsets $\Delta \subset M^{s(\Delta)+1}$ so that every $\Delta \in \mathcal{D}$ gives rise to a definable topology on M and so that for any $\Delta, \Gamma \in \mathcal{D}$, we can find a $\Theta \in \mathcal{D}$ with the following property. If $\mathbf{u} \in M^{s(\Delta)}$ and $\mathbf{v} \in M^{s(\Gamma)}$, and if $a \in \Delta(\mathbf{u}) \cap \Gamma(\mathbf{v})$, then we can find $\mathbf{w} \in M^{s(\Theta)}$, such that

$$a \in \Theta(\mathbf{w}) \subset \Delta(\mathbf{u}) \cap \Gamma(\mathbf{v}). \tag{26}$$

By abuse of terminology, we will also call $\mathcal{D}$ an *ind-definable topology.* We call any set of the form $\Delta(\mathbf{u})$ with $\Delta \in \mathcal{D}$ and $\mathbf{u} \in M^{s(\Delta)}$, a $\mathcal{D}$-fiber. As a basis of opens for the topology on M we take finite intersections of $\mathcal{D}$-fibers.

Note that these conditions give an axiom scheme for ind-definable topologies, with a sentence for each $\Delta \in \mathcal{D}$ expressing that it is a definable topology and a sentence for each pair $\Delta, \Gamma \in \mathcal{D}$ expressing the existence of $\Theta \in \mathcal{D}$ satisfying condition (26). In particular, if M^* is elementary equivalent with M, then $\mathcal{D}$ gives rise to an ind-definable topology on M^*.

7.2. Definition. Let M be a first order structure in some language $\mathcal{L}$ with an ind-definable topology $\mathcal{D}$. We say that M is *t-minimal*, if every definable subset A of M is a finite union of $\mathcal{D}$-fibers and singletons. In other words, we can find $\Delta_i \in \mathcal{D}$, $\mathbf{u}_i \in M^{s(\Delta_i)}$ and a finite set $F \subset A$, such that

$$A = \Delta_1(\mathbf{u}_1) \cup \cdots \cup \Delta_p(\mathbf{u}_p) \cup F.$$

We say that M is *strongly t-minimal*, if every structure elementary equivalent with M is t-minimal in the induced ind-definable topology. By the same argument as before, in order for M to be strongly t-minimal, it suffices to check that every ultrapower of M is t-minimal.

We call $\mathcal{D}$ *locally dense*, if each $\Delta \in \mathcal{D}$ is.

7.3. Definition. Cells and continuous cells are defined in an analogous way using at each step some $\Delta \in \mathcal{D}$. More precisely, we employ the following recursive definition for $C \subset M^m$ to be a *(continuous) cell*. If $m = 1$ then any $\mathcal{D}$-fiber is a continuous cell. If $m > 1$, then C is a (continuous) cell of type 1, if there exists a (continuous) cell D in M^{m-1}, a $\Delta \in \mathcal{D}$ and a (continuous) definable map $a\colon D \to M^{s(\Delta)}$, such that C consists of all tuples $(\mathbf{x}, y) \in M^m$, for which $\mathbf{x} \in D$ and $y \in \Delta(a(\mathbf{x}))$. The dimension of C is defined to be the dimension of D plus one. It is a (continuous) cell of type 0 and dimension equal to $\dim D$, if instead a has range M and C consists of all tuples $(\mathbf{x}, y) \in M^m$, for which $\mathbf{x} \in D$ and $y = a(\mathbf{x})$.

Note that different members of $\mathcal{D}$ can occur in a cell, but only one in each dimension.

The reader is invited to check that all the proofs work in this more general setup, so that the following theorem holds.

7.4. Theorem. *Let M be a first order structure with an ind-definable topology. If M has definable Skolem functions and is strongly t-minimal, then any definable set A can be written as a finite union of cells. If $\mathcal{D}$ is moreover locally dense and f is a definable map with domain A, then we can choose a decomposition of A in continuous cells C_i, such that the restriction of f to each C_i is continuous.*

Moreover, each of these decompositions is uniform in definable families and all properties proven so far for definable topologies remain true for this extended notion.

7.5. ***p*-adic Fields.** Let us show how the p-adics are strongly t-minimal in this more general sense. Let us denote the valuation of an element by $v(a)$. Observe that $v(a) \leq v(b)$ if, and only if, $a^2 + pb^2$ is a square. Therefore, we can express the clause $v(a) \leq v(b)$ in the language of fields, so that we need not to consider a second sort for the value group (in fact, below we will introduce a predicate symbol P_2 for the set of non-zero squares, so that we can use this predicate to express the clause $v(a) \leq v(b)$). However, for notational clarity, we will keep using the valuation symbol. We put $v(0) = +\infty$, with the usual assumptions that $+\infty$ is larger than any integer.

Let Δ be the subset of $\mathbb{Q}_p^5$ consisting of all tuples $(x_1, \dots, x_4, y)$, satisfying one (and only one) of the following three conditions

$$v(x_3) \le v(x_1 y + x_2) \le v(x_4) \quad \wedge \quad v(x_3) \le v(x_4), \tag{27}$$

$$v(x_1 y + x_2) \le v(x_4) \quad \wedge \quad v(x_3) = v(x_4) + 1, \tag{28}$$

$$v(x_3) \le v(x_1 y + x_2) \quad \wedge \quad v(x_4) + 1 < v(x_3). \tag{29}$$

We will call an arbitrary fiber of Δ an *annulus*. One checks that Δ is a locally dense definable topology, inducing the p-adic topology on $\mathbb{Q}_p$. Unfortunately, $\mathbb{Q}_p$ is not t-minimal in this topology. The simplest counterexample is given by the set P_2 of non-zero squares; it cannot be written as a finite union of annuli and singletons. If one adjoins new unary predicate symbols P_n to the language (with $n = 2, 3, \dots$) interpreting the non-zero n-th powers, then $\mathbb{Q}_p$ admits Quantifier Elimination in this expanded language $\mathcal{L}_{\mathrm{Mac}}$ by [9].

Let $\Delta_n \subset \mathbb{Q}_p^5$, for $n = 2, 3, \dots$, be defined as the set of all tuples $(x_1, \dots, x_4, y) \in \Delta$ for which $x_1 y + x_2$ is a non-zero n-th power. Let $\delta_n(x_1, \dots, x_4, y)$ be the corresponding formula in $\mathcal{L}_{\mathrm{Mac}}$ defining Δ_n. I claim that the collection $\mathcal{D}_{\mathrm{Mac}}$ of all Δ_n is a locally dense strongly t-minimal ind-definable topology. Therefore, since $\mathbb{Q}_p$ also admits definable Skolem functions by [4] (in the language of fields or in its definable expansion $\mathcal{L}_{\mathrm{Mac}}$), we obtain that Theorem 7.4 is true for the field of p-adics. I will simply indicate the arguments for proving the claim since these are well-known among specialists (see for instance [5, 9, 11, 15]).

Firstly, $\mathcal{D}_{\mathrm{Mac}}$ is an ind-definable topology, since the P_n-cosets are open, whence contain arbitrary small annuli. Next, one easily verifies that each Δ_n is locally dense. Therefore, remains to give the argument that $\mathbb{Q}_p$ is strongly t-minimal. As in Theorem 3.4, strong t-minimality is equivalent with showing that the structure is t-minimal in a uniform way. In the case of the p-adics this will follow from our argument below, since the number of disjuncts will only depend on the degrees of the polynomials in the original formula (and p). Let A be a definable subset of $\mathbb{Q}_p$. By Quantifier Elimination, it is defined by a quantifier free $\mathcal{L}_{\mathrm{Mac}}$-formula $\alpha(y)$ in a single variable y. Our goal is to describe A as a union of $\mathcal{D}_{\mathrm{Mac}}$-fibers (or, alternatively, show that α is equivalent with a disjunction of formulae $\delta_n(a_1, \dots, a_4, y)$). Let $\mathfrak{R}_n$ be a complete set of coset representatives of the subgroup P_n of the multiplicative group $\mathbb{Q}_p^\times$, with the understanding that 1 is the representative of the coset P_n. Then $\neg P_n(a)$ is equivalent with the disjunction of all $P_n(\xi a)$, where $1 \neq \xi \in \mathfrak{R}_n$. In this way, we may replace any negation by a disjunction.

Let $f(y)$ be a polynomial. I will explain briefly how to rewrite $P_n(f(y))$ and $v(f(y))$ using only linear factors. Write $f(y)$ as a product of a nowhere vanishing polynomial $g(y)$ and of powers of linear terms $(y - a_i)^{m_i}$. One can find an annulus of the form $v(y) \le N$ not containing any root of f (in an algebraic closure of $\mathbb{Q}_p$). Consequently, $v(f(y)) = v(y)$ for all such y. Moreover, the coset of $f(y)$ modulo P_n depends only on y^d, where d is the degree of f. In turn, the P_n-coset of y^d only depends on the P_m-coset of y, where m is obtained from n by dividing out the greatest common factor with d. Around each root $a = a_i$ one can find a small enough annulus, so that the same is true for $f(y)/(y - a)^m$ in terms of $y - a$, so that one only has to consider on that disk the behavior of $(y - a)^m$. By a similar argument as before, statements about $P_n(f(y))$ and $v(f(y))$ can be translated into statements about $v(y - a)$ and P_m-cosets of $y - a$.

Finally, we have to deal with conjunctions of the form $P_n(ay+b) \wedge P_m(cy+d)$ (or similar conjunctions of norm-inequalities between linear factors). Using the least common multiple of m and n, we may assume that $m = n$ (actually, we can only assume that $m^k = n^l$ for some k, l, but this does not alter the argument much). For $v(ay + b)$ large or small enough, the P_n-coset of $cy + d$ only depends on the P_n-coset of $ay + b$ (and conversely). The intermediate annulus can be written as a union of smaller annulus on which $ay + b$ and $cy + d$ lie in fixed cosets. A similar argument can be used for conjunctions of norm-inequalities. This completes the argument that $\mathbb{Q}_p$ is strongly t-minimal.

To give a non-trivial example, observe that $\mathbb{Q}_p - \{0, 1\}$ is the union of the following Δ_2-fibers:

$$v(x) < 0 \tag{30}$$

$$v(x) \geq 1 \quad \wedge \quad P_2(x^2) \tag{31}$$

$$v(x-1) \geq 1 \quad \wedge \quad P_2((x-1)^2) \tag{32}$$

$$v(x-j) \geq 1, \qquad \text{for } j = 2, \dots, p-1. \tag{33}$$

Note that $P_2(x^2)$ holds for all x except for $x = 0$.

8. Connected Components

A topological space X is called *connected*, if it cannot be written as the union of two disjoint, non-empty open subsets. A subset A of X is called *connected*, if A with its induced topology is connected. In other words, A is connected if for any two open sets U and V which are not disjoint from A and such that $A \subset U \cup V$, we have that $A \cap U \cap V \neq \emptyset$. If A is connected, then so is its closure $\bar{A}$.

A subset S of A is called a *connected component*, if S is connected and S is maximal with this property (i.e., if $S \subsetneq T \subset A$ implies that T is not connected). One verifies easily (using for instance Lemma 8.2 below) that S is a connected component, if and only if, S is connected and has the property that any connected set which is not disjoint from it, must belong to it. A connected component is therefore closed, for its closure is again connected whence contained in it.

8.1. Definition (Pseudo-intersection)**.** Let X be a topological space and let A and B be subsets of X. We introduce the *pseudo-intersection* of A and B as the set

$$(\overline{A} \cap B) \cup (A \cap \overline{B})$$

and we denote it by $A \overline{\cap} B$.

8.2. Lemma. *Let X be a topological space and let A and B be non-empty connected subsets of X. Then $A \cup B$ is connected, if and only if, $A \overline{\cap} B \neq \emptyset$.*

Proof. Set $S = A \cup B$. We prove the contrapositive of the assertion. Hence assume first that $A \overline{\cap} B = \emptyset$. Since $\overline{A} \cap B$ is empty, we can find for every $x \in B$ an open U_x containing x, for which $U_x \cap A = \emptyset$. Let U be the union of all U_x, so that U is an open with $B \subset U$ and $A \cap U = \emptyset$. Similarly, we can find an open V containing A and disjoint from B. In particular, $S \cap U = A$ and $S \cap V = B$. Therefore $S \subset U \cup V$ and $S \cap U \cap V = A \cap B \subset A \overline{\cap} B = \emptyset$, showing that S is not connected. (Moreover, A and B are its two connected components).

Conversely, assume that S is not connected, so that there exists opens U and V with $S \subset U \cup V$ and $S \cap U \cap V = \emptyset$, whereas S is not contained in U nor in V. Since

A is connected, it must be contained in either U or V, say $A \subset U$. The same holds true for B, so that necessarily $B \subset V$ (the case $B \subset U$ is excluded since otherwise $S \subset U$). Suppose $x \in \overline{A} \cap B$. From $B \subset V$ it follows that x lies in the open V. As $x \in \overline{A}$, we must therefore have that $A \cap V$ is non-empty. On the other hand, since $A \subset U$, we have that $A \cap V \subset S \cap U \cap V = \emptyset$, contradiction. This shows that $\overline{A} \cap B$ must be empty and so must $A \cap \overline{B}$ be, by symmetry. □

8.3. Definition (Proximity Graph)**.** Let X be a topological space and let $\mathcal{U}$ be a finite covering of X (we do not require that the members of $\mathcal{U}$ be open). To this data we attach a graph $\Gamma_{\mathcal{U}}$, called the *proximity graph* of the covering $\mathcal{U}$. Its vertices are the members of $\mathcal{U}$ and there is an edge between any two members U and V of $\mathcal{U}$ precisely when $U \overline{\cap} V \neq \emptyset$.

8.4. Proposition. *Let X be a topological space and let $\mathcal{U}$ be a finite covering of X. Assume that each member of $\mathcal{U}$ is connected. Let Ψ be a connected component of the proximity graph $\Gamma_{\mathcal{U}}$ of $\mathcal{U}$. Let $X(\Psi)$ be the union of all $A \in \mathcal{U}$ which occur as a vertex in Ψ. Then $X(\Psi)$ is a connected component of X and every connected component occurs in this way. More precisely, if $\Psi_1, \dots, \Psi_n$ are all connected components of $\Gamma_{\mathcal{U}}$, then $X(\Psi_1) \sqcup \cdots \sqcup X(\Psi_n)$ is the decomposition of X in its connected components.*

Proof. With a *connected component* of a finite graph Γ we mean a full subgraph Ψ with the property that any two of its vertices can be connected by a path (of edges) and Ψ is maximal with this property. Let Ψ be any full subgraph of $\Gamma_{\mathcal{U}}$ and let $X(\Psi)$ be the union of all $A \in \mathcal{U}$ which occur as vertex in Ψ. I claim that if Ψ is connected (i.e., there is an edge-path between any two vertices), then $X(\Psi)$ is connected. To prove the claim, we induct on the number m of vertices in Ψ. There is nothing to prove if $m = 1$, for we assumed that all members of $\mathcal{U}$ are connected. Assume $m > 1$. Let A be a vertex of Ψ and let Ψ' be the full subgraph of Ψ obtained by deleting the vertex A and all edges ending in A. Since all graphs under consideration are finite, we can choose A in such way that Ψ' remains connected. By induction $X(\Psi')$ is connected. Since there was at least one edge between A and some vertex B of Ψ', we have that $A \overline{\cap} B \neq \emptyset$. Therefore also $A \overline{\cap} X(\Psi')$ is non-empty. By Lemma 8.2, it then follows that $A \cup X(\Psi')$ is connected, and since the latter is just $X(\Psi)$, we proved the claim.

Assume next that Ψ is a connected component of $\Gamma_{\mathcal{U}}$. I claim that $X(\Psi)$ is closed. Assume not, so that there exists a point $x \notin X(\Psi)$ belonging to the closure of $X(\Psi)$. In particular, since $X(\Psi)$ is a finite union of members of $\mathcal{U}$, there must exist an $A \in \mathcal{U}$ which is a vertex of Ψ and for which $x \in \overline{A}$. On the other hand, there is some $B \in \mathcal{U}$ containing x. By assumption, B is not a vertex of Ψ. However, the existence of x shows that $A \overline{\cap} B \neq \emptyset$, so that A and B are related by an edge in $\mathcal{U}$, contradicting that Ψ is a connected component.

Finally, let $\Psi_1, \dots, \Psi_n$ be the connected components of $\Gamma_{\mathcal{U}}$. Since $\mathcal{U}$ is a covering, we have that

$$X = X(\Psi_1) \cup \cdots \cup X(\Psi_n). \tag{34}$$

From the above it follows that each $X(\Psi_i)$ is connected and closed. If $X(\Psi_i) \cap X(\Psi_j) \neq \emptyset$, for $i \neq j$, then there would be a vertex A_i of Ψ_i and a vertex A_j of Ψ_j which are not disjoint (considered as subsets of X). Therefore A_i and A_j would be connected by an edge in $\mathcal{U}$, which is impossible. In other words, (34) is a disjoint union of closed sets. Taking complements we see that each $X(\Psi_i)$ is also open.

Since the $X(\Psi_i)$ are also connected, it follows that (34) is the decomposition in connected components, as required. □

8.5. Lemma. *Let M be a t-minimal structure and let A be a definable subset of M. Assume that every Δ-fiber is connected. Then every connected component of A is also definable.*

Proof. Suppose

$$A = \{a_1, \dots, a_p\} \cup \Delta(\mathbf{u}_1) \cup \dots \cup \Delta(\mathbf{u}_p)$$

Let S be a connected component of A. If some $\Delta(\mathbf{u}_i)$ meets S, then it must be contained in S, since we assumed that every Δ-fiber is connected. It follows that S is the union of some of the $\Delta(\mathbf{u}_i)$ and some of the points a_i, showing that S is definable. □

8.6. Theorem. *Let M be a strongly t-minimal structure with definable Skolem functions and connected Δ-fibers. Let $f\colon A \subset M^{m+1} \to M^m$ be a definable map. There exist definable maps $g_i\colon B_i \subset M^{m+1} \to M^m$, for $i = 1, \dots, p$, such that for each $\mathbf{u} \in M^m$, the non-empty sets among the $g_1^{-1}(\mathbf{u}), \dots, g_p^{-1}(\mathbf{u})$ are the connected components of $f^{-1}(\mathbf{u})$.*

Proof. By strong t-minimality, there exists a $p \in \mathbb{N}$, so that each fiber $f^{-1}(\mathbf{u})$ can be written as the union of at most p sets C_i which are either singletons or Δ-fibers. Since any connected component of $f^{-1}(\mathbf{u})$ is then a union of some of the C_i by the proof of Lemma 8.5, $f^{-1}(\mathbf{u})$ has at most p connected components. Choose p as small as possible. We will induct on p. If $p = 1$, then there is nothing to prove, so assume $p > 1$.

Choose a definable section $s\colon f(A) \to M^{m+1}$. For each $\mathbf{u} \in f(A)$, let $S(\mathbf{u})$ be the connected component of $f^{-1}(\mathbf{u})$ containing $s(\mathbf{u})$. Fix $\mathbf{u} \in f(A)$ and let $\varphi(x, \mathbf{u})$ be the formula expressing that there exist $\mathbf{u}_1, \dots, \mathbf{u}_p \in M^s$ and $a_1, \dots, a_t \in M$ with the following properties. Let us temporarily denote by $\mathcal{U}$ the collection of all $\Delta(\mathbf{u}_i)$ and of all singletons $\{a_i\}$. Let S be the union of the members of $\mathcal{U}$. The formula $\varphi(x, \mathbf{u})$ states that $x \in S$ with the additional requirements that $s(\mathbf{u}) \in S \subset f^{-1}(\mathbf{u})$ and that the proximity graph of $\mathcal{U}$ is connected. Moreover, S has to be maximal with these properties. It follows from Proposition 8.4 that the set defined by the formula $\varphi(x, \mathbf{u})$ is $S(\mathbf{u})$. Let B_1 be the subset of M^{m+1} consisting of all $(\mathbf{u}, x)$ with $\mathbf{u} \in f(A)$ and $x \in S(\mathbf{u})$ and let g_1 be the restriction of f to B_1. By construction, each (non-empty) $g_1^{-1}(\mathbf{u})$ is a connected component of $f^{-1}(\mathbf{u})$. Let f_1 be the restriction of f to $A - B_1$. Each fiber $f_1^{-1}(\mathbf{u})$ has at most $p - 1$ connected components, and the proof is now finished by induction on p. □

8.7. Lemma. *Let M be a topological space and D a connected subset of M^m. If a map $f\colon D \to M^n$ is continuous, then its graph is connected.*

Proof. Let $\Gamma \subset D \times M^n$ be the graph of f. Towards a contradiction, assume that there exist two opens $U_1, U_2 \subset M^{m+n}$ not containing Γ, but such that $\Gamma \subset U_1 \cup U_2$ and $\Gamma \cap U_1 \cap U_2 = \emptyset$. Let D_i be the collection of all $x \in D$ for which $(x, f(x)) \in U_i$. Our assumptions imply that D is the disjoint union of D_1 and D_2 and that neither one of them is empty. I claim that each D_i is open in D, which then would contradict the fact that D is connected. Let us show that D_1 is open. Let $x \in D_1$, so that $(x, f(x)) \in U_1$. Hence there exist opens $X \subset M^m$ and $Y \subset M^n$ with $(x, f(x)) \in X \times Y \subset U_1$. Since f is continuous, $f^{-1}(Y)$ is open, so that after shrinking X is necessary, we may assume that $X \cap D \subset f^{-1}(Y)$. Take an arbitrary $a \in D \cap X$. Then $f(a) \in Y$, so that $(a, f(a)) \in X \times Y \subset U_1$. Therefore, $a \in D_1$. In other words, we showed that $D \cap X \subset D_1$, whence that D_1 is open in D. □

8.8. Proposition. *Let M be a first order structure with a definable topology Δ. If Δ has a definable and continuous section and if each Δ-fiber is connected, then any continuous cell is connected.*

Proof. Let $C \subset M^m$ be a continuous cell. We induct on m to show that C is connected. If $m = 1$, then this is part of our hypothesis, so that we may assume that $m > 1$. If C is of type 0, then we are done by Lemma 8.7 and our induction hypothesis. Therefore assume that C is of type 1, so that it consists of all tuples $(\mathbf{x}, y) \in M^m$ such that $\mathbf{x} \in D$ and $y \in \Delta(a(\mathbf{x}))$, for some continuous cell D in M^{m-1} and some continuous definable map $a\colon D \to M^s$. We have to show that given opens $U_1, U_2 \subset M^m$, for which $C \subset U_1 \cup U_2$ and $C \cap U_1 \cap U_2 = \emptyset$, then C lies entirely inside one of U_i. Let $\mathbf{x} \in D$. Since Δ-fibers are connected, we must have that $C(\mathbf{x}) = \Delta(a(\mathbf{x}))$ lies entirely in U_1 or in U_2. Let $\delta\colon M^s \to \Delta$ be a continuous and definable section of Δ. Consider the composite map

$$D \xrightarrow{a} M^s \xrightarrow{\delta} \Delta \xrightarrow{\pi^{(s+1)}_{\{s+1\}}} M.$$

This is a continuous map, so that by Lemma 8.7 and our induction hypothesis, its graph Γ is connected. By construction, $\Gamma \subset C$, so that Γ must lie entirely in one of the two opens, say in U_1. But this means that any fiber $C(\mathbf{x})$ has one point in common with U_1, so, by our previous observation, must lie entirely in U_1. Therefore, $C \subset U_1$, as required. □

8.9. Corollary. *Let M be a locally dense, strongly t-minimal structure with definable Skolem functions. Suppose moreover that Δ has a definable and continuous section and that each Δ-fiber is connected. If $A \subset M^m$ is a definable set, then any connected component of A is also definable.*

Proof. By Theorem 6.4, we can write A as a finite union of continuous cells C_i. Let A_0 be a connected component of A. Since by Proposition 8.8 each C_i is connected, it must be either entirely inside A_0 or disjoint from it. In other words, A_0 is the union of some of the C_i, whence is definable. □

8.10. Theorem. *Let M be a strongly t-minimal structure with definable Skolem functions. Suppose moreover that Δ has the following properties.*

8.10.1. *Each Δ-fiber is connected.*
8.10.2. *The union of two non-disjoint Δ-fibers is again a Δ-fiber.*
8.10.3. *Two Δ-fibers $\Delta(\mathbf{u})$ and $\Delta(\mathbf{v})$ are equal if, and only if, $\mathbf{u} = \mathbf{v}$.*

Then M has weak uniform elimination of imaginaries.

Proof. By an argument similar to [8, Lemma 4.4.3], it suffices to show that M is *uniformly weakly* 1*-eliminable*. With this we mean that for every definable set $A \subset M^{m+1}$, there exists a definable subset $B \subset M^{n+1}$ with the following property. For each $\mathbf{x} \in M^m$, there exists a finite subset $\Lambda_{\mathbf{x}} \subset M^n$, such that for an arbitrary $\mathbf{y} \in M^n$, we have an equality $A(\mathbf{x}) = B(\mathbf{y})$ if, and only if, $\mathbf{y} \in \Lambda_{\mathbf{x}}$. In other words, a definable family of definable subsets of M can be reparametrized, such that each member in the family has a finite set of parameters. Note that $A = \Delta$ already satisfies these requirements in view of (8.10.3).

Let C be an arbitrary definable subset of M; later we will take C to be an A-fiber. By t-minimality, we can write C as a finite union

$$C = \Delta(\mathbf{u}_1) \cup \cdots \cup \Delta(\mathbf{u}_p) \cup \{v_1, \dots, v_q\}. \tag{35}$$

If some v_i is an interior point of C, then we can find $\Delta(\mathbf{w}) \subset C$ containing v_i. In this way, we may replace all interior points v_i by some Δ-fiber and still maintain a decomposition of C. So, we may assume that in (35), all the v_i are not interior. After applying (8.10.2), we may furthermore assume that all the $\Delta(\mathbf{u}_i)$ are disjoint. Using (8.10.1), we conclude that each $\Delta(\mathbf{u}_i)$ is a connected component of the interior $\operatorname{int} C$ of C. Together with (8.10.3) this implies that the set $\{\mathbf{u}_1, \dots, \mathbf{u}_p, v_1, \dots, v_q\}$ is uniquely determined by C.

I claim that by strong t-minimality, we can do this uniformly in $\mathbf{x} \in M^m$, for all $C = A(\mathbf{x})$. More precisely, we can find $p, q \in \mathbb{N}$ and definable maps $f_i \colon V_i \subset M^m \to M^s$ and $g_i \colon W_i \subset M^m \to M$, such that for each $\mathbf{x} \in M^m$, we have that

$$A(\mathbf{x}) = \Delta(f_1(\mathbf{x})) \cup \cdots \cup \Delta(f_p(\mathbf{x})) \cup \{g_1(\mathbf{x}), \dots, g_q(\mathbf{x})\}. \tag{36}$$

with the understanding that we omit in this union the contribution of $\Delta(f_i(\mathbf{x}))$ or $g_i(\mathbf{x})$ whenever $\mathbf{x}$ does not lie in the domain of f_i or g_i. Indeed, by strong t-minimality, both the cardinality of the boundary and the number of connected components of the interior are uniformly bounded in the family of A-fibers. By an argument similar to the one used in Lemma 8.5, we can easily express in a first order way that $\Delta(\mathbf{u})$ is a connected component of the interior, or that v is a point in $A(\mathbf{x}) - \operatorname{int} A(\mathbf{x})$. Since we have definable Skolem functions, the claim then follows readily.

By partitioning M^m according to whether $\mathbf{x}$ lies in the domain of some f_i or g_i or not, we may assume that all f_i and g_i have the same domain V (which is then necessarily $\pi^{(m+1)}_{\{1,\dots,m\}}(A)$). In other words, each non-empty fiber $A(\mathbf{x})$ of A has a decomposition (36) for fixed p and q. Let B be the definable subset of M^{sp+q+1} given as the collection of all tuples $(\mathbf{u}_1, \dots, \mathbf{u}_p, v_1, \dots, v_q, x)$, where $\mathbf{u}_i \in M^s$ and $v_i, x \in M$, with the property that $x \in C$, where C is as in (35) with all the assumptions previously made, that is to say, so that the $\Delta(\mathbf{u}_i)$ are the connected components of $\operatorname{int} C$ and v_i are distinct points in $C - \operatorname{int} C$. Now, for $\mathbf{x} \in M^m$, let $\Lambda_{\mathbf{x}}$ consist of all $sp + q$-tuples of the form

$$(f_{\sigma(1)}(\mathbf{x}), \dots, f_{\sigma(p)}(\mathbf{x}), v_{\tau(1)}, \dots, v_{\tau(q)})$$

with σ a permutation of $\{1, \dots, p\}$ and τ a permutation of $\{1, \dots, q\}$. From (36), it follows, for $\mathbf{x} \in M^m$ and $\mathbf{y} \in M^{sp+q}$, that $A(\mathbf{x}) = B(\mathbf{y})$ if, and only if, $\mathbf{y} \in \Lambda_{\mathbf{x}}$. □

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