where the first module vanishes by induction. As above, the kernel of g is easily seen to be $B/(IB:a)$, so that our assumption on the colon ideals implies that δ is the zero map, whence $\mathrm{Tor}_1^A(B, A/J) = 0$ as we wanted to show. □

Here is a nice 'descent type' application of this criterion:

Corollary 3.3.15. *Let $A \to B \to C$ be homomorphisms whose composition is flat. If $B \to C$ is cyclically pure, then $A \to B$ is flat. In fact, it suffices that $B \to C$ is* cyclically pure with respect to ideals extended from A, *that is to say, that $JB = JC \cap B$ for all ideals $J \subseteq A$.*

Proof. Given an ideal $I \subseteq A$ and an element $a \in A$, we need to show in view of Theorem 3.3.14 that $(IB : a) = (I : a)B$. One inclusion is immediate, so take y in $(IB : a)$. By the same theorem, we have $(IC : a) = (I : a)C$, so that y lies in $(I : a)C \cap B$ whence in $(I : a)B$ by cyclical purity. □

The next criterion will be useful when dealing with non-Noetherian algebras in the next chapter. We call an ideal J in a ring B *finitely related*, if it is of the form $J = (I : b)$ with $I \subseteq B$ a finitely generated ideal and $b \in B$.

Theorem 3.3.16. *Let A be a Noetherian ring and B an arbitrary A-algebra. Suppose $\mathscr{P}$ is a collection of prime ideals in B such that every proper, finitely related ideal of B is contained in some prime ideal belonging to $\mathscr{P}$. If $A \to B_{\mathfrak{p}}$ is flat for every $\mathfrak{p} \in \mathscr{P}$, then $A \to B$ is flat.*

Proof. By Theorem 3.3.14, we need to show that $(IB : a) = (I : a)B$ for all $I \subseteq A$ and $a \in A$. Put $J := (I : a)$. Towards a contradiction, let x be an element in $(IB : a)$ but not in JB. Hence $(JB : x)$ is a proper, finitely related ideal, and hence contained in some $\mathfrak{p} \in \mathscr{P}$. However, $(IB_{\mathfrak{p}} : a) = JB_{\mathfrak{p}}$ by flatness and another application of Theorem 3.3.14, so that $x \in JB_{\mathfrak{p}}$, contradicting that $(JB : x) \subseteq \mathfrak{p}$. □

We can also derive a coherency criterion due to Chase ([21]):

Corollary 3.3.17. *A ring is coherent if and only if every finitely related ideal is finitely generated.*

Proof. The direct implication is a simple application of the coherency condition. For the converse, suppose every finitely related ideal is finitely generated. We will prove that $R \to R_\natural$ is flat, where $R_\natural$ is an ultrapower of R, from which it follows that R is coherent by Theorem 3.3.4. To prove flatness, we use the Colon Criterion, Theorem 3.3.14. To this end, let $I \subseteq R$ be finitely generated and let $a \in R$. We have to show that if b lies in $(IR_\natural : a)$ then it already lies in $(I : a)R_\natural$. Let b_w be an approximation of b. By Łoś' Theorem, almost each $b_w \in (I : a)$. By assumption, the colon ideal $(I : a)$ is finitely generated, say by $f_1, \dots, f_s$, and hence we can find c_{iw} such that $b_w = c_{1w}f_1 + \dots + c_{sw}f_s$. Let $c_i \in R_\natural$ be the ultraproduct of the c_{iw}, for each $i = 1, \dots, s$. By Łoś' Theorem, $b = c_1 f_1 + \dots + c_s f_s$, showing that it belongs to $(I : a)R_\natural$. □

3.3.6 *Local criterion for flatness.*

For finitely generated modules, we have the following criterion:

Theorem 3.3.18 (Local flatness theorem–finitely generated case). *Let R be a Noetherian local ring with residue field k. If M is a finitely generated R-module whose first Betti number vanishes, that is to say, if $\mathrm{Tor}_1^R(M,k) = 0$, then M is flat.*

Proof. Take a minimal free resolution

$$\cdots \to F_1 \to F_0 \to M \to 0$$

of M, that is to say, such that the kernel of each boundary map $d_i\colon F_i \to F_{i-1}$ lies inside $\mathfrak{m}F_i$. Therefore, since tensoring this complex with k yields the zero complex, the rank of F_i is equal to the *i-th Betti number* of M, that is to say, the vector space dimension of $\mathrm{Tor}_i^R(M,k)$. In particular, F_1 has rank zero, so that $M \cong F_0$ is free whence flat. □

There is a much stronger version of this result, where we may replace the condition that M is finitely generated over R by the condition that M is finitely generated over a Noetherian local R-algebra S (see for instance [69, Theorem 22.3] or [27, Theorem 6.8]). We will present here a new proof, for which we need to make some further definitions. The method is an extension of the work in [93], which primarily dealt with detecting finite projective dimension.

Let A be a (not necessarily Noetherian) ring, and let $\mathbf{mod}_A$ be the class of all finitely presented A-modules. We will call a subclass $\mathbf{N} \subseteq \mathbf{mod}_R$ a *deformation class* if it is closed under isomorphisms, direct summands, extensions, and deformations, that is to say, if it is closed under the following respective rules:[2]

- 3.3.18.i. if N belongs to $\mathbf{N}$ and $M \cong N$, then M belongs to $\mathbf{N}$;
- 3.3.18.ii. if $N \cong M \oplus M'$ belongs to $\mathbf{N}$, then so do M and M';
- 3.3.18.iii. if $0 \to K \to M \to N \to 0$ is an exact sequence in $\mathbf{mod}_R$ with $K, N \in \mathbf{N}$, then also $M \in \mathbf{N}$;
- 3.3.18.iv. if x is an M-regular element in the Jacobson radical of A and M/xM belongs to $\mathbf{N}$, then so does M.

Recall that the *Jacobson radical* of A is the intersection of all its maximal ideals; equivalently, it is the ideal of all x such that $1+ax$ is unit for all a. Condition 3.3.18.iv holds vacuously, if the Jacobson radical is equal to the *nilradical*, the ideal of all nilpotent elements. Clearly, $\mathbf{mod}_A$ itself is a deformation class. We leave it as an easy exercise to show that:

3.3.19 *Any intersection of deformation classes is again a deformation class. In particular, any class $\mathbf{K} \subseteq \mathbf{mod}_A$ sits inside a smallest deformation class, called the* deformation class of $\mathbf{K}$. □

[2] A class satisfying the first three conditions is called a *net* in [93].

Let us call a subclass $\mathbf{K} \subseteq \mathbf{mod}_A$ *deformationally generating*, if its deformation class is equal to $\mathbf{mod}_A$, and *quasi-deformationally generating*, if its deformation class contains all cyclic modules of the form A/I with $I \subseteq A$ finitely generated. One easily shows, by induction on the number of generators, that if A is coherent, deformationally generating and quasi-deformationally generating are equivalent notions.

Proposition 3.3.20. *If R is a Noetherian local ring, then its residue field is deformationally generating.*

Proof. We need to show that any finitely generated module M belongs to the deformation class $\mathbf{N}$ generated by the residue field. Since any module generated by n elements is the extension of two modules generated by less than n elements, an induction on n using (3.3.18.iii) reduces to the case $n = 1$, that is to say, $M = R/\mathfrak{a}$. Suppose the assertion is false, and let $\mathfrak{a}$ be a maximal counterexample. If $\mathfrak{a}$ is not prime, then for $\mathfrak{p}$ a minimal prime ideal $\mathfrak{p}$ of $\mathfrak{a}$, we have an exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{a} \to R/\mathfrak{a}' \to 0$$

for some $\mathfrak{a}' \subseteq R$ strictly containing $\mathfrak{a}$. The two outer modules belong to $\mathbf{N}$ by maximality, whence so does the inner one by (3.3.18.iii), contradiction. Hence $\mathfrak{a}$ is a prime ideal, which therefore must be different from the maximal ideal of R. Let x be an element in the maximal ideal not in $\mathfrak{a}$. By maximality $R/(\mathfrak{a}+xR)$ belongs to $\mathbf{N}$, whence so does $R/\mathfrak{a}$ by (3.3.18.iv), since x is $R/\mathfrak{a}$-regular, contradiction again. □

The main flatness criterion of this section is:

Theorem 3.3.21. *Let $A \to B$ be a homomorphism sending the Jacobson radical of A inside that of B, and let $\mathbf{K} \subseteq \mathbf{mod}_A$ be quasi-deformationally generating. A coherent B-module Q is flat over A if and only if $\mathrm{Tor}_1^A(Q, M) = 0$ for all $M \in \mathbf{K}$.*

Proof. One direction is immediate, so we only need to show the direct implication. Define a functor $\mathscr{F}$ on $\mathbf{mod}_R$, by $\mathscr{F}(M) := \mathrm{Tor}_1^A(Q, M)$. By Theorem 3.1.5, it suffices to show that $\mathscr{F}$ vanishes on each A/I with $I \subseteq A$ finitely generated. This will follow as soon as we can show that $\mathscr{F}(M) = 0$ for all M in the deformation class $\mathbf{N}$ of $\mathbf{K}$. By induction on the rules (3.3.18.i)–(3.3.18.iv), it will suffice to show that each new module M in $\mathbf{N}$ obtained from an application of one of these rules vanishes again on $\mathscr{F}$. The case of rule (3.3.18.i) is trivial; for (3.3.18.ii), we use that $\mathscr{F}$ is additive; and for (3.3.18.iii), we are done by the long exact sequence of Tor (3.1.4). So remains to verify the claim for rule (3.3.18.iv), that is to say, assume x is an M-regular element in the Jacobson radical of A such that $\mathscr{F}(M/xM) = 0$. Applying 3.1.4 to the exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

we get part of a long exact sequence

$$\mathscr{F}(M) \xrightarrow{x} \mathscr{F}(M) \to \mathscr{F}(M/xM) = 0. \tag{3.14}$$

Since M is finitely presented, we have an exact sequence

$$F \to A^m \to A^n \to M \to 0$$

with F some (possibly infinitely generated) free A-module. Tensoring with Q yields a complex

$$F \otimes_A Q \to Q^m \to Q^n \to M \otimes_A Q \to 0 \tag{3.15}$$

whose first homology is by definition $\mathscr{F}(M)$. Since Q is a coherent module, so is any direct sum of Q by [35, Corollary 2.2.3], and hence the kernel of the morphism $Q^m \to Q^n$ in (3.15) is finitely generated by [35, Corollary 2.2.2]. Since $\mathscr{F}(M)$ is a quotient of this kernel, it, too, is finitely generated. By (3.14), we have an equality $\mathscr{F}(M) = x\mathscr{F}(M)$. By assumption, x belongs to the Jacobson radical of B, and hence, by Nakayama's Lemma, $\mathscr{F}(M) = 0$, as we needed to show. □

Combining Proposition 3.3.20 with Theorem 3.3.21 immediately gives the following well-known flatness criterion:

Corollary 3.3.22 (Local Flatness Criterion). *Let $R \to S$ be a local homomorphism of Noetherian local rings, and let k be the residue field of R. If M is a finitely generated S-module such that* $\operatorname{Tor}_1^R(M,k) = 0$*, then M is flat over R.* □

To extend this local flatness criterion to a larger class of rings, we make the following definition. Let us call a local ring R *ind-Noetherian*, if it is a direct limit of Noetherian local subrings R_i, indexed by a directed poset I, such that each $R_i \to R$ is a scalar extension (that is to say, faithfully flat and unramified; see §3.2.3). Clearly, any Noetherian local ring is ind-Noetherian (by taking $R_i = R$).

Lemma 3.3.23. *An ind-Noetherian local ring is coherent and has finite embedding dimension.*

Proof. Let $(R,\mathfrak{m})$ be ind-Noetherian. Since $\mathfrak{m}$ is in particular extended from a Noetherian local ring, it is finitely generated. We use Corollary 3.3.17 to prove coherency. To this end we must show that a finitely related ideal $(\mathfrak{a} : \mathfrak{b})$ is finitely generated. Since $\mathfrak{a}$ and $\mathfrak{b}$ are finitely generated, there exists a Noetherian local subring $S \subseteq R$ and ideals $I, J \subseteq S$ such that $S \to R$ is a scalar extension, and $\mathfrak{a} = IR$ and $\mathfrak{b} = JR$. Theorem 3.3.14 yields that $(I : J)R = (IR : JR) = (\mathfrak{a} : \mathfrak{b})$, whence in particular, is finitely generated. □

3.3.24 *If $R \to S$ is essentially of finite type and R is ind-Noetherian, then so is S.*

Indeed, S is isomorphic to the localization of $R[x]/(f_1,\dots,f_s)R[x]$ with respect to the ideal generated by the variables and by the maximal ideal of R. Hence, there is a directed subset $J \subseteq I$ such that $f_1,\dots,f_s$ are defined over each R_j with $j \in J$. It is now easy to see that the appropriate localization S_j of $R_j[x]/(f_1,\dots,f_s)R_j[x]$ forms a directed system with union equal to S, and each $S_j \to S$ is a scalar extension. □

Corollary 3.3.25. *Let $R \to S$ be a local homomorphism of ind-Noetherian rings. If Q is a finitely presented S-module such that $\mathrm{Tor}_1^R(Q,k)=0$, where k is the residue field of R, then Q is flat over R. If Q is moreover Noetherian, then so is R.*

Proof. In view of Theorem 3.3.21, to prove the first assertion, we need to show that k is quasi-deformationally generating (note that S is coherent by Lemma 3.3.23, whence so is the finitely presented S-module Q). Let $\mathfrak{a} \subseteq R$ be a finitely generated ideal. Choose a Noetherian local subring T and an ideal $I \subseteq T$ such that $T \subseteq R$ is a scalar extension, and $IR = \mathfrak{a}$. By Proposition 3.3.20, the module T/I belongs to the deformation class of T-modules generated by the residue field l of T. Since each of the rules (3.3.18.i)–(3.3.18.iv) are preserved by faithfully flat extensions, $T/I \otimes_T R = R/\mathfrak{a}$ lies in the deformation class of $l \otimes_T R \cong k$, where the latter isomorphism follows from the unramifiedness of $T \to R$.

To prove that R is Noetherian, under the additional assumption that Q is Noetherian, let $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ be a chain of ideals in R. Choose i such that $\mathfrak{a}_i Q = \mathfrak{a}_j Q$ for all $j \geq i$. Hence $\mathfrak{a}_i/\mathfrak{a}_j \otimes Q = 0$, for $j \geq i$, and since Q is faithfully flat, as it is non-degenerated by 3.2.1, we get $\mathfrak{a}_i/\mathfrak{a}_j = 0$ by 3.2.3. □

3.3.7 Dimension criterion for flatness

If $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ is a local homomorphism of Noetherian local rings, then we have the following dimension inequality, with equality when $R \to S$ is flat (see [69, Theorem 15.1]):

$$\dim(S) \leq \dim(R) + \dim(S/\mathfrak{m}S). \tag{3.16}$$

Recall that we call $S/\mathfrak{m}S$ the *closed fiber* of $R \to S$: it defines the locus of all prime ideals in S which lie above $\mathfrak{m}$. Conversely, equality in (3.16) often implies flatness. We first discuss one well-known criterion, and then prove one new one.

Theorem 3.3.26. *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a homomorphism of Noetherian local rings, with R regular and S Cohen-Macaulay. Then $R \to S$ is flat if and only if we have equality in (3.16).*

Proof. One direction holds always, as we discussed above. So assume we have equality in (3.16), that is to say, $e = d + h$ where d, h, and e, are the respective dimension of R, the closed fiber $S/\mathfrak{m}S$, and S. Let $(x_1,\dots,x_d)$ be a system of parameters of R. Since $S/\mathfrak{m}S$ has dimension $h = e - d$, there exist $x_{d+1},\dots,x_e$ in S such that their image in $S/\mathfrak{m}S$ is a system of parameters. Hence $(x_1,\dots,x_e)$ is a system of parameters in S, whence is an S-regular sequence. In particular, $(x_1,\dots,x_d)$ is S-regular, showing that S is a balanced big Cohen-Macaulay R-module, and therefore is flat by Theorem 3.3.9. □

For our last criterion, which generalizes a flatness criterion due to Kollár [62, Theorem 8], we impose some regularity condition on the closed fiber, weakening instead the conditions on the rings themselves.

Theorem 3.3.27. *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume R is either an excellent normal local domain with perfect residue field, or an analytically irreducible domain with algebraically closed residue field. If the closed fiber is regular, of dimension* $\dim(S) - \dim(R)$, *then $R \to S$ is faithfully flat.*

Proof. Let d and e be the respective dimensions of R and S. We will induct on the dimension $h := e - d$ of the closed fiber. If $h = 0$, then $R \to S$ is in fact unramified. It suffices to prove this case under the additional assumption that both R and S are complete. Indeed, if $R \to S$ is arbitrary, then $\widehat{R} \to \widehat{S}$ satisfies again the hypotheses of the theorem and therefore would be faithfully flat. Hence $R \to S$ is faithfully flat by Proposition 3.2.11.

So assume R and S are complete and let l be the residue field of S. Either assumption on R implies that $\widehat{R_l}$ is again a domain, of the same dimension as R (we leave this as an exercise to the reader; see [102, Corollary 3.10 and Proposition 3.11]). By the universal property of complete scalar extensions (Theorem 3.2.13—note that this result also holds in mixed characteristic, although we did not provide a proof in these notes; see [102, Corollary 3.3]), we get a local R-algebra homomorphism $\widehat{R_l} \to S$. By [69, Theorem 8.4], this homomorphism is surjective. It is also injective, since $\widehat{R_l}$ and S have the same dimension and $\widehat{R_l}$ is a domain. Hence $\widehat{R_l} \cong S$, so that $R \to S$ is a scalar extension, whence faithfully flat.

For the general case, $h > 0$, let $\tilde{R}$ be the localization of $R[\xi]$ at the ideal $\tilde{\mathfrak{m}}$ generated by $\mathfrak{m}$ and the variables $\xi := (\xi_1, \dots, \xi_h)$. By assumption, $\tilde{R}$ has the same dimension as S. Let $\mathbf{y}$ be an h-tuple whose image in the closed fiber is a regular system of parameters, that is to say, which generates $\mathfrak{n}(S/\mathfrak{m}S)$. Let $\tilde{R} \to S$ be the R-algebra homomorphism given by sending ξ to $\mathbf{y}$. Hence $\mathfrak{n} = \mathfrak{m}S + \mathbf{y}S = \tilde{\mathfrak{m}}S$, so that by the case $h = 0$, the homomorphism $\tilde{R} \to S$ is flat, whence so is $R \to S$. $\square$

The requirement on R that we really need is that any complete scalar extension is again a domain, and for this, it suffices that the complete scalar extension over the algebraic closure of the residue field of R is a domain (see [102, Proposition 3.11]).

Chapter 4
Uniform bounds

In this chapter, we will discuss our first application of ultraproducts: the existence of uniform bounds over polynomial rings. The method goes back to A. Robinson, but really gained momentum by the work of Schmidt and van den Dries in [86], where they brought in flatness as an essential tool. Most of our applications will be concerned with affine algebras over an ultra-field. For such an algebra, we construct its ultra-hull as a certain faithfully flat ultra-ring. As we will also use this construction in our alternative definition of tight closure in characteristic zero in Chapter 6, we study it in detail in §4.3. In particular, we study transfer between the affine algebra and its approximations. We conclude in §4.4 with some applications to uniform bounds, in the spirit of Schmidt and van den Dries.

4.1 Ultra-hulls

Let us fix an ultra-field K, realized as the ultraproduct of fields K_w for $w \in W$. For a concrete example, one may take $K := \mathbb{C}$ and $K_p := \mathbb{F}_p^{\mathrm{alg}}$ by Theorem 2.4.3 (with W the set of prime numbers). We make the construction of the ultra-hull in three stages.

4.1.1 Ultra-hull of a polynomial ring.

In this section, we let $A := K[\xi]$, where $\xi := (\xi_1, \dots, \xi_n)$ are indeterminates. We define the *ultra-hull* (called the *non-standard hull* in the earlier papers [88, 89, 92]) of A as the ultraproduct of the $A_w := K_w[\xi]$, and denote it $U(A)$. The inclusions $K_w \subseteq A_w$ induce an inclusion $K \subseteq U(A)$. Let ξ_i also denote the ultraproduct $\operatorname{ulim}_w \xi_i$ of the constant sequence ξ_i. By Łoś' Theorem, Theorem 2.3.1, the ξ_i are algebraically independent over K. Hence, we may view them as indeterminates over K in $U(A)$, thus yielding an embedding $A = K[\xi] \subseteq U(A)$. To see why this is called an ultra-hull, let us introduce the category of ultra-K-algebras: a K-algebra $B_\natural$ is called an *ultra-K-algebra* if it is the ultraproduct of K_w-algebras B_w; a morphism of ultra-K-algebras $B_\natural \to C_\natural$ is any K-algebra homomorphism obtained as the ultraproduct of K_w-algebra homomorphisms $B_w \to C_w$. It follows that any ultra-K-algebra is a K-algebra. The ultra-hull $U(A)$ is clearly an ultra-K-algebra. We have:

4.1.1 *The ultra-hull $U(A)$ satisfies the following universal property: given an ultra-K-algebra $B_\natural$, and a K-algebra homomorphism $A \to B_\natural$, there exists a unique ultra-K-algebra homomorphism $U(A) \to B_\natural$ extending $A \to B$.*

Indeed, by assumption, $B_\natural$ is the ultraproduct of K_w-algebras B_w. Let $b_{i\natural}$ be the image of ξ_i under the the homomorphism $A \to B_\natural$, and choose $b_{iw} \in B_w$ whose ultraproduct equals $b_{i\natural}$. Define K_w-algebra homomorphisms $A_w \to B_w$ by the rule $\xi_i \mapsto b_{iw}$. The ultraproduct of these homomorphisms is then the required ultra-K-algebra homomorphism $U(A) \to B_\natural$. Its uniqueness follows by an easy application of Łoś' Theorem. $\square$

An intrinsic characterization of A as a subset of $U(A)$ is provided by the next result (in the terminology of Chapter 9, this exhibits A as a certain *protoproduct*):

4.1.2 *An ultraproduct $f_\natural \in U(A)$ belongs to A if and only if its approximations $f_w \in A_w$ have bounded degree, meaning that there is a d such that almost all f_w have degree at most d.*

Indeed, if $f \in A$ has degree d, then we can write it as $f = \sum_\nu u_\nu \xi^\nu$ for some $u_\nu \in K$, where ν runs over all n-tuples with $|\nu| \leq d$. Choose $u_{\nu w} \in K_w$ such that their ultraproduct is u_ν, and put

$$f_w := \sum_{|\nu| \leq d} u_{\nu w} \xi^\nu. \tag{4.1}$$

An easy calculation shows that the ultraproduct of the f_w is equal to f, viewed as an element in $U(A)$. Conversely, if almost each f_w has degree at most d, so that we can write it in the form (4.1), then

$$\operatorname*{ulim}_{w\to\infty} f_w = \sum_{|\nu| \leq d} (\operatorname*{ulim}_{w\to\infty} u_{\nu w}) \xi^\nu$$

is a polynomial (of degree at most d). $\square$

4.1.2 Ultra-hull of an affine algebra.

More generally, let C be a *K-affine algebra*, that is to say, a finitely generated K-algebra, say of the form $C = A/I$ for some ideal $I \subseteq A$. We define the *ultra-hull* of C to be $U(A)/IU(A)$, and denote it $U(C)$. It is clear that the diagonal embedding $A \subseteq U(A)$ induces by base change a homomorphism $C \to U(C)$. Less obvious is that this is still an injective map, which we will prove in Corollary 4.2.3 below. To show that the construction of $U(C)$ does not depend on the choice of presentation $C = A/I$, we verify that $U(C)$ satisfies the same universal property 4.1.1 as $U(A)$: any K-algebra homomorphism $C \to B_\natural$ to some ultra-K-algebra $B_\natural$ extends uniquely to a homomorphism $U(C) \to B_\natural$ of ultra-K-algebras (recall that any solution to a universal property is necessarily unique). To see why the universal property holds, apply 4.1.1 to the composition $A \twoheadrightarrow A/I = C \to B_\natural$ to get a unique extension $U(A) \to B_\natural$. Since any element in I is sent to zero under the composition $A \to B_\natural$, this homomorphism factors through $U(A)/IU(A)$, yielding the required homomorphism $U(C) \to B_\natural$ of ultra-K-algebras. Uniqueness follows from the uniqueness of $U(A) \to B_\natural$.

Since $IU(A)$ is finitely generated, it is an ultra-ideal by 2.4.12, that is to say, an ultraproduct of ideals $I_w \subseteq A_w$, and the ultraproduct of the $C_w := A_w/I_w$ is equal to $U(C) = U(A)/IU(A)$ by 2.1.6. If $C = A'/I'$ is a different presentation of C as a K-algebra (with A' a polynomial ring in finitely many indeterminates), and $C'_w := A'_w/I'_w$ the corresponding K_w-algebras, then their ultraproduct $U(A')/I'U(A')$ is another way of defining the ultra-hull of C, whence it must be isomorphic to $U(C)$. Without loss of generality, we may assume $A \subseteq A'$ and hence $A_w \subseteq A'_w$. Since $U(A)/IU(A) \cong U(C) \cong U(A')/I'U(A')$, the homomorphisms $A_w \subseteq A'_w$ induce homomorphisms $C_w \to C'_w$, and by Łoś' Theorem, almost all are isomorphisms. This justifies the usage of calling the C_w *approximations* of C (in spite of the fact that they are not uniquely determined by C).

4.1.3 *The ultra-hull $U(\cdot)$ is a functor from the category of K-affine algebras to the category of ultra-K-algebras.*

The only thing which remains to be verified is that an arbitrary K-algebra homomorphism $C \to D$ of K-affine rings induces a homomorphism of ultra-K-algebras $U(C) \to U(D)$. However, this follows from the universal property applied to the composition $C \to D \to U(D)$, admitting a unique extension so that the following diagram is commutative

$$\begin{tikzcd} C \arrow[r] \arrow[d] & D \arrow[d] \\ U(C) \arrow[r] & U(D). \end{tikzcd} \tag{4.2}$$

□

4.1.3 Ultra-hull of a local affine algebra.

Recall that a K-affine local ring R is simply the localization $C_{\mathfrak{p}}$ of a K-affine algebra C at a prime ideal $\mathfrak{p}$. Let us call R *geometric*, if $\mathfrak{p}$ is a maximal ideal $\mathfrak{m}$ of C. A geometric K-affine local ring, in other words, is the local ring of a closed point on an affine scheme of finite type over K. Note that a local K-affine algebra is in general not finitely generated as a K-algebra; one usually says that R is *essentially of finite type* over K. The next result will enable us to define the ultra-hull of a geometric affine local ring; we shall discuss the general case in §4.3.2 below (see Remark 4.3.5):

4.1.4 *Let C be a K-affine algebra. If $\mathfrak{m}$ is a maximal ideal in C, then $\mathfrak{m}U(C)$ is a maximal ideal in $U(C)$, and $C/\mathfrak{m} \cong U(C)/\mathfrak{m}U(C)$.*

By our previous discussion, $U(L) := U(C)/\mathfrak{m}U(C)$ is the ultra-hull of the field $L := C/\mathfrak{m}$. By the Nullstellensatz, the extension $K \subseteq L$ is finite, and from this, it is easy to see that L is an ultra-field. By the universal property, L is equal to its own ultra-hull, and hence $\mathfrak{m}U(C)$ is a maximal ideal. □

We can now define the ultra-hull of a local K-affine algebra $R = C_{\mathfrak{m}}$ as the localization $U(R) := U(C)_{\mathfrak{m}U(C)}$. Note that $U(R)$ is again an ultra-ring: let C_w be approximations of C, and let $\mathfrak{m}_w \subseteq C_w$ be ideals whose ultraproduct is equal to $\mathfrak{m}U(C)$. Since the latter is maximal, so are almost all $\mathfrak{m}_w$. For those w, set $R_w := (C_w)_{\mathfrak{m}_w}$ (and arbitrary for the remaining w). One easily verifies that $U(R)$ is then isomorphic to the ultraproduct of the R_w, and for this reason we call the R_w again an *approximation* of R. We can formulate a similar universal property which is satisfied by $U(R)$, and then show that any local homomorphism $R \to S$ of local K-affine algebras induces a unique homomorphism $U(R) \to U(S)$. Moreover, any two approximations agree almost everywhere. In particular, for homomorphic images we have:

4.1.5 *If $I \subseteq C$ is an ideal in a K-affine (local) ring, then $U(C/I) = U(C)/IU(C)$.*

We extend our nomenclature also to elements and ideals: if $a \in C$ is an element or $I \subseteq C$ is an ideal, and $a_w \in C_w$ and $I_w \subseteq C_w$ are such that their ultraproduct equals $a \in U(C)$ and $IU(C)$ respectively, then we call the a_w and the I_w *approximations* of a and I respectively. In particular, by 4.1.4, the approximations of a maximal ideal are almost all maximal. The same holds true with 'prime' instead of 'maximal', but the proof is more involved, and we have to postpone it until Theorem 4.3.4 below.

4.2 The Schmidt-van den Dries theorem

The ring $U(A)$ is highly non-Noetherian. In particular, although each $\mathfrak{m}U(A)$ is a maximal ideal for $\mathfrak{m}$ a maximal ideal of $A = K[\xi]$, these are not the only maximal ideals of $U(A)$. To see an example, choose, for each w, a polynomial $f_w \in A_w$ in ξ_1 of degree w with distinct roots in K_w (assuming K_w has at least size w), and let $f \in U(A)$ be their ultraproduct. Let $\mathfrak{a}$ be the ideal generated by all f/h where h runs over all elements in A such that $f \in hU(A)$. Since f has infinitely many roots,[1] $\mathfrak{a}$ is not the unit ideal, and hence is contained in some maximal ideal $\mathfrak{M}$ of $U(A)$. However, for the same reason, $\mathfrak{a}$ cannot be inside a maximal ideal of the form $\mathfrak{m}U(A)$ with $\mathfrak{m} \subseteq A$, showing that $\mathfrak{M}$ is not of the latter form. In fact, $\mathfrak{M}$ is not even an ultra-ideal.

Nonetheless, the maximal ideals that are extended from A 'cover' enough of $U(A)$ in order to apply Theorem 3.3.16. More precisely:

4.2.1 *If almost all K_w are algebraically closed, then any proper finitely related ideal of $U(A)$ is contained in some $\mathfrak{m}U(A)$ with $\mathfrak{m} \subseteq A$ a maximal ideal.*

Indeed, this is even true for any proper ultra-ideal $I \subseteq U(A)$. Namely, let I be the ultraproduct of ideals $I_w \subseteq A_w$. By Łoś' Theorem, almost each I_w is a proper ideal whence contained in some maximal ideal $\mathfrak{m}_w$. By the Nullstellensatz, we can write $\mathfrak{m}_w$ as $(\xi_1 - u_{1w}, \dots, \xi_n - u_{nw})A_w$ for some $u_{iw} \in K_w$. Let $u_i \in K$ be the ultraproduct of the u_{iw}, so that the ultraproduct of the $\mathfrak{m}_w$ is equal to $(\xi_1 - u_1, \dots, \xi_n - u_n)U(A)$, and by Łoś' Theorem it contains I. □

It is necessary that almost all K_w are algebraically closed. For instance, if all K_w are equal to $\mathbb{Q}$ (whence K is the ultrapower $\mathbb{Q}_\natural$), and we let $\mathfrak{m}_w$ be the ideal in $\mathbb{Q}[\xi]$ generated by $\xi^{2w} + 1$, then the ultraproduct $\mathfrak{m}_\natural$ of the $\mathfrak{m}_w$ is principal but contains no non-zero element of $\mathbb{Q}_\natural[\xi]$.

Theorem 4.2.2. *For any K-affine algebra, the diagonal embedding $C \to U(C)$ is faithfully flat, whence in particular injective.*

Proof. If we have proven this result for the ultra-hull $U(A)$ of A, then it will follow from 3.1.3 for any $C \to U(C)$, since the latter is just a base change $C = A/I \to U(A)/IU(A) = U(C)$, where $C = A/I$ is some presentation of C. The non-degeneratedness of $U(A)$ is immediate from 4.1.4. So remains to show the flatness of $A \to U(A)$, and for this we may assume that K and all K_w are algebraically closed. Indeed, if K' is the ultraproduct of the algebraic closures of the K_w, then $A \to A' := K'[\xi]$ is flat by 3.1.3. By Łoś' Theorem, the canonical homomorphism $U(A) \to U(A')$ is cyclically pure with respect to ideals extended from A, where $U(A')$ is the ultra-K'-hull of A. Hence if we showed that $A' \to U(A')$ is flat, then so is $A \to U(A)$ by Corollary 3.3.15. Hence we may assume all K_w are algebraically closed. By Theorem 3.3.16 in conjunction with 4.2.1, we only need to show that $R := A_\mathfrak{m} \to U(R) = U(A)_{\mathfrak{m}U(A)}$ is flat for every maximal ideal $\mathfrak{m} \subseteq A$. After a translation, we may assume $\mathfrak{m} = (\xi_1, \dots, \xi_n)A$. By Łoś' Theorem, $(\xi_1, \dots, \xi_n)$ is an $U(A)$-regular sequence whence $U(R)$-regular. This proves that $U(R)$ is a big Cohen-Macaulay R-module. By Proposition 3.3.8 it is therefore a balanced big

[1] Notwithstanding that f is only an *ultra-polynomial*, we may view it by 2.1.2 as a function on $K_\natural$, and a *root* of f then means an element $u \in K_\natural$ such that $f(u) = 0$ (which means that, for any approximation $u_w \in K_w$ of u, almost all $f_w(u_w) = 0$ are zero).

Cohen-Macaulay module, since any regular sequence in $U(R)$ is permutable by Łoś' Theorem, because this is so in the Noetherian local rings $(A_w)_{\mathfrak{m}_w}$ (see [69, Corollary to Theorem 16.3]). Hence $U(R)$ is flat over R by Theorem 3.3.9. $\square$

Immediately from this and the cyclic purity of faithfully flat homomorphisms (Proposition 3.2.5) we get:

Corollary 4.2.3. *The diagonal embedding $C \to U(C)$ is injective, and $IU(C) \cap C = I$ for any ideal $I \subseteq C$.* $\square$

4.3 Transfer of structure

We will use ultra-hulls in our definition of tight closure in characteristic zero (see §6), and to this end, we need to investigate more closely the relation between an affine algebra and its approximations. We start with the following far reaching generalization of 4.1.4.

4.3.1 Finite extensions.

Proposition 4.3.1. *If $C \to D$ is a finite homomorphism of K-affine algebras, then $U(D) \cong U(C) \otimes_C D$, and hence $U(C) \to U(D)$ is also finite.*

Proof. Since D is finite as a module over C, the tensor product $U(C) \otimes_C D$ is finite over $U(C)$, whence an ultra-K-algebra. By the universal property of the ultra-hull of D, we therefore have a unique homomorphism $U(D) \to U(C) \otimes_C D$ of ultra-K-algebras. On the other hand, by the universal property of tensor products, we have a unique homomorphism $U(C) \otimes_C D \to U(D)$. It is no hard to see that the latter is in fact a morphism of ultra-K-algebras. By uniqueness of both homomorphisms, they must therefore be each other's inverse. $\square$

Corollary 4.3.2. *If C is an Artinian K-affine algebra, then $C \cong U(C)$.*

Proof. Since C is a direct product of local Artinian rings ([27, Corollary 9.1]), and since ultra-hulls are easily seen to commute with direct products, we may assume C is moreover local, with maximal ideal $\mathfrak{m}$, say. Let $L := C/\mathfrak{m}$, so that $L \cong U(L)$ by 4.1.4. Note that the vector space dimension of C over L is equal to the length of C. In any case, C is a finite L-module, so that by Proposition 4.3.1 we get $U(C) = U(L) \otimes_L C = C$. $\square$

Corollary 4.3.3. *The dimension of a K-affine algebra is equal to the dimension of almost all of its approximations.*

Proof. Let C be an n-dimensional K-affine algebra, with approximations C_w. The assertion is trivial for $C = A$ a polynomial ring. By Noether normalization (see for instance [27, Theorem 13.3]), there exists a finite extension $A \subseteq C$. The induced homomorphism $U(A) \to U(C) \cong U(A) \otimes_A C$ is finite, by Proposition 4.3.1, and injective since $A \to U(A)$ is flat by Theorem 4.2.2. By Łoś' Theorem, almost all $A_w \to C_w$ are finite and injective. Hence almost all C_w have dimension n by [27, Proposition 9.2.]. □

4.3.2 Prime ideals.

We return to our discussion on the behavior of prime ideals under the ultra-hull, and we are ready to prove the promised generalization of 4.1.4 (this was originally proven in [86] by different means).

Theorem 4.3.4. *A K-affine algebra C is a domain if and only if $U(C)$ is, if and only if almost all of its approximations are. In particular, if $\mathfrak{p}$ is a prime ideal in an arbitrary K-affine algebra D, then $\mathfrak{p}U(D)$ is again a prime ideal, and so are almost all of its approximations $\mathfrak{p}_w$.*

Proof. By Łoś' Theorem, almost all C_w are domains if and only if $U(C)$ is a domain. If this holds, then C too is a domain since it is a subring of $U(C)$ by Corollary 4.2.3. Conversely, assume C is a domain, and let $A \subseteq C$ be a Noether normalization of C, that is to say a finite and injective extension. Let $A_w \subseteq C_w$ be the corresponding approximations implied by Proposition 4.3.1. Let $\mathfrak{p}_w$ be a prime ideal in C_w of maximal dimension, and let $\mathfrak{p}_\natural$ be their ultraproduct, a prime ideal in $U(C)$. An easy dimension argument shows that $\mathfrak{p}_w \cap A_w = (0)$ and hence by Łoś' Theorem, $\mathfrak{p}_\natural \cap U(A) = (0)$. Let $\mathfrak{p} := \mathfrak{p}_\natural \cap C$. Since $\mathfrak{p} \cap A$ is contained in $\mathfrak{p}_\natural \cap U(A)$, it is also zero. Hence $A \to C/\mathfrak{p}$ is again finite and injective. Since C is a domain, an easy dimension argument yields that $\mathfrak{p} = 0$. On the other hand, we have an isomorphism $U(C) = U(A) \otimes_A C$, so that by general properties of tensor products

$$U(C)/\mathfrak{p}_\natural = U(A)/(\mathfrak{p}_\natural \cap U(A)) \otimes_{A/(\mathfrak{p}_\natural \cap A)} C/(\mathfrak{p}_\natural \cap C) = U(A) \otimes_A C = U(C)$$

showing that $\mathfrak{p}_\natural$ is zero, whence so are almost all $\mathfrak{p}_w$. Hence almost all C_w are domains, and hence by Łoś' Theorem, so is $U(C)$.

The last assertion is immediate from the first applied to $C := D/\mathfrak{p}$. □

Remark 4.3.5. This allows us to define the ultra-hull of an arbitrary local K-affine algebra $C_\mathfrak{p}$ as the localization $U(C)_{\mathfrak{p}U(C)}$.

To show that a local affine algebra has the same dimension as almost all of its approximations, we introduce a new dimension notion. Let $(R, \mathfrak{m})$ be a local ring of finite embedding dimension (that is to say, with a finitely generated maximal ideal).

Definition 4.3.6 (Geometric dimension). We define the *geometric dimension* of R, denoted $\operatorname{geodim}(R)$, as the least number of elements generating an $\mathfrak{m}$-primary ideal.

As the (Krull) dimension $\dim(R)$ equals the dimension of the topological space $V := \operatorname{Spec}(R)$, it is essentially a topological invariant. On the other hand, $\operatorname{geodim}(R)$ is the least number of equations defining the closed point x corresponding to the maximal ideal $\mathfrak{m}$, and hence is a geometric invariant. It is a well-known result from commutative algebra that for Noetherian local rings (Krull) dimension equals geometric dimension (see, for instance, [69, Theorem 13.4]). The next result shows that this is no longer true if one drops the Noetherianity condition, since ultra-rings are in general infinite dimensional (for some calculations of their prime spectrum, see [72, 73, 74]).

Proposition 4.3.7. *If $(R,\mathfrak{m})$ is a d-dimensional local K-affine algebra, then $U(R)$ has geometric dimension d.*

Proof. We induct on the dimension d, where the case $d = 0$ follows from Corollary 4.3.2. So assume $d > 0$, and let x be a parameter in R. Hence, R/xR has dimension $d-1$, so that by induction, $U(R/xR)$ has geometric dimension $d-1$. Since $U(R/xR) = U(R)/xU(R)$ by 4.1.5, we see that $U(R)$ has geometric dimension at most d. By way of contradiction, suppose its geometric dimension is at most $d-1$. In particular, there exists an $\mathfrak{m}U(R)$-primary ideal $\mathfrak{N}$ generated by $d-1$ elements. Put $\mathfrak{n} := \mathfrak{N} \cap R$, and let n be such that $\mathfrak{m}^n U(R) \subseteq \mathfrak{N}$. By faithful flatness, that is to say, by Corollary 4.2.3, we have an inclusion $\mathfrak{m}^n \subseteq \mathfrak{n}$, showing that $\mathfrak{n}$ is $\mathfrak{m}$-primary. Hence $R/\mathfrak{n} \cong U(R/\mathfrak{n}) = U(R)/\mathfrak{n}U(R)$ by Corollary 4.3.2. Hence $U(R)/\mathfrak{N}$ is a homomorphic image of $R/\mathfrak{n}$ whence equal to it by definition of $\mathfrak{n}$. In conclusion, $\mathfrak{N} = \mathfrak{n}U(R)$. Since R has geometric dimension d, the $\mathfrak{m}$-primary ideal $\mathfrak{n}$ requires at least d generators. Since $R \to U(R)$ is flat by Theorem 4.2.2, also $\mathfrak{n}U(R)$ requires at least d generators by 3.2.7, contradiction. □

We can now extend the result from Corollary 4.3.3 to the local case as well:

Corollary 4.3.8. *The dimension of a local K-affine algebra R is equal to the dimension of almost all of its approximations R_w. Moreover, if $\mathbf{x}$ is a sequence in R with approximations $\mathbf{x}_w$, then $\mathbf{x}$ is a system of parameters if and only if almost all $\mathbf{x}_w$ are.*

Proof. The second assertion follows immediately from the first and Łoś' Theorem. By Proposition 4.3.7, the geometric dimension of $U(R)$ is equal to $d := \dim(R)$. Let R_w be approximations of R, so that their ultraproduct equals $U(R)$. If I is an $\mathfrak{m}U(R)$-primary ideal generated by d elements, then its approximation I_w is an $\mathfrak{m}_w$-primary ideal generated by d elements for almost all w by 2.4.11. Hence almost all R_w have (geometric) dimension at most d.

Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{m}$ be a chain of prime ideals in R of maximal length. By faithful flatness (in the form of Corollary 4.2.3), this chain remains strict when extended to $U(R)$, and by Theorem 4.3.4, it consists again of prime ideals. Hence if $\mathfrak{p}_{iw} \subseteq R_w$ are approximations of $\mathfrak{p}_i$, then by Łoś' Theorem, we get a strict chain of prime ideals $\mathfrak{p}_{0w} \subsetneq \cdots \subsetneq \mathfrak{p}_{dw} = \mathfrak{m}_w$ for almost all w, proving that almost all R_w have dimension at least d. □

Note that it is not true that if $\mathbf{x}_w$ are systems of parameters in the approximations, then their ultraproduct (which in general lies outside R) does not necessarily generate an $\mathfrak{m}U(R)$-primary ideal.

4.3.3 Singularities.

Now that we know how dimension behaves under ultra-hulls, we can investigate singularities.

Theorem 4.3.9. *A local K-affine algebra is respectively regular or Cohen-Macaulay if and only if almost all its approximations are.*

Proof. Let R be a d-dimensional local K-affine algebra, and let R_w be its approximations. If R is regular, then its embedding dimension is d, whence so is the embedding dimension of $U(R)$, and by Łoś' Theorem, then so is the embedding dimension of R_w for almost each w, and conversely. Corollary 4.3.8 then proves the assertion for regularity. As for the Cohen-Macaulay condition, let $\mathbf{x}$ be a system of parameters with approximation $\mathbf{x}_w$. Hence almost each $\mathbf{x}_w$ is a system of parameters in R_w by Corollary 4.3.8. If R is Cohen-Macaulay, then $\mathbf{x}$ is an R-regular sequence, hence $U(R)$-regular by flatness (see Proposition 3.2.9), whence almost each $\mathbf{x}_w$ is R_w-regular by Łoś' Theorem, and almost all R_w are Cohen-Macaulay. The converse follows along the same lines. □

4.4 Uniform bounds

In this last section, we discuss some applications of ultraproducts to the study of rings. The results as well as the proof method via ultraproducts are due to Schmidt and van den Dries from their seminal paper [86], and were further developed in [84, 88, 89, 98].

4.4.1 Linear equations.

The proof of the next result is very typical for an argument based on ultraproducts, and will be the template for all future proofs.

Theorem 4.4.1 (Schmidt-van den Dries). *For any pair of positive integers (d,n), there exists a uniform bound $b := b(d,n)$ with the following property. Let k be a field,* *and let $f_0,\dots,f_s \in k[\xi]$ be polynomials of degree at most d in at most n indeterminates ξ such that $f_0 \in (f_1,\dots,f_s)k[\xi]$. Then there exist $g_1,\dots g_s \in k[\xi]$ of degree at most b such that $f_0 = g_1f_1+\dots+g_sf_s$.*

Proof. By way of contradiction, suppose this result is false for some pair (d,n). This means that we can produce counterexamples requiring increasingly high degrees. Before we write these down, observe that the number s of polynomials in these counterexamples can be taken to be the same by Lemma 4.4.2 below (by adding zero polynomials if necessary). So, for each $w \in \mathbb{N}$, we can find counterexamples consisting of a field K_w, and polynomials $f_{0w},\dots,f_{sw} \in A_w := K_w[\xi]$ of degree at most d, such that f_{0w} can be written as an A_w-linear combination of the $f_{1w},\dots,f_{sw}$, but any such linear combination involves a polynomial of degree at least w. Let f_i be the ultraproduct of the f_{iw}. This is again a polynomial of degree d in A by 4.1.2. Moreover, by Łoś' Theorem, $f_0 \in (f_1,\dots,f_d)U(A)$. We use the flatness of $A \to U(A)$ via its corollary in 4.2.3, to conclude that $f_0 \in (f_1,\dots,f_s)U(A)\cap A = (f_1,\dots,f_s)A$. Hence we can find polynomials $g_i \in A$ such that

$$f_0 = g_1f_1+\dots+g_sf_s. \tag{4.3}$$

Let e be the maximum of the degrees of the g_i. By 4.1.2 again, we can choose approximations $g_{iw} \in A_w$ of each g_i, of degree at most e. By Łoś' Theorem, (4.3) yields for almost all w that $f_{0w} = \sum_i g_{iw}f_{iw}$, contradicting our assumption. □

Lemma 4.4.2. *Any ideal in A generated by polynomials of degree at most d requires at most $b := \binom{d+n}{n}$ generators.*

Proof. Note that b is equal to the number of monomials of degree at most d in n variables. Let $I := (f_1,\dots,f_s)A$ be an ideal in A with each f_i of degree at most d. Choose some (total) ordering $<$ on these monomials (e.g., the lexicographical ordering on the exponent vectors), and let $l(f)$ denote the largest monomial appearing in f with non-zero coefficient, for any $f \in A$ of degree at most d (where we put $l(0) := -\infty$). If $l(f_i) = l(f_j)$ for some non-zero f_i, f_j with $i<j$, then $l(uf_i - vf_j) < l(f_i)$ for some non-zero elements $u,v \in K$, and we may replace the generator f_j by the new generator $uf_i - vf_j$. Doing this recursively for all i, we arrive at a situation in which all non-zero f_i have different $l(f_i)$, and hence there can be at most b of these. □

We can reformulate the result in Theorem 4.4.1 to arrive at some further generalizations. The ideal membership condition in that theorem is really about solving an (inhomogeneous) linear equation in A: the equation $f_0 = f_1t_1+\dots+f_st_s$, where the t_i are the unknowns of this equation (as opposed to ξ, which are indeterminates). One can then easily extend the previous argument to arbitrary systems of equations: there exists a uniform bound $b := b(d,n)$ such that for any field k, and for any linear system of equations $\lambda_1 = \dots = \lambda_s = 0$ with $\lambda_i \in k[\xi,t]$ of ξ-degree at most d and t-degree at most one, where ξ is an n-tuple of indeterminates and t is a finite tuple of variables, if the system admits a solution in $k[\xi]$, then it admits a solutions all of whose entries have degree at most b. In the homogeneous case we can say even more:

Theorem 4.4.3. *For any pair of positive integers (d,n), there exists a uniform bound $b := b(d,n)$ with the following property. Over a field k, any homogeneous system of equations with coefficients in $k[\xi_1,\dots,\xi_n]$ all of whose coefficients have degree at most d, admits a finite number of solutions of degree at most b such that any other solution is a linear combination of these finitely many solutions.*

Proof. The proof once more is by contradiction. Assume the statement is false for the pair (n,d). Hence we can find for each $w \in \mathbb{N}$, a field K_w, and a linear system of homogeneous equations

$$\lambda_{1w}(t) = \cdots = \lambda_{sw}(t) = 0 \qquad (\mathscr{L}_w)$$

in the variables $t = (t_1,\dots,t_m)$ with coefficients in A_w, such that the module of solutions $\mathrm{Sol}_{A_w}(\mathscr{L}_w) \subseteq A_w^k$ requires at least one generator one of whose entries is a polynomial of degree at least w. Here, we may again take the number m of t-variables as well as the number s of equations to be the same in all counterexamples, by another use of Lemma 4.4.2. The ultraproduct of each λ_{iw} is, as before by 4.1.2, an element $\lambda_i \in A[t]$ which is a linear form in the t-variables (and has degree at most d in ξ). By the equational criterion for flatness, Theorem 3.3.1, the flatness of $A \to U(A)$, proven in Theorem 4.2.2, amounts to the existence of solutions $\mathbf{b}_1,\dots,\mathbf{b}_l \in \mathrm{Sol}_A(\mathscr{L})$ such that any solution of the homogeneous linear system $\mathscr{L}$ of equations $\lambda_1 = \cdots = \lambda_s = 0$ in $U(A)$ lies in the $U(A)$-module generated by the $\mathbf{b}_i$. Let e be the maximum of the degrees occurring in the $\mathbf{b}_i$. In particular, we can find approximations $\mathbf{b}_{iw} \in A_w^m$ of $\mathbf{b}_i$ whose entries all have degree at most e. I claim that almost each $\mathrm{Sol}_{A_w}(\mathscr{L}_w)$ is equal to the submodule H_w generated by $\mathbf{b}_{1w},\dots,\mathbf{b}_{lw}$, which would then contradict our assumption.

To prove the claim, one inclusion is clear, so assume by way of contradiction that we can find for almost all w a solution $\mathbf{q}_w \in \mathrm{Sol}_{A_w}(\mathscr{L}_w)$ outside H_w. Let $\mathbf{q}_\natural \in U(A)^m$ be its ultraproduct (note that this time, we cannot guarantee that its entries lie in A since the degrees might be unbounded). By Łoś' Theorem, $\mathbf{q}_\natural \in \mathrm{Sol}_{U(A)}(\mathscr{L})$, whence can be written as an $U(A)$-linear combination of the $\mathbf{b}_i$. Writing this out and using Łoś' Theorem once more, we conclude that $\mathbf{q}_w$ lies in H_w for almost all w, contradiction. □

4.4.2 Primality testing.

The next result, with a slightly different proof from the original, is also due to Schmidt and van den Dries.

Theorem 4.4.4 (Schmidt-van den Dries). *For any pair of positive integers (d,n), there exists a uniform bound $b := b(d,n)$ with the following property. Let k be a field, and let $\mathfrak{p}$ be an ideal in $k[\xi_1,\dots,\xi_n]$ generated by polynomials of degree at most d.*

Then $\mathfrak{p}$ is a prime ideal if and only if for any two polynomials f, g of degree at most b which do not belong to $\mathfrak{p}$, neither does their product.

Proof. One direction in the criterion is obvious. Suppose the other is false for the pair (d,n), so that we can find for each $w \in \mathbb{N}$, a field K_w and a non-prime ideal $\mathfrak{a}_w \subseteq A_w$ generated by polynomials of degree at most d, such that any two polynomials of degree at most w not in $\mathfrak{a}_w$ have their product also outside $\mathfrak{a}_w$. Taking ultraproducts of the generators of the $\mathfrak{a}_w$ of degree at most d gives polynomials of degree at most d in A by 4.1.2, and by Łoś' Theorem if $\mathfrak{a} \subseteq A$ is the ideal they generate, then $\mathfrak{a}U(A)$ is the ultraproduct of the $\mathfrak{a}_w$. I claim that $\mathfrak{a}$ is a prime ideal. However, this implies that almost all $\mathfrak{a}_w$ must be prime ideals by Theorem 4.3.4, contradiction.

To verify the claim, let $f, g \notin \mathfrak{a}$. We want to show that $fg \notin \mathfrak{a}$. Let e be the maximum of the degrees of f and g. Choose approximations $f_w, g_w \in A_w$ of degree at most e, of f and g respectively. By Łoś' Theorem, $f_w, g_w \notin \mathfrak{a}_w$ for almost all w. For $w \geq e$, our assumption then implies that $f_w g_w \notin \mathfrak{a}_w$, whence by Łoś' Theorem, their ultraproduct $fg \notin \mathfrak{a}U(A)$. A fortiori, then neither does fg belong to $\mathfrak{a}$, as we wanted to show. □

The pattern by now must become clear: prove that a particular property of ideals is preserved under ultra-hulls, and use this to deduce uniform bounds. For instance, one can easily derive from Theorem 4.3.4 that:

Proposition 4.4.5. *The image of a radical ideal in the ultra-hull remains radical.* □

Since the radical of an ideal is the intersection of its minimal overprimes, we derive from this the following uniform bounds property:

Theorem 4.4.6. *For any pair of positive integers (d,n), there exists a uniform bound $b := b(d,n)$ with the following property. Let k be a field, and let I be an ideal in $k[\xi_1, \dots, \xi_n]$ generated by polynomials of degree at most d. Then its radical $J := \operatorname{rad}(I)$ is generated by polynomials of degree at most b. Moreover, $J^b \subseteq I$ and I has at most b distinct minimal overprimes, all of which are generated by polynomials of degree at most b.* □

Similarly, we can use Theorem 3.3.14, the Colon Criterion, to show that there exists a uniform bound $b := b(d,n)$ such that for any field k, any ideal $I \subseteq k[\xi]$ generated by polynomials of degree at most d, and any $a \in k[\xi]$ of degree at most d in the n indeterminates ξ, the ideal $(I : a)$ is generated by polynomials of degree at most b.

Realizing a finitely generated module as the cokernel of a matrix (acting on a free module) and using that ultraproducts commute with homology (Theorem 3.1.1), one can extend all the previous bounds to modules as well. This was the route taken in [88]. Without proof, I state one of the results of that paper proven by this technique.

Theorem 4.4.7 ([88, Theorem 4.5]). *For any triple of positive integers (d,n,i), there exists a uniform bound $b := b(d,n,i)$ with the following property. Let k be a field,* *let B be an affine algebra of the form $k[\xi_1,\dots,\xi_n]/I$ with I an ideal generated by polynomials of degree at most d, and let M and N be finitely generated B-modules realized as the cokernel of matrices of size at most d and with entries of degree at most d. If $M \otimes_A N$ has finite length, then the i-th Betti number, that is to say, the length of $\operatorname{Tor}_i^A(M,N)$, is bounded by b. Similarly, if $\operatorname{Hom}_A(M,N)$ has finite length, then the i-th Bass number, that is to say, the length of $\operatorname{Ext}^i_A(M,N)$, is at most b.* □

4.4.3 Comments

Our proof of the flatness of ultra-hulls (Theorem 4.2.2) is entirely different from the original proof of Schmidt and van den Dries, which uses an induction on the number of indeterminates based on classical arguments of Hermann from constructive commutative algebra. The present approach via big Cohen-Macaulay algebras has the advantage that one can extend this method to many other situations, like Theorem 7.1.6 below. Yet another approach, through a coherency result due to Vasconcelos, can be found in [5].

As already mentioned, some of the bounds proven here were already established by Hermann [42], based on work of Seidenberg [103, 104, 105] on constructions in algebra. Using Groebner bases, Buchberger obtained in [18] the same result, but by an explicit description of an algorithm (e.g., one that calculates the polynomials g_i in Theorem 4.4.1). This led to a direct implementation into various algebraic software programs, which was not practically feasible in the case of Hermann's explicit proof using elimination theory, in view of the exponential growth of degrees of polynomials involved in this elimination process. Model-theoretic proofs, such as the ones in this book, lack even more practical implementation, but they provide sometimes extra information. For instance, we show that there exist uniform bounds that are independent of the base field. With some additional work, it is sometimes possible to show that the bounds are recursive (see, for instance, [11]). But even these abstract methods can sometimes lead to explicit bounds, as is evident from Schmidt's work [84, 85].

Chapter 5
Tight closure in positive characteristic

In this chapter, p is a fixed prime number, and all rings are assumed to have characteristic p, unless explicitly mentioned otherwise. We review the notion of tight closure due to Hochster and Huneke (as a general reference, we will use [59]). The main protagonist in this elegant theory is the p-th power Frobenius map. We will focus on five key properties of tight closure, which will enable us to prove, virtually effortlessly, several beautiful theorems. Via these five properties, we can give a more axiomatic treatment, which lends itself nicely to generalization, and especially to a similar theory in characteristic zero (see Chapters 6 and 7).

5.1 Frobenius

The major advantage of rings of positive characteristic is the presence of an algebraic endomorphism: the Frobenius. More precisely, let A be a ring of characteristic p, and let $\mathbf{F}_p$, or more accurately, $\mathbf{F}_{p,A}$, be the ring homomorphism $A \to A\colon a \mapsto a^p$, called the *Frobenius* on A. Recall that this is indeed a ring homomorphism, where the only thing to note is that the coefficients in the binomial expansion

$$\mathbf{F}_p(a+b) = \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i} = \mathbf{F}_p(a) + \mathbf{F}_p(b)$$

are divisible by p for all $0 < i < p$ whence zero in A, proving that $\mathbf{F}_p$ is additive.

When A is reduced, $\mathbf{F}_p$ is injective whence yields an isomorphism with its image $A^p := \mathrm{Im}(\mathbf{F}_p)$ consisting of all p-th powers of elements in A (and not to be confused with the p-th Cartesian power of A). The inclusion $A^p \subseteq A$ is isomorphic with the Frobenius on A because we have a commutative diagram

$$
\begin{tikzcd}
& A \arrow[dl, "\cong"'] \arrow[dr, "\mathbf{F}_p"] & \\
A^p \arrow[rr, "\subseteq"'] & & A
\end{tikzcd}
\tag{5.1}
$$

When A is a domain, then we can also define the ring $A^{1/p}$ as the subring of the algebraic closure of the field of fractions of A consisting of all elements b satisfying $b^p \in A$. Hence $A \subseteq A^{1/p}$ is integral. Since, $\mathbf{F}_p(A^{1/p}) = A$ and $\mathbf{F}_p$ is injective, we get $A^{1/p} \cong A$. Moreover, we have a commutative diagram

$$
\begin{tikzcd}
& A \arrow[dl, "\subseteq"'] \arrow[dr, "\mathbf{F}_p"] & \\
A^{1/p} \arrow[rr, "\cong"'] & & A
\end{tikzcd}
\tag{5.2}
$$

showing that the Frobenius on A is also isomorphic to the inclusion $A \subseteq A^{1/p}$. It is sometimes easier to work with either of these inclusions rather than with the Frobenius itself, especially to avoid notational ambiguity between source and target of the Frobenius (instances where this approach would clarify the argument are the proofs of Theorem 5.1.2 and Corollary 5.1.3 below).

Often, the inclusion $A^p \subseteq A$ is even finite, and hence so is the Frobenius itself. One can show, using Noether normalization or Cohen's Structure Theorems that this is true when A is respectively a k-affine algebra or a complete Noetherian local ring with residue field k, and k is perfect, or more generally, $(k : k^p) < \infty$.

5.1.1 Frobenius transforms.

Given an ideal $I \subseteq A$, we will denote its extension under the Frobenius by $\mathbf{F}_p(I)A$, and call it the *Frobenius transform* of I. Note that $\mathbf{F}_p(I)A \subseteq I^p$, but the inclusion is in general strict. In fact, one easily verifies that

5.1.1 *If* $I = (x_1, \dots, x_n)A$, *then* $\mathbf{F}_p(I)A = (x_1^p, \dots, x_n^p)A$.

If we repeat this process, we get the *iterated Frobenius transforms* $\mathbf{F}_p^n(I)A$ of I, generated by the p^n-th powers of elements in I, and in fact, of generators of I. In tight closure theory, the simplified notation

$$I^{[p^n]} := \mathbf{F}_p^n(I)A$$

is normally used, but for reasons that will become apparent once we defined tight closure as a difference closure (see §6.1.1), we will use the 'heavier' notation. On the other hand, since we fix the characteristic, we may omit p from the notation and simply write $\mathbf{F}\colon A \to A$ for the Frobenius.

5.1.2 Kunz's theorem.

The next result, due to Kunz's Theorem, characterizes regular local rings in positive characteristic via the Frobenius. We will only prove the direction that we need.

Theorem 5.1.2 (Kunz). *Let R be a Noetherian local ring. If R is regular, then $\mathbf{F}_p$ is flat. Conversely, if R is reduced and $\mathbf{F}_p$ is flat, then R is regular.*

Proof. We only prove the direct implication; for the converse see [68, §42]. Let $\mathbf{x}$ be a system of parameters of R, whence an R-regular sequence. Since $\mathbf{F}(\mathbf{x})$ is also a system of parameters, it too is R-regular. Hence, R, viewed as an R-algebra via $\mathbf{F}$, is a balanced big Cohen-Macaulay algebra, whence is flat by Theorem 3.3.9. □

Corollary 5.1.3. *If R is a regular local ring, $I \subseteq R$ an ideal, and $a \in R$ an arbitrary element, then $a \in I$ if and only if $\mathbf{F}(a) \in \mathbf{F}(I)R$.*

Proof. One direction is of course trivial, so assume $\mathbf{F}(a) \in \mathbf{F}(I)R$. However, since $\mathbf{F}$ is flat by Theorem 5.1.2, the contraction of the extended ideal $\mathbf{F}(I)R$ along $\mathbf{F}$ is again I by Proposition 3.2.5, and a lies in this contraction (recall that $\mathbf{F}(I)R \cap R$ stands really for $\mathbf{F}^{-1}(\mathbf{F}(I)R)$.) □

5.2 Tight closure

The definition of tight closure, although not complicated, is not that intuitive either. The idea is inspired by the ideal membership test of Corollary 5.1.3. Unfortunately, that test only works over regular local rings, so that it will be no surprise that whatever test we design, it will have to be more involved. Moreover, the proposed test will in fact fail in general, that is to say, the elements satisfying the test form an ideal which might be strictly bigger than the original ideal. But not too much bigger, so that we may view this bigger ideal as a closure of the original ideal, and as such, it is a 'tight' fit.

In the remainder of this section, A is a Noetherian ring, of characteristic p. A first obvious generalization of the ideal membership test from Corollary 5.1.3 is to allow iterates of the Frobenius: we could ask, given an ideal $I \subseteq A$, what are the elements x such that $\mathbf{F}^n(x) \in \mathbf{F}^n(I)A$ for some power n? They do form an ideal and the resulting closure operation is called the *Frobenius closure*. However, its properties are not sufficiently strong to derive all the results tight closure can.

The adjustment to make in the definition of Frobenius closure, although minor, might at first be a little surprising. To make the definition, we will call an element $a \in A$ a *multiplier*, if it is either a unit, or otherwise generates an ideal of positive height (necessarily one by Krull's Principal Ideal Theorem). Put differently, a is a multiplier if it does not belong to any minimal prime ideal of A. In particular, the product of two multipliers is again a multiplier. In a domain, a situation we can often reduce to, a multiplier is simply a non-zero element.

The name 'multiplier' comes from the fact that we will use such elements to multiply our test condition with. However, for this to make sense, we cannot just take one iterate of the Frobenius, we must take all of them, or at least all but finitely many. So we now define: an element $x \in A$ belongs to the *tight closure* $\mathrm{cl}_A(I)$ of an ideal $I \subseteq A$, if there exists a multiplier $c \in A$ and a positive integer N such that

$$c\,\mathbf{F}^n(x) \in \mathbf{F}^n(I)A \tag{5.3}$$

for all $n \geq N$. Note that the multiplier c and the bound N may depend on x and I, but not on n. We will write $\mathrm{cl}(I)$ for $\mathrm{cl}_A(I)$ if the ring A is clear from the context. In the literature, tight closure is invariably denoted I^*, but again for reasons that will become clear in the next chapter, our notation better suits our purposes. Let us verify some elementary properties of this closure operation:

5.2.1 *The tight closure of an ideal I in a Noetherian ring A is again an ideal, it contains I, and it is equal to its own tight closure. Moreover, we can find a multiplier c and a positive integer N which works simultaneous for all elements in* $\mathrm{cl}(I)$ *in criterion* (5.3).

It is easy to verify that $\mathrm{cl}(I)$ is closed under multiples, and contains I. To show that it is closed under sums, whence an ideal, assume $x, x' \in A$ both lie in $\mathrm{cl}(I)$, witnessed by the equations (5.3) for some multipliers c and c', and some positive integers N and N' respectively. However, $cc'\mathbf{F}^n(x+x')$ then lies in $\mathbf{F}^n(I)A$ for all $n \geq \max\{N, N'\}$, showing that $x + x' \in \mathrm{cl}(I)$ since cc' is again a multiplier. Let $J := \mathrm{cl}(I)$ and choose generators $y_1, \dots, y_s$ of J. Let c_i and N_i be the corresponding multiplier and bound for y_i. It follows that $c := c_1c_2\cdots c_s$ is a multiplier such that (5.3) holds for all $n \geq N := \max\{N_1, \dots, N_s\}$ and all $x \in J$, since any such element is a linear combination of the y_i. In particular, $c\,\mathbf{F}^n(J)A \subseteq \mathbf{F}^n(I)A$ for all $n \geq N$. Hence if z lies in the tight closure of J, so that $d\mathbf{F}^n(z) \in \mathbf{F}^n(J)A$ for some multiplier d and for all $n \geq M$, then $cd\,\mathbf{F}^n(z) \in \mathbf{F}^n(I)A$ for all $n \geq \max\{M, N\}$, whence $z \in \mathrm{cl}(I) = J$. The last assertion now easily follows from the above analysis. In the sequel, we will therefore no longer make the bound N explicit and instead of "for all $n \geq N$" we will just write "for all $n \gg 0$".

Example 5.2.2. It is instructive to look at some examples. Let K be a field of characteristic $p > 3$, and let $A := K[\xi, \zeta, \eta]/(\xi^3 - \zeta^3 - \eta^3)K[\xi, \zeta, \eta]$ be the projective coordinate ring of the *cubic Fermat curve*. Let us show that ξ^2 is in the tight closure of $I := (\zeta, \eta)A$. For a fixed e, write $2p^e = 3h + r$ for some $h \in \mathbb{N}$ and some remainder $r \in \{1, 2\}$, and let c be the multiplier ξ^3. Hence

$$c\mathbf{F}^e(\xi^2) = \xi^{3(h+1)+r} = \xi^r(\zeta^3+\eta^3)^{h+1}.$$

A quick calculation shows that any monomial in the expansion of $(\zeta^3+\eta^3)^{h+1}$ is a multiple of either $\mathbf{F}^e(\zeta)$ or $\mathbf{F}^e(\eta)$, showing that (5.3) holds for all e, and hence that $(\xi^2,\zeta,\eta)A \subseteq \mathrm{cl}(I)$.

It is often much harder to show that an element does not belong to the tight closure of an ideal. Shortly, we will see in Theorem 5.3.6 that any element outside the integral closure is also outside the tight closure. Since $(\xi^2,\zeta,\eta)A$ is integrally closed, we conclude that it is equal to $\mathrm{cl}(I)$.

Example 5.2.3. Let A be the coordinate ring of the hypersurface in $\mathbb{A}^3_K$ given by the equation $\xi^2-\zeta^3-\eta^7=0$. By a similar calculation as in the previous example, one can show that ξ lies in the tight closure of $(\zeta,\eta)A$.

A far more difficult result is to show that this is not true if we replace η^7 by η^5 in the above equation. In fact, in this new coordinate ring A', any ideal is tightly closed, that is to say, in the terminology from Definition 5.2.7 below, A' is F-regular, but this is a deep fact, following from it being log-terminal (see the discussion following Theorem 5.5.6).

It is sometimes cumbersome to work with multipliers in arbitrary rings, but in domains they are just non-zero elements. Fortunately, we can always reduce to the domain case when calculating tight closure:

Proposition 5.2.4. *Let A be a Noetherian ring, let $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ be its minimal primes, and put $\bar{A}_i := A/\mathfrak{p}_i$. For all ideals $I\subseteq A$ we have*

$$\mathrm{cl}_A(I) = \bigcap_{i=1}^{s} \mathrm{cl}_{\bar{A}_i}(I\bar{A}_i)\cap A. \tag{5.4}$$

Proof. The same equations which exhibit x as en element of $\mathrm{cl}_A(I)$ also show that it is in $\mathrm{cl}_{\bar{A}_i}(I\bar{A}_i)$ since any multiplier in A remains, by virtue of its definition, a multiplier in $\bar{A}_i$ (moreover, the converse also holds: by prime avoidance, we can lift any multiplier in $\bar{A}_i$ to one in A). So one inclusion in (5.4) is clear.

Conversely, suppose x lies in the intersection on the right hand side of (5.4). Let $c_i\in A$ be a multiplier in A (so that its image is a multiplier in $\bar{A}_i$), such that

$$c_i\mathbf{F}^n_{\bar{A}_i}(x)\in \mathbf{F}^n_{\bar{A}_i}(I)\bar{A}_i$$

for all $n\gg 0$. This means that each $c_i\mathbf{F}^n_A(x)$ lies in $\mathbf{F}^n_A(I)A+\mathfrak{p}_i$ for $n\gg 0$. Choose for each i, an element $t_i\in A$ inside all minimal primes except $\mathfrak{p}_i$, and let $c := c_1t_1+\dots+c_st_s$. A moment's reflection yields that c is again a multiplier. Moreover, since $t_i\mathfrak{p}_i\subseteq\mathfrak{n}$, where $\mathfrak{n} := \mathrm{nil}(R)$ is the nilradical of A, we get

$$c\,\mathbf{F}^n_A(x)\in\mathbf{F}^n_A(I)A+\mathfrak{n}$$

for all $n\gg 0$. Choose m such that $\mathfrak{n}^{p^m}$ is zero, whence also the smaller ideal $\mathbf{F}_A(\mathfrak{n})$. Applying $\mathbf{F}^m_A$ to the previous equations, yields

$$\mathbf{F}_A^m(c)\mathbf{F}_A^{m+n}(x) \in \mathbf{F}_A^{m+n}(I)A$$

for all $n \gg 0$, which means that $x \in \mathrm{cl}_A(I)$ since $\mathbf{F}_A^m(c)$ is again a multiplier. □

We will encounter many operations similar to tight closure, and so we formally define:

Definition 5.2.5 (Closure operation). A *closure operation* on a ring A is any order-preserving, increasing, idempotent endomorphism on the set of ideals of A ordered by inclusion.

For instance, taking the radical of an ideal is a closure operation, and so is *integral closure* discussed below. Tight closure too is a closure operation on A, since it clearly also preserves inclusion: if $I \subseteq I'$, then $\mathrm{cl}(I) \subseteq \mathrm{cl}(I')$. An ideal that is equal to its own tight closure is called *tightly closed*. Recall that the *colon ideal* $(I : J)$ is the ideal of all elements $a \in A$ such that $aJ \subseteq I$; here $I \subseteq A$ is an ideal, but $J \subseteq A$ can be any subset, which, however, most of the time is either a single element or an ideal. Almost immediately from the definitions, we get

5.2.6 *If I is tightly closed, then so is $(I : J)$ for any $J \subseteq A$.* □

One of the longest outstanding open problems in tight closure theory was its behavior under localization: do we always have

$$\mathrm{cl}_A(I)A_{\mathfrak{p}} \stackrel{?}{=} \mathrm{cl}_{A_{\mathfrak{p}}}(IA_{\mathfrak{p}}) \tag{5.5}$$

for every prime ideal $\mathfrak{p} \subseteq A$. Recently, Brenner and Monsky have announced (see [15]) a negative answer to this question. The full extent of this phenomenon is not yet understood, and so one has proposed the following two definitions (the above cited counterexample still does not contradict that both notions are the same).

Definition 5.2.7. A Noetherian ring A is called *weakly F-regular* if each of its ideals is tightly closed. If all localizations of A are weakly F-regular, then A is called *F-regular*.

5.3 Five key properties of tight closure

In this section we derive five key properties of tight closure, all of which admit fairly simple proofs. It is important to keep this in mind, since these five properties will already suffice to prove in the next section some deep theorems in commutative algebra. In fact, as we will see, any closure operation with these five properties on a class of Noetherian local rings would establish these deep theorems for that particular class (and there are still classes for which these statements remain conjectural). Moreover, the proofs of the five properties themselves rest on a few simple facts about the Frobenius, so that this will allow us to also carry over our arguments to characteristic zero in Chapters 6 and 7.

The first property, stated here only in its weak version, is merely an observation. Namely, any equation (5.3) in a ring A extends to a similar equation in any A-algebra B. In order for the latter to calculate tight closure, the multiplier $c \in A$ should remain a multiplier in B, and so we proved:

Theorem 5.3.1 (Weak Persistence). *Let $A \to B$ be a ring homomorphism, and let $I \subseteq A$ be an ideal. If $A \to B$ is injective and B is a domain, or more generally, if $A \to B$ preserves multipliers, then* $\mathrm{cl}_A(I) \subseteq \mathrm{cl}_B(IB)$. □

The remarkable fact is that this is also true if $A \to B$ is arbitrary and A is of finite type over an excellent Noetherian local ring (see [59, Theorem 2.3]). We will not need this stronger version, the proof of which requires another important ingredient of tight closure theory: the notion of a test element. A multiplier $c \in A$ is called a *test element* for A, if for every $a \in \mathrm{cl}(I)$, we have $c\mathbf{F}^n(a) \in \mathbf{F}^n(I)A$ for all n. The existence of test elements is not easy, and lies outside the scope of these notes, but once one has established their existence, many arguments become even more streamlined.

Theorem 5.3.2 (Regular closure). *In a regular local ring, every ideal is tightly closed. In fact, a regular ring is F-regular.*

Proof. Let R be a regular local ring. Since any localization of R is again regular, the second assertion follows from the first. To prove the first, let I be an ideal and $x \in \mathrm{cl}(I)$. Towards a contradiction, assume $x \notin I$. In particular, we must have $(I : x) \subseteq \mathfrak{m}$. Choose a non-zero element c such that (5.3) holds for all $n \gg 0$. This means that c lies in the colon ideal $(\mathbf{F}^n(I)R : \mathbf{F}^n(x))$, for all $n \gg 0$. Since $\mathbf{F}$ is flat by Theorem 5.1.2, the colon ideal is equal to $\mathbf{F}^n(I : x)R$ by Theorem 3.3.14. Since $(I : x) \subseteq \mathfrak{m}$, we get $c \in \mathbf{F}^n(\mathfrak{m})R \subseteq \mathfrak{m}^{p^n}$. Since this holds for all $n \gg 0$, we get $c = 0$ by Theorem 2.4.14, clearly a contradiction. □

Theorem 5.3.3 (Colon Capturing). *Let R be a Noetherian local domain which is a homomorphic image of a Cohen-Macaulay local ring, and let $(x_1, \dots, x_d)$ be a system of parameters in R. Then for each i, the colon ideal $((x_1, \dots, x_i)R : x_{i+1})$ is contained in* $\mathrm{cl}((x_1, \dots, x_i)R)$.

Proof. Let S be a local Cohen-Macaulay ring such that $R = S/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq S$ of height h. By prime avoidance, we can lift the x_i to elements in S, again denoted for simplicity by x_i, and find elements $y_1, \dots, y_h \in \mathfrak{p}$ such that $(y_1, \dots, y_h, x_1, \dots, x_d)$ is a system of parameters in S, whence an S-regular sequence. Since $\mathfrak{p}$ contains the ideal $J := (y_1, \dots, y_h)S$ of the same height, it is a minimal prime of J. Let $J = \mathfrak{g}_1 \cap \dots \cap \mathfrak{g}_s$ be a minimal primary decomposition of J, with $\mathfrak{g}_1$ the $\mathfrak{p}$-primary component of J. In particular, some power of $\mathfrak{p}$ lies in $\mathfrak{g}_1$, and we may assume that this power is of the form p^m for some m. Choose c inside all $\mathfrak{g}_i$ with $i > 1$, but outside $\mathfrak{p}$ (note that this is possible by prime avoidance). Putting everything together, we have

$$c\mathbf{F}^m(\mathfrak{p}) \subseteq c\,\mathfrak{p}^{p^m} \subseteq J. \tag{5.6}$$

Fix some i, let $I := (x_1, \dots, x_i)S$ and assume $zx_{i+1} \in IR$, for some $z \in S$. Lifting this to S, we get $zx_{i+1} \in I + \mathfrak{p}$. Applying the n-th power of Frobenius to this for $n > m$, we get $\mathbf{F}^n(z)\mathbf{F}^n(x_{i+1}) \in \mathbf{F}^n(I)S + \mathbf{F}^n(\mathfrak{p})S$. By (5.6), this means that $c\,\mathbf{F}^n(z)\mathbf{F}^n(x_{i+1})$ lies in $\mathbf{F}^n(I)S + \mathbf{F}^{n-m}(J)S$. Since the $\mathbf{F}^{n-m}(y_j)$ together with the $\mathbf{F}^n(x_j)$ form again an S-regular sequence, we conclude that

$$c\,\mathbf{F}^n(z) \in \mathbf{F}^n(I)S + \mathbf{F}^{n-m}(J)S \subseteq \mathbf{F}^n(I)S + J$$

whence $c\,\mathbf{F}^n(z) \in \mathbf{F}^n(I)R$ for all $n > m$. By the choice of c, it is non-zero in R, so that the latter equations show that $z \in \mathrm{cl}(IR)$. □

The condition that R is a homomorphic image of a regular local ring is satisfied either if R is a local affine algebra, or, by Cohen's Structure Theorems, if R is complete. These are the two only cases in which we will apply the previous theorem. With a little effort, one can extend the proof without requiring R to be a domain (see for instance [59, Theorem 3.1]).

Theorem 5.3.4 (Finite extensions). *If $A \to B$ is a finite, injective homomorphism of domains, and $I \subseteq A$ be an ideal, then $\mathrm{cl}_B(IB) \cap A = \mathrm{cl}_A(I)$.*

Proof. One direction is immediate by Theorem 5.3.1. For the converse, there exists an A-module homomorphism $\varphi\colon B \to A$ such that $c := \varphi(1) \neq 0$, by Lemma 5.3.5 below. Suppose $x \in \mathrm{cl}_B(IB) \cap A$, so that for some non-zero $d \in B$, we have $d\,\mathbf{F}^n(x) \in \mathbf{F}^n(I)B$ for $n \gg 0$. Since B is finite over A, some non-zero multiple of d lies in A, and hence without loss of generality, we may assume $d \in A$. Applying φ to these equations, we get

$$cd\,\mathbf{F}^n(x) \in \mathbf{F}^n(I)A$$

showing that $x \in \mathrm{cl}_A(I)$, since cd is a multiplier. □

Lemma 5.3.5. *If $A \subseteq B$ is a finite extension of domains, then there exists an A-linear map $\varphi\colon B \to A$ with $\varphi(1) \neq 0$.*

Proof. Suppose B is generated over A by the elements $b_1, \dots, b_s$. Let K and L be the fields of fractions of A and B respectively. Since B is a domain, it lies inside the K-vector subspace $V \subseteq L$ generated by the b_i. Choose an isomorphism $\gamma\colon V \to K^t$ of K-vector spaces. After renumbering, we may assume that the first entry of $\gamma(1)$ is non-zero. Let $\pi\colon K^t \to K$ be the projection onto the first coordinate, and let $d \in A$ be the common denominator of the $\pi(\gamma(b_i))$ for $i = 1, \dots, s$. Now define an A-linear homomorphism φ by the rule $\varphi(y) = d\pi(\gamma(y))$ for $y \in B$. Since y is an A-linear combination of the b_i and since $d\pi(\gamma(b_i)) \in A$, also $\varphi(y) \in A$. Moreover, by construction, $\varphi(1) \neq 0$. □

Note that a special case of Theorem 5.3.4 is the fact that tight closure measures the extent to which an extension of domains $A \subseteq B$ fails to be cyclically pure: $IB \cap A$ is contained in the tight closure of I, for any ideal $I \subseteq A$. In particular, in view of Theorem 5.3.2, this reproves the well-known fact that if $A \subseteq B$ is an extension of domains with A regular, then $A \subseteq B$ is cyclically pure. The next and last property involves another closure operation, integral closure. It will be discussed in more detail below (§5.4), and here we just state its relationship with tight closure:

Theorem 5.3.6 (Integral closure). *For every ideal $I \subseteq A$, its tight closure is contained in its integral closure. In particular, radical ideals, and more generally integrally closed ideals, are tightly closed.*

Proof. The second assertion is an immediate consequence of the first. We verify condition (5.4.1.iv) below to show that if x belongs to the tight closure $\mathrm{cl}_A(I)$, then it also belongs to the integral closure $\bar{I}$. Let $A \to V$ be a homomorphism into a discrete valuation ring V, such that its kernel is a minimal prime of A. We need to show that $x \in IV$. However, this is clear since $x \in \mathrm{cl}_V(IV)$ by Theorem 5.3.1 (note that $A \to V$ preserves multipliers), and since $\mathrm{cl}_V(IV) = IV$, by Theorem 5.3.2 and the fact that V is regular. □

It is quite surprising that there is no proof, as far as I am aware of, that a prime ideal is tightly closed without reference to integral closure.

5.4 Integral closure

The *integral closure* $\bar{I}$ of an ideal I is the collection of all elements $x \in A$ satisfying an integral equation of the form

$$x^d + a_1 x^{d-1} + \cdots + a_d = 0 \tag{5.7}$$

with $a_j \in I^j$ for all $j = 1, \dots, d$. We say that I is *integrally closed* if $I = \bar{I}$. Since clearly $\bar{I} \subseteq \mathrm{rad}(I)$, radical ideals are integrally closed. It follows from either characterization (5.4.1.ii) or (5.4.1.iv) below that $\bar{I}$ is an ideal.

Theorem 5.4.1. *Let A be an arbitrary Noetherian ring (not necessarily of characteristic p). For an ideal $I \subseteq A$ and an element $x \in A$, the following are equivalent*

- 5.4.1.i. *x belongs to the integral closure, $\bar{I}$;*
- 5.4.1.ii. *there is a finitely generated A-module M with zero annihilator such that $xM \subseteq IM$;*
- 5.4.1.iii. *there is a multiplier $c \in A$ such that $cx^n \in I^n$ for infinitely many (respectively, for all sufficiently large) n;*
- 5.4.1.iv. *for every homomorphism $A \to V$ into a discrete valuation ring V with kernel equal to a minimal prime of A, we have $x \in IV$.*

Proof. We leave it to the reader to show that x lies in the integral closure of an ideal I if and only if it lies in the integral closure of each $I(A/\mathfrak{p})$, for $\mathfrak{p}$ a minimal prime of A. Hence we may moreover assume that A is a domain. Suppose x satisfies an integral equation (5.7), and let $J := x^{d-1}A + x^{d-2}I + \cdots + I^d$. An easy calculation shows that $xJ \subseteq IJ$, proving (5.4.1.i) $\Rightarrow$ (5.4.1.ii). Moreover, by induction, $x^nJ \subseteq I^nJ$, and hence for any non-zero element $c \in J$, we get $cx^n \in I^n$, proving (5.4.1.iii). Note that in particular, $x^nI^d \subseteq I^n$ for all n. The implication (5.4.1.ii) $\Rightarrow$ (5.4.1.i) is proven by a 'determinantal trick': apply [69, Theorem 2.1] to the multiplication with x on M. To prove (5.4.1.iii) $\Rightarrow$ (5.4.1.iv), suppose there is some non-zero $c \in A$ such that $cx^n \in I^n$ for infinitely many n. Let $A \subseteq V$ be an injective homomorphism into a discrete valuation ring V, and let v be the valuation on V. Hence $v(c) + nv(x) \geq nv(I)$ for infinitely many n, where $v(I)$ is the minimum of all $v(a)$ with $a \in I$. It follows that $v(x) \geq v(I)$, and hence $x \in IV$.

Remains to prove (5.4.1.iv) $\Rightarrow$ (5.4.1.i), so assume $x \in IV$ for every embedding $A \subseteq V$ into a discrete valuation ring V. Let $I = (a_1, \dots, a_n)A$, and consider the homomorphism $A[\xi] \to A_x$ given by $\xi_i \mapsto a_i/x$, where $\xi := (\xi_1, \dots, \xi_n)$. Let B be its image, so that $A \subseteq B \subseteq A_x$ (one calls B the *blowing-up* of $I + xA$ at x). Let $\mathfrak{m} := (\xi_1, \dots, \xi_n)A[\xi]$. I claim that $\mathfrak{m}B = B$. Assuming the claim, we can find $f \in \mathfrak{m}$ such that $f(\mathbf{a}/x) = 1$ in A_x, where $\mathbf{a} := (a_1, \dots, a_n)$. Write $f = f_1 + \cdots + f_d$ in its homogeneous parts f_j of degree j, so that

$$1 = x^{-1}f_1(\mathbf{a}) + \cdots + x^{-d}f_d(\mathbf{a}).$$

Multiplying with x^d, and observing that $f_j(\mathbf{a}) \in I^j$, we see that x satisfies an integral equation (5.7), and hence $x \in \bar{I}$.

To prove the claim ex absurdum, suppose $\mathfrak{m}B$ is not the unit ideal, whence is contained in a maximal ideal $\mathfrak{n}$ of B. Let $(x_1, \dots, x_n)$ be a generating tuple of $\mathfrak{n}$. Let R be the $B_\mathfrak{n}$-algebra generated by the fractions x_i/x_1 with $i = 1, \dots, n$ (the blowing-up of $B_\mathfrak{n}$ at $\mathfrak{n}$). Since $\mathfrak{n}R = x_1R$, there exists a height one prime ideal $\mathfrak{p}$ in R containing $\mathfrak{n}R$. Let V be the normalization of $R_\mathfrak{p}$. It follows that V is a discrete valuation ring (see [69, Theorem 11.2]) containing $B_\mathfrak{n}$ as a local subring. In particular, $A \subseteq V$, and $\mathfrak{m}V$ lies in the maximal ideal πV. Since $\xi_i \mapsto a_i/x$, we get $a_i \in x\pi V$ for all i, and hence $IV \subseteq x\pi V$, contradicting that $x \in IV$. □

From this we readily deduce:

Corollary 5.4.2. *A domain A is normal if and only if each principal ideal is integrally closed if and only if each principal ideal is tightly closed.* □

In one of our applications below (Theorem 5.5.1), we will make use of the following nice application of the chain rule:

Proposition 5.4.3. *Let K be a field of characteristic zero, and let R be either the power series ring $K[[\xi]]$, the ring of convergent power series $K\{\xi\}$ (assuming K is a normed field), or the localization of $K[\xi]$ at the ideal generated by the indeterminates $\xi := (\xi_1, \dots, \xi_n)$. If f is a non-unit, then it lies in the integral closure of its Jacobian ideal $\mathrm{Jac}(f) := (\partial f/\partial\xi_1, \dots, \partial f/\partial\xi_n)R$.*

Proof. Recall that $K\{\xi\}$ consists of all formal power series f such that $f(\mathbf{u})$ is a convergent series for all $\mathbf{u}$ in a small enough neighborhood of the origin. Put $J := \mathrm{Jac}(f)$. In view of (5.4.1.iv), we need to show that given an embedding $R \subseteq V$

into a discrete valuation ring V, we have $f \in JV$. Since completion is faithfully flat, we may replace V by its completion, and hence already assume V is complete. By Cohen's Structure Theorems, V is a power series ring $\kappa[[\zeta]]$ in a single variable over a field extension κ of K. Viewing the image of f in $\kappa[[\zeta]]$ as a power series in ζ, the multi-variate chain rule yields

$$\frac{df}{d\zeta} = \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i} \cdot \frac{d\xi_i}{d\zeta} \in JV.$$

However, since f has order $e \geq 1$ in V, its derivative $df/d\zeta$ has order $e-1$, and hence $f \in (df/d\zeta)V \subseteq JV$. Note that for this to be true, however, the characteristic needs to be zero. For instance, in characteristic p, the power series ξ^p would already be a counterexample to the proposition. $\square$

Since the integral closure is contained in the radical closure, we get that some power of f lies in its Jacobian ideal $\mathrm{Jac}(f)$. A famous theorem due to Briançon-Skoda states that in fact already the n-th power lies in the Jacobian, where n is the number of variables. We will prove this via an elegant tight closure argument in Theorem 5.5.1 below).

5.5 Applications

We will now discuss three important applications of tight closure. Perhaps surprisingly, the original statements all were in characteristic zero (with some of them in their original form plainly false in positive characteristic), and their proofs required deep and involved arguments, some even based on transcendental/analytic methods. However, they each can be reformulated so that they also make sense in positive characteristic, and then can be established by surprisingly elegant tight closure arguments. As for the proofs of their characteristic zero counterparts, they must wait until we have developed the theory in characteristic zero in Chapters 6 and 7 (or one can use the 'classical' tight closure in characteristic zero discussed in §5.6).

5.5.1 The Briançon-Skoda theorem.

We already mentioned this famous result, proven first in [16].

Theorem 5.5.1 (Briançon-Skoda). *Let R be either the ring of formal power series $\mathbb{C}[[\xi]]$, or the ring of convergent power series $\mathbb{C}\{\xi\}$, or the localization of the polynomial ring $\mathbb{C}[\xi]$ at the ideal generated by ξ, where $\xi := (\xi_1, \dots, \xi_n)$ are some indeterminates. If f is not a unit, then $f^n \in \mathrm{Jac}(f) := (\partial f/\partial \xi_1, \dots, \partial f/\partial \xi_n)R$.*

This theorem will follow immediately from the characteristic zero analogue of the next result (with $l = 1$), in view of Proposition 5.4.3; we will do this in Theorem 6.2.5 below.

Theorem 5.5.2 (Briançon-Skoda—tight closure version). *Let A be a Noetherian ring of characteristic p, and $I \subseteq A$ an ideal generated by n elements. Then we have for all $l \geq 1$ an inclusion*

$$\overline{I^{n+l-1}} \subseteq \mathrm{cl}(I^l).$$

In particular, if A is a regular local ring, then the integral closure of I^{n+l-1} lies inside I^l for $l \geq 1$.

Proof. For simplicity, I will only prove the case $l = 1$ (which gives the original Briançon-Skoda theorem). Assume z lies in the integral closure of I^n. By (5.4.1.iii), there exists a multiplier $c \in A$ such that $cz^k \in I^{kn}$ for all $k \gg 0$. Since $I := (f_1, \dots, f_n)A$, we have an inclusion $I^{kn} \subseteq (f_1^k, \dots, f_n^k)A$. Hence with k equal to p^m, we get $c\mathbf{F}^m(z) \in \mathbf{F}^m(I)A$ for all $m \gg 0$. In conclusion, $z \in \mathrm{cl}(I)$. The last assertion then follows from Theorem 5.3.2. □

5.5.2 *The Hochster-Roberts theorem.*

We will formulate the next result without defining in detail all the concepts involved, except when we get to its algebraic formulation. A *linear algebraic group G* is an affine subscheme of the general linear group $\mathrm{GL}(K,n)$ over an algebraically closed field K such that its K-rational points form a subgroup of the latter group. When G acts (as a group) on a closed subscheme $X \subseteq \mathbb{A}^n_K$ (more precisely, for each algebraically closed field L containing K, there is an action of the L-rational points of $G(L)$ on $X(L)$), we can define the *quotient space X/G*, consisting of all orbits under the action of G on X, as the affine space $\mathrm{Spec}(R^G)$, where R^G denotes the subring of G-invariant sections in $R := \Gamma(X, \mathscr{O}_X)$ (the action of G on X induces an action on the sections of X, and hence in particular on R). For this to work properly, we also need to impose a certain finiteness condition: G has to be *linearly reductive*. Although not usually its defining property, we will here take this to mean that there exists an R^G-linear map $R \to R^G$ which is the identity on R^G, called the *Reynolds operator* of the action. For instance, if $K = \mathbb{C}$, then an algebraic group is linearly reductive if and only if it is the complexification of a real Lie group, where the Reynolds operator is obtained by an integration process. This is the easiest to understand if G is finite, when the integration is just a finite sum

$$\rho : R \to R^G : a \mapsto \frac{1}{|G|} \sum_{\sigma \in G} a^\sigma,$$

where a^σ denotes the result of $\sigma \in G$ acting on $a \in R$. In fact, as indicated by the above formula, a finite group is linearly reductive over a field of positive characteristic, provided its cardinality is not divisible by the characteristic. If X is nonsingular and G is linearly reductive, then we will call X/G a *quotient singularity*.[1] The celebrated Hochster-Roberts theorem now states:

Theorem 5.5.3. *Any quotient singularity is Cohen-Macaulay.*

To state a more general result, we need to take a closer look at the Reynolds operator. A ring homomorphism $A \to B$ is called *split*, if there exists an A-linear map $\sigma\colon B \to A$ which is the identity on A (note that σ need not be multiplicative, that is to say, is not a ring homomorphism, only a module homomorphism). We call σ the *splitting* of $A \to B$. Hence the Reynolds operator is a splitting of the inclusion $R^G \subseteq R$. The only property of split maps that will matter is the following:

5.5.4 *A split homomorphism $A \to B$ is cyclically pure.*

See the discussion at the beginning of §2.4.3 for the definition of cyclic purity. Let $a \in IB \cap A$ with $I = (f_1, \dots, f_s)A$ an ideal in A. Hence $a = f_1b_1 + \cdots + f_sb_s$ for some $b_i \in B$. Applying the splitting σ, we get by A-linearity $a = f_1\sigma(b_1) + \cdots + f_s\sigma(b_s) \in I$, proving that A is cyclically pure in B. □

We also need the following result on the preservation of cyclic purity under completions:

Lemma 5.5.5. *Let R and S be Noetherian local rings with respective completions $\widehat{R}$ and $\widehat{S}$. If $R \to S$ is cyclically pure, then so is its completion $\widehat{R} \to \widehat{S}$.*

Proof. The homomorphism $S \to \widehat{S}$ is faithfully flat, hence cyclically pure; thus the composition $R \to S \to \widehat{S}$ is cyclically pure. So from now on we may suppose that $S = \widehat{S}$. It suffices to show that $\widehat{R} \to S$ is injective, since the completion of $R/\mathfrak{a}$ is equal to $\widehat{R}/\mathfrak{a}\widehat{R}$, for any ideal $\mathfrak{a}$ in R. Let $a \in \widehat{R}$ be such that $a = 0$ in S, and for each i choose $a_i \in R$ such that $a \equiv a_i \bmod \mathfrak{m}^i\widehat{R}$, where $\mathfrak{m}$ is the maximal ideal of R. Then a_i lies in $\mathfrak{m}^iS$, hence by cyclical purity, in $\mathfrak{m}^i$. Therefore $a \in \mathfrak{m}^i\widehat{R}$ for all i, showing that $a = 0$ in $\widehat{R}$ by Krull's Intersection Theorem (Theorem 2.4.14). □

We can now state a far more general result, of which Theorem 5.5.3 is just a special case.

Theorem 5.5.6. *If $R \to S$ is a cyclically pure homomorphism and if S is regular, then R is Cohen-Macaulay.*

Proof. The problem is clearly local, and so we assume that $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are local. By Lemma 5.5.5, we may further reduce to the case that R and S are both complete. We split the proof in two parts: we first show that R is F-regular (see Definition 5.2.7), and then show that any complete local F-regular domain is Cohen-Macaulay.

[1] The reader should be aware that other authors might use the term more restrictively, only allowing X to be affine space $\mathbb{A}^n_K$, or G to be finite.

5.5.7 *A cyclically pure subring of a regular ring is F-regular.*

Indeed, since both cyclic purity and regularity are preserved under localization, we only need to show that every ideal in R is tightly closed. To this end, let $I \subseteq R$ and $x \in \mathrm{cl}(I)$. Hence x lies in the tight closure of IS by (weak) persistence (Theorem 5.3.1), and therefore in IS by Theorem 5.3.2. Hence by cyclic purity, $x \in I = IS \cap R$, proving that R is weakly F-regular. Note that we actually proved that a cyclically pure subring of a (weakly) F-regular ring is again (weakly) F-regular.

5.5.8 *A complete local F-regular domain is Cohen-Macaulay.*

Assume R is F-regular and let $(x_1,\dots,x_d)$ be a system of parameters in R. To show that x_{i+1} is $R/(x_1,\dots,x_i)R$-regular, assume $zx_{i+1} \in (x_1,\dots,x_i)R$. Colon Capturing (Theorem 5.3.3) yields that z lies in the tight closure of $(x_1,\dots,x_i)R$, whence in the ideal itself since R is F-regular. □

Remark 5.5.9. In fact, R is then also normal (this follows easily from 5.5.7 and Corollary 5.4.2). A far more difficult result is that R is then also *pseudo-rational* (a concept that lies beyond the scope of these notes; see for instance [59, 99] for a discussion of what follows). This was first proven by Boutot in [14] for $\mathbb{C}$-affine algebras by means of deep vanishing theorems. The positive characteristic case was proven by Smith in [108] by tight closure methods, where she also showed that pseudo-rationality is in fact equivalent with the weaker notion of F-rationality (a local ring is *F-rational* if some parameter ideal is tightly closed). I proved the general characteristic zero case in [99] by means of ultraproducts. In fact, being F-regular is equivalent under the $\mathbb{Q}$-Gorenstein assumption with having log-terminal singularities (see [38, 95]; for an example see Example 5.2.3). It should be noted that 'classical' tight closure theory in characteristic zero (see §5.6 below) is not sufficiently versatile to derive these results: so far, only our present ultraproduct method seems to work.

5.5.3 The Ein-Lazardsfeld-Smith theorem.

The next result, although elementary in its formulation, was only proven recently in [26] using quite complicated methods (which only work over $\mathbb{C}$), but then soon after in [55] by an elegant tight closure argument (see also [90]), which proves the result over any field K.

Theorem 5.5.10. *Let $V \subseteq K^2$ be a finite subset with ideal of definition $I := \mathfrak{I}(V)$. For each k, let $J_k(V)$ be the ideal of all polynomials f having multiplicity at least k at each point $x \in V$. Then $J_{2k}(V) \subseteq I^k$, for all k.*

To formulate the more general result of which this is just a corollary, we need to introduce symbolic powers. We first do this for a prime ideal $\mathfrak{p}$: its k-th *symbolic* *power* is the contracted ideal $\mathfrak{p}^{(k)} := \mathfrak{p}^k R_{\mathfrak{p}} \cap R$. In general, the inclusion $\mathfrak{p}^k \subseteq \mathfrak{p}^{(k)}$ may be strict, and, in fact, $\mathfrak{p}^{(k)}$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^k$. If $\mathfrak{a}$ is a radical ideal (we will not treat the more general case), then we define its k-th *symbolic power* $\mathfrak{a}^{(k)}$ as the intersection $\mathfrak{p}_1^{(k)} \cap \cdots \cap \mathfrak{p}_s^{(k)}$, where the $\mathfrak{p}_i$ are all the minimal overprimes of $\mathfrak{a}$. The connection with Theorem 5.5.10 is given by:

5.5.11 *The k-th symbolic power of the ideal of definition $I := \mathfrak{I}(V)$ of a finite subset $V \subseteq K^2$ is equal to the ideal $J_k(V)$ of all polynomials that have multiplicity at least k at any point of V.*

Indeed, for $\mathbf{x} \in V$, let $\mathfrak{m} := \mathfrak{m}_{\mathbf{x}}$ be the corresponding maximal ideal. By definition, a polynomial f has multiplicity at least k at each $\mathbf{x} \in V$, if $f \in \mathfrak{m}^k A_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ containing I. The latter condition simply means that $f \in \mathfrak{m}^{(k)}$, so that the claim follows from the definition of symbolic power. □

Hence, in view of this, Theorem 5.5.10 is an immediate consequence of the following theorem (at least in positive characteristic; for the characteristic zero case, see Theorems 6.2.6 and 7.2.4 below):

Theorem 5.5.12. *Let A be a regular domain of characteristic p. Let $\mathfrak{a} \subseteq A$ be a radical ideal and let h be the maximal height of its minimal overprimes. Then we have an inclusion $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$, for all n.*

Proof. We start with proving the following useful inclusion:

$$\mathfrak{a}^{(hp^e)} \subseteq \mathbf{F}^e(\mathfrak{a})A \tag{5.8}$$

for all e. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals of $\mathfrak{a}$. We first prove (5.8) locally at one of these minimal primes $\mathfrak{p}$. Since $A_{\mathfrak{p}}$ is regular and $\mathfrak{a}A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, we can find $f_i \in \mathfrak{a}$ such that $\mathfrak{a}A_{\mathfrak{p}} = (f_1, \dots, f_h)A_{\mathfrak{p}}$. By definition of symbolic powers, $\mathfrak{a}^{(hp^e)}A_{\mathfrak{p}} = \mathfrak{a}^{hp^e}A_{\mathfrak{p}}$. On the other hand, $\mathfrak{a}^{hp^e}A_{\mathfrak{p}}$ consists of monomials in the f_i of degree hp^e, and hence any such monomial lies in $\mathbf{F}^e(\mathfrak{a})A_{\mathfrak{p}}$. This establishes (5.8) locally at $\mathfrak{p}$. To prove this globally, take $z \in \mathfrak{a}^{(hp^e)}$. By what we just proved, there exists $s_i \notin \mathfrak{p}_i$ such that $s_i z \in \mathbf{F}^e(\mathfrak{a})A$ for each $i = 1, \dots, m$. For each i, choose an element t_i in all $\mathfrak{p}_j$ except $\mathfrak{p}_i$, and put $s := t_1 s_1 + \cdots + s_m t_m$. It follows that s multiplies z inside $\mathbf{F}^e(\mathfrak{a})A$, whence a fortiori, so does $\mathbf{F}^e(s)$. Hence

$$z \in (\mathbf{F}^e(\mathfrak{a})A : \mathbf{F}^e(s)) = \mathbf{F}^e(\mathfrak{a} : s)A$$

where we used Theorem 3.3.14 and the fact that $\mathbf{F}$ is flat on A by Theorem 5.1.2. However, s does not lie in any of the $\mathfrak{p}_i$, whence $(\mathfrak{a} : s) = \mathfrak{a}$, proving (5.8).

To prove the theorem, let $f \in \mathfrak{a}^{(hn)}$, and fix some e. We may write $p^e = an + r$ for some $a, r \in \mathbb{N}$ with $0 \le r < n$. Since the usual powers are contained in the symbolic powers, and since $r < n$, we have inclusions

$$\mathfrak{a}^{hn} f^a \subseteq \mathfrak{a}^{hr} f^a \subseteq \mathfrak{a}^{(han+hr)} = \mathfrak{a}^{(hp^e)} \subseteq \mathbf{F}^e(\mathfrak{a})A \tag{5.9}$$

where we used (5.8) for the last inclusion. Taking n-th powers in (5.9) shows that $\mathfrak{a}^{hn^2} f^{an}$ lies in the n-th power of $\mathbf{F}^e(\mathfrak{a})A$, and this in turn lies inside $\mathbf{F}^e(\mathfrak{a}^n)A$.

Choose some non-zero c in $\mathfrak{a}^{hn^2}$. Since $p^e \geq an$, we get $c\mathbf{F}^e(f) \in \mathbf{F}^e(\mathfrak{a}^n)A$ for all e. In conclusion, f lies in $\mathrm{cl}(\mathfrak{a}^n)$ whence in $\mathfrak{a}^n$ by Theorem 5.3.2. □

One might be tempted to try to prove a more general form which does not assume A to be regular, replacing $\mathfrak{a}^n$ by its tight closure. However, we used the regularity assumption not only via Theorem 5.3.2 but also via Kunz's Theorem that the Frobenius is flat. Hence the above proof does not work in arbitrary rings.

5.6 Classical tight closure in characteristic zero

To prove the previous three theorems in a ring of equal characteristic zero, Hochster and Huneke also developed tight closure theory for such rings. One of the precursors to tight closure theory was the proof of the Intersection Theorem by Peskine and Szpiro in [75]. They used properties of the Frobenius together with a method to transfer results from characteristic p to characteristic zero, which was then generalized by Hochster in [43]. This same technique is also used to obtain a tight closure theory in equal characteristic zero, as we will discuss briefly in this section. However, using ultraproducts, we will bypass in Chapters 6 and 7 this rather heavy-duty machinery, to arrive much quicker at proofs in equal characteristic zero.

Let A be a Noetherian ring containing the rationals. The idea is to associate to A some rings in positive characteristic, its *reductions modulo p*, and calculate tight closure in the latter. More precisely, let $\mathfrak{a} \subseteq A$ be an ideal, and $z \in A$. We say that z lies in the *HH-tight closure* of $\mathfrak{a}$ (where "HH" stands for Hochster-Huneke), if there exists a $\mathbb{Z}$-affine subalgebra $R \subseteq A$ containing z, such that (the image of) z lies in the tight closure of $I(R/pR)$ for all primes numbers p, where $I := \mathfrak{a} \cap R$.

It is not too hard to show that this yields a closure operation on A (in the sense of Definition 5.2.5). Much harder is showing that it satisfies all the necessary properties from §5.3. For instance, to prove the analogue of Theorem 5.3.2, one needs some results on generic flatness, and some deep theorems on Artin Approximation (see for instance [59, Appendix 1] or [54]; for a brief discussion of Artin Approximation, see §7.1 below). In contrast, using ultraproducts, one can avoid all these complications in the affine case (Chapter 6), or get by with a more elementary version of Artin Approximation in the general case (Chapter 7).